Symbolic Models for a Class of Impulsive Systems

Abdalla Swikir, Antoine Girard, and Majid Zamani

***Abstract*—Symbolic models have been used as the basis of a systematic framework to address control design of several classes of hybrid systems with sophisticated control objectives. However, results available in the literature are not concerned with impulsive systems which are an important modeling framework of many applications. In this paper, we provide an approach for constructing symbolic models for a class of impulsive systems possessing some stability properties. We formally relate impulsive systems and their symbolic models using a notion of so-called alternating simulation function. We show that behaviors of the constructed symbolic models are approximately equivalent to those of the impulsive systems. Finally, we illustrate the effectiveness of our results through a case study.**

***Index Terms*—Hybrid systems; Quantized systems; Supervisory control**

I. Introduction

SYMBOLIC models have been the aim of intensive study in the last two decades since they provide a mechanism for reducing complexity in the analysis and control of cyber-physical systems [1], [2]. They serve as abstract mathematical models where each symbolic state and input represent a collection of continuous states and inputs in the original concrete model. As they have finite number of states and inputs, they enable the use of correct-by-construction methods from the computer science community to design controllers for a wide variety of systems. For instance, they allow one to use automata-theoretic methods [3] to design controllers for hybrid systems with respect to logic specifications such as those expressed as linear temporal logic (LTL) formulae [4]. In such frameworks, controllers designed for symbolic models can be refined to ones for concrete systems based on some behavioral relation between original systems and their symbolic models such as approximate alternating simulation relations [5] or feedback refinement relations [6].

The synthesis of symbolic models for different classes of systems has been investigated, among many others, in the following papers: for incrementally stable and incrementally forward complete nonlinear control systems in [7] and [8], respectively; for nonlinear switched systems in [9], [10], [11]; for nonlinear control systems with known constant time delays and time-varying delays in [12], [13]; for networked control systems in [14], [15], and finally for incrementally stable infinite-dimensional systems with finite-dimensional input spaces in [16], [17]. All the aforementioned approaches essentially take a monolithic view of the systems while constructing symbolic models. On the other hand, different compositional methods for constructing symbolic models have been recently introduced in the literature with or without imposing stability assumptions over the network; see [18], [19], [20], [21] and references therein. Although the literature on symbolic models is very rich, unfortunately, there are no results so far on constructing symbolic models for impulsive systems.

Impulsive systems are an important class of hybrid systems that contain discontinuities or jumps (also referred to as impulses) in the state and input trajectories of the system governed by discrete dynamics [22], [23]. They serve as an important modeling framework for a very large variety of applications, e.g. power electronics, sample-data systems, bursting rhythm models in medicine, and some models in economics; see [24], [25] and the references therein. Hence, constructing symbolic models for impulsive systems enlarges the class of systems for which designing correct-by-construction controllers enforcing complex logic specifications is possible. In this work, we consider time-dependent impulsive systems in which the distance between the impulses is assumed to belong to a finite set. Such a class of systems is well studied in the literature; see [26] and references therein. For example, this class of systems models the dynamics of the estimation error in networked control systems [27, Section 8.2.] while assuming time instants of the reception of measurements are nondetermined but lie in a finite set.

This paper provides for the first time an approach for synthesizing symbolic models for a class of impulsive systems. The symbolic models constructed in this work are complete as their behaviors are approximately equivalent to those of the concrete systems [1]. First, we introduce a class of transition systems which allows us to model impulsive systems and their symbolic models in a common framework. Then we recall a notion of so-called alternating simulation function to relate two transition systems. Such a function allows one to determine quantitatively the mismatch between the observed behavior of two systems, and implies the existence of an approximate alternating simulation relation between them [5]. Second, we provide a methodology for constructing symbolic models together with their alternating simulation functions for impulsive systems possessing some incremental stability properties. In particular, we require that either the continuous or the discrete dynamic of the impulsive system to be incrementally input-to-state stable [28] while the other one is forward complete [8]. Given such an incremental property, we show that the constructed symbolic model is indeed a complete one, i.e., the

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behaviors of the symbolic models are (approximately) equivalent to those of the concrete systems [1]. This completeness roughly implies that there exists a controller enforcing the desired specifications on the symbolic model *if and only if* there exists a controller enforcing the same specifications on the concrete system. Finally, we apply our results to a model of storage-delivery process by constructing its symbolic model under different stability properties. We also design a controller maintaining the number of items in the storage in a desired range.

II. Notation and Preliminaries

A. Notation

We denote by $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$ the set of real numbers, integers, and non-negative integers, respectively. These symbols are annotated with subscripts to restrict them in the obvious way, e.g., $\mathbb{R}_{>0}$ denotes the positive real numbers. We denote the closed, open, and half-open intervals in $\mathbb{R}$ by $[a,b]$, (a,b), $[a,b)$, and $(a,b]$, respectively. For $a,b \in \mathbb{N}$ and $a \leq b$, we use $[a;b]$, $(a;b)$, $[a;b)$, and $(a;b]$ to denote the corresponding intervals in $\mathbb{N}$. Given any $a \in \mathbb{R}$, $|a|$ denotes the absolute value of a. Given any $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$, the infinity norm of u is defined by $\|u\| = \max_{1\leq i\leq n} |u_i|$. Given a function $\nu : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, the supremum of ν is denoted by $\|\nu\|_\infty$; we recall that $\|\nu\|_\infty := \sup_{t\in\mathbb{R}_{\geq 0}}\|\nu(t)\|$. Given $\mathbf{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n, \forall t,s \in \mathbb{R}_{\geq 0}$ with $t \geq s$, we define $\mathbf{x}(^-t) = \lim_{s\to t}\mathbf{x}(s)$. We denote by $\mathrm{card}(\cdot)$ the cardinality of a given set and by $\emptyset$ the empty set. Given sets X and Y, we denote by $f : X \to Y$ an ordinary map of X into Y, whereas $f : X \rightrightarrows Y$ denotes a set-valued map [29]. For any set $S \subseteq \mathbb{R}^n$ of the form $S = \bigcup_{j=1}^{M} S_j$ for some $M \in \mathbb{N}$, where $S_j = \prod_{i=1}^{n}[c_i^j, d_i^j] \subseteq \mathbb{R}^n$ with $c_i^j < d_i^j$, and nonnegative constant $\eta \leq \tilde{\eta}$, where $\tilde{\eta} = \min_{j=1,\ldots,M} \eta_{S_j}$ and $\eta_{S_j} = \min\{|d_1^j - c_1^j|, \ldots, |d_n^j - c_n^j|\}$, we define $[S]_\eta = \{a \in S \mid a_i = k_i\eta, k_i \in \mathbb{Z}, i = 1, \ldots, n\}$ if $\eta \neq 0$, and $[S]_\eta = S$ if $\eta = 0$. The set $[S]_\eta$ will be used as a finite approximation of the set S with precision $\eta \neq 0$. Note that $[S]_\eta \neq \emptyset$ for any $\eta \leq \tilde{\eta}$. We use notations $\mathcal{K}$ and $\mathcal{K}_\infty$ to denote different classes of comparison functions, as follows: $\mathcal{K} = \{\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} | \ \alpha$ is continuous, strictly increasing, and $\alpha(0) = 0\}$; $\mathcal{K}_\infty = \{\alpha \in \mathcal{K} | \lim\limits_{s\to\infty} \alpha(s) = \infty\}$. For $\alpha, \gamma \in \mathcal{K}_\infty$ we write $\alpha \leq \gamma$ if $\alpha(r) \leq \gamma(r)$, and, by abuse of notation, $\alpha = c$ if $\alpha(r) = cr \ \forall r \in \mathbb{R}_{\geq 0}$. Finally, we denote by $\mathcal{I}_d$ the identity function over $\mathbb{R}_{\geq 0}$, i.e. $\mathcal{I}_d(r) = r, \forall r \in \mathbb{R}_{\geq 0}$.

