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## ASYMPTOTIC BEHAVIOR OF ORTHOGONAL POLYNOMIALS WITHOUT THE CARLEMAN CONDITION

D. R. YAFAEV

**ABSTRACT.** Our goal is to find an asymptotic behavior as  $n \rightarrow \infty$  of orthogonal polynomials  $P_n(z)$  defined by the Jacobi recurrence coefficients  $a_n, b_n$ . We suppose that the off-diagonal coefficients  $a_n$  grow so rapidly that the series  $\sum a_n^{-1}$  converges, that is, the Carleman condition is violated. With respect to diagonal coefficients  $b_n$  we assume that  $-b_n(a_n a_{n-1})^{-1/2} \rightarrow 2\beta_\infty$  for some  $\beta_\infty \neq \pm 1$ . The asymptotic formulas obtained for  $P_n(z)$  are quite different from the case  $\sum a_n^{-1} = \infty$  when the Carleman condition is satisfied. In particular, if  $\sum a_n^{-1} < \infty$ , then the phase factors in these formulas do not depend on the spectral parameter  $z \in \mathbb{C}$ . The asymptotic formulas obtained in the cases  $|\beta_\infty| < 1$  and  $|\beta_\infty| > 1$  are also qualitatively different from each other. As an application of these results, we find necessary and sufficient conditions for the essential self-adjointness of the corresponding minimal Jacobi operator.

### 1. INTRODUCTION

**1.1. Overview.** Orthogonal polynomials  $P_n(z)$  can be defined by a recurrence relation

$$a_{n-1}P_{n-1}(z) + b_nP_n(z) + a_nP_{n+1}(z) = zP_n(z), \quad n \in \mathbb{Z}_+, \quad z \in \mathbb{C}, \quad (1.1)$$

with the boundary conditions  $P_{-1}(z) = 0$ ,  $P_0(z) = 1$ . We always suppose that  $a_n > 0$ ,  $b_n = \bar{b}_n$ . Determining  $P_n(z)$ ,  $n = 1, 2, \dots$ , successively from (1.1), we see that  $P_n(z)$  is a polynomial of degree  $n$ :  $P_n(z) = \gamma_n z^n + \dots$  where  $\gamma_n = (a_0 a_1 \dots a_{n-1})^{-1}$ .

We are interested in the asymptotic behavior of the polynomials  $P_n(z)$  as  $n \rightarrow \infty$ . This is a classical problem investigated under various assumptions including usually the condition

$$\sum_{n=0}^{\infty} a_n^{-1} = \infty \quad (1.2)$$

introduced by T. Carleman in his book [3]. We are aware of only one paper [14] where the asymptotics of  $P_n(z)$  was studied without assumption (1.2); this paper is discussed at the end of this subsection. On the contrary, under assumption (1.2) there is an enormous literature on this subject. Here we mention Nevai's approach (see his book [8]) which allowed the authors of [7] to treat the case  $a_n \rightarrow a_\infty > 0$ ,  $b_n \rightarrow 0$  as  $n \rightarrow \infty$  in such a way that

$$\sum_{n=0}^{\infty} (|a_{n+1} - a_n| + |b_{n+1} - b_n|) < \infty. \quad (1.3)$$

The asymptotics of the polynomials  $P_n(z)$  for real  $z$  in case of the coefficients  $a_n \rightarrow \infty$  but satisfying the Carleman condition (1.2) was studied in the papers [5] (see Theorem 3.2) and [2] (see Theorem 3). The authors of [5] solved equations (1.1) for  $P_n(z)$  successively for  $n = 1, 2, \dots$  (the transfer matrix method) which yielded a representation for  $(P_n(z), P_{n+1}(z))$  as a product of

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$n \times 2$ -matrices. The conditions on the coefficients were rather restrictive in [5]; in particular, it was assumed there that  $b_n = 0$  for all  $n$ . Under broader assumptions this problem was considered in [2] where Nevai's method was used.

Under very general assumptions on the coefficients  $a_n, b_n$  the asymptotics of the polynomials  $P_n(z)$  was studied in [14] where however it was assumed that  $z \in \mathbb{R}$  and that the coefficients  $b_n$  are small compared to  $a_n$ . The Carleman and non-Carleman cases were treated in [14] at an equal footing so that the difference in the corresponding asymptotic formulas for  $P_n(z)$  was not quite visible. Among the results of the present paper, Corollary 4.3 seems to be the closest to the main result, Theorem C, of [14].

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**1.2. Main results.** Our goal is to find asymptotic formulas for  $P_n(z)$  as  $n \rightarrow \infty$  without the Carleman condition (1.2), that is, in the case

$$\sum_{n=0}^{\infty} a_n^{-1} < \infty; \quad (1.4)$$

then the coefficients  $a_n \rightarrow \infty$  faster than  $n$ . Astonishingly, the asymptotics of the orthogonal polynomials in this a priori highly singular case is particularly simple and general. Let us briefly describe some of our main results omitting minor technical assumptions. In addition to (1.4), suppose that there exists a finite limit

$$-\frac{b_n}{2\sqrt{a_{n-1}a_n}} =: \beta_n \rightarrow \beta_\infty \quad \text{where} \quad |\beta_\infty| \neq 1 \quad (1.5)$$

as  $n \rightarrow \infty$ . We distinguish two cases:  $|\beta_\infty| < 1$  and  $|\beta_\infty| > 1$ .

If  $|\beta_\infty| < 1$ , we set

$$\phi_n = \sum_{m=0}^{n-1} \arccos \beta_m, \quad n \geq 1, \quad (1.6)$$

where the sum is restricted to  $m$  such that  $|\beta_m| \leq 1$ . For example,  $\phi_n = \pi n/2$  if  $b_n = 0$  for all  $n \in \mathbb{Z}_+$ . We show in Theorem 4.1 that, for an arbitrary  $z \in \mathbb{C}$ , all solutions  $F_n(z)$  of the Jacobi equation

$$a_{n-1}F_{n-1}(z) + b_nF_n(z) + a_nF_{n+1}(z) = zF_n(z), \quad n \in \mathbb{Z}_+, \quad (1.7)$$

have asymptotic behavior

$$F_n(z) = a_n^{-1/2} \left( k_+ e^{-i\phi_n} + k_- e^{i\phi_n} + o(1) \right), \quad n \rightarrow \infty, \quad (1.8)$$

with some constants  $k_\pm \in \mathbb{C}$  (depending of course on the solution  $\{F_n(z)\}$ ). The equalities  $k_+ = k_- = 0$  are possible for the trivial solution  $F_n(z) = 0$  only. Conversely, for all  $k_\pm \in \mathbb{C}$ , there exists a solution  $F_n(z)$  of equation (1.7) with asymptotics (1.8). In particular, formula (1.8) with some constants  $k_\pm(z)$  is true for the orthogonal polynomials  $P_n(z)$ . Note that due to (1.4) and (1.8) all solutions of the equation (1.7) belong to  $\ell^2(\mathbb{Z}_+)$ .

In the case  $|\beta_\infty| > 1$  we set

$$\varphi_n = \sum_{m=0}^{n-1} \operatorname{arccosh} |\beta_m|, \quad n \geq 1, \quad (1.9)$$

where the sum is restricted to  $m$  such that  $|\beta_m| \geq 1$ . Then for all  $z \in \mathbb{C}$ , we have (see Theorem 4.13) an asymptotic relation

$$P_n(z) = k(z) a_n^{-1/2} (\operatorname{sgn} \beta_\infty)^n e^{\varphi_n} (1 + o(1)). \quad (1.10)$$

If the Jacobi operator with the coefficients  $a_n, b_n$  is essentially self-adjoint, then the coefficient  $k(z) \neq 0$  unless  $z$  is an eigenvalue of this operator. Note that  $e^{\varphi_n}$  can be, alternatively, written as a product

$$e^{\varphi_n} = \prod_{m=0}^{n-1} (|\beta_m| + \sqrt{\beta_n^2 - 1}), \quad n \geq 1.$$

According to (1.8) and (1.10) the asymptotic behavior of the polynomials  $P_n(z)$  is the same for all  $z \in \mathbb{C}$ , both for real  $z$  and for  $z$  with  $\operatorname{Im} z \neq 0$ . The coefficients  $k_{\pm}(z)$  and  $k(z)$  can be expressed via the Wronskians of the solution  $\{P_n(z)\}$  and the Jost solution  $\{f_n(z)\}$  (introduced below) of the Jacobi equation (1.7); these coefficients depend on  $z$ .

We emphasize that the asymptotic formulas for the cases (1.2) and (1.4) are qualitatively different from each other. This is discussed in Sect. 5.2 and 5.3.

**1.3. Scheme of the approach.** We are motivated by an analogy of the difference (1.7) and differential

$$-(a(x)f'(x, z))' + b(x)f(x, z) = zf(x, z), \quad x > 0, \quad (1.11)$$

equations where  $a(x) > 0$  and  $b(x)$  is real. To a large extent,  $x$ ,  $a(x)$  and  $b(x)$  here play the roles of the parameters  $n$ ,  $a_n$  and  $b_n$  in the Jacobi equation (1.7). In the case  $a(x) = 1$ ,  $b \in L^1(\mathbb{R}_+)$ , equation (1.11) has a solution  $f(x, z)$ , known as the Jost solution, behaving like  $e^{i\sqrt{z}x}$ ,  $\operatorname{Im} \sqrt{z} \geq 0$ , as  $x \rightarrow \infty$ . For rather general coefficients  $a(x)$ ,  $b(x)$ , equation (1.11) has solutions  $f(x, z)$  with asymptotics given by the classical Liouville-Green formula (see Chapter 6 of the book [10]). In the case  $a(x) \rightarrow a_{\infty} > 0$ ,  $b(x) \rightarrow 0$ , the Liouville-Green formula was simplified in [18] which yields solutions  $f(x, z)$  of (1.11) with asymptotics

$$f(x, z) \sim \exp\left(-\int_0^x \left(\frac{b(y) - z}{a(y)}\right)^{1/2} dy\right) =: Q(x, z), \quad \operatorname{Re}\left(\frac{b(y) - z}{a(y)}\right)^{1/2} \geq 0, \quad (1.12)$$

as  $x \rightarrow \infty$ . Note that the function  $Q(x, z)$  (the Ansatz for the Jost solution  $f(x, z)$ ) satisfies equation (1.11) with a sufficiently good accuracy. Formula (1.12) was modified in [19] for Jacobi equations (1.7) where  $a_n \rightarrow a_{\infty} > 0$ ,  $b_n \rightarrow 0$  as  $n \rightarrow \infty$  and condition (1.3) is satisfied. This permitted to find asymptotics of the orthogonal polynomials  $P_n(z)$  for such coefficients  $a_n, b_n$  in a very natural way.

We are applying the same scheme in the non-Carleman case. Here, we accept conditions (1.4) and (1.5). Let us briefly discuss the main steps of our approach.

A. First, we forget about the orthogonal polynomials  $P_n(z)$  and distinguish solutions (the Jost solutions)  $f_n(z)$  of the difference equation (1.7) by their asymptotics as  $n \rightarrow \infty$ . This requires a construction of an Ansatz  $Q_n(z)$  for the Jost solutions.

