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# BOTTOM OF THE $L^2$ SPECTRUM OF THE LAPLACIAN ON LOCALLY SYMMETRIC SPACES

JEAN-PHILIPPE ANKER & HONG-WEI ZHANG

ABSTRACT. We estimate the bottom of the  $L^2$  spectrum of the Laplacian on locally symmetric spaces in terms of the critical exponents of appropriate Poincaré series. Our main result is the higher rank analog of a characterization due to Elstrodt, Patterson, Sullivan and Corlette in rank one. It improves upon previous results obtained by Leuzinger and Weber in higher rank.

## 1. INTRODUCTION

We adopt the standard notation and refer to [12] for more details. Let  $G$  be a semi-simple Lie group, connected, noncompact, with finite center, and  $K$  be a maximal compact subgroup of  $G$ . The homogenous space  $X = G/K$  is a Riemannian symmetric space of noncompact type. Let  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  be the Cartan decomposition of the Lie algebra of  $G$ . Fix a maximal abelian subspace  $\mathfrak{a}$  in  $\mathfrak{p}$ . Let  $\Gamma$  be a discrete torsion-free subgroup of  $G$  that acts freely and properly discontinuously on  $X$ . Then  $Y = \Gamma \backslash X$  is a locally symmetric space, whose Riemannian structure is inherited from  $X$ . We denote by  $n$  the joint dimension of  $X$  and  $Y$ , and by  $\ell$  the joint rank of  $X$  and  $Y$ , which is the dimension of  $\mathfrak{a}$ .

Let  $\Sigma \subset \mathfrak{a}$  be the root system of  $(\mathfrak{g}, \mathfrak{a})$  and let  $W$  be the associated Weyl group. Choose a positive Weyl chamber  $\mathfrak{a}^+ \subset \mathfrak{a}$  and let  $\Sigma^+ \subset \Sigma$  be the corresponding subsystem of positive roots. Denote by  $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$  the half sum of positive roots counted with their multiplicities. Occasionally we shall need the reduced root system  $\Sigma_{\text{red}} = \{\alpha \in \Sigma \mid \frac{\alpha}{2} \notin \Sigma\}$ .

Consider the classical Poincaré series

$$(1.1) \quad P_s(xK, yK) = \sum_{\gamma \in \Gamma} e^{-sd(xK, \gamma yK)} \quad \forall s > 0, \forall x, y \in G$$

and denote by  $\delta(\Gamma) = \inf\{s > 0 \mid P_s(xK, yK) < +\infty\}$  its critical exponent. Recall that  $\delta(\Gamma) \in [0, 2\|\rho\|]$  may be also defined by

$$\delta(\Gamma) = \limsup_{R \rightarrow +\infty} \frac{\log N_R(xK, yK)}{R} \quad \forall x, y \in G,$$

where  $N_R(xK, yK) = |\{\gamma \in \Gamma \mid d(xK, \gamma yK) \leq R\}|$  denotes the orbital counting function. Finally, let  $\Delta_Y$  be the Laplace-Beltrami operator on  $Y$  and let  $\lambda_0(Y)$  be the bottom of the  $L^2$  spectrum of  $-\Delta_Y$ .

The following celebrated result, due to Elstrodt [9, 10, 11], Patterson [16], Sullivan [18] and Corlette [7], expresses  $\lambda_0(Y)$  in terms of  $\rho$  and  $\delta(\Gamma)$  in rank one.

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**Theorem 1.1.** *In the rank one case, we have*

$$(1.2) \quad \lambda_0(Y) = \begin{cases} \|\rho\|^2 & \text{if } 0 \leq \delta(\Gamma) \leq \|\rho\|, \\ \|\rho\|^2 - (\delta(\Gamma) - \|\rho\|)^2 & \text{if } \|\rho\| \leq \delta(\Gamma) \leq 2\|\rho\|. \end{cases}$$

This result was extended in higher rank as follows by Leuzinger [15] and Weber [19]. Let  $\rho_{\min} = \min_{H \in \overline{\mathfrak{a}^+}, \|H\|=1} \langle \rho, H \rangle \in (0, \|\rho\|]$ .