B. Nonlinear Impulsive Systems

Among several classes of impulsive systems studied in the literature, e.g., [22], [23], [27], in this work, we study a class of time-dependent nonlinear impulsive systems as defined next.

Definition 1: A nonlinear impulsive system Σ is defined by the tuple $\Sigma = (\mathbb{R}^n, \mathbb{U}, \mathcal{U}, f, g)$, where $\mathbb{R}^n$ is the state space, $\mathbb{U} \subseteq \mathbb{R}^m$ is the input set, $\mathcal{U}$ is the set of all measurable bounded input functions $\nu : \mathbb{R}_{\geq 0} \to \mathbb{U}$, and $f, g : \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^n$ are locally Lipschitz functions;

The nonlinear impulsive system Σ is described by differential and difference equations of the form

$$\Sigma : \begin{cases} \dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \nu(t)), & t \in \mathbb{R}_{\geq 0}\backslash\Omega, \\ \mathbf{x}(t) = g(\mathbf{x}(^-t), \nu(t)), & t \in \Omega, \end{cases} \tag{II.1}$$

where $\Omega = \{t_k\}_{k\in\mathbb{N}}$ with $t_{k+1} - t_k \in \{p_1\tau, \ldots, p_2\tau\}$ for fixed jump parameters $\tau \in \mathbb{R}_{>0}$ and $p_1, p_2 \in \mathbb{N}_{\geq 1}$, $p_1 \leq p_2$; and, $\mathbf{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is the state signal, which is assumed to be right-continuous for all $t \in \mathbb{R}_{\geq 0}$, and $\nu \in \mathcal{U}$ is the input signal. We will use $\mathbf{x}_{x,\nu}(t)$ to denote a point reached at time $t \in \mathbb{R}_{\geq 0}$ from initial state $x = \mathbf{x}(0)$ under input signal $\nu \in \mathcal{U}$. The Lipschitz condition imposed on f ensures the existence and uniqueness of a solution of system Σ in (II.1); see [30], [31] for more details. We denote by Σ_c and Σ_d the continuous and discrete dynamics of system Σ, i.e., $\Sigma_c : \dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \nu(t))$, and $\Sigma_d : \mathbf{x}(t) = g(\mathbf{x}(^-t), \nu(t))$.

III. Transition Systems and Alternating Simulation Functions

We start by introducing the class of transition systems [1] which allows us to model impulsive and symbolic systems in a common framework.

Definition 2: A transition system is a tuple $T = (X, X_0, U, \mathcal{F}, Y, \mathcal{H})$ consisting of:

- a set of states X;
- a set of initial states $X_0 \subseteq X$;
- a set of inputs U;
- transition function $\mathcal{F} : X \times U \rightrightarrows X$;
- an output set Y;
- an output map $\mathcal{H} : X \to Y$.

The transition $x^+ \in \mathcal{F}(x,u)$ means that the system can evolve from state x to state x^+ under the input u. Thus, the transition function defines the dynamics of the transition system. Sets X, U, and Y are assumed to be subsets of normed vector spaces with appropriate finite dimensions. If for all $x \in X, u \in U$, $\mathrm{card}(\mathcal{F}(x,u)) \leq 1$ we say that T is deterministic, and non-deterministic otherwise. Additionally, T is called finite if X, U are finite sets and infinite otherwise. Furthermore, if for all $x \in X$ there exists $u \in U$ such that $\mathrm{card}(\mathcal{F}(x,u)) \neq 0$ we say that T is non-blocking. In this work, we only deal with non-blocking transition systems.

Next we introduce a notion of so-called alternating simulation functions, inspired by [32, Definition 1], which quantitatively relates two transition systems.

Definition 3: Let $T = (X, X_0, U, \mathcal{F}, Y, \mathcal{H})$ and $\hat{T} = (\hat{X}, \hat{X}_0, \hat{U}, \hat{\mathcal{F}}, \hat{Y}, \hat{\mathcal{H}})$ be transition systems with $\hat{Y} \subseteq Y$. A function $\tilde{\mathcal{S}} : X \times \hat{X} \to \mathbb{R}_{\geq 0}$ is called an alternating simulation function from $\hat{T}$ to T if there exist $\tilde{\alpha} \in \mathcal{K}_\infty$, $0 < \tilde{\sigma} < 1$, $\tilde{\rho}_u \in \mathcal{K}_\infty \cup \{0\}$, and some $\tilde{\varepsilon} \in \mathbb{R}_{\geq 0}$ so that the following hold:

- For every $x \in X, \hat{x} \in \hat{X}$, one has

$$\tilde{\alpha}(\|\mathcal{H}(x) - \hat{\mathcal{H}}(\hat{x})\|) \leq \tilde{\mathcal{S}}(x,\hat{x}). \tag{III.1}$$

- For every $x \in X, \hat{x} \in \hat{X}, \hat{u} \in \hat{U}$ there exists $u \in U$ such that for every $x^+ \in \mathcal{F}(x,u)$ there exists $\hat{x}^+ \in \hat{\mathcal{F}}(\hat{x},\hat{u})$ so that

$$\tilde{\mathcal{S}}(x^+,\hat{x}^+) \leq \tilde{\sigma}\tilde{\mathcal{S}}(x,\hat{x}) + \tilde{\rho}_u(\|\hat{u}\|_\infty) + \tilde{\varepsilon}. \tag{III.2}$$

The next lemma is adapted from [33, Theorem 1] and stated without a proof. This lemma is needed in the proof of Proposition 1.