B. Under assumption (1.4) this construction (see Sect. 2.3) is very explicit and, in particular, does not depend on  $z \in \mathbb{C}$ . In the case  $|\beta_{\infty}| < 1$ , we set

$$Q_n = a_n^{-1/2} e^{-i\phi_n} \quad (1.13)$$

with the phase  $\phi_n$  defined by formula (1.6). In the case  $|\beta_{\infty}| > 1$ , the Ansatz equals

$$Q_n = a_n^{-1/2} (\operatorname{sgn} \beta_{\infty})^n e^{-\varphi_n} \quad (1.14)$$

where the phase  $\varphi_n$  is given by (1.9). It is shown in Sect. 2.4 that in both cases the relative remainder

$$r_n(z) := (\sqrt{a_{n-1}a_n}Q_n)^{-1}(a_{n-1}Q_{n-1} + (b_n - z)Q_n + a_nQ_{n+1}), \quad n \geq 1, \quad (1.15)$$

belongs to  $\ell^1(\mathbb{Z}_+)$ . At an intuitive level, the fact that the Ansätze (1.13) and (1.14) do not depend on  $z \in \mathbb{C}$  can be explained by the fast growth of the coefficients  $a_n$  which makes the spectral parameter  $z$  negligible.

Actually, the Ansätze we use (especially, the amplitude factor  $a_n^{-1/2}$ ) are only distantly similar to the Liouville-Green Ansatz for the Schrödinger equation (1.11).

C. Then we make in Sect. 2.5 a multiplicative change of variables

$$f_n(z) = Q_n u_n(z) \quad (1.16)$$

which permits us to reduce the difference equation (1.7) for  $f_n(z)$  to a Volterra “integral” equation for the sequence  $u_n(z)$ . This equation depends of course on the parameters  $a_n, b_n$ . In particular, it is somewhat different in the cases  $|\beta_\infty| < 1$  and  $|\beta_\infty| > 1$ . However in both cases this Volterra equation is standardly solved by iterations in Sect. 3.1 and 3.2. This allows us to prove the existence of solutions  $u_n(z)$  of this equation such that  $u_n(z) \rightarrow 1$  as  $n \rightarrow \infty$ . According to (1.13) or (1.14) and (1.16) this yields (see Sect. 3.3) asymptotics of the Jost solutions  $f_n(z)$ . The functions  $f_n(z)$  turn out to be analytic in  $z \in \mathbb{C}$ .

D. We come back to the orthogonal polynomials  $P_n(z)$  in Sect. 4. First, we observe that the sequence

$$\tilde{f}_n(z) = \overline{f_n(\bar{z})}$$

also satisfies the equation (1.7). In the case  $|\beta_\infty| < 1$ , the solutions  $f_n(z)$  and  $\tilde{f}_n(z)$  are linearly independent. Therefore it follows from (1.13) that all solutions of the Jacobi equation (1.7) have asymptotic behavior (1.8). In particular, formula (1.8) is true for the orthogonal polynomials  $P_n(z)$ .

In the case  $|\beta_\infty| > 1$ , a solution  $g_n(z)$  of (1.7) linearly independent with  $f_n(z)$  can be constructed by an explicit formula

$$g_n(z) = f_n(z) \sum_{m=n_0}^n (a_{m-1} f_{m-1}(z) f_m(z))^{-1}, \quad n \geq n_0, \quad (1.17)$$

where  $n_0 = n_0(z)$  is a sufficiently large number. This solution grows exponentially as  $n \rightarrow \infty$ ,

$$g_n(z) = \frac{1}{2\sqrt{\beta_\infty^2 - 1}} a_n^{1/2} (\operatorname{sgn} \beta_\infty)^{n+1} e^{\varphi_n} (1 + o(1)).$$

Since  $g_n(z)$  is linearly independent with  $f_n(z)$ , the polynomials  $P_n(z)$  are linear combinations of  $f_n(z)$  and  $g_n(z)$  which leads to the formula (1.10). Note that (1.17) is a discrete analogue of formula (1.36) in Chapter 4 of the book [16] for the Schrödinger equation.

The scheme described briefly above seems to be quite different from [14] where the first step was a study of the Turán determinants  $P_n(z)^2 - P_{n-1}(z)P_{n+1}(z)$ .

**1.4. Jacobi operators.** It is natural (see the book [1]) to associate with the coefficients  $a_n, b_n$  a three-diagonal matrix

$$\mathcal{J} = \begin{pmatrix} b_0 & a_0 & 0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & 0 & \cdots \\ 0 & 0 & a_2 & b_3 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} \quad (1.18)$$

known as the Jacobi matrix. Then equation (1.1) with the boundary condition  $P_{-1}(z) = 0$  is equivalent to the equation  $\mathcal{J}P(z) = zP(z)$  for the vector  $P(z) = \{P_n(z)\}_{n=0}^\infty$ . Thus  $P(z)$  is the “eigenvector” of the matrix  $\mathcal{J}$  corresponding to the “eigenvalue”  $z$ .

Let us now consider Jacobi operators defined by matrix (1.18) in the canonical basis  $e_0, e_1, \dots$  of the space  $\ell^2(\mathbb{Z}_+)$ . The minimal Jacobi operator  $J_0$  is defined by the formula  $J_0 f = \mathcal{J}f$  on a set  $\mathcal{D} \subset \ell^2(\mathbb{Z}_+)$  of vectors  $f = \{f_n\}_{n=0}^\infty$  such that  $f_n = 0$  for sufficiently large  $n$ . It is symmetric in the space  $\ell^2(\mathbb{Z}_+)$ , and its adjoint operator  $J_0^*$  is given by the same formula  $J_0^* f = \mathcal{J}f$  on all

vectors  $f \in \ell^2(\mathbb{Z}_+)$  such that  $\mathcal{J}f \in \ell^2(\mathbb{Z}_+)$ . The deficiency indices of the operator  $J_0$  are either  $(0, 0)$  (the limit point case) or  $(1, 1)$  (the limit circle case). If the Carleman condition (1.2) holds, then (see, e.g., the book [1]) for all  $b_n$  the operator  $J_0$  is essentially self-adjoint on  $\mathcal{D}$  so that  $J_0$  has the unique self-adjoint extension  $J = \text{clos } J_0$  (the closure of  $J_0$ ).

On the contrary, under assumption (1.4) the deficiency indices of  $J_0$  depend on the value of  $|\beta_\infty|$ . If  $|\beta_\infty| < 1$ , then it follows from (1.8) that all solutions of equation (1.7) are in  $\ell^2(\mathbb{Z}_+)$ , and hence the deficiency indices of the operator  $J_0$  are  $(1, 1)$ . In this case the operator  $J_0$  has a one-parameter family of self-adjoint extensions  $J \subset J_0^*$ . Their domains can be described explicitly (see Sect. 6.5 of [12] or §2 of [13]) in terms of the orthogonal polynomials  $P_n(z)$  (of first kind) and  $\tilde{P}_n(z)$  (of second kind). We recall that  $\tilde{P}_n(z)$  are defined by equations (1.1) where  $n \geq 1$  with the boundary conditions  $\tilde{P}_0(z) = 0$ ,  $\tilde{P}_1(z) = a_0^{-1}$ ; clearly,  $\tilde{P}_n(z)$  is a polynomial of degree  $n - 1$ .

In the case  $|\beta_\infty| > 1$  we show that the operator  $J_0$  is essentially self-adjoint if and only if

$$\sum_{n=0}^{\infty} a_n^{-1} e^{2\varphi_n} = \infty \quad (1.19)$$

where  $\varphi_n$  is defined by (1.9); otherwise the deficiency indices of  $J_0$  are  $(1, 1)$ . This result may be compared with the Berezanskii theorem (see, e.g., page 26 in the book [1]) stating that the Carleman condition (1.2) is necessary for the essential self-adjointness of  $J_0$  provided  $b_n = 0$  and  $a_{n-1}a_{n+1} \leq a_n^2$ .

The spectra of all self-adjoint Jacobi operators  $J$  are simple with  $e_0 = (1, 0, 0, \dots)^\top$  being a generating vector. Therefore it is natural to define the spectral measure of  $J$  by the relation  $d\rho_J(\lambda) = d(E_J(\lambda)e_0, e_0)$  where  $dE_J(\lambda)$  is the spectral family of the operator  $J$ . For all extensions  $J$  of the operator  $J_0$ , the polynomials  $P_n(\lambda)$  are orthogonal and normalized in the spaces  $L^2(\mathbb{R}; d\rho_J)$ :

$$\int_{-\infty}^{\infty} P_n(\lambda) P_m(\lambda) d\rho_J(\lambda) = \delta_{n,m};$$

as usual,  $\delta_{n,n} = 1$  and  $\delta_{n,m} = 0$  for  $n \neq m$ .

We also note a link with a moment problem

$$s_n = \int_{-\infty}^{\infty} \lambda^n d\rho(\lambda)$$

where  $s_n$  are given and the measure  $d\rho(\lambda)$  has to be found. A moment problem is called determinate if its solution  $d\rho(\lambda)$  is unique. In the opposite case it is called indeterminate. Suppose that  $s_n = (\mathcal{J}_0^n e_0, e_0)$ . Then

$$s_n = (J^n e_0, e_0) = \int_{-\infty}^{\infty} \lambda^n d\rho_J(\lambda)$$

for all self-adjoint extensions  $J$  of the operator  $J_0$ . It is known (see, e.g., Theorem 2 in [13]) that the moment problem with the coefficients  $s_n = (J_0^n e_0, e_0)$  is determinate if and only if the operator  $J_0$  is essentially self-adjoint.

The comprehensive presentation of the results described in this subsection can be found in the books [1, 12] or the survey [13]. We rely essentially on a direct study of the difference equation (1.7) and practically do not use operator methods. So, the above information was given mainly to put our results into the right framework.

## 2. ANSATZ

In this section, we calculate the remainder (1.15) for the Ansatz  $Q_n$  defined by formulas (1.13) or (1.14). Then we make substitution (1.16).

**2.1. Preliminaries.** Let us consider equation (1.7). Note that the values of  $F_{N-1}$  and  $F_N$  for some  $N \in \mathbb{Z}_+$  determine the whole sequence  $F_n$  satisfying the difference equation (1.7).

Let  $f = \{f_n\}_{n=-1}^\infty$  and  $g = \{g_n\}_{n=-1}^\infty$  be two solutions of equation (1.7). A direct calculation shows that their Wronskian

$$\{f, g\} := a_n(f_n g_{n+1} - f_{n+1} g_n) \quad (2.1)$$

does not depend on  $n = -1, 0, 1, \dots$ . In particular, for  $n = -1$  and  $n = 0$ , we have

$$\{f, g\} = 2^{-1}(f_{-1}g_0 - f_0g_{-1}) \quad \text{and} \quad \{f, g\} = a_0(f_0g_1 - f_1g_0)$$

(we put  $a_{-1} = 1/2$ ). Clearly, the Wronskian  $\{f, g\} = 0$  if and only if the solutions  $f$  and  $g$  are proportional.