**Theorem 1.2.** *In the general case, the following estimates hold:*

• *Upper bound:*

$$\lambda_0(Y) \leq \begin{cases} \|\rho\|^2 & \text{if } 0 \leq \delta(\Gamma) \leq \|\rho\|, \\ \|\rho\|^2 - (\delta(\Gamma) - \|\rho\|)^2 & \text{if } \|\rho\| \leq \delta(\Gamma) \leq 2\|\rho\|. \end{cases}$$

• *Lower bound:*

$$\lambda_0(Y) \geq \begin{cases} \|\rho\|^2 & \text{if } 0 \leq \delta(\Gamma) \leq \rho_{\min}, \\ \max\{0, \|\rho\|^2 - (\delta(\Gamma) - \rho_{\min})^2\} & \text{if } \rho_{\min} \leq \delta(\Gamma) \leq 2\|\rho\|. \end{cases}$$

*In other terms,*

$$\begin{cases} \lambda_0(Y) = \|\rho\|^2 & \text{if } \delta(\Gamma) \in [0, \rho_{\min}], \\ \lambda_0(Y) \in [\|\rho\|^2 - (\delta(\Gamma) - \rho_{\min})^2, \|\rho\|^2] & \text{if } \delta(\Gamma) \in [\rho_{\min}, \|\rho\|], \\ \lambda_0(Y) \in [\|\rho\|^2 - (\delta(\Gamma) - \rho_{\min})^2, \|\rho\|^2 - (\delta(\Gamma) - \|\rho\|)^2] & \text{if } \delta(\Gamma) \in [\|\rho\|, \|\rho\| + \rho_{\min}], \\ \lambda_0(Y) \in [0, \|\rho\|^2 - (\delta(\Gamma) - \|\rho\|)^2] & \text{if } \delta(\Gamma) \in [\|\rho\| + \rho_{\min}, 2\|\rho\|]. \end{cases}$$

Notice that  $\rho_{\min} = \|\rho\|$  in rank one and thus Theorem 1.2 reduces Theorem 1.1.

In this paper, we sharpen Theorem 1.2 by considering appropriate Poincaré series. We first improve in Theorem 2.9 the lower bound of  $\lambda_0(Y)$  by a slight modification of the classical Poincaré series (1.1). We obtain next in Theorem 3.1 the proper analog of Theorem 1.1 by considering a more involved family of Poincaré series. In the last section, we use these results to restate the existing heat kernel bounds on  $Y$  [19].

## 2. FIRST IMPROVEMENT

In this section, we replace the Riemannian distance  $d$  on  $X$  by a polyhedral distance  $d'$ , which reflects the volume growth at infinity and which was already used by the first author in [2, 3, 4]. Specifically, let

$$(2.1) \quad d'(xK, yK) = \langle \frac{\rho}{\|\rho\|}, (y^{-1}x)^+ \rangle \quad \forall x, y \in G,$$

where  $(y^{-1}x)^+$  denotes the  $\overline{\mathfrak{a}^+}$ -component of  $y^{-1}x$  in the Cartan decomposition  $G = K(\exp \overline{\mathfrak{a}^+})K$ .

**Proposition 2.1.**  *$d'$  is a  $G$ -invariant distance on  $X$ .*

*Proof.* The  $G$ -invariance of  $d'$  is straightforward from the definition (2.1). The symmetry

$$d'(xK, yK) = d'(yK, xK)$$

follows from

$$(2.2) \quad (y^{-1})^+ = -w_0 \cdot y^+ \quad \text{and} \quad -w_0 \cdot \rho = \rho,$$

where  $w_0$  denotes the longest element in the Weyl group. Let us check the triangular inequality

$$d'(xK, yK) \leq d'(xK, zK) + d'(zK, yK).$$

By  $G$ -invariance, we may reduce to the case where  $zK = eK$ . According to Lemma 2.2 below,

$$x^+ + (y^{-1})^+ - (y^{-1}x)^+$$

belongs to the cone generated by the positive roots.  $\square$

**Lemma 2.2.** *For every  $x, y \in G$ , we have the following inclusion*

$$(2.3) \quad \text{co}[W.(xy)^+] \subset \text{co}[W.(x^+ + y^+)]$$

between convex hulls.