Lemma 1: Let $\tilde{\mathcal{S}}$ be an alternating simulation function from $\hat{T}$ to T as in Definition 3. Then for every $x \in X, \hat{x} \in \hat{X}, \hat{u} \in \hat{U}$, there exists $u \in U$ such that for every $x^+ \in \mathcal{F}(x,u)$ there exists $\hat{x}^+ \in \hat{\mathcal{F}}(\hat{x},\hat{u})$ so that

$$\tilde{\mathcal{S}}(x^+,\hat{x}^+) \le \max\{\sigma\tilde{\mathcal{S}}(x,\hat{x}), \rho(\|\hat{u}\|_\infty), \varepsilon\}, \tag{III.3}$$

where $\sigma = 1-(1-\psi)(1-\tilde{\sigma})$, $\rho = \frac{2}{(1-\tilde{\sigma})\psi}\tilde{\rho}_u$, and $\varepsilon = \frac{2\tilde{\varepsilon}}{(1-\tilde{\sigma})\psi}$, for an arbitrarily chosen positive constant $\psi < 1$, and $\tilde{\sigma}, \tilde{\varepsilon}, \tilde{\rho}_u$ are constants and function appearing in Definition 3.

Before showing the next result, let us recall the definition of an alternating simulation relation introduced in [5].

Definition 4: Let $T = (X, X_0, U, \mathcal{F}, Y, \mathcal{H})$ and $\hat{T} = (\hat{X}, \hat{X}_0, \hat{U}, \hat{\mathcal{F}}, \hat{Y}, \hat{\mathcal{H}})$ be transition systems with $\hat{Y} \subseteq Y$. A relation $R \subseteq X \times \hat{X}$ is called an $\hat{\varepsilon}$-approximate alternating simulation relation from $\hat{T}$ to T if for any $(x,\hat{x}) \in R$

- (i) $\|\mathcal{H}(x) - \hat{\mathcal{H}}(\hat{x})\| \le \hat{\varepsilon}$;
- (ii) For any $\hat{u} \in \hat{U}$, there exists $u \in U$ such that for all $x^+ \in \mathcal{F}(x,u)$ there exists $\hat{x}^+ \in \hat{\mathcal{F}}(\hat{x},\hat{u})$ satisfying $(x^+,\hat{x}^+) \in R$.

In addition, if (ii) still holds when reversing the role of T and $\hat{T}$, the relation R is in fact an $\hat{\varepsilon}$-approximate alternating bisimulation relation between T and $\hat{T}$ [5] (see Remark 4).

The next result shows that the existence of an alternating simulation function for transition systems implies the existence of an approximate alternating simulation relation between them as as defined above

Proposition 1: Let $T = (X, X_0, U, \mathcal{F}, Y, \mathcal{H})$ and $\hat{T} = (\hat{X}, \hat{X}_0, \hat{U}, \hat{\mathcal{F}}, \hat{Y}, \hat{\mathcal{H}})$ be transition systems with $\hat{Y} \subseteq Y$. Assume $\tilde{\mathcal{S}}$ is an alternating simulation function from $\hat{T}$ to T as in Definition 3 and that there exists $r \in \mathbb{R}_{>0}$ such that $\|\hat{u}\|_\infty \le r$ for all $\hat{u} \in \hat{U}$. Then, relation $R \subseteq X \times \hat{X}$ defined by $R = \left\{(x,\hat{x}) \in X \times \hat{X} | \tilde{\mathcal{S}}(x,\hat{x}) \le \max\{\rho(r), \varepsilon\}\right\}$, where ρ, ε as in Lemma 1, is an $\hat{\varepsilon}$-approximate alternating simulation relation from $\hat{T}$ to T with $\hat{\varepsilon} = \tilde{\alpha}^{-1}(\max\{\rho(r), \varepsilon\})$.

Proof: Item (i) in Definition 4 is a simple consequence of the definition of R and condition (III.1) (i.e. $\tilde{\alpha}(\|\mathcal{H}(x) - \hat{\mathcal{H}}(\hat{x})\|) \le \tilde{\mathcal{S}}(x,\hat{x}) \le \max\{\rho(r), \varepsilon\}$), which results in $\|\mathcal{H}(x) - \hat{\mathcal{H}}(\hat{x})\| \le \tilde{\alpha}^{-1}(\max\{\rho(r), \varepsilon\}) = \hat{\varepsilon}$. Item (ii) in Definition 4 follows immediately from the definition of R, condition (III.3) in Lemma 1, and the fact that $0 < \sigma < 1$. In particular, we have $\tilde{\mathcal{S}}(x^+,\hat{x}^+) \le \max\{\rho(r), \varepsilon\}$ which implies $(x^+,\hat{x}^+) \in R$. ■

The approximate alternating simulation relation guarantees that for each output behavior of T there exists one of $\hat{T}$ such that the distance between these output behaviors is uniformly bounded by $\hat{\varepsilon}$.

Remark 1: Since the input set in all practical applications is bounded, requiring the control inputs to be bounded is not restrictive at all. Moreover, under certain properties of impulsive systems (see Section IV), one can choose function $\tilde{\rho}_u$ the definition of R to be identically zero which cancels the dependency to the size of control inputs in Proposition 1. ⋄

IV. Construction of Symbolic Models

This section contains the main contribution of this work and its results rely on additional assumption on $\mathcal{U}$ that we now describe. Consider impulsive system $\Sigma = (\mathbb{R}^n, \mathbb{U}, \mathcal{U}, f, g)$ with jump parameters τ, p_1 and p_2. We restrict attention to sampled-data impulsive systems, where input curves belong to $\mathcal{U}_\tau$ containing only curves, constant in duration τ, i.e.

$$\mathcal{U}_\tau = \{\nu : \mathbb{R}_{\ge 0} \to \mathbb{U} | \nu(t) = \nu((k-1)\tau), \quad t \in [(k-1)\tau, k\tau), k \in \mathbb{N}_{\ge 1}\}. \tag{IV.1}$$

Next we define sampled-data impulsive systems as a transition system. Such a transition system would be the bridge that relates impulsive systems to their symbolic models.