It is convenient to introduce a notation

$$x'_n = x_{n+1} - x_n \quad (2.2)$$

for the “derivative” of a sequence  $x_n$ . Note the Abel summation formula (“integration by parts”):

$$\sum_{n=N}^M x_n y'_n = x_M y_{M+1} - x_{N-1} y_N - \sum_{n=N}^M x'_{n-1} y_n; \quad (2.3)$$

here  $M \geq N \geq 0$  are arbitrary, but we have to set  $x_{-1} = 0$  so that  $x'_{-1} = x_0$  if  $N = 0$ .

It follows from equation (1.1) that if  $P_n(z)$  are the orthogonal polynomials corresponding to coefficients  $(a_n, b_n)$ , then the polynomials  $(-1)^n P_n(-z)$  correspond to the coefficients  $(a_n, -b_n)$ . Therefore without loss of generality, we could have assumed that  $\beta_\infty \geq 0$ .

To emphasize the analogy between differential and difference operators, we often use “continuous” terminology (Volterra integral equations, integration by parts, etc.) for sequences labelled by the discrete variable  $n$ . Below  $C$ , sometimes with indices, and  $c$  are different positive constants whose precise values are of no importance.

In all our constructions below, it suffices to consider the Jacobi equation (1.7) for large  $n$  only.

**2.2. Assumptions.** In addition to (1.4) and (1.5), we need some mild conditions on a regularity of behavior of the sequences  $a_n$  and  $b_n$  as  $n \rightarrow \infty$ . Let us set

$$\varkappa_n = \sqrt{\frac{a_{n+1}}{a_n}}, \quad k_n = \frac{\varkappa_{n-1}}{\varkappa_n} = \frac{a_n}{\sqrt{a_{n-1}a_{n+1}}}. \quad (2.4)$$

With respect to  $a_n$ , we assume that

$$\{k_n - 1\} \in \ell^1(\mathbb{Z}_+) \quad (2.5)$$

which implies also the following properties of the numbers  $\varkappa_n$ .

**Lemma 2.1.** *Under assumption (2.5) there exists a finite limit*

$$\lim_{n \rightarrow \infty} \varkappa_n =: \varkappa_\infty, \quad (2.6)$$

and  $\varkappa_\infty \geq 1$  if condition (1.4) is satisfied. Moreover,

$$\{\varkappa'_n\} \in \ell^1(\mathbb{Z}_+) \quad (2.7)$$

if both (1.4) and (2.5) are true.

*Proof.* By definition (2.4) of  $k_n$ , we have  $\ln k_n = \ln \varkappa_{n-1} - \ln \varkappa_n$  whence

$$\ln \varkappa_n = \ln \varkappa_0 - \sum_{m=1}^n \ln k_m.$$

It follows from (2.5) that the series on the right converges which implies the existence of the limit (2.6). If  $\varkappa_\infty < 1$ , then, by definition (2.4) of  $\varkappa_n$ , we would have  $a_n \leq \gamma a_{n-1}$  for some  $\gamma < 1$  and all  $n \geq n_0$  if  $n_0$  is sufficiently large. Thus,  $a_n^{-1} \geq a_{n_0}^{-1} \gamma^{n_0} \gamma^{-n}$  so that the series in (1.4) diverges.

Since

$$\varkappa_n - \varkappa_{n-1} = \frac{\varkappa_n^2}{\varkappa_n + \varkappa_{n-1}} (1 + k_n)(1 - k_n),$$

relation (2.7) is a direct consequence of assumption (2.5).  $\square$

**Example 2.2.** Both conditions (1.4) and (2.5) are satisfied for  $a_n = \gamma n^p$  where  $\gamma > 0$ ,  $p > 1$  and for  $a_n = \gamma x^{n^q}$  where  $x > 1$ ,  $q < 1$ . In these cases  $\varkappa_\infty = 1$ . For  $a_n = \gamma x^n$ , conditions (1.4) and (2.5) are also satisfied but  $\varkappa_\infty = \sqrt{x}$ . On the contrary, condition (2.5) fails if  $a_n = \gamma x^{n^q}$  with  $q > 1$ .

With respect to the coefficients  $b_n$  or, more precisely, the ratios  $\beta_n$  defined by (1.5), we assume that

$$\{\beta'_n\} \in \ell^1(\mathbb{Z}_+). \quad (2.8)$$

**2.3. Construction.** Let us construct an Ansatz  $Q_n$  such that the corresponding remainder (1.15) belongs to  $\ell^1(\mathbb{Z}_+)$ . Put

$$\zeta_n = \varkappa_n Q_{n+1} Q_n^{-1}$$

and introduce notation

$$\alpha_n = \frac{1}{2\sqrt{a_{n-1}a_n}}, \quad \beta_n = -\frac{b_n}{2\sqrt{a_{n-1}a_n}}, \quad n \in \mathbb{Z}_+;$$

for definiteness, we set  $a_{-1} = 1/2$  and  $Q_0 = a_0^{-1/2}$ . Then

$$Q_n = \frac{1}{\sqrt{a_n}} \zeta_0 \cdots \zeta_{n-1}, \quad n \geq 1, \quad (2.9)$$

and (1.15) can be rewritten as

$$r_n(z) = \zeta_{n-1}^{-1} - 2\beta_n + k_n \zeta_n - 2z\alpha_n. \quad (2.10)$$

We have to find  $\zeta_n$  such that  $\{r_n(z)\} \in \ell^1(\mathbb{Z}_+)$ . In principle, the numbers  $\zeta_n$  can be successively determined from the equations  $r_n(z) = 0$  which leads of course to very complicated expressions. Fortunately the construction of  $\zeta_n$  becomes quite explicit if one neglects in (2.10) the terms from  $\ell^1(\mathbb{Z}_+)$ . Obviously, the term  $2z\alpha_n$  in the right-hand side of (2.10) can be omitted if condition (1.4) is satisfied. Furthermore, under the assumptions below  $\zeta_{n-1}^{-1}$  can be replaced by  $\zeta_n^{-1}$  and  $k_n - 1$  by 1 which allows us to define  $\zeta_n$  from the equation

$$\zeta_n + \zeta_n^{-1} = 2\beta_n. \quad (2.11)$$

Putting together relations (2.10) and (2.11), we can state an intermediary result.

**Lemma 2.3.** *Let numbers  $\zeta_n$  satisfy equation (2.11), and let  $Q_n$  be defined by relation (2.9). Then the remainder (1.15) admits the representation*

$$r_n(z) = (\zeta_{n-1}^{-1} - \zeta_n^{-1}) + (k_n - 1)\zeta_n - 2z\alpha_n. \quad (2.12)$$

Solutions of (2.11) are obviously given by the equalities

$$\zeta_n = \beta_n - i\sqrt{1 - \beta_n^2} = (\beta_n + i\sqrt{1 - \beta_n^2})^{-1} \quad (2.13)$$

if  $|\beta_n| \leq 1$  and

$$\zeta_n = \operatorname{sgn} \beta_n (|\beta_n| - \sqrt{\beta_n^2 - 1}) = \operatorname{sgn} \beta_n (|\beta_n| + \sqrt{\beta_n^2 - 1})^{-1} \quad (2.14)$$



if  $|\beta_n| > 1$ . Of course, the numbers  $\zeta_n^{-1}$  also satisfy (2.11). In the case  $|\beta_n| \leq 1$ , we choose formula (2.13) for definiteness only. On the contrary, in the case  $|\beta_n| > 1$ , the condition  $|\zeta_n| \leq 1$  will be important below. It is convenient to set

$$\theta_n = \arccos \beta_n \in [0, \pi] \quad (2.15)$$

for  $|\beta_n| \leq 1$  and

$$\vartheta_n = \operatorname{arccosh} |\beta_n| = \ln (|\beta_n| + \sqrt{\beta_n^2 - 1}) > 0 \quad (2.16)$$

for  $|\beta_n| > 1$ . Then

$$\zeta_n = e^{-i\theta_n} \text{ for } |\beta_n| \leq 1 \quad \text{and} \quad \zeta_n = \operatorname{sgn} \beta_n e^{-\vartheta_n} \text{ for } |\beta_n| \geq 1.$$

It follows from condition (1.5) that

$$\zeta_n \rightarrow \beta_\infty - i\sqrt{1 - \beta_\infty^2} =: \zeta_\infty = e^{-i\theta_\infty} \quad \text{if } |\beta_\infty| < 1 \quad (2.17)$$

and

$$\zeta_n \rightarrow \operatorname{sgn} \beta_\infty (|\beta_\infty| - \sqrt{\beta_\infty^2 - 1}) =: \zeta_\infty = \operatorname{sgn} \beta_\infty e^{-\vartheta_\infty} \quad \text{if } |\beta_\infty| > 1 \quad (2.18)$$

as  $n \rightarrow \infty$ . Here  $\theta_\infty$  and  $\vartheta_\infty$  are defined by formulas (2.15) and (2.16) where  $n = \infty$ , that is,

$$\theta_\infty = \arccos \beta_\infty \in (0, \pi) \quad \text{and} \quad \vartheta_\infty = \ln (|\beta_\infty| + \sqrt{\beta_\infty^2 - 1}) > 0. \quad (2.19)$$

Obviously,  $|\zeta_\infty| = 1$  and  $\zeta_\infty^2 \neq 1$  if  $|\beta_\infty| < 1$  and  $\zeta_\infty \in (-1, 1)$  but  $\zeta_\infty \neq 0$  if  $|\beta_\infty| > 1$ . For  $|\beta_n| \geq 1$  in the case  $|\beta_\infty| < 1$  and for  $|\beta_n| \leq 1$  in the case  $|\beta_\infty| > 1$ , the numbers  $\theta_n$  and  $\vartheta_n$  can be chosen in an arbitrary way; for definiteness, we set  $\theta_n = 0$  and  $\vartheta_n = 0$ .

Now the Ansatz (2.9) can be written as

$$Q_n = \frac{1}{\sqrt{a_n}} e^{-i\phi_n} \quad \text{if } |\beta_\infty| < 1 \quad (2.20)$$

and

$$Q_n = \frac{1}{\sqrt{a_n}} (\operatorname{sgn} \beta_\infty)^n e^{-\varphi_n} \quad \text{if } |\beta_\infty| > 1. \quad (2.21)$$

Here

$$\phi_n = \sum_{m=0}^{n-1} \theta_m, \quad \varphi_n = \sum_{m=0}^{n-1} \vartheta_m \quad (2.22)$$

and  $\theta_n, \vartheta_n$  are defined by (2.15), (2.16). Obviously, formulas (2.20), (2.21) coincide with (1.13), (1.14), respectively.

Since  $\theta_n \rightarrow \theta_\infty$  and  $\vartheta_n \rightarrow \vartheta_\infty$  as  $n \rightarrow \infty$ , it follows from (2.22) that

$$\phi_n = \theta_\infty n + o(n) \quad \text{and} \quad \varphi_n = \vartheta_\infty n + o(n) \quad \text{as } n \rightarrow \infty \quad (2.23)$$

with  $\theta_\infty$  and  $\vartheta_\infty$  given by formulas (2.19). Moreover, we have the following statement.