*Proof.* The inclusion (2.3) amounts to the fact that

$$x^+ + y^+ - (xy)^+$$

belongs to the cone generated by the positive roots or, equivalently, to the inequality

$$(2.4) \quad \langle \lambda, (xy)^+ \rangle \leq \langle \lambda, x^+ \rangle + \langle \lambda, y^+ \rangle \quad \forall \lambda \in \overline{\mathfrak{a}^+}.$$

It is enough to prove (2.4) for all highest weights  $\lambda$  of irreducible finite-dimensional complex representations  $\pi : G \rightarrow \text{GL}(V)$  with  $K$ -fixed vectors. As usual, consider an inner product on  $V$  such that

$$\begin{cases} \pi(k) \text{ is unitary} & \forall k \in K, \\ \pi(a) \text{ is self-adjoint} & \forall a \in \exp \mathfrak{a}. \end{cases}$$

Then

$$e^{\langle \lambda, (xy)^+ \rangle} = \|\pi(xy)\| \leq \|\pi(x)\| \|\pi(y)\| = e^{\langle \lambda, x^+ \rangle} e^{\langle \lambda, y^+ \rangle} = e^{\langle \lambda, x^+ + y^+ \rangle}.$$

$\square$

**Remark 2.3.** The distance  $d'$  is comparable to the Riemannian distance  $d$ . Specifically,

$$(2.5) \quad \frac{\rho_{\min}}{\|\rho\|} d(xK, yK) \leq d'(xK, yK) \leq d(xK, yK) \quad \forall x, y \in G.$$

This follows indeed from

$$\frac{\rho_{\min}}{\|\rho\|} \|H\| \leq \langle \frac{\rho}{\|\rho\|}, H \rangle \leq \|H\| \quad \forall H \in \overline{\mathfrak{a}^+}.$$

The volume of balls

$$B'_r(xK) = \{yK \in X \mid d'(yK, xK) \leq r\}$$

was determined in [2, Lemma 6]. For the reader's convenience, we recall the statement and its proof.

**Lemma 2.4.** *For every  $x \in G$  and  $r > 0$ , we have <sup>(1)</sup>*

$$|B'_r(xK)| \asymp \begin{cases} r^{n-1} & \text{if } r \text{ is small,} \\ r^{\ell-1} e^{2\|\rho\|r} & \text{if } r \text{ is large.} \end{cases}$$

<sup>1</sup>The symbol  $f \asymp g$  between two non-negative expressions means that there exist constants  $0 < A \leq B < +\infty$  such that  $Ag \leq f \leq Bg$ .

**Remark 2.5.** Notice the different large scale behavior, in comparison with the classical ball volume

$$|B_r(xK)| \asymp \begin{cases} r^{n-1} & \text{if } r \text{ is small,} \\ r^{\frac{\ell-1}{2}} e^{2\|\rho\|r} & \text{if } r \text{ is large.} \end{cases}$$

(see for instance [17] or [13]).

*Proof.* By translation invariance, we may assume that  $x = e$ . Recall the integration formula

$$(2.6) \quad \int_X dx f(x) = \text{const.} \int_K dk \int_{\mathfrak{a}^+} dH \omega(H) f(k(\exp H)K),$$

in the Cartan decomposition, with density

$$\omega(H) = \prod_{\alpha \in \Sigma^+} (\sinh \langle \alpha, H \rangle)^{m_\alpha} \asymp \prod_{\alpha \in \Sigma^+} \left( \frac{\langle \alpha, H \rangle}{1 + \langle \alpha, H \rangle} \right)^{m_\alpha} e^{2\langle \rho, H \rangle}.$$

Thus

$$|B'_r(eK)| \asymp \int_{\{H \in \mathfrak{a}^+ \mid \|H\| \leq r\}} dH \prod_{\alpha \in \Sigma^+} \langle \alpha, H \rangle^{m_\alpha} \asymp r^n$$

if  $r$  is small. Let us turn to  $r$  large. On the one hand, we estimate from above

$$|B'_r(eK)| \lesssim \int_{\{H \in \mathfrak{a}^+ \mid \langle \rho, H \rangle \leq \|\rho\|r\}} dH e^{2\langle \rho, H \rangle} \asymp \int_0^{2\|\rho\|r} ds s^{\ell-1} e^s \asymp r^{\ell-1} e^{2\|\rho\|r}.$$

On the other hand, let  $H_0 \in \mathfrak{a}^+$ . As

$$\omega(H) \asymp e^{2\langle \rho, H \rangle} \quad \forall H \in H_0 + \overline{\mathfrak{a}^+},$$

we estimate from below

$$|B'_r(eK)| \gtrsim \int_{\{H \in H_0 + \mathfrak{a}^+ \mid \langle \rho, H \rangle \leq \|\rho\|r\}} dH e^{2\langle \rho, H \rangle} \gtrsim \int_{C_0}^{2\|\rho\|r} ds s^{\ell-1} e^s \asymp r^{\ell-1} e^{2\|\rho\|r},$$

where  $C_0 > 0$  is a constant depending on  $H_0$ .  $\square$

Consider now the modified Poincaré series

$$(2.7) \quad P'_s(xK, yK) = \sum_{\gamma \in \Gamma} e^{-sd'(xK, \gamma yK)} \quad \forall s > 0, \forall x, y \in G$$

associated with  $d'$ , its critical exponent

$$(2.8) \quad \delta'(\Gamma) = \inf\{s > 0 \mid P'_s(xK, yK) < +\infty\}$$

and the modified orbital counting function

$$(2.9) \quad N'_R(xK, yK) = |\{\gamma \in \Gamma \mid d'(xK, \gamma yK) \leq R\}| \quad \forall R \geq 0, \forall x, y \in G.$$