Definition 5: Given an impulsive system $\Sigma = (\mathbb{R}^n, \mathbb{U}, \mathcal{U}_\tau, f, g)$, with jump parameters (τ, p_1, p_2), we define the associated transition system $T_\tau(\Sigma) = (X, X_0, U, \mathcal{F}, Y, \mathcal{H})$ where:

- $X = \mathbb{R}^n \times \{0, \ldots, p_2\}$;
- $X_0 = \mathbb{R}^n \times \{0\}$;
- $U = \mathcal{U}_\tau$;
- $(x^+, l^+) \in \mathcal{F}((x,l), \nu)$ if and only if one of the following scenarios hold:
  - Flow scenario: $0 \le l \le p_2 - 1$, $x^+ = \mathbf{x}_{x,\nu}(^-\tau)$, and $l^+ = l+1$;
  - Jump scenario: $p_1 \le l \le p_2$, $x^+ = g(x, \nu(0))$, and $l^+ = 0$;
- $Y = \mathbb{R}^n$;
- $\mathcal{H} : X \to Y$, defined as $\mathcal{H}(x,l) = x$.

In order to construct a symbolic model for $T_\tau(\Sigma)$, we introduce the following assumptions and lemma.

Assumption 1: Consider impulsive system $\Sigma = (\mathbb{R}^n, \mathbb{U}, \mathcal{U}_\tau, f, g)$ with jump parameters τ, p_1 and p_2. Assume that there exist a locally Lipschitz function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\ge 0}$, $\mathcal{K}_\infty$ functions $\underline{\alpha}, \overline{\alpha}, \rho_{u_c}, \rho_{u_d}$, and constants $\kappa_c \in \mathbb{R}$, $\kappa_d \in \mathbb{R}_{>0}$, such that the following hold

- $\forall x, \hat{x} \in \mathbb{R}^n$
$$\underline{\alpha}(\|x - \hat{x}\|) \le V(x,\hat{x}) \le \overline{\alpha}(\|x - \hat{x}\|). \tag{IV.2}$$
- $\forall x, \hat{x} \in \mathbb{R}^n$ a.e, and $\forall u, \hat{u} \in \mathbb{U}$
$$\frac{\partial V(x,\hat{x})}{\partial x} f(x,u) + \frac{\partial V(x,\hat{x})}{\partial \hat{x}} f(\hat{x},\hat{u}) \le -\kappa_c V(x,\hat{x}) + \rho_{u_c}(\|u - \hat{u}\|). \tag{IV.3}$$
- $\forall x, \hat{x} \in \mathbb{R}^n$ and $\forall u, \hat{u} \in \mathbb{U}$
$$V(g(x,u), g(\hat{x},\hat{u})) \le \kappa_d V(x,\hat{x}) + \rho_{u_d}(\|u - \hat{u}\|). \tag{IV.4}$$

Assumption 2: There exists a $\mathcal{K}_\infty$ function $\hat{\gamma}$ such that

$$\forall x,y,z \in \mathbb{R}^n, \quad V(x,y) \le V(x,z) + \hat{\gamma}(\|y - z\|). \tag{IV.5}$$

Remark 2: Assumption 1 has different implications based on the values of κ_c and κ_d as the following. Given (IV.2) holds: (i) the existence of function V satisfying (IV.3) and (IV.4) with $k_c \le 0$ and $k_d \ge 1$ results in incremental forward completeness of the continuous and discrete dynamics of Σ, respectively, and we say Σ_c and Σ_d are δ-FC [8]; (ii) the existence of function V satisfying (IV.3) and (IV.4) with $k_c > 0$ and $k_d < 1$ results in incremental input-to-state stability of the continuous and discrete dynamics of Σ, respectively, and we say Σ_c and Σ_d are δ-ISS [28], [34]. In addition, Assumptions 2 is non-restrictive conditions provided that one is interested to work on a compact subset of $\mathbb{R}^n$ [35]. ⋄

Remark 3: In condition (IV.3), "$\forall x, \hat{x} \in \mathbb{R}^n$ a.e." should be interpreted as "for every $x, \hat{x} \in \mathbb{R}^n$ except on a set of zero Lebesgue-measure in $\mathbb{R}^n$". From Rademacher's theorem [36], the local Lipschitz assumption on function V ensures that

$\dfrac{\partial V(x,\hat{x})}{\partial x}f(x,u)+\dfrac{\partial V(x,\hat{x})}{\partial \hat{x}}f(\hat{x},\hat{u})$ is well defined, except on a set of measure zero. $\diamond$

The following lemma provides a bound to the evolution of function V in Assumption 1 which is needed in the proof of Theorem 1.

Lemma 2: Consider impulsive system $\Sigma = (\mathbb{R}^n, \mathbb{U}, \mathcal{U}_\tau, f, g)$ with jump parameters τ, p_1 and p_2, where $\mathcal{U}_\tau$ is given by (IV.1). Let (IV.3) in Assumption 1 holds. Then for all $x, \hat{x} \in \mathbb{R}^n$, for all $\nu, \hat{\nu} \in \mathcal{U}_\tau$, and for any two consecutive impulses (t_k, t_{k+1}), one has

$$\begin{aligned} &V(\mathbf{x}_{x,\nu}(^{-}t_{k+1}), \mathbf{x}_{\hat{x},\hat{\nu}}(^{-}t_{k+1})) \\ &\quad\le e^{-\kappa_c(t_{k+1}-t_k)}V(\mathbf{x}_{x,\nu}(t_k), \mathbf{x}_{\hat{x},\hat{\nu}}(t_k)) \\ &\qquad + \frac{1-e^{-\kappa_c(t_{k+1}-t_k)}}{\kappa_c}\rho_{u_c}(\|\nu-\hat{\nu}\|_\infty). \end{aligned} \tag{IV.6}$$

The proof follows by unitizing Lemma 1 in [27] and following similar arguments to the ones in [37, Section 2]. The proof is omitted here due to lack of space, and it can be found in [38].

We now have all the ingredients to construct a symbolic model $\hat{T}_\tau(\Sigma)$ of transition system $T_\tau(\Sigma)$ associated to the impulsive system Σ admitting a function V that satisfies Assumption 1 as follows.