**Lemma 2.4.** *Assume that*

$$\sum_{n=0}^{\infty} |\beta_n - \beta_\infty| < \infty. \quad (2.24)$$

*Then the sequences  $\phi_n - \theta_\infty n$  and  $\varphi_n - \vartheta_\infty n$  have finite limits as  $n \rightarrow \infty$ .*

*Proof.* Observe that

$$\phi_n - \theta_\infty n = \sum_{m=0}^{n-1} (\theta_m - \theta_\infty) \quad (2.25)$$

where, by definition (2.15),

$$|\theta_n - \theta_\infty| = |\arccos \beta_n - \arccos \beta_\infty| \leq C |\beta_n - \beta_\infty|.$$

Here  $n$  is sufficiently large and the condition  $|\beta_\infty| < 1$  is taken into account. Therefore assumption (2.24) implies that the series in the right-hand side of (2.25) converges. The difference  $\varphi_n - \vartheta_\infty n$  can be considered quite similarly.  $\square$

**2.4. Estimates of the remainder.** Our goal here is to estimate expression (2.12). Under assumption (1.5) we have

$$|1 - \beta_n^2| \geq c > 0 \quad (2.26)$$

for sufficiently large  $n$ . It follows from definitions (2.13) and (2.14) that

$$\zeta_{n-1}^{-1} - \zeta_n^{-1} = (\beta_{n-1} - \beta_n) \left( 1 - i \frac{\beta_{n-1} + \beta_n}{\sqrt{1 - \beta_{n-1}^2} + \sqrt{1 - \beta_n^2}} \right), \quad |\beta_\infty| < 1,$$

and

$$\zeta_{n-1}^{-1} - \zeta_n^{-1} = (\beta_{n-1} - \beta_n) \left( 1 + \operatorname{sgn} \beta_\infty \frac{\beta_{n-1} + \beta_n}{\sqrt{\beta_{n-1}^2 - 1} + \sqrt{\beta_n^2 - 1}} \right), \quad |\beta_\infty| > 1.$$

According to (2.26) each of these identities yields an estimate

$$|\zeta_{n-1}^{-1} - \zeta_n^{-1}| \leq C |\beta_{n-1} - \beta_n|. \quad (2.27)$$

Therefore (2.12) implies that

$$|r_n(z)| \leq C |\beta_{n-1} - \beta_n| + |k_n - 1| + 2\alpha_n |z| \quad (2.28)$$

where the constant  $C$  does not depend on  $z$ . This leads to the following assertion.

**Lemma 2.5.** *Let conditions (1.4) and (1.5) be satisfied. Then estimate (2.28) for the remainder (2.12) is true for all  $z \in \mathbb{C}$ . Under additional assumptions (2.5) and (2.8), we have*

$$\{r_n(z)\} \in \ell^1(\mathbb{Z}_+). \quad (2.29)$$

**2.5. Multiplicative substitution.** Let the sequence  $Q_n$  be given by formulas (2.20) or (2.21). We are looking for solutions  $f_n(z)$  of the difference equation (1.7) satisfying the condition

$$f_n(z) = Q_n(1 + o(1)), \quad n \rightarrow \infty. \quad (2.30)$$

The uniqueness of such solutions is almost obvious.

**Lemma 2.6.** *Equation (1.7) may have only one solution  $f_n(z)$  satisfying condition (2.30).*

*Proof.* Let  $\mathbf{f}_n(z)$  be another solution of (1.7) satisfying (2.30). It follows from (2.20) or (2.21) that the Wronskian (2.1) of these solutions calculated for  $n \rightarrow \infty$  equals

$$\{f, \mathbf{f}\} = a_n Q_n Q_{n+1} o(1) = o(1) \varkappa_n^{-1} \begin{cases} e^{-i(\phi_{n-1} + \phi_n)}, & |\beta_\infty| < 1 \\ \operatorname{sgn} \beta_\infty e^{-\varphi_{n-1} - \varphi_n}, & |\beta_\infty| > 1 \end{cases}$$

so that  $\{f, \mathbf{f}\} = 0$  according to (2.6). Thus  $\mathbf{f} = Cf$  where  $C = 1$  by virtue again of condition (2.30).  $\square$

**Remark 2.7.** For the calculation of the Wronskian  $\{f, \mathbf{f}\}$ , it is essential that the power of  $a_n$  in (2.9) equals  $-1/2$ .

For construction of  $f_n(z)$ , we will reformulate the problem introducing a sequence

$$u_n(z) = Q_n^{-1} f_n(z), \quad n \in \mathbb{Z}_+. \quad (2.31)$$

Then (2.30) is equivalent to the condition

$$\lim_{n \rightarrow \infty} u_n(z) = 1. \quad (2.32)$$

Let us derive a difference equation for  $u_n(z)$ .

**Lemma 2.8.** *Let  $z \in \mathbb{C}$ , let  $\zeta_n$  be defined by formulas (2.13) or (2.14), and let the remainder  $r_n(z)$  be given by formula (2.12). Then equation (1.7) for a sequence  $f_n(z)$  is equivalent to the equation*

$$k_n \zeta_n (u_{n+1}(z) - u_n(z)) - \zeta_{n-1}^{-1} (u_n(z) - u_{n-1}(z)) = -r_n(z) u_n(z), \quad n \in \mathbb{Z}_+, \quad (2.33)$$

for sequence (2.31).

*Proof.* Substituting expression  $f_n = Q_n u_n$  into (1.7) and using the equality

$$\frac{Q_{n+1}}{Q_n} = \sqrt{\frac{a_n}{a_{n+1}}} \zeta_n,$$

we see that

$$\begin{aligned} & (\sqrt{a_{n-1} a_n} Q_n)^{-1} (a_{n-1} f_{n-1} + (b_n - z) f_n + a_n f_{n+1}) \\ &= \sqrt{\frac{a_{n-1}}{a_n}} \frac{Q_{n-1}}{Q_n} u_{n-1} - 2(\alpha_n z + \beta_n) u_n + \sqrt{\frac{a_n}{a_{n-1}}} \frac{Q_{n+1}}{Q_n} u_{n+1} \\ &= \zeta_{n-1}^{-1} u_{n-1} - 2(\alpha_n z + \beta_n) u_n + k_n \zeta_n u_{n+1}. \end{aligned}$$

In view of (2.11) the right-hand side here equals

$$k_n \zeta_n (u_{n+1} - u_n) - \zeta_{n-1}^{-1} (u_n - u_{n-1}) + r_n u_n$$

with  $r_n$  defined by (2.12). Therefore equations (1.7) and (2.33) are equivalent.  $\square$

### 3. MODIFIED JOST SOLUTIONS

In this section, we reduce the Jacobi difference equation (1.7) for Jost solutions  $f_n(z)$  to a Volterra equation for functions  $u_n(z)$  defined by formula (2.31) and satisfying the condition (2.32). Solutions  $u_n(z)$  of the Volterra equation can be constructed by iterations. At this point, there is almost no difference between the cases  $|\beta_\infty| < 1$  and  $|\beta_\infty| > 1$ . Finally, formula (2.31) yields asymptotics of  $f_n(z)$  as  $n \rightarrow \infty$ .

**3.1. Volterra integral equation.** The sequence  $u_n(z)$  satisfying the difference equation (2.33) will be constructed as a solution of an appropriate ‘‘Volterra integral’’ equation. We set

$$\sigma_n = \zeta_n \zeta_{n-1}, \quad S_n = \sigma_1 \sigma_2 \cdots \sigma_{n-1}, \quad n \geq 2, \quad (3.1)$$

and, for  $n < m$ ,

$$G_{n,m} = -(\kappa_{m-1} \zeta_m)^{-1} S_{m+1} \sum_{p=n+1}^m \kappa_{p-1} S_p^{-1} = -(\kappa_{m-1} \zeta_m)^{-1} \sum_{p=n+1}^m \kappa_{p-1} \sigma_p \cdots \sigma_m. \quad (3.2)$$

Note that the kernel  $G_{n,m}$  does not depend on  $z$ . Let us consider an equation

$$u_n(z) = 1 + \sum_{m=n+1}^{\infty} G_{n,m} r_m(z) u_m(z) \quad (3.3)$$

where the sequence  $r_m(z)$  is defined by formula (2.12).

Our first goal is to estimate the matrix elements  $G_{n,m}$ . This is quite straightforward in the case  $|\beta_\infty| > 1$ .

**Lemma 3.1.** *Let the assumptions (1.4), (1.5) where  $|\beta_\infty| > 1$  and (2.5) be satisfied. Then kernel (3.2) is bounded uniformly in  $m > n \geq 0$ :*

$$|G_{n,m}| \leq C < \infty. \quad (3.4)$$

*Proof.* By virtue of (2.18), we have

$$|\sigma_n| \leq s_\infty < 1 \quad (3.5)$$

if  $s_\infty \in (\zeta_\infty^2, 1)$  and  $n$  is sufficiently large, say,  $n \geq n_0$ . Moreover,  $\{\varkappa_n\} \in \ell^\infty(\mathbb{Z}_+)$  and  $\{\varkappa_n^{-1}\} \in \ell^\infty(\mathbb{Z}_+)$  according to Lemma 2.1. Therefore it follows from (3.2) that

$$|G_{n,m}| \leq C \sum_{p=n+1}^m s_\infty^{m-p+1} = C \frac{1 - s_\infty^{m-n}}{s_\infty^{-1} - 1} \leq C \frac{s_\infty}{1 - s_\infty}, \quad n < m,$$

if  $n \geq n_0$ .

Let now  $n < n_0$ . If  $n < m \leq n_0$ , then the sum (3.2) consists of at most  $n_0$  terms. If  $m > n$ , then

$$G_{n,m} = G_{n,n_0} + G_{n_0,m}$$

is a sum of two bounded terms. □

In the case  $|\beta_\infty| < 1$  we have to “integrate by parts”.