The following proposition shows that (2.7), (2.8) and (2.9) share the properties of their classical counterparts.

**Proposition 2.6.** *The following assertions hold:*

- (i)  $\delta'(\Gamma)$  does not depend on the choice of  $x$  and  $y$ .
- (ii)  $0 \leq \delta'(\Gamma) \leq 2\|\rho\|$ .
- (iii) For every  $x, y \in G$ ,

$$(2.10) \quad \delta'(\Gamma) = \limsup_{R \rightarrow +\infty} \frac{\log N'_R(xK, yK)}{R}.$$

**Remark 2.7.** It follows from (2.5) that

$$P_s(xK, yK) \leq P'_s(xK, yK) \leq P_{\frac{\rho_{\min}}{\|\rho\|}}(xK, yK)$$

and

$$N_R(xK, yK) \leq N'_R(xK, yK) \leq N_{\frac{\|\rho\|}{\rho_{\min}}R}(xK, yK).$$

Hence

$$0 \leq \delta(\Gamma) \leq \delta'(\Gamma) \leq \frac{\|\rho\|}{\rho_{\min}} \delta(\Gamma).$$

*Proof.* (i) follows from the triangular inequality. More precisely, let  $x_1, y_1, x_2, y_2 \in G$  and  $s > 0$ . Then

$$d'(x_2K, \gamma y_2K) \leq d'(x_2K, x_1K) + d'(x_1K, \gamma y_1K) + d'(\gamma y_1K, \gamma y_2K) \quad \forall \gamma \in \Gamma,$$

hence

$$\underbrace{\sum_{\gamma \in \Gamma} e^{-s d'(x_1K, \gamma y_1K)}}_{P'_s(x_1K, y_1K)} \leq e^{s\{d'(x_1K, x_2K) + d'(y_1K, y_2K)\}} \underbrace{\sum_{\gamma \in \Gamma} e^{-s d'(x_2K, \gamma y_2K)}}_{P'_s(x_2K, y_2K)}.$$

(ii) Let us show that  $P'_s(eK, eK) < +\infty$  for every  $s > 2\|\rho\|$ . According to Lemma 2.8 below, there exists  $r > 0$  such that the balls  $B'_r(\gamma K)$ , with  $\gamma \in \Gamma$ , are pairwise disjoint in  $G/K$ . Let us apply the integration formula (2.6) to the function

$$f(xK) = \sum_{\gamma \in \Gamma} e^{-s d'(xK, eK)} \mathbf{1}_{B'_r(\gamma K)}(xK).$$

On the one hand, as

$$|d'(xK, eK) - d'(\gamma K, eK)| \leq r \quad \forall xK \in B'_r(\gamma K),$$

we have

$$\int_X d(xK) f(xK) \asymp \sum_{\gamma \in \Gamma} e^{-s d'(\gamma K, eK)} \underbrace{|B'_r(\gamma K)|}_{|B'_r(eK)|} \asymp P'_s(eK, eK).$$

On the other hand,

$$\begin{aligned} \int_X d(xK) f(xK) &\leq \int_X d(xK) e^{-s d'(xK, eK)} \\ &\asymp \int_{\mathfrak{a}^+} dH \omega(H) e^{-s \langle \frac{\rho}{\|\rho\|}, H \rangle} \lesssim \int_{\mathfrak{a}^+} dH e^{-(\frac{s}{\|\rho\|} - 2) \langle \rho, H \rangle} \end{aligned}$$

is finite if  $s > 2\|\rho\|$ . Thus  $P'_s(eK, eK) < +\infty$  and consequently  $\delta'(\Gamma) \leq 2\|\rho\|$ .