Definition 6: Consider a transition system $T_\tau(\Sigma) = (X, X_0, U, \mathcal{F}, Y, \mathcal{H})$, associated to the impulsive system $\Sigma = (\mathbb{R}^n, \mathbb{U}, \mathcal{U}_\tau, f, g)$. Assume Σ admits a function V that satisfies Assumption 1. Then one can construct symbolic model $\hat{T}_\tau(\Sigma) = (\hat{X}, \hat{X}_0, \hat{U}, \hat{\mathcal{F}}, \hat{Y}, \hat{\mathcal{H}})$ where:

- $\hat{X} = \hat{\mathbb{R}}^n \times \{0, \cdots, p_2\}$, where $\hat{\mathbb{R}}^n = [\mathbb{R}^n]_\eta$ and η is the state space quantization parameter;
- $\hat{X}_0 = \hat{\mathbb{R}}^n \times \{0\}$;
- $\hat{U} = [\mathbb{U}]_\mu$, where μ is the input set quantization parameter;
- $(\hat{x}^+, l^+) \in \hat{\mathcal{F}}((\hat{x}, l), \hat{u})$ if and only if one of the following scenarios hold:
  - Flow scenario: $0 \le l \le p_2-1$, $\|\hat{x}^+ - \mathbf{x}_{\hat{x},\hat{u}}(^{-}\tau)\| \le \eta$, $l^+ = l+1$;
  - Jump scenario: $p_1 \le l \le p_2$, $\|\hat{x}^+ - g(\hat{x},\hat{u})\| \le \eta$, $l^+ = 0$;
- $\hat{Y} = Y$;
- $\hat{\mathcal{H}} : \hat{X} \to \hat{Y}$, defined as $\hat{\mathcal{H}}(\hat{x}, l) = \hat{x}$.

In the definition of the transition function, and in the remainder of the paper, we abuse notation by identifying $\hat{u}$ with the constant input curve with domain $[0, \tau)$ and value $\hat{u}$. Now, we establish the relation between $T_\tau(\Sigma)$ and $\hat{T}_\tau(\Sigma)$, introduced above, via the notion of alternating simulation function as in Definition 3.

Theorem 1: Consider an impulsive system $\Sigma = (\mathbb{R}^n, \mathbb{U}, \mathcal{U}_\tau, f, g)$ with its associated transition system $T_\tau(\Sigma) = (X, X_0, U, \mathcal{F}, Y, \mathcal{H})$. Let Assumptions 1, and 2 hold. Consider symbolic model $\hat{T}_\tau(\Sigma) = (\hat{X}, \hat{X}_0, \hat{U}, \hat{\mathcal{F}}, \hat{Y}, \hat{\mathcal{H}})$ constructed as in Definition 6. If inequality

$$\ln(\kappa_d) - \kappa_c \tau l < 0 \tag{IV.7}$$

holds for $l \in \{p_1, p_2\}$, then function $\mathcal{V}$ defined as

$$\mathcal{V}((x,l),(\hat{x},l)) := \begin{cases} V(x,\hat{x}) & \text{if } \kappa_d < 1 \,\&\, \kappa_c > 0, \\ V(x,\hat{x})e^{\kappa_c \tau \epsilon l} & \text{if } \kappa_d \ge 1 \,\&\, \kappa_c > 0, \\ V(x,\hat{x})\kappa_d^{\frac{l}{\delta}} & \text{if } \kappa_d < 1 \,\&\, \kappa_c \le 0, \end{cases} \tag{IV.8}$$

for some $0 < \epsilon < 1$ and $\delta > p_2$, is an alternating simulation function from $\hat{T}_\tau(\Sigma)$ to $T_\tau(\Sigma)$.

After proving Theorem 1, we will provide additional insight into condition (IV.7). Note that for the case in which $\kappa_d \ge 1$ and $\kappa_c \le 0$, this condition cannot hold at all. Hence this case is excluded from the definition of $\mathcal{V}$ in (IV.8).

Proof: By using (IV.2), $\forall (x,l) \in X$ and $\forall (\hat{x}, l) \in \hat{X}$, we have

$$\|\mathcal{H}(x,l) - \hat{\mathcal{H}}(\hat{x},l)\| = \|x-\hat{x}\| \le \underline{\alpha}^{-1}(V(x,\hat{x})) \le \hat{\alpha}\left(\mathcal{V}((x,l),(\hat{x},l))\right),$$

where

$$\hat{\alpha}(s) = \begin{cases} \underline{\alpha}^{-1}(s) & \text{if } \kappa_d < 1 \,\&\, \kappa_c > 0, \\ \underline{\alpha}^{-1}(e^{-\kappa_c \tau \epsilon p_1} s) & \text{if } \kappa_d \ge 1 \,\&\, \kappa_c > 0, \\ \underline{\alpha}^{-1}(\kappa_d^{-\frac{p_2}{\delta}} s) & \text{if } \kappa_d < 1 \,\&\, \kappa_c \le 0, \end{cases}$$

for all $s \in \mathbb{R}_{\ge 0}$. Hence, (III.1) holds with $\tilde{\alpha} = \hat{\alpha}^{-1}$.

Now we show that inequality (III.2) holds as well. Consider any $\hat{u} \in \hat{U}$ and choose $\nu(\cdot) = \hat{u}$. Then, using (IV.5), for all $x, \hat{x} \in \mathbb{R}^n$, for all $\hat{u} \in \hat{U}$, we have in the flow scenario the following inequality:

$$V(\mathbf{x}_{x,\hat{u}}(^{-}\tau), \hat{x}^+) \le V(\mathbf{x}_{x,\hat{u}}(^{-}\tau), \mathbf{x}_{\hat{x},\hat{u}}(^{-}\tau)) + \hat{\gamma}(\|\hat{x}^+ - \mathbf{x}_{\hat{x},\hat{u}}(^{-}\tau)\|).$$

Now, from Definition 6, the above inequality reduces to

$$V(\mathbf{x}_{x,\hat{u}}(^{-}\tau), \hat{x}^+) \le V(\mathbf{x}_{x,\hat{u}}(^{-}\tau), \mathbf{x}_{\hat{x},\hat{u}}(^{-}\tau)) + \hat{\gamma}(\eta),$$

for any $\hat{x}^+$ such that $(\hat{x}^+, l^+) \in \hat{\mathcal{F}}((\hat{x}, l), \hat{u})$. From (IV.6) with $t_{k+1} = \tau, t_k = 0$, one gets

$$\begin{aligned} V(\mathbf{x}_{x,\hat{u}}(^{-}\tau), \mathbf{x}_{\hat{x},\hat{u}}(^{-}\tau)) &\le e^{-\kappa_c \tau} V(\mathbf{x}_{x,\hat{u}}(0), \mathbf{x}_{\hat{x},\hat{u}}(0)) \\ &= e^{-\kappa_c \tau} V(x, \hat{x}) \end{aligned}$$