**Lemma 3.2.** *Let the assumptions (1.4), (1.5) where  $|\beta_\infty| < 1$  and (2.5), (2.8) be satisfied. Then kernel (3.2) is bounded uniformly in  $m > n \geq 0$ , that is, estimate (3.4) holds.*

*Proof.* As in the previous lemma, we can suppose that  $n \geq n_0$  where  $n_0$  is sufficiently large. Since  $|\zeta_p| = 1$  according to (2.13), it follows from definition (3.1) that

$$|S_{m+1}S_p^{-1}| = |\sigma_p \cdots \sigma_m| = 1, \quad p \leq m. \quad (3.6)$$

Relation (2.17) where  $\zeta_\infty^2 \neq 1$  implies that

$$|\sigma_n - 1| \geq c > 0. \quad (3.7)$$

By definitions (2.2) and (3.1) we have

$$(S_p^{-1})' = S_{p+1}^{-1} - S_p^{-1} = (\sigma_p^{-1} - 1)S_p^{-1},$$

and hence integrating by parts (that is, using formula (2.3)), we find that

$$\begin{aligned} \sum_{p=n+1}^m \varkappa_{p-1} S_p^{-1} &= \sum_{p=n+1}^m \varkappa_{p-1} (\sigma_p^{-1} - 1)^{-1} (S_p^{-1})' \\ &= \varkappa_{m-1} (\sigma_m^{-1} - 1)^{-1} S_{m+1}^{-1} - \varkappa_{n-1} (\sigma_n^{-1} - 1)^{-1} S_{n+1}^{-1} - \sum_{p=n+1}^m (\varkappa_{p-2} (\sigma_{p-2}^{-1} - 1)^{-1})' S_p^{-1}. \end{aligned} \quad (3.8)$$

Note that

$$((\sigma_{p-1}^{-1} - 1)^{-1})' = (\sigma_{p-1} - 1)^{-1} (\sigma_p - 1)^{-1} \sigma_{p-1}'$$

where  $\sigma_p' \in \ell^1(\mathbb{Z}_+)$  by virtue of (2.27) and (2.8). Using also (2.7) and (3.7), we see that this sequence belongs to  $\ell^1(\mathbb{Z}_+)$  whence

$$(\varkappa_{p-1} (\sigma_{p-1}^{-1} - 1)^{-1})' \in \ell^1(\mathbb{Z}_+).$$

Let us multiply identity (3.8) by  $S_{m+1}$ . According to (3.6) and (3.7) all three terms in the right-hand side of the equality obtained are bounded for  $n \geq n_0$ . □

**3.2. Successive approximations.** Let us come back to the Volterra equation (3.3). Lemma 2.5 ensures inclusion (2.29), and Lemmas 3.1 or 3.2 show that the kernels  $G_{n,m}$  are bounded. This allows us to estimate iterations of equation (3.3) and then solve it.

**Lemma 3.3.** *Let the assumptions (1.4), (1.5) as well as (2.5), (2.8) be satisfied. Set  $u_n^{(0)} = 1$  and*

$$u_n^{(k+1)}(z) = \sum_{m=n+1}^{\infty} G_{n,m} r_m(z) u_m^{(k)}(z), \quad k \geq 0, \quad (3.9)$$

for all  $n \in \mathbb{Z}_+$ . Then the estimates

$$|u_n^{(k)}(z)| \leq \frac{C^k}{k!} \left( \sum_{m=n+1}^{\infty} |r_m(z)| \right)^k, \quad \forall k \in \mathbb{Z}_+. \quad (3.10)$$

are true for all sufficiently large  $n$  with the same constant  $C$  as in (3.4).

*Proof.* Suppose that (3.10) is satisfied for some  $k \in \mathbb{Z}_+$ . We have to check the same estimate (with  $k$  replaced by  $k+1$  in the right-hand side) for  $u_n^{(k+1)}$ . Set

$$R_m = \sum_{p=m+1}^{\infty} |r_p|.$$

According to definition (3.9), it follows from estimate (3.4) and (3.10) that

$$|u_n^{(k+1)}| \leq \frac{C^{k+1}}{k!} \sum_{m=n+1}^{\infty} |r_m| R_m^k. \quad (3.11)$$

Observe that

$$R_m^{k+1} + (k+1)|r_m|R_m^k \leq R_{m-1}^{k+1},$$

and hence, for all  $N \in \mathbb{Z}_+$ ,

$$(k+1) \sum_{m=n+1}^N |r_m| R_m^k \leq \sum_{m=n+1}^N (R_{m-1}^{k+1} - R_m^{k+1}) = R_n^{k+1} - R_N^{k+1} \leq R_n^{k+1}.$$

Substituting this bound into (3.11), we obtain estimate (3.10) for  $u_n^{(k+1)}$ .  $\square$

Now we are in a position to solve equation (3.3) by iterations.

**Theorem 3.4.** *Let the assumptions (1.4), (1.5) as well as (2.5), (2.8) be satisfied. Then equation (3.3) has a unique bounded solution  $u_n(z)$ . Moreover,*

$$|u_n(z) - 1| \leq e^{C\varepsilon_n(z)} - 1, \quad n \geq 0, \quad (3.12)$$

where the constant  $C$  does not depend on  $z \in \mathbb{C}$  and

$$\varepsilon_n(z) = \sum_{m=n+1}^{\infty} (|\beta_{m-1} - \beta_m| + |k_m - 1| + \alpha_m |z|) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.13)$$

For all  $n \in \mathbb{Z}_+$ , the functions  $u_n(z)$  are entire functions of  $z \in \mathbb{C}$ .

*Proof.* Set

$$u_n = \sum_{k=0}^{\infty} u_n^{(k)} \quad (3.14)$$

where  $u_0^{(k)} = 1$  and  $u_n^{(k)}$ ,  $k \geq 1$ , are defined by recurrence relations (3.9). Estimate (3.10) shows that this series is absolutely convergent. Using the Fubini theorem to interchange the order of summations in  $m$  and  $k$ , we see that

$$\sum_{m=n+1}^{\infty} G_{n,m} r_m u_m = \sum_{k=0}^{\infty} \sum_{m=n+1}^{\infty} G_{n,m} r_m u_m^{(k)} = - \sum_{k=0}^{\infty} u_n^{(k+1)} = 1 - \sum_{k=0}^{\infty} u_n^{(k)} = 1 - u_n.$$

This is equation (3.3) for sequence (3.14). Estimate (3.12) for the sequence (3.14) also follows from (3.10) where  $r_n(z)$  satisfies (2.28).

According to (2.12) the remainder  $r_m(z)$  and hence the kernels  $G_{n,m} r_m(z)$  are linear functions of  $z$ . Therefore recurrence arguments show that all successive approximations  $u_n^{(k)}(z)$  are analytic functions of  $z \in \mathbb{C}$ . The same assertion is of course true for the series (3.14).  $\square$

**Corollary 3.5.** *We have an estimate*

$$|u_n(z) - 1| \leq C \varepsilon_n(z) \quad (3.15)$$

where  $\varepsilon_n(z) \rightarrow 0$  uniformly on compact subsets of  $\mathbb{C}$ .

It turns out that the construction above yields a solution of the difference equation (2.33).

**Lemma 3.6.** *Let  $r_n(z)$  and  $G_{n,m}$  be given by formulas (2.12) and (3.2), respectively. Then a solution  $u_n(z)$  of integral equation (3.3) satisfies also the difference equation (2.33).*

*Proof.* It follows from (3.3) that

$$u_{n+1} - u_n = \sum_{m=n+2}^{\infty} (G_{n+1,m} - G_{n,m}) r_m u_m - G_{n,n+1} r_{n+1} u_{n+1}. \quad (3.16)$$

Since according to (3.2)

$$G_{n+1,m} - G_{n,m} = -\varkappa_n(\varkappa_{m-1}\zeta_m)^{-1} S_{n+1}^{-1} S_{m+1} \quad \text{and} \quad G_{n,n+1} = \zeta_{n+1}^{-1} S_{n+2} S_{n+1}^{-1},$$

equality (3.16) can be rewritten as

$$\varkappa_n^{-1}(u_{n+1} - u_n) = - \sum_{m=n+1}^{\infty} S_{n+1}^{-1} S_{m+1} (\varkappa_{m-1}\zeta_m)^{-1} r_m u_m. \quad (3.17)$$

Putting together this equality with the same equality for  $n+1$  replaced by  $n$ , we see that

$$\begin{aligned} & \varkappa_n^{-1}(u_{n+1} - u_n) - \sigma_n^{-1} \varkappa_{n-1}^{-1}(u_n - u_{n-1}) \\ &= \sum_{m=n+1}^{\infty} S_{n+1}^{-1} S_{m+1} r_m (\varkappa_{m-1}\zeta_m)^{-1} u_m - \sigma_n^{-1} \sum_{m=n}^{\infty} S_n^{-1} S_{m+1} r_m (\varkappa_{m-1}\zeta_m)^{-1} u_m. \end{aligned}$$

Since  $S_{n+1} = \sigma_n S_n$ , the right-hand side here equals  $-(\varkappa_{n-1}\zeta_n)^{-1} r_n u_n$ , and hence the equation obtained coincides with (2.33).  $\square$

The above arguments show also that the functions  $u_n(z)$  are of minimal exponential type.

**Lemma 3.7.** *Under the assumptions of Theorem 3.4, for an arbitrary  $\epsilon > 0$  and some constants  $C_n(\epsilon)$  (that do not depend on  $z \in \mathbb{C}$ ), every function  $u_n(z)$  satisfies an estimate*

$$|u_n(z)| \leq C_n(\epsilon) e^{\epsilon|z|}. \quad (3.18)$$

*Proof.* It suffices to check (3.18) for  $|z| \geq 1$ . According to (3.12) and (3.13) we have an estimate

$$|u_m(z)| \leq e^{\epsilon_m |z|} \quad (3.19)$$

where  $\epsilon_m \rightarrow 0$  as  $m \rightarrow \infty$ . On the other hand, it follows from equation (1.7) for function (1.16) that

$$|u_n(z)| \leq C_n |z| (|u_{n+1}(z)| + |u_{n+2}(z)|) \leq \dots \leq C_{n,k} |z|^k (|u_{n+k}(z)| + |u_{n+k+1}(z)|) \quad (3.20)$$

for every  $k = 1, 2, \dots$ . For a given  $\epsilon > 0$ , choose  $k$  such that  $2\epsilon_{n+k} \leq \epsilon$ ,  $2\epsilon_{n+k+1} \leq \epsilon$ . Then putting estimates (3.19) and (3.20) together, we see that

$$|u_n(z)| \leq 4C_{n,k} |z|^k e^{\epsilon |z|/2}.$$

Since  $|z|^k \leq c_k(\epsilon) e^{\epsilon |z|/2}$ , this proves (3.18).  $\square$

In the case  $|\beta_\infty| > 1$ , we also need estimates on  $u'_n$ .

**Lemma 3.8.** *Let  $|\beta_\infty| > 1$ . Under the assumptions of Theorem 3.4, we have  $\{u'_n\} \in \ell^1(\mathbb{Z}_+)$ .*

*Proof.* Let us proceed from expression (3.17) for  $u_{n+1} - u_n$ . It follows from (3.1) and (3.5) that

$$|S_{n+1}^{-1} S_{m+1}| = |\sigma_{n+1} \cdots \sigma_m| \leq s_\infty^{m-n}$$

for sufficiently large  $n$  and  $m \geq n$  whence

$$|u_{n+1} - u_n| \leq C \sum_{m=n+1}^{\infty} s_\infty^{m-n} |r_m u_m|.$$

So we only have to take the sum over  $n$  here. Then we use that  $s_\infty < 1$  and  $\{r_m u_m\} \in \ell^1(\mathbb{Z}_+)$ .  $\square$

**3.3. Jost solutions.** Now we are in a position to construct solutions of the difference equation (1.7) with asymptotics (2.30) as  $n \rightarrow \infty$ . Recall that the sequence  $Q_n$  was defined by formulas (2.20) or (2.21). According to Lemma 2.6, equation (1.7) may have only one solution with such asymptotics. By Theorem 3.4 and Corollary 3.5, equation (3.3) has a solution  $u_n(z)$  satisfying estimate (3.12) with the remainder  $\varepsilon_n(z)$  given by formula (3.13). By Lemma 3.6 the difference equation (2.33) is also true for this solution. It follows from Lemma 2.8 that

$$f_n(z) = Q_n u_n(z) \quad (3.21)$$

satisfies equation (1.7).