(iii) Denote the right hand side of (2.10) by  $L(xK, yK)$  and let us first show that  $L(xK, yK)$  is finite. By applying Lemma 2.8 below to  $y^{-1}\Gamma y$ , we deduce that there exists  $r > 0$  such that the balls  $B'_r(\gamma yK)$ , with  $\gamma \in \Gamma$ , are pairwise disjoint. Set

$$\Gamma'_R(xK, yK) = \{\gamma \in \Gamma \mid d'(xK, \gamma yK) \leq R\} \quad \forall R \geq 0, \forall x, y \in G.$$

Then the ball  $B'_{R+r}(xK)$  contains the disjoint balls  $B'_r(\gamma yK)$ , with  $\gamma \in \Gamma'_R(xK, yK)$ . By computing volumes, we estimate

$$N'_R(xK, yK) = |\Gamma'_R(xK, yK)| \leq \frac{|B'_{R+r}(xK)|}{|B'_r(eK)|} \asymp (1+R)^{\ell-1} e^{2\|\rho\|R}.$$

Hence  $L(xK, yK) \leq 2\|\rho\|$ . Let us next show that  $L(xK, yK)$  is actually independent of  $x, y \in G$ .

Given  $x_1, y_1, x_2, y_2 \in G$  and  $R_1 > 0$ , let

$$R_2 = R_1 + d'(x_1K, x_2K) + d'(y_1K, y_2K).$$

Then the triangular inequality

$$d'(x_2K, \gamma y_2K) \leq d'(x_2K, x_1K) + d'(x_1K, \gamma y_1K) + d'(\gamma y_1K, \gamma y_2K)$$

implies successively

$$\begin{aligned} \Gamma'_{R_1}(x_1K, y_1K) &\subset \Gamma'_{R_2}(x_2K, y_2K), \\ N'_{R_1}(x_1K, y_1K) &\leq N'_{R_2}(x_2K, y_2K), \\ L(x_1K, y_1K) &\leq L(x_2K, y_2K). \end{aligned}$$

Let us finally prove the equality between  $\delta'(\Gamma)$  and  $L = L(eK, eK)$ . For this purpose, observe that

$$\begin{aligned} (2.11) \quad P'_s &= 1 + \sum_{R \in \mathbb{N}^*} \sum_{\gamma \in \Gamma'_R \setminus \Gamma'_{R-1}} e^{-s d'(eK, \gamma K)} \\ &\asymp 1 + \sum_{R \in \mathbb{N}^*} (N'_R - N'_{R-1}) e^{-sR} \\ &\asymp \sum_{R \in \mathbb{N}} N'_R e^{-sR}, \end{aligned}$$

where we have written for simplicity

$$P'_s = P'_s(eK, eK), \quad \Gamma'_R = \Gamma'_R(eK, eK) \quad \text{and} \quad N'_R = N'_R(eK, eK).$$

On the one hand, let  $s > L$  and set  $\varepsilon = \frac{s-L}{2}$ . By definition of  $L$ ,

$$N'_R \lesssim e^{(L+\varepsilon)R} \quad \forall R \geq 0.$$

Hence

$$P'_s \lesssim \sum_{R \in \mathbb{N}} e^{-\varepsilon R} < +\infty.$$

On the other hand, let  $s < L$ . By definition of  $L$ , there exists a sequence of integers  $1 < R_1 < R_2 < \dots \rightarrow +\infty$  such that

$$N'_{R_j} \geq e^{sR_j} \quad \forall j \in \mathbb{N}^*.$$

Hence the series (2.11) diverges.  $\square$

**Lemma 2.8.** *There exists  $r > 0$  such that the balls  $B'_r(\gamma K)$ , with  $\gamma \in \Gamma$ , are pairwise disjoint in  $G/K$ .*

*Proof.* Let  $r > 0$ . As  $\Gamma$  is discrete in  $G$ , its intersection with the compact subset

$$G'_r = \{y \in G \mid d'(yK, eK) \leq r\} = K(\exp\{H \in \overline{\mathfrak{a}^+} \mid \langle \rho, H \rangle \leq \|\rho\|r\})K$$

is finite. Moreover, as  $\Gamma$  is torsion-free,

$$\gamma^+ \neq 0 \quad \forall \gamma \in \Gamma \setminus \{e\}.$$

Hence there exists  $r > 0$  such that  $\Gamma \cap G'_{2r} = \{e\}$ , which implies that the sets  $\gamma G'_r$  are pairwise disjoint in  $G$ . In other words, the balls  $B'_r(\gamma K)$  are pairwise disjoint in  $G/K$ .  $\square$

By using  $\delta'(\Gamma)$ , we now improve the lower bound in Theorem 1.2.

