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► **To cite this version:**

Jean Honorio. Unanimous Approval is Seldom Optimal. [Technical Report] Purdue University. 2020. ⟨hal-02865806⟩

**HAL Id: hal-02865806**

**<https://hal.science/hal-02865806v1>**

Submitted on 12 Jun 2020

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# Unanimous Approval is Seldom Optimal

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## Abstract

In a voting process, a set of voters decide on a particular topic. In several real world scenarios, it has become customary to take the final decision to approve if and only if voters unanimously approve. This is the case of academic peer-reviewed journals and conferences, for instance. We formalize this problem and show that under some conditions, *unanimous approval* is rarely optimal.

## 1 Preliminaries

In what follows, we formalize our problem. We define a random variable  $Y \in \{0, 1\}$ , to represent the *true nature* of the topic. That is,  $Y = 1$  denotes that the topic must be approved, and  $Y = 0$  denotes that the topic must be rejected. We assume  $Y = 1$  with probability  $p \in [0, 1]$ , and  $Y = 0$  with probability  $1 - p$ . (We refer the reader to [1] for an introduction to the field of Probability.)

For concreteness and clarity, we assume three voters. We define random variables  $X_1, X_2, X_3$  to represent the decision of the voters. That is,  $X_i = 1$  denotes that voter  $i$  approves the topic, and  $X_i = 0$  denotes that voter  $i$  rejects the topic. Furthermore, we assume each voter *independently* agrees with the true nature of the topic with probability  $q$ , and disagrees otherwise. That is, we assume  $X_i = Y$  with probability  $q \in [0, 1]$ , and  $X_i = 1 - Y$  with probability  $1 - q$ , for all  $i \in \{1, 2, 3\}$ .

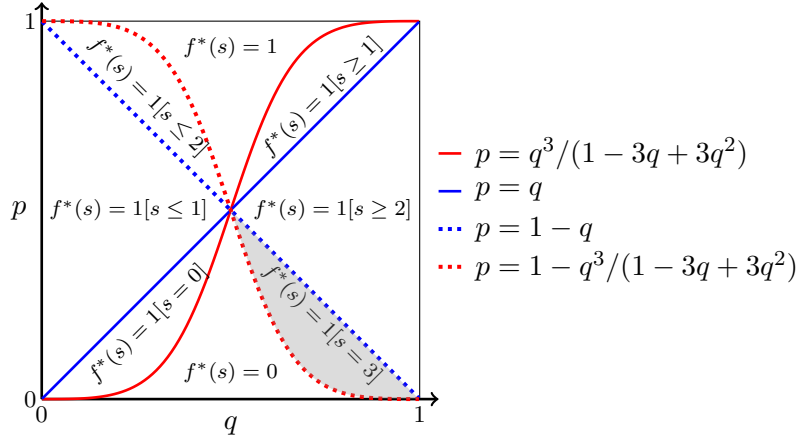
Note that while the votes could be described as a binary vector, i.e.,  $(X_1, X_2, X_3) \in \{0, 1\} \times \{0, 1\} \times \{0, 1\}$ , we can more efficiently describe the votes by their sum, i.e.,  $X_1 + X_2 + X_3 \in \{0, 1, 2, 3\}$ . Based on this observation, we provide the following definitions. A *policy* is a function  $f : \{0, 1, 2, 3\} \rightarrow \{0, 1\}$ . Given  $s \in \{0, 1, 2, 3\}$  and a policy  $f$ , we call  $f(s) \in \{0, 1\}$  a *final decision*. The *unanimous approval* policy is defined as  $f(s) = 1[s = 3]$ , where  $1[\text{true}] = 1$  and  $1[\text{false}] = 0$ . We call a policy  $f^*$  *optimal* if and only if  $f^*$  maximizes the probability of the final decision  $f(X_1 + X_2 + X_3)$  agreeing with the true nature  $Y$  of the topic, i.e.,

$$f^* = \arg \max_{f: \{0,1,2,3\} \rightarrow \{0,1\}} \mathbb{P}[f(X_1 + X_2 + X_3) = Y]. \quad (1)$$

## 2 Main Result

Next we state our main result.

**Theorem 1. (i)** Depending on the values of  $p$  and  $q$ , the optimal policies are:



**(ii)** The unanimous approval policy (highlighted in gray in the figure above) is optimal if and only if  $p \in (1 - q^3/(1 - 3q + 3q^2), 1 - q)$ .

**(iii)** Furthermore, if  $p$  and  $q$  are independent random variables, uniformly distributed on  $[0, 1]$ , the unanimous approval policy is optimal with probability  $(8 \log 2 - 3)/36 \approx 0.071$ .