Hence, for all $x, \hat{x} \in \mathbb{R}^n$, for all $\hat{u} \in \hat{U}$, one obtains

$$V(\mathbf{x}_{x,\hat{u}}(^{-}\tau), \hat{x}^+) \le e^{-\kappa_c \tau} V(x, \hat{x}) + \hat{\gamma}(\eta), \tag{IV.9}$$

for any $\hat{x}^+$ such that $(\hat{x}^+, l^+) \in \hat{\mathcal{F}}((\hat{x}, l), \hat{u})$. By following similar argument to the previous one and using (IV.4), one also obtains the following inequality in the jump scenario for all $x, \hat{x} \in \mathbb{R}^n$, and for all $\hat{u} \in \hat{U}$

$$V(g(x,\hat{u}), \hat{x}^+) \le \kappa_d V(x, \hat{x}) + \hat{\gamma}(\eta), \tag{IV.10}$$

for any $\hat{x}^+$ such that $(\hat{x}^+, l^+) \in \hat{\mathcal{F}}((\hat{x}, l), \hat{u})$.

Now, in order to show function $\mathcal{V}$ defined in (IV.8) satisfies (III.2), we consider the different scenarios in Definition 6 and different cases for values of κ_d and κ_c as follows:

- $\kappa_d < 1$ & $\kappa_c > 0$ (**case 1**):
  - Flow scenario ($l^+ = l+1$):
  $$\begin{aligned} &\mathcal{V}((x^+, l^+), (\hat{x}^+, l^+)) = V(x^+, \hat{x}^+) \\ &\quad \le e^{-\kappa_c \tau} V(x, \hat{x}) + \hat{\gamma}(\eta) = e^{-\kappa_c \tau} \mathcal{V}((x,l),(\hat{x},l)) + \hat{\gamma}(\eta). \end{aligned}$$
  - Jump scenario ($l^+ = 0$):
  $$\begin{aligned} &\mathcal{V}((x^+, l^+), (\hat{x}^+, l^+)) = V(x^+, \hat{x}^+) \\ &\quad \le \kappa_d V(x, \hat{x}) + \hat{\gamma}(\eta) = \kappa_d \mathcal{V}((x,l),(\hat{x},l)) + \hat{\gamma}(\eta). \end{aligned}$$

  Let $\lambda_f = \max\{e^{-\kappa_c \tau}, \kappa_d\}$, and $\gamma_f = \hat{\gamma}$, then
  $$\mathcal{V}((x^+, l^+), (\hat{x}^+, l^+)) \le \lambda_f \mathcal{V}((x,l),(\hat{x},l)) + \gamma_f(\eta).$$
- $\kappa_d \ge 1$ & $\kappa_c > 0$ (**case 2**):
  - Flow scenario ($l^+ = l+1$):
  $$\begin{aligned} &\mathcal{V}((x^+, l^+), (\hat{x}^+, l^+)) = V(x^+, \hat{x}^+) e^{\kappa_c \tau \epsilon l^+} \\ &= V(x^+, \hat{x}^+) e^{\kappa_c \tau \epsilon (l+1)} \\ &\le (e^{-\kappa_c \tau} V(x, \hat{x}) + \hat{\gamma}(\eta)) e^{\kappa_c \tau \epsilon (l+1)} \\ &= e^{-\kappa_c \tau} e^{\kappa_c \tau \epsilon} e^{\kappa_c \tau \epsilon l} V(x, \hat{x}) + \frac{\hat{\gamma}(\eta)}{e^{-\kappa_c \tau \epsilon (l+1)}} \\ &= e^{-\kappa_c \tau (1-\epsilon)} \mathcal{V}((x,l),(\hat{x},l)) + \frac{\hat{\gamma}(\eta)}{e^{-\kappa_c \tau \epsilon (l+1)}}. \end{aligned}$$

– Jump scenario ($l^+ = 0$):

$$\begin{aligned}\mathcal{V}((x^+,l^+),(\hat{x}^+,l^+)) &= V(x^+,\hat{x}^+)e^{\kappa_c\tau\epsilon l^+} = V(x^+,\hat{x}^+)\\ &\le \kappa_d V(x,\hat{x})+\hat{\gamma}(\eta) = \frac{e^{\kappa_c\tau\epsilon l}}{e^{\kappa_c\tau\epsilon l}}\kappa_d V(x,\hat{x})+\hat{\gamma}(\eta)\\ &= e^{-\kappa_c\tau\epsilon l}\kappa_d\mathcal{V}((x,l),(\hat{x},l))+\hat{\gamma}(\eta).\end{aligned}$$

Let $\lambda_f = \max\{e^{-\kappa_c\tau(1-\epsilon)}, e^{-\kappa_c\tau\epsilon p_1}\kappa_d\}$, and $\gamma_f = e^{\kappa_c\tau\epsilon(p_2+1)}\hat{\gamma}$, then

$$\mathcal{V}((x^+,l^+),(\hat{x}^+,l^+)) \le \lambda_f\mathcal{V}((x,l),(\hat{x},l)) + \gamma_f(\eta).$$

- $\kappa_d < 1$ & $\kappa_c \le 0$ (**case 3**):
  – Flow scenario ($l^+ = l+1$):

$$\begin{aligned}&\mathcal{V}((x^+,l^+),(\hat{x}^+,l^+)) = V(x^+,\hat{x}^+)\kappa_d^{\frac{l^+}{\delta}}\\ &= V(x^+,\hat{x}^+)\kappa_d^{\frac{(l+1)}{\delta}} \le (e^{-\kappa_c\tau}V(x,\hat{x})+\hat{\gamma}(\eta))\kappa_d^{\frac{(l+1)}{\delta}}\\ &= e^{-\kappa_c\tau}\kappa_d^{\frac{l}{\delta}}\kappa_d^{\frac{1}{\delta}}V(x,\hat{x})+\hat{\gamma}(\eta)\kappa_d^{\frac{(l+1)}{\delta}}\\ &= e^{-\kappa_c\tau}\kappa_d^{\frac{1}{\delta}}\mathcal{V}((x,l),(\hat{x},l))+\hat{\gamma}(\eta)\kappa_d^{\frac{(l+1)}{\delta}}.\end{aligned}$$