**Theorem 2.9.** *The following lower bound holds for the bottom  $\lambda_0(Y)$  of the  $L^2$  spectrum of  $-\Delta$  on  $Y = \Gamma \backslash G/K$ :*

$$\lambda_0(Y) \geq \begin{cases} \|\rho\|^2 & \text{if } 0 \leq \delta'(\Gamma) \leq \|\rho\|, \\ \|\rho\|^2 - (\delta'(\Gamma) - \|\rho\|)^2 & \text{if } \|\rho\| \leq \delta'(\Gamma) \leq 2\|\rho\|. \end{cases}$$

Next statement is obtained by combining this lower bound with the upper bound in Theorem 1.2.

**Corollary 2.10.** *The following estimates hold for  $\lambda_0(Y)$ :*

$$\begin{cases} \lambda_0(Y) = \|\rho\|^2 & \text{if } \delta'(\Gamma) \leq \|\rho\|, \\ \|\rho\|^2 - (\delta'(\Gamma) - \|\rho\|)^2 \leq \lambda_0(Y) \leq \|\rho\|^2 & \text{if } \delta(\Gamma) \leq \|\rho\| \leq \delta'(\Gamma), \\ \|\rho\|^2 - (\delta'(\Gamma) - \|\rho\|)^2 \leq \lambda_0(Y) \leq \|\rho\|^2 - (\delta(\Gamma) - \|\rho\|)^2 & \text{if } \|\rho\| \leq \delta(\Gamma). \end{cases}$$

*Proof of Theorem 2.9 and Corollary 2.10.* Let us resume the approach in [7, Section 4] and [15, Section 3]. It consists in studying the convergence of the positive series

$$(2.12) \quad g_\zeta^\Gamma(\Gamma xK, \Gamma yK) = \sum_{\gamma \in \Gamma} g_\zeta(Ky^{-1}\gamma^{-1}xK),$$

which expresses the kernel  $g_\zeta^\Gamma$  of  $(-\Delta - \|\rho\|^2 + \zeta^2)^{-1}$  on the locally symmetric space  $Y = \Gamma \backslash G/K$  in terms of the corresponding Green function  $g_\zeta$  on the symmetric space  $X = G/K$ . Here  $\zeta > 0$  and  $\Gamma xK \neq \Gamma yK$ . Recall [5, Theorem 4.2.2] that

$$(2.13) \quad g_\zeta(\exp H) \asymp \left\{ \prod_{\alpha \in \Sigma_{\text{red}}^+} (1 + \langle \alpha, H \rangle) \right\} \|H\|^{-\frac{\ell-1}{2} - |\Sigma_{\text{red}}^+|} e^{-\langle \rho, H \rangle - \zeta \|H\|}$$

for  $H \in \overline{\mathfrak{a}^+}$  large, let say  $\|H\| \geq \frac{1}{2}$ , while

$$g_\zeta(\exp H) \asymp \begin{cases} \|H\|^{-(n-2)} & \text{if } n > 2 \\ \log \frac{1}{\|H\|} & \text{if } n = 2 \end{cases}$$

for  $H$  small, let say  $0 < \|H\| \leq \frac{1}{2}$ . Thus (2.12) converges if and only if

$$(2.14) \quad \sum_{\gamma \in \Gamma} \left\{ \prod_{\alpha \in \Sigma_{\text{red}}^+} (1 + \langle \alpha, (y^{-1}\gamma^{-1}x)^+ \rangle) \right\} \times \\ \times d(xK, \gamma yK)^{-\frac{\ell-1}{2} - |\Sigma_{\text{red}}^+|} e^{-\|\rho\|d'(xK, \gamma yK) - \zeta d(xK, \gamma yK)}$$

converges. Let us compare the series (2.14) with the Poincaré series (1.1) and (2.7). On the one hand, as  $\|(y^{-1}\gamma^{-1}x)^+\| = d(xK, \gamma yK)$ , (2.14) is bounded from above by  $P'_{\|\rho\|+\zeta}(xK, yK)$ . On the other hand, as

$$d(xK, \gamma yK)^{-\frac{\ell-1}{2} - |\Sigma_{\text{red}}^+|} \gtrsim e^{-\varepsilon d(xK, \gamma yK)}$$

for every  $\varepsilon > 0$ , (2.14) is bounded from below by  $P_{\|\rho\|+\zeta+\varepsilon}(xK, yK)$ . Hence (2.14) converges if  $\|\rho\| + \zeta > \delta'(\Gamma)$ , i.e.,  $\zeta > \delta'(\Gamma) - \|\rho\|$ , while (2.14) diverges if  $\zeta < \delta(\Gamma) - \|\rho\|$ . We conclude by using the fact [7, Section 4] that  $\lambda_0(Y)$  is the supremum of  $\|\rho\|^2 - \zeta^2$  over all  $\zeta > 0$  such that (2.12) converges.  $\square$