*Proof.* In order to make eq.(1) have a unique minimizer, we assume that the pair  $(p, q)$  fulfills:

$$p \notin \{q^3/(1 - 3q + 3q^2), q, 1 - q, 1 - q^3/(1 - 3q + 3q^2)\}. \quad (2)$$

Note that the objective function of eq.(1) can be written as:

$$\begin{aligned} & \mathbb{P}[f(X_1 + X_2 + X_3) = Y] \\ &= \mathbb{P}[Y = 0] \mathbb{P}[f(X_1 + X_2 + X_3) = 0 \mid Y = 0] + \mathbb{P}[Y = 1] \mathbb{P}[f(X_1 + X_2 + X_3) = 1 \mid Y = 1] \\ &= (1 - p) \sum_{s=0}^3 1[f(s) = 0] \mathbb{P}[X_1 + X_2 + X_3 = s \mid Y = 0] + p \sum_{s=0}^3 1[f(s) = 1] \mathbb{P}[X_1 + X_2 + X_3 = s \mid Y = 1] \\ &= (1 - p) \sum_{s=0}^3 1[f(s) = 0] q^{3-s} (1 - q)^s + p \sum_{s=0}^3 1[f(s) = 1] q^s (1 - q)^{3-s} \\ &= (1 - p) \sum_{s=0}^3 (1 - f(s)) q^{3-s} (1 - q)^s + p \sum_{s=0}^3 f(s) q^s (1 - q)^{3-s} \\ &= \sum_{s=0}^3 f(s) h(p, q, s) + (1 - p) \sum_{s=0}^3 q^{3-s} (1 - q)^s, \end{aligned}$$

where  $h(p, q, s) = -(1 - p)q^{3-s}(1 - q)^s + pq^s(1 - q)^{3-s}$ . Note that we can minimize the function

above independently for each *final decision*  $f^*(s)$  of the *policy*  $f^*$ , i.e.,

$$\begin{aligned}
f^* &= \arg \max_{f:\{0,1,2,3\}\rightarrow\{0,1\}} \sum_{s=0}^3 f(s) h(p, q, s) + (1-p) \sum_{s=0}^3 q^{3-s}(1-q)^s \\
\Leftrightarrow f^*(s) &= \arg \max_{f(s)\in\{0,1\}} f(s) h(p, q, s) \quad \text{for all } s \in \{0, 1, 2, 3\} \\
\Leftrightarrow f^*(s) &= 1[h(p, q, s) > 0] \quad \text{for all } s \in \{0, 1, 2, 3\}.
\end{aligned}$$

By eq.(2), we have that  $h(p, q, s) \neq 0$ . Finally:

$$f^*(s) = \begin{cases} 1[p > q^3/(1-3q+3q^2)] & \text{if } s = 0, \\ 1[p > q] & \text{if } s = 1, \\ 1[p > 1-q] & \text{if } s = 2, \\ 1[p > 1-q^3/(1-3q+3q^2)] & \text{if } s = 3. \end{cases} \quad (3)$$

Now, consider the following policies:

Policy $f : \{0, 1, 2, 3\} \rightarrow \{0, 1\}$		$f(0)$	$f(1)$	$f(2)$	$f(3)$
Always reject	$f(s) = 0$	0	0	0	0
Approve if every voter rejects	$f(s) = 1[s = 0]$	1	0	0	0
Minority vote	$f(s) = 1[s \leq 1]$	1	1	0	0
Approve if at least one voter rejects	$f(s) = 1[s \leq 2]$	1	1	1	0
Always approve	$f(s) = 1$	1	1	1	1
Approve if at least one voter approves	$f(s) = 1[s \geq 1]$	0	1	1	1
Majority vote	$f(s) = 1[s \geq 2]$	0	0	1	1
<i>Unanimous approval</i>	$f(s) = 1[s = 3]$	0	0	0	1

Considering eq.(3) and the table above, different policies are optimal, depending on the values of  $p$  and  $q$ , which proves our claim (i). In particular, consider the unanimous approval policy  $f(s) = 1[s = 3]$  and eq.(3). The latter corresponds to  $1[p > 1 - q] = 0$  and  $1[p > 1 - q^3/(1 - 3q + 3q^2)] = 1$ . This is equivalent to  $p \in (1 - q^3/(1 - 3q + 3q^2), 1 - q)$ , which proves our claim (ii). Finally, note that the set  $(1 - q^3/(1 - 3q + 3q^2), 1 - q)$  is not empty if and only if  $q \geq 1/2$  and thus:

$$\int_{1/2}^1 ((1-q) - (1 - q^3/(1 - 3q + 3q^2))) dq = (8 \log 2 - 3)/36,$$

which proves our claim (iii). □

## References

- [1] Sheldon Ross. *Introduction to Probability Models*. Academic Press, 11th edition, 2014.