  – Jump scenario ($l^+ = 0$):

$$\begin{aligned}&\mathcal{V}((x^+,l^+),(\hat{x}^+,l^+)) = V(x^+,\hat{x}^+)\kappa_d^{\frac{l^+}{\delta}}\\ &= V(x^+,\hat{x}^+) \le \kappa_d V(x,\hat{x})+\hat{\gamma}(\eta)\\ &= \frac{\kappa_d^{\frac{l}{\delta}}}{\kappa_d^{\frac{l}{\delta}}}\kappa_d V(x,\hat{x})+\hat{\gamma}(\eta) = \kappa_d^{\frac{\delta-l}{\delta}}\mathcal{V}((x,l),(\hat{x},l))+\hat{\gamma}(\eta).\end{aligned}$$

Let $\lambda_f = \max\{e^{-\kappa_c\tau}\kappa_d^{\frac{1}{\delta}}, \kappa_d^{\frac{\delta-p_2}{\delta}}\}$, and $\gamma_f = \hat{\gamma}$, then

$$\mathcal{V}((x^+,l^+),(\hat{x}^+,l^+)) \le \lambda_f\mathcal{V}((x,l),(\hat{x},l)) + \gamma_f(\eta).$$

To continue with the proof, we need to show that $\lambda_f < 1$ for **case 2** and **case 3** (**case 1** is trivial). In **case 2**, note that $e^{-\kappa_c\tau(1-\epsilon)} < 1$ since $0 < \epsilon < 1$ and $\kappa_c > 0$. Additionally, $e^{-\kappa_c\tau\epsilon p_1}\kappa_d < 1 \Leftrightarrow \ln(\kappa_d) - \kappa_c\tau\epsilon p_1 < 0$. By continuity of the real number, we can always find some $0 < \epsilon < 1$ such that $\ln(\kappa_d) - \kappa_c\tau l < 0$, $l \in \{p_1, p_2\}$, implies $\ln(\kappa_d) - \kappa_c\tau\epsilon p_1 < 0$. Hence, $\lambda_f < 1$. Similarly, in **case 3**, we have $\kappa_d^{\frac{\delta-p_2}{\delta}} < 1$ since $\delta > p_2$ and $\kappa_d < 1$. Moreover, $e^{-\kappa_c\tau}\kappa_d^{\frac{1}{\delta}} < 1 \Leftrightarrow \ln(\kappa_d) - \kappa_c\tau\delta < 0$. By continuity of the real number, we can always find some $\delta > p_2$ such that $\ln(\kappa_d) - \kappa_c\tau l < 0$, $l \in \{p_1, p_2\}$, implies $\ln(\kappa_d) - \kappa_c\tau\delta < 0$. Therefore, $\lambda_f < 1$.

Hence, for all $((x,l),(\hat{x},l)) \in X \times \hat{X}$, for all $\hat{u} \in \hat{U}$, for any $\hat{x}^+$ such that $(\hat{x}^+, l^+) \in \hat{\mathcal{F}}((\hat{x},l),\hat{u})$, $\mathcal{V}$ satisfies inequality (III.2) with $\nu = \hat{u}$, $\tilde{\sigma} = \lambda_f$, $\tilde{\varepsilon} = \gamma_f(\eta)$, and $\tilde{\rho}_u = 0$. Thus, $\mathcal{V}$ is an alternating simulation function from $\hat{T}_\tau(\Sigma)$ to $T_\tau(\Sigma)$. ■

Remark 4: One can also verify that function $\mathcal{V}$ given by (IV.8) is also an alternating simulation function from $T_\tau(\Sigma)$ to $\hat{T}_\tau(\Sigma)$. In particular, $\mathcal{V}$ satisfies (III.1) and (III.2) with choosing $\hat{u}$ satisfying[1] $\|\hat{u} - \nu\| \le \mu$, same $\tilde{\sigma}$, $\tilde{\rho}_u$ defined in Theorem 1, $\tilde{\varepsilon} = \gamma_f(\eta) + \max\left\{\frac{1-e^{-\kappa_c(t_{k+1}-t_k)}}{\kappa_c}\rho_{u_c}, \rho_{u_d}\right\}(\mu)$ for **case 1** and **3**, and $\tilde{\varepsilon} = \gamma_f(\eta) + \max\left\{e^{\kappa_c\tau\epsilon(p_2+1)}\frac{1-e^{-\kappa_c(t_{k+1}-t_k)}}{\kappa_c}\rho_{u_c}, \rho_{u_d}\right\}(\mu)$ for **case 2**. Observe that the existence of a function $\mathcal{V}$ serving as an alternating simulation function in both directions, i.e. from $T_\tau(\Sigma)$ to $\hat{T}_\tau(\Sigma)$ and from $\hat{T}_\tau(\Sigma)$ to $T_\tau(\Sigma)$, implies the existence of an approximate alternating bisimulation relation between $T_\tau(\Sigma)$ and $\hat{T}_\tau(\Sigma)$ as introduced in [5]. Consequently, $\hat{T}_\tau(\Sigma)$ is a complete symbolic model for $T_\tau(\Sigma)$. ⋄

Remark 5: The symbolic model $\hat{T}_\tau(\Sigma)$ has a countably infinite set of states. However, in practical applications, the physical variables are restricted to a compact set. Hence, we are usually interested in the dynamics of the impulsive system only on a compact subset $\mathbb{X} \subseteq \mathbb{R}^n$. Then, we can restrict the set of states of $\hat{T}_\tau(\Sigma)$ to the sets $([\mathbb{R}^n]_\eta \cap \mathbb{X}) \times \{0, \cdots, p_2\}$ which is finite. We refer the interested readers to the explanation provided after Remark 4.1 in [8] for more details. ⋄

Finally, we would like to provide a discussion on condition (IV.7) in Theorem 1. In the case when $\kappa_d < 1$ and $\kappa_c > 0$, the continuous and discrete dynamics of Σ are δ-ISS, and, clearly, (IV.7) always holds. For the case when $\kappa_c > 0$ and $\kappa_d \ge 1$, the continuous dynamic Σ_c is δ-ISS while the discrete dynamic Σ_d is δ-FC. In order for condition (IV.7) to hold in this case, κ_c should be large enough to accommodate the undesirable effect of κ_d and that the impulses do not happen too frequently. Finally, $\kappa_c \le 0$ and $\kappa_d < 1$ corresponds to the case that the continuous dynamic Σ_c is δ-FC while the discrete one Σ_d is δ-ISS. Here, we require the impulses to happen very often and κ_d to be small enough to accommodate the undesirable effect of κ_c. Note that condition (IV.7) ensures that an increase in the value of function V in Assumption 1 during flows is compensated by a decrease at jumps and vice versa. A similar argument was used in [27, Sections 4,5,6] to reason about input-to-state stability of impulsive systems, and we expect that by utilizing Assumption 1 with condition (IV.7), one can get δ-ISS for system Σ in (II.1).