## 3. SECOND IMPROVEMENT

In this section, we obtain the proper higher rank analog of Theorem 1.1 by considering a further family of distances on  $X$ , which reflects the large scale behavior (2.13) of the Green function. Specifically, for every  $s > 0$  and  $x, y \in G$ , let

$$(3.1) \quad \begin{aligned} d_s(xK, yK) &= \min\{s, \|\rho\|\} d'(xK, yK) + \max\{s - \|\rho\|, 0\} d(xK, yK) \\ &= \begin{cases} s d'(xK, yK) & \text{if } 0 < s \leq \|\rho\|, \\ \|\rho\| d'(xK, yK) + (s - \|\rho\|) d(xK, yK) & \text{if } s \geq \|\rho\|. \end{cases} \end{aligned}$$

Then (3.1) defines a  $G$ -invariant distance on  $X$  such that

$$(3.2) \quad s d'(xK, yK) \leq d_s(xK, yK) \leq s d(xK, yK) \quad \forall s > 0, \forall x, y \in G.$$

Consider the associated Poincaré series

$$(3.3) \quad P_s''(xK, yK) = \sum_{\gamma \in \Gamma} e^{-d_s(xK, \gamma yK)} \quad \forall s > 0, \forall x, y \in G$$

and its critical exponent

$$\delta''(\Gamma) = \inf\{s > 0 \mid P_s''(xK, yK) < +\infty\}.$$

It follows from (3.2) that

$$(3.4) \quad 0 \leq \delta(\Gamma) \leq \delta''(\Gamma) \leq \delta'(\Gamma) \leq 2\|\rho\|.$$

**Theorem 3.1.** *The following characterization holds for the bottom  $\lambda_0(Y)$  of the  $L^2$  spectrum of  $-\Delta$  on  $Y = \Gamma \backslash G/K$ :*

$$(3.5) \quad \lambda_0(Y) = \begin{cases} \|\rho\|^2 & \text{if } 0 \leq \delta''(\Gamma) \leq \|\rho\|, \\ \|\rho\|^2 - (\delta''(\Gamma) - \|\rho\|)^2 & \text{if } \|\rho\| \leq \delta''(\Gamma) \leq 2\|\rho\|. \end{cases}$$

*Proof.* In the proof of Theorem 2.9 and Corollary 2.10, we compared the series (2.12), or equivalently (2.14), with the Poincaré series (1.1) and (2.7). If we consider instead the Poincaré series (3.3), we obtain in the same way that (2.14) is bounded from above by  $P_{\|\rho\|+\zeta}''(xK, yK)$  and from below by  $P_{\|\rho\|+\zeta+\varepsilon}''(xK, yK)$ , for every  $\varepsilon > 0$ . Hence (2.14) converges if  $\zeta > \delta''(\Gamma) - \|\rho\|$ , while (2.14) diverges if  $\zeta < \delta''(\Gamma) - \|\rho\|$ . We conclude as in the above-mentioned proof.  $\square$

**Remark 3.2.** If  $\Gamma$  is a lattice, i.e.,  $Y = \Gamma \backslash G/K$  has finite volume, then  $\lambda_0(Y) = 0$  and  $\delta''(\Gamma) = 2\|\rho\|$ , hence  $\delta'(\Gamma) = 2\|\rho\|$ . Furthermore  $\delta(\Gamma) = 2\|\rho\|$  [1, Theorem 7.4]. As pointed out by Corlette [7] in rank one and by Leuzinger [14] in higher rank, if  $G$  has Kazhdan's property (T), then the following conditions are actually equivalent:

- (a)  $\Gamma$  is a lattice, (b)  $\lambda_0(Y) = 0$ , (c)  $\delta(\Gamma) = 2\|\rho\|$ , (d)  $\delta''(\Gamma) = 2\|\rho\|$ .

## 4. FURTHER RESULTS ABOUT HEAT KERNEL BOUNDS

As for the Green function, the heat kernel

$$(4.1) \quad h_t^Y(\Gamma xK, \Gamma yK) = \sum_{\gamma \in \Gamma} h_t(Ky^{-1}\gamma^{-1}xK)$$

on a locally symmetric space  $Y = \Gamma \backslash G/K$  can be expressed and estimated by using the heat kernel  $h_t$  on the symmetric space  $X = G/K$ , whose behavior is well understood [6]. The following Gaussian bounds were obtained this way by Davies & Mandouvalos [8] in rank one and by Weber in higher rank [19].