This leads to the following results. We state them separately for the cases  $|\beta_\infty| < 1$  and  $|\beta_\infty| > 1$ .

**Theorem 3.9.** *Let the assumptions (1.4), (1.5) with  $|\beta_\infty| < 1$  as well as (2.5), (2.8) be satisfied. Then, for all  $z \in \mathbb{C}$ , the equation (1.7) has a solution  $\{f_n(z)\}$  with asymptotics*

$$f_n(z) = \frac{1}{\sqrt{a_n}} e^{-i\phi_n} (1 + O(\varepsilon_n(z))), \quad n \rightarrow \infty, \quad (3.22)$$

where  $\phi_n$  and  $\varepsilon_n(z)$  are given by formulas (1.6) and (3.13), respectively. Asymptotics (3.22) is uniform in  $z$  from compact subsets of the complex plane  $\mathbb{C}$ . For all  $n \in \mathbb{Z}_+$ , the functions  $f_n(z)$  are entire functions of  $z \in \mathbb{C}$  of minimal exponential type.

**Corollary 3.10.** *Suppose that  $b_n = 0$ , at least for sufficiently large  $n$ . Let the assumptions (1.4) and (2.5) be satisfied. Then, for all  $z \in \mathbb{C}$ , the equation (1.7) has a solution  $\{f_n(z)\}$  with asymptotics*

$$f_n(z) = \frac{1}{\sqrt{a_n}} (-i)^n (1 + O(\varepsilon_n(z))), \quad n \rightarrow \infty.$$

By analogy with the continuous case, the sequence  $\{f_n(z)\}_{n=-1}^\infty$  will be called the (modified) Jost solution of equation (1.7). Additionally, we define the conjugate Jost solution by the formula

$$\tilde{f}_n(z) = \overline{f_n(\bar{z})}. \quad (3.23)$$

It also satisfies equation (1.7) because the coefficients  $a_n$  and  $b_n$  are real, and it has the asymptotics

$$\tilde{f}_n(z) = \frac{1}{\sqrt{a_n}} e^{i\phi_n} (1 + O(\varepsilon_n)), \quad n \rightarrow \infty, \quad (3.24)$$

Using asymptotic formulas (3.22), (3.24) and definition (2.15), we can calculate the Wronskian (2.1) of the solutions  $f(z)$ ,  $\tilde{f}(z)$ :

$$\{f(z), \tilde{f}(z)\} = \lim_{n \rightarrow \infty} \varkappa_n^{-1} (e^{i\theta_n} - e^{-i\theta_n}) = \varkappa_\infty^{-1} (e^{i\theta_\infty} - e^{-i\theta_\infty}) = 2i\varkappa_\infty^{-1} \sqrt{1 - \beta_\infty^2} \neq 0 \quad (3.25)$$

according to (2.17). Note that the amplitude factors  $a_n^{-1/2}$  in (3.22) and (3.24) are quite natural (cf. Remark 2.7) because their product should cancel  $a_n$  in the Wronskian (3.25).

Thus, in the case  $|\beta_\infty| < 1$  for all  $z \in \mathbb{C}$ , we have two linearly independent oscillating solutions  $f_n(z)$  and  $\tilde{f}_n(z)$ . Under assumption (1.4) both of them belong to  $\ell^2(\mathbb{Z}_+)$ . This leads to the following result.

**Corollary 3.11.** *Under the assumptions of Theorem 3.9 the minimal Jacobi operator  $J_0$  has deficiency indices  $(1, 1)$ .*

Let us show that, even in the case  $b_n = 0$ , assumption (2.5) cannot be omitted in Theorem 3.9. The example below was considered in [6], Example 2, and in [4], Lemma 2.3.

**Example 3.12.** Suppose that  $b_n = 0$  and that  $a_n = n^p(1 + c_1 n^{-1})$  if  $n$  is odd and  $a_n = n^p(1 + c_2 n^{-1})$  if  $n$  is even for  $p > 1$  and sufficiently large  $n$ . Corollary 3.11 does not apply in this case because

$$\frac{a_n}{\sqrt{a_{n-1}a_{n+1}}} = 1 + (-1)^n \frac{c_2 - c_1}{n} + O\left(\frac{1}{n^2}\right), \quad n \rightarrow \infty, \quad (3.26)$$

so that condition (2.5) breaks down unless  $c_1 = c_2$ . Actually, it was proven in [6, 4] that the corresponding Jacobi operator  $J_0$  is essentially adjoint if  $|c_2 - c_1| \geq p - 1$ .

In the case  $|\beta_\infty| > 1$ , we have the following result.

**Theorem 3.13.** *Let the assumptions (1.4), (1.5) with  $|\beta_\infty| > 1$  as well as (2.5), (2.8) be satisfied. Then, for all  $z \in \mathbb{C}$ , the equation (1.7) has a solution  $\{f_n(z)\}$  with asymptotics*

$$f_n(z) = \frac{1}{\sqrt{a_n}} (\operatorname{sgn} \beta_\infty)^n e^{-\varphi_n} (1 + O(\varepsilon_n)), \quad n \rightarrow \infty, \quad (3.27)$$

where  $\varphi_n$  and  $\varepsilon_n(z)$  are given by formulas (1.9) and (3.13), respectively. Asymptotics (3.27) is uniform in  $z$  from compact subsets of the complex plane  $\mathbb{C}$ . For all  $n \in \mathbb{Z}_+$ , the functions  $f_n(z)$  are entire functions of  $z \in \mathbb{C}$  of minimal exponential type.

According to (2.19) and (2.23) the solution (3.27) tends to zero exponentially as  $n \rightarrow \infty$ . It will also be called the (modified) Jost solution of equation (1.7). However, in contrast to the case  $|\beta_\infty| < 1$ , for  $|\beta_\infty| > 1$  the construction of Theorem 3.13 yields only one solution  $f_n(z)$  of the Jacobi equation (1.7).

**Remark 3.14.** Suppose that instead of (2.8), a stronger assumption (2.24) holds and use Lemma 2.4. Then taking into account relations (2.15) and (2.16), we can replace  $e^{-i\phi_n}$  in (3.22) and  $e^{-\varphi_n}$  in (3.27) by

$$e^{-in\theta_\infty} = (\beta_\infty - i\sqrt{1 - \beta_\infty^2})^n \quad \text{and} \quad e^{-n\vartheta_\infty} = (|\beta_\infty| + \sqrt{\beta_\infty^2 - 1})^{-n}, \quad (3.28)$$

respectively. In this case the remainders  $O(\varepsilon_n)$  should be replaced by  $o(1)$ .



**3.4. Example.** Suppose that (for sufficiently large  $n$ )

$$a_n = \gamma n^p \quad \text{and} \quad b_n = -\delta n^p(1 + \sigma n^{-1}), \quad p > 1,$$

where  $\gamma > 0$  and  $\delta, \sigma$  are arbitrary real numbers. Then

$$\beta_n = \frac{\delta}{2\gamma}(1 + \sigma n^{-1})(1 - n^{-1})^{-p/2} = \frac{\delta}{2\gamma}\left(1 + \left(\sigma + \frac{p}{2}\right)n^{-1} + O(n^{-2})\right)$$

and  $\beta_\infty = \delta(2\gamma)^{-1}$ . The assumptions of Theorem 3.9 (of Theorem 3.13) are satisfied if  $|\delta| < 2\gamma$  (resp.,  $|\delta| > 2\gamma$ ). Since

$$\beta_n - \beta_\infty = (2\gamma)^{-1}\delta(\sigma + p/2)n^{-1} + O(n^{-2}), \quad (3.29)$$

the condition (2.24) fails unless  $\sigma + p/2 = 0$ .

According to (2.15) for  $|\beta_\infty| < 1$ , we now have

$$\theta_n = \arccos \beta_\infty + \nu n^{-1} + O(n^{-2}), \quad \nu = -\frac{\delta(\sigma + p/2)}{\sqrt{|\delta^2 - 4\gamma^2|}},$$

so that according to (2.22) the sequence

$$\phi_n - n \arccos \beta_\infty - \nu \ln n$$

has a finite limit as  $n \rightarrow \infty$ . Therefore formula (3.22) in Theorem 3.9 can be replaced by a simpler one

$$f_n(z) = n^{-p/2 - i\nu} (\beta_\infty - i\sqrt{1 - \beta_\infty^2})^n (1 + o(1)), \quad n \rightarrow \infty.$$

Quite similarly, in the case  $|\beta_\infty| > 1$ , it follows from definitions (2.16), (2.22) and formula (3.29) that the sequence

$$\varphi_n - n \operatorname{arccosh} |\beta_\infty| - \nu \operatorname{sgn} \beta_\infty \ln n$$

has a finite limit as  $n \rightarrow \infty$ . Therefore formula (3.27) in Theorem 3.13 can be replaced by a simpler one

$$f_n(z) = n^{-p/2 + \nu \operatorname{sgn} \delta} (\operatorname{sgn} \delta)^n (|\beta_\infty| + \sqrt{\beta_\infty^2 - 1})^{-n} (1 + o(1)), \quad n \rightarrow \infty.$$

#### 4. ORTHOGONAL POLYNOMIALS

**4.1. Small diagonal elements.** Here we suppose that  $|\beta_\infty| < 1$ . Recall that the Jost solution  $f(z) = \{f_n(z)\}$  was constructed in Theorem 3.9 and  $\tilde{f}(z)$  was defined by relation (3.23). By virtue of (3.25), an arbitrary solution  $F(z) = \{F_n(z)\}$  of the Jacobi equation (1.7) is a linear combination of the Jost solutions  $f(z)$  and  $\tilde{f}(z)$ , that is

$$F_n(z) = k_+(z)f_n(z) + k_-(z)\tilde{f}_n(z), \quad (4.1)$$

where the constants can be expressed via the Wronskians:

$$k_+(z) = \kappa_\infty \frac{\{F(z), \tilde{f}(z)\}}{2i\sqrt{1 - \beta_\infty^2}}, \quad k_-(z) = -\kappa_\infty \frac{\{F(z), f(z)\}}{2i\sqrt{1 - \beta_\infty^2}}. \quad (4.2)$$

According to (4.1) the following result is a direct consequence of Theorem 3.9. Recall that the phase  $\phi_n$  is defined by (1.6) and the remainder  $\varepsilon_n(z)$  is given by (3.13).

**Theorem 4.1.** *Let the assumptions of Theorem 3.9 be satisfied. Choose some  $z \in \mathbb{C}$ . Then an arbitrary solution  $F_n$  of the Jacobi equation (1.7) has asymptotics*

$$F_n = a_n^{-1/2} (k_+ e^{-i\phi_n} + k_- e^{i\phi_n}) (1 + O(\varepsilon_n(z))), \quad n \rightarrow \infty, \quad (4.3)$$

for some  $k_\pm \in \mathbb{C}$ . Conversely, for arbitrary  $k_\pm \in \mathbb{C}$ , there exists a solution  $F_n$  of the equation (1.7) with asymptotics (4.3).

Recall that the polynomials  $P_n(z)$  are solutions of the Jacobi equation (1.7) satisfying the conditions  $P_{-1}(z) = 0$ ,  $P_0(z) = 1$ . Relations (4.1), (4.2) and Theorem 4.1 remain of course true in this case. Moreover, the corresponding asymptotic coefficients

$$k_+(z) = \varkappa_\infty \frac{\{P(z), \tilde{f}(z)\}}{2i\sqrt{1-\beta_\infty^2}}, \quad k_-(z) = -\varkappa_\infty \frac{\{P(z), f(z)\}}{2i\sqrt{1-\beta_\infty^2}} \quad (4.4)$$

satisfy the relations  $k_-(z) = \overline{k_+(\bar{z})}$  because  $P_n(z) = \overline{P_n(\bar{z})}$ . Of course,  $k_+(z)$  and  $k_-(z)$  are analytic functions of  $z \in \mathbb{C}$ . Set  $\kappa(z) = |k_+(z)|$ . Then

$$k_+(z) = \kappa(z)e^{i\eta(z)}, \quad \eta(z) \in \mathbb{R}/2\pi\mathbb{Z} \quad \text{and} \quad k_-(z) = \kappa(\bar{z})e^{-i\eta(\bar{z})}.$$

Observe that the equalities  $\kappa(z) = \kappa(\bar{z}) = 0$  are impossible, since otherwise relation (4.1) for  $P_n(z)$  would imply that  $P_n(z) = 0$  for all  $n$  but  $P_0 = 1$ .

The next result is a particular case of Theorem 4.1.

**Theorem 4.2.** *Let the assumptions of Theorem 3.9 be satisfied. Then, for all  $z \in \mathbb{C}$ , the sequence of the orthogonal polynomials  $P_n(z)$  has asymptotics*

$$P_n(z) = a_n^{-1/2} \left( \kappa(z)e^{-i\phi_n+i\eta(z)} + \kappa(\bar{z})e^{i\phi_n-i\eta(\bar{z})} + O(\varepsilon_n(z)) \right), \quad n \rightarrow \infty. \quad (4.5)$$

Asymptotics (4.5) is uniform in  $z$  from compact subsets of the complex plane  $\mathbb{C}$ .

If  $z = \lambda \in \mathbb{R}$ , then relation (4.1) for  $P_n(z)$  reduces to

$$P_n(\lambda) = 2 \operatorname{Re} (k_+(\lambda)f_n(\lambda)), \quad (4.6)$$

and (4.5) yields the following result.

**Corollary 4.3.** *If  $\lambda \in \mathbb{R}$ , then*

$$P_n(\lambda) = 2a_n^{-1/2} \left( \kappa(\lambda) \cos(\phi_n - \eta(\lambda)) + O(\varepsilon_n) \right), \quad n \rightarrow \infty. \quad (4.7)$$

Asymptotics (4.7) is uniform in  $\lambda$  from compact subsets of the line  $\mathbb{R}$ .

Observe that  $\kappa(\lambda) \neq 0$ , since otherwise (4.6) would imply that  $P_n(\lambda) = 0$  for all  $n$  but  $P_0 = 1$ .

**4.2. An identity for asymptotic amplitudes.** Here we suppose that  $\operatorname{Im} z \neq 0$ . Multiplying equation (1.1) for  $P_n(z)$  by its complex conjugate  $\bar{P}_n(z)$  and taking the sum over  $n = 0, 1, \dots, N$ , we find that

$$\sum_{n=0}^N a_{n-1} P_{n-1}(z) \bar{P}_n(z) + \sum_{n=0}^N a_n P_{n+1}(z) \bar{P}_n(z) + \sum_{n=0}^N b_n |P_n(z)|^2 = z \sum_{n=0}^N |P_n(z)|^2. \quad (4.8)$$

Since  $P_{-1}(z) = 0$ , the first sum on the left equals

$$\sum_{n=0}^N a_n P_n(z) \bar{P}_{n+1}(z) - a_N P_N(z) \bar{P}_{N+1}(z).$$

Therefore taking the imaginary part of (4.8), we see that

$$a_N \operatorname{Im} (P_{N+1}(z) \bar{P}_N(z)) = \operatorname{Im} z \sum_{n=0}^N |P_n(z)|^2. \quad (4.9)$$

According to (4.5) the left-hand side here equals

$$\varkappa_N^{-1} \operatorname{Im} \left( (\kappa(z)e^{-i\phi_{N+1}+i\eta(z)} + \kappa(\bar{z})e^{i\phi_{N+1}-i\eta(\bar{z})}) (\kappa(z)e^{i\phi_N-i\eta(z)} + \kappa(\bar{z})e^{-i\phi_N+i\eta(\bar{z})}) \right) + o(1) \quad (4.10)$$

with  $\varkappa_N$  defined by (2.4). Observe that the sum

$$\begin{aligned} \kappa(z)e^{-i\phi_{N+1}+i\eta(z)}\kappa(\bar{z})e^{-i\phi_N+i\eta(\bar{z})} + \kappa(\bar{z})e^{i\phi_{N+1}-i\eta(\bar{z})}\kappa(z)e^{i\phi_N-i\eta(z)} \\ = 2\kappa(z)\kappa(\bar{z})\cos(\phi_N + \phi_{N+1} - \eta(z) - \eta(\bar{z})) \end{aligned}$$

is real. Therefore taking relations and (2.6), (2.13) and (2.22) into account, we see that expression (4.10) equals

$$\varkappa_N^{-1} \operatorname{Im}(\kappa(z)^2 e^{-i\theta_N} + \kappa(\bar{z})^2 e^{i\theta_N}) + o(1) = \varkappa_\infty^{-1}(\kappa(\bar{z})^2 - \kappa(z)^2) \sin \theta_\infty + o(1)$$

where  $\sin \theta_\infty = \sqrt{1 - \beta_\infty^2}$ . Replacing the left-hand side of (4.9) by this expression, we obtain an identity for the asymptotic amplitude  $\kappa(z) = |k_+(z)|$  where  $k_+(z)$  is defined by (4.4). We also recall that the number  $\varkappa_\infty$  is given by (2.6).

**Theorem 4.4.** *Under the assumptions of Theorem 3.9 the identity*

$$\kappa(\bar{z})^2 - \kappa(z)^2 = \operatorname{Im} z \varkappa_\infty (1 - \beta_\infty^2)^{-1/2} \sum_{n=0}^{\infty} |P_n(z)|^2$$

holds.

**Corollary 4.5.** *If  $\operatorname{Im} z > 0$ , then  $\kappa(z) < \kappa(\bar{z})$ . Equivalently,  $\kappa(z) > \kappa(\bar{z})$  if  $\operatorname{Im} z < 0$ .*

**4.3. Large diagonal elements.** Here we consider the case  $|\beta_\infty| > 1$ . Choose an arbitrary  $z \in \mathbb{C}$ . By Theorem 3.13, the sequence  $f_n(z)$  defined by equality (3.21) satisfies equation (1.7), and it has exponentially decaying asymptotics (3.27) where the phase  $\varphi_n$  is given by (1.9). In particular,  $f_n(z) \neq 0$  for sufficiently large  $n$ , say,  $n \geq n_0 = n_0(z)$ . Now we have to construct a solution  $g_n(z)$  of (1.7) linearly independent with  $f_n(z)$ . We define it by the formula (1.17), that is,

$$g_n(z) = f_n(z)G_n(z) \tag{4.11}$$

where

$$G_n(z) = \sum_{m=n_0}^n (a_{m-1}f_{m-1}(z)f_m(z))^{-1}, \quad n \geq n_0. \tag{4.12}$$

**Theorem 4.6.** *Suppose that a sequence  $f(z) = \{f_n(z)\}$  satisfies the Jacobi equation (1.7). Then the sequence  $g(z) = \{g_n(z)\}$  defined by formulas (4.11) and (4.12) satisfies the same equation and the Wronskian  $\{f(z), g(z)\} = 1$ . In particular, the solutions  $f(z)$  and  $g(z)$  are linearly independent.*

*Proof.* First, we check equation (1.7) for  $g_n$ . Observe that

$$\begin{aligned} a_{n-1}f_{n-1}G_{n-1} + (b_n - z)f_nG_n + a_nf_{n+1}G_{n+1} \\ = (a_{n-1}f_{n-1} + (b_n - z)f_n + a_nf_{n+1})G_n + a_{n-1}f_{n-1}(G_{n-1} - G_n) + a_nf_{n+1}(G_{n+1} - G_n). \end{aligned}$$

The first term here is zero because equation (1.7) is true for the sequence  $f_n$ . According to definition (4.12)

$$G_{n+1} = G_n + (a_nf_nf_{n+1})^{-1}, \tag{4.13}$$

so that the second and third terms equal  $-f_n^{-1}$  and  $f_n^{-1}$ , respectively. This proves equation (1.7) for  $g_n$ .

It also follows from (4.13) that the Wronskian (2.1) equals

$$\{f(z), g(z)\} = a_nf_n(z)f_{n+1}(z)(G_{n+1}(z) - G_n(z)) = 1,$$

whence the solutions  $f(z)$  and  $g(z)$  are linearly independent.  $\square$

Theorem 4.11 below shows that the solution  $g_n(z)$  of equation (1.7) exponentially grows as  $n \rightarrow \infty$ . For a proof of this result, we will have to integrate by parts in (4.12). It is convenient to start with a simple technical assertion. Recall that  $\sigma_m = \zeta_{m-1}\zeta_m = e^{-\vartheta_{m-1}-\vartheta_m}$ , the numbers  $\varkappa_n$  were defined in formula (2.4) and the sequence  $u_n(z)$  was constructed in Theorem 3.4.

**Lemma 4.7.** *Let*

$$v_m(z) = \operatorname{sgn} \beta_\infty \varkappa_{m-1} (\sigma_m^{-1} - 1)^{-1} (u_{m-1}(z) u_m(z))^{-1} \quad (4.14)$$

and

$$t_m = e^{\varphi_{m-1} + \varphi_m}. \quad (4.15)$$

Then

$$(a_{m-1} f_{m-1}(z) f_m(z))^{-1} = v_m(z) t'_m. \quad (4.16)$$

*Proof.* It follows from formulas (2.21) and (3.21) that

$$a_{m-1} f_{m-1} f_m = \operatorname{sgn} \beta_\infty \frac{a_{m-1}}{\sqrt{a_{m-1} a_m}} u_{m-1} u_m t_m^{-1},$$

whence

$$(a_{m-1} f_{m-1} f_m)^{-1} = \operatorname{sgn} \beta_\infty \varkappa_{m-1} (u_{m-1} u_m)^{-1} t_m.$$

By definitions (2.22) and (4.15), we have

$$t'_m = t_m (\sigma_m^{-1} - 1).$$

Putting together the last two formulas with the definition (4.14) of  $v_m$ , we arrive at (4.16).  $\square$

Using (4.16) and integrating by parts in (4.12), we find that

$$G_n = \sum_{m=n_0}^n v_m t'_m = v_n t_{n+1} - v_{n_0-1} t_{n_0} - \widetilde{G}_n \quad \text{where} \quad \widetilde{G}_n = \sum_{m=n_0}^n v'_{m-1} t_m. \quad (4.17)$$

We will see that the asymptotics of sequence (4.11) as  $n \rightarrow \infty$  is determined by the first term in the right-hand side of (4.17). Let us calculate it.

**Lemma 4.8.** *The asymptotic relation*

$$\lim_{n \rightarrow \infty} \sqrt{a_n} e^{-\varphi_n} (\operatorname{sgn} \beta_\infty)^n f_n(z) v_n(z) t_{n+1} = \frac{\operatorname{sgn} \beta_\infty}{2\sqrt{\beta_\infty^2 - 1}} \varkappa_\infty \quad (4.18)$$

holds.

*Proof.* It follows from Theorem 3.13 that

$$\sqrt{a_n} (\operatorname{sgn} \beta_\infty)^n f_n(z) = e^{-\varphi_n} (1 + o(1)).$$

Using definition (4.14) of  $v_n$ , Lemma 2.1 and Theorem 3.4 we find that

$$v_n(z) = \operatorname{sgn} \beta_\infty \varkappa_\infty (\zeta_\infty^{-2} - 1)^{-1} (1 + o(1)).$$

Since according to (4.15)

$$t_{n+1} = e^{2\varphi_n + \vartheta_n} = e^{2\varphi_n} \zeta_\infty^{-1} (1 + o(1)),$$

the limit (4.18) equals  $\operatorname{sgn} \beta_\infty \varkappa_\infty \zeta_\infty^{-1} (\zeta_\infty^{-2} - 1)^{-1}$  which in view of (2.18) coincides with the right-hand side of (4.18).  $\square$

Next we show that the remainder  $\widetilde{G}_n(z)$  in (4.17) is negligible. We use the following technical assertions.

**Lemma 4.9.** *The sequence  $v_n(z)$  defined by (4.14) satisfies the condition  $\{v'_n\} \in \ell^1(\mathbb{Z}_+)$ .*

*Proof.* According to (2.14) we have

$$|\zeta_n| = (|\beta_n| + \sqrt{\beta_n^2 - 1})^{-1} < |\beta_\infty|^{-1} < 1$$

for sufficiently large  $n$ . Moreover, it follows from (2.27) that  $\{\zeta'_n\} \in \ell^1(\mathbb{Z}_+)$  if  $\{\beta'_n\} \in \ell^1(\mathbb{Z}_+)$  whence

$$((\sigma_n^{-1} - 1)^{-1})' \in \ell^1(\mathbb{Z}_+).$$

Putting together Theorem 3.4 and Lemma 3.8 we see that

$$(u_{n-1}^{-1} u_n^{-1})' \in \ell^1(\mathbb{Z}_+).$$

Now the inclusion  $\{v'_n\} \in \ell^1(\mathbb{Z}_+)$  is a direct consequence of definition (4.14).  $\square$

**Lemma 4.10.** *Let  $\tilde{G}_n(z)$  be defined in formula (4.17). Then*

$$\lim_{n \rightarrow \infty} \sqrt{a_n} e^{-\varphi_n} f_n(z) \tilde{G}_n(z) = 0. \quad (4.19)$$

*Proof.* Using Theorem 3.13 and definition (4.15), we see that

$$\begin{aligned} \sqrt{a_n} e^{-\varphi_n} |f_n \tilde{G}_n| &\leq 2e^{-2\varphi_n} \sum_{m=n_0}^n |v'_{m-1}| e^{\varphi_{m-1} + \varphi_m} \\ &\leq 2e^{-\varphi_n} \sum_{n_0 \leq m < [n/2]} |v'_{m-1}| + 2 \sum_{[n/2] \leq m \leq n} |v'_{m-1}|. \end{aligned}$$

Since  $\varphi_n \rightarrow \infty$  as  $n \rightarrow \infty$ , relation (4.19) follows from Lemma 4.9.  $\square$

Let us come back to the representations (4.11) and (4.17). Putting together Lemmas 4.8, 4.10 and taking into account that the term  $v_{n_0-1} t_{n_0}$  is negligible, we obtain the asymptotics of  $g_n(z)$ .

**Theorem 4.11.** *Under the assumptions of Theorem 3.13 the solution (1.17) of the Jacobi equation (1.7) satisfies for all  $z \in \mathbb{C}$  the relation*

$$\lim_{n \rightarrow \infty} \sqrt{a_n} e^{-\varphi_n} (\operatorname{sgn} \beta_\infty)^{n+1} g_n(z) = \frac{\varkappa_\infty}{2\sqrt{\beta_\infty^2 - 1}}. \quad (4.20)$$

*Convergence here is uniform on compact subsets of  $\mathbb{C}$ .*

It follows from (4.20) that  $\{g_n(z)\} \in \ell^2(\mathbb{Z}_+)$  (for all  $z \in \mathbb{C}$ ) if and only if

$$\sum_{n=0}^{\infty} a_n^{-1} e^{2\varphi_n} < \infty. \quad (4.21)$$

In this case all solutions of equation (1.7) are in  $\ell^2(\mathbb{Z}_+)$ . Therefore we can state

**Corollary 4.12.** *Under the assumptions of Theorem 3.13 the minimal Jacobi operator  $J_0$  is essentially self-adjoint if and only if condition (1.19) is satisfied. Otherwise, that is under assumption (4.21), the operator  $J_0$  has deficiency indices (1, 1).*

For an arbitrary  $\varepsilon > 0$  and sufficiently large  $n$ , it follows from (2.23) that  $\vartheta_\infty n - \varepsilon \leq \varphi_n \leq \vartheta_\infty n + \varepsilon$ . Therefore under the assumptions of Theorem 3.13, the operator  $J_0$  is essentially self-adjoint if  $a_n \leq cn^p$  for some  $c > 0$ ,  $p < \infty$  and all  $n \geq 1$ . On the contrary,  $J_0$  has deficiency indices (1, 1) if  $a_n \geq cx^{n^p}(1 + o(1))$  where  $x > 1$  and  $p > 1$ .

Since the Wronskian  $\{f(z), g(z)\} = 1$  by Theorem 4.6, we see that

$$P_n(z) = \omega(z) f_n(z) - \Omega(z) g_n(z)$$

where  $\omega(z) = \{P(z), g(z)\}$  and

$$\Omega(z) := \{P(z), f(z)\} = -2^{-1} f_{-1}(z) \quad (4.22)$$

is the Jost function. Obviously,  $\omega(z) \neq 0$  if  $\Omega(z) = 0$ . Therefore Theorems 3.13 and 4.11 imply the following result.

**Theorem 4.13.** *Under the assumptions of Theorem 3.13 the relation*

$$P_n(z) = -\kappa_\infty \Omega(z) \frac{(\operatorname{sgn} \beta_\infty)^{n+1}}{2\sqrt{\beta_\infty^2 - 1}} \frac{e^{\varphi_n}}{\sqrt{a_n}} (1 + o(1)), \quad n \rightarrow \infty, \quad (4.23)$$

is true for all  $z \in \mathbb{C}$  with convergence uniform on compact subsets of  $\mathbb{C}$ . Moreover, if  $\Omega(z) = 0$ , then

$$\lim_{n \rightarrow \infty} \sqrt{a_n} e^{\varphi_n} (\operatorname{sgn} \beta_\infty)^n P_n(z) = \{P(z), g(z)\} \neq 0.$$

**Remark 4.14.** Let assumption (2.24) be satisfied. Similarly to Remark 3.14, we can replace  $e^{-i\phi_n}$  and  $e^{-\varphi_n}$  by expressions (3.28) in all assertions of this section. In particular, this is true for asymptotic formulas for orthogonal polynomials  $P_n(z)$  and condition (4.21) reduces to

$$\sum_{n=0}^{\infty} a_n^{-1} (|\beta_\infty| + \sqrt{\beta_\infty^2 - 1})^{2n} < \infty.$$

**4.4. Resolvent.** If condition (1.19) is satisfied, then the closure  $J$  of the minimal Jacobi operator  $J_0$  is self-adjoint. In this case  $\Omega(z) = 0$  if and only if  $z$  is an eigenvalue of  $J$ . The resolvent of the operator  $J$  can be constructed by the standard (cf. Lemma 2.6 in [17] or Lemma 5.1 in [19]) formulas. Recall that  $e_n$ ,  $n \in \mathbb{Z}_+$ , is the canonical basis in  $\ell^2(\mathbb{Z}_+)$ ,  $f_n(z)$  is the Jost solution of the equation (1.7) constructed in Theorem 3.13 and  $\Omega(z)$  is the Wronskian (4.22).

**Proposition 4.15.** *Under the assumptions of Theorem 3.13 suppose also that condition (1.19) holds. Then the resolvent  $(J - z)^{-1}$  of the Jacobi operator  $J$  satisfies the relations*

$$((J - z)^{-1} e_n, e_m) = \Omega(z)^{-1} P_n(z) f_m(z), \quad \operatorname{Im} z \neq 0, \quad (4.24)$$

if  $n \leq m$  and  $((J - z)^{-1} e_n, e_m) = ((J - z)^{-1} e_m, e_n)$ .

Since in view of Theorem 3.13,  $f_n(z)$  and hence  $\Omega(z)$  are entire functions of  $z \in \mathbb{C}$ , we can state

**Corollary 4.16.** *The spectrum of the operator  $J$  is discrete, and its eigenvalues  $\lambda_1, \dots, \lambda_k, \dots$  are given by the equation  $\Omega(\lambda_k) = 0$ . The resolvent  $(J - z)^{-1}$  is an analytic function of  $z \in \mathbb{C}$  with poles in the points  $\lambda_1, \dots, \lambda_k, \dots$ .*

A much more general result of this type is stated below as Proposition 5.1.

In view of formula (4.22) and equation (1.7) where  $n = 0$  for the Jost solution  $f_n(z)$ , the equation for eigenvalues of  $J$  can be also written as

$$(b_0 - \lambda_k) f_0(\lambda_k) + a_0 f_1(\lambda_k) = 0.$$

It follows from representation (4.24) where  $n = m = 0$  and formula (4.22) that, up to terms regular in a neighborhood of the point  $\lambda_k$ ,

$$((J - z)^{-1} e_0, e_0) = 2 \frac{f_0(\lambda_k)}{f_{-1}(\lambda_k)} \frac{1}{\lambda_k - z}, \quad \text{where} \quad f_{-1}(\lambda) = \frac{d f_{-1}(\lambda)}{d \lambda}. \quad (4.25)$$

On the other hand, since

$$((J - z)^{-1} e_0, e_0) = \int_{-\infty}^{\infty} \frac{d\rho(\lambda)}{\lambda - z}$$

by the definition of the spectral measure, we see that

$$((J - z)^{-1} e_0, e_0) = \frac{\rho(\{\lambda_k\})}{\lambda_k - z},$$

again up to terms regular