V. Case study: A storage-delivery process model

In this case study, we apply our approach to a variant of the storage-delivery process model from [39]. Let the number $x \in \mathbb{R}_{\ge 0}$ of goods in a storage be continuously evolving proportionally to the number of items with rate coefficient a. At every time instant $t \in \Omega = \{t_k\}_{k\in\mathbb{N}}$, with $t_{k+1} - t_k \in \{p_1\tau, \ldots, p_2\tau\}$ for a fixed jump parameters $\tau \in \mathbb{R}_{>0}$ and $p_1, p_2 \in \mathbb{N}_{\ge 1}$, $p_1 \le p_2$, a truck comes to the storage and delivers $(b-1)\%$, or picks up $(1-b)\%$ of the current items. Let c denote the number of items per time unit that can be added, through lineside delivery from the factory to the storage, or taken out, from the storage to other locations during $t \in (t_{k+1}, t_k)$. Similarly, let d be the number of items that can be added, or taken out, from the storage at time instants $t \in \Omega$. The delivery and picking-up process is controlled by the input $\nu(t) = \nu(0) \in \{-1, 0, 1\}, t \in [0, \tau)$. The evolution of this process can be modeled as

$$\Sigma : \begin{cases} \dot{\mathbf{x}}(t) = a\mathbf{x} + c\nu(t), & t \in \mathbb{R}_{\ge 0}\backslash\Omega, \\ \mathbf{x}(t) = b\mathbf{x}(^-t) + d\nu(t), & t \in \Omega. \end{cases} \tag{V.1}$$

In order to construct a symbolic model for impulsive system Σ, we start by checking Assumptions 1 and 2. It can be shown that conditions (IV.2), (IV.3), and (IV.4) hold with $V(x,x') = \|x - x'\|$ with $\underline{\alpha} = \overline{\alpha} = \mathcal{I}_d$, $\rho_{u_c} = |c|$, $\rho_{u_d} = |d|$, $\kappa_c = -a$, and $\kappa_d = |b|$. Moreover, condition (IV.5) holds with $\hat{\gamma} = \mathcal{I}_d$. Given that (IV.7) holds for $l \in \{1, p\}$, and, with a proper choice[2] of ϵ and δ, function $\mathcal{V}(x,\hat{x})$ given by (IV.8) is an alternating simulation function from $\hat{T}_\tau(\Sigma)$, constructed as in Definition

[1]By the structure of $\hat{U}$, there always exists $\hat{u}$ satisfying $\|\hat{u} - \nu\| \le \mu$.

[2]$\epsilon = 1 - \varsigma$, and $\delta = p_2 + \varsigma$ with ς sufficiently small.

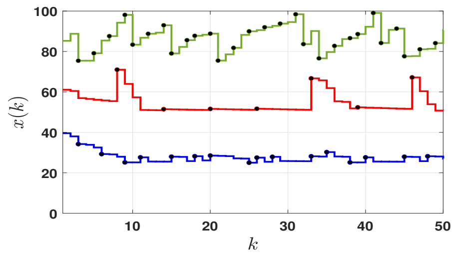

Fig. 1: Trajectories of system Σ for different values of p_1, p_2, a, b,c, d, Ψ: blue (bottom) ($p_1 = 1$, $p_2 = 5$, $a = -0.2$, $b = 0.9$, $c = d = 5$, $\Psi = \{25, 50\}$), red (middle) ($p_1 = 5$, $p_2 = 7$, $a = -0.3$, $b = 1.01$, $c = d = 15$, $\Psi = \{50, 75\}$), green (top) ($p_1 = 1$, $p_2 = 2$, $a = 0.2$, $b = 0.85$, $c{=}d{=}15$, $\Psi{=}\{75, 100\}$). The jumps are indicated by $\bullet$.

6, to $T_\tau(\Sigma)$. In particular, $\mathcal{V}$ satisfies conditions (III.1) and (III.2) with functions $\tilde{\alpha}, \tilde{\rho}_u$ and constants $\tilde{\sigma}$, ε given below based on the value of a and b, with $\psi = 0.99$.

- $|b| < 1$ & $a < 0$: $\tilde{\alpha} = \mathcal{I}_d$, $\tilde{\rho}_u = 0$, $\tilde{\sigma} = \max\{e^{a\tau}, |b|\}$, $\tilde{\varepsilon} = \eta$.
- $|b| \geq 1$ & $a < 0$: $\tilde{\alpha} = e^{-a\tau\epsilon p_1}$, $\tilde{\rho}_u = 0$, $\tilde{\sigma} = \max\{e^{a\tau(1-\epsilon)}, e^{a\tau\epsilon p_1}|b|\}$, and $\tilde{\varepsilon} = e^{-a\tau\epsilon(p_2+1)}\eta$.
- $|b| < 1$ & $a \geq 0$: $\tilde{\alpha} = |b|^{\frac{p_2}{\delta}}$, $\tilde{\rho}_u = 0$, $\tilde{\sigma} = \max\{e^{a\tau}|b|^{\frac{1}{\delta}}, |b|^{\frac{\delta-p_2}{\delta}}\}$, and $\tilde{\varepsilon} = \eta$.

The control objective here is to maintain the number of items in a desired range Ψ given by $\Psi = [\psi_l, \psi_u]$ (a safety specification). For the sake of numerical illustration, we choose different combinations of p_1, p_2, a, b, c, d, Ψ, and leverage software tool SCOTS [40] for constructing symbolic models $\hat{T}_\tau(\Sigma)$ and controller u for $T_\tau(\Sigma)$ with $\tau = 0.2$, and $\eta = 0.01$. The controllers for all cases with their domains are available in [38]. In addition, Figure 1 shows trajectories of system Σ for different values of p_1, p_2, a, b, c, d, Ψ. Finally, one can compute the mismatch between the output behavior of $T_\tau(\Sigma)$ and its symbolic model $\hat{T}_\tau(\Sigma)$ by utilizing Proposition 1. In particular, we have $\hat{\varepsilon} = 0.25$ for **case 1**, $\hat{\varepsilon} = 0.75$ for **case 2**, and $\hat{\varepsilon} = 0.65$ for **case 3**.

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