**Theorem 4.1.** *The following estimates hold for all  $t > 0$  and all  $x, y \in G$ :*

(i) *Assume that  $\delta(\Gamma) < \rho_{\min}$  and let  $\delta(\Gamma) < s < \rho_{\min}$ . Then*

$$h_t^Y(\Gamma xK, \Gamma yK) \lesssim t^{-\frac{n}{2}} (1+t)^{\frac{n-D}{2}} e^{-\|\rho\|^2 t} e^{-\frac{d(\Gamma xK, \Gamma yK)^2}{4t}} P_s(xK, yK),$$

where  $D = \ell + 2|\Sigma_{\text{red}}^+|$  is the so-called dimension at infinity of  $X$ .

(ii) *Assume that  $\rho_{\min} \leq \delta(\Gamma) < \rho_{\min} + \|\rho\|$  and let  $\delta(\Gamma) - \rho_{\min} < s_1 < s_2 < \|\rho\|$ . Then*

$$h_t^Y(\Gamma xK, \Gamma yK) \lesssim t^{-\frac{n}{2}} e^{-(\|\rho\|^2 - s_2^2)t} P_{\rho_{\min} + s_1}(xK, yK).$$

(iii) *Assume that  $\delta(\Gamma) < \rho_{\min} + \|\rho\|$ . Let  $s > \delta(\Gamma)$  and  $\varepsilon > 0$ . Then*

$$h_t^Y(\Gamma xK, \Gamma yK) \lesssim t^{-\frac{n}{2}} e^{-(\lambda_0(Y) - 2\varepsilon)t} e^{-\frac{d(\Gamma xK, \Gamma yK)^2}{4(1+\varepsilon)t}} \times \\ \times P_s(xK, xK)^{\frac{1}{2}} P_s(yK, yK)^{\frac{1}{2}}.$$

By using Theorem 3.1 instead of Theorem 1.1 and Theorem 1.2, we obtain the following alternative statement in term of  $\delta''(\Gamma)$ .

**Theorem 4.2.** *The following estimates hold for all  $t > 0$  and all  $x, y \in G$ :*

(i) *Assume that  $\delta''(\Gamma) < \|\rho\|$  and let  $\delta''(\Gamma) < s < \|\rho\|$ . Then*

$$h_t^Y(\Gamma xK, \Gamma yK) \lesssim t^{-\frac{n}{2}} (1+t)^{\frac{n-D}{2}} e^{-\|\rho\|^2 t} e^{-\frac{d(\Gamma xK, \Gamma yK)^2}{4t}} P_s''(xK, yK).$$

(ii) *Assume that  $\|\rho\| \leq \delta''(\Gamma) < 2\|\rho\|$  and let  $\delta''(\Gamma) - \|\rho\| < s_1 < s_2 < \|\rho\|$ . Then*

$$h_t^Y(\Gamma xK, \Gamma yK) \lesssim t^{-\frac{n}{2}} e^{-(\|\rho\|^2 - s_2^2)t} P_{\|\rho\| + s_1}''(xK, yK).$$

(iii) *Assume that  $\delta''(\Gamma) < 2\|\rho\|$ . Let  $s > \delta''(\Gamma)$  and  $\varepsilon > 0$ . Then*

$$h_t^Y(\Gamma xK, \Gamma yK) \lesssim t^{-\frac{n}{2}} e^{-(\|\rho\|^2 - (\delta''(\Gamma) - \|\rho\|)^2 - 2\varepsilon)t} e^{-\frac{d(\Gamma xK, \Gamma yK)^2}{4(1+\varepsilon)t}} \times \\ \times P_s''(xK, xK)^{\frac{1}{2}} P_s''(yK, yK)^{\frac{1}{2}}.$$

We omit the *proof*, which is straightforwardly adapted from [8] and [19].

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INSTITUT DENIS POISSON (UMR 7013), UNIVERSITÉ D'ORLÉANS, UNIVERSITÉ DE TOURS  
& CNRS, BÂTIMENT DE MATHÉMATIQUES - RUE DE CHARTRES, B.P. 6759 - 45067 ORLÉANS  
CEDEX 2 - FRANCE

*Email address:* anker@univ-orleans.fr, hong-wei.zhang@univ-orleans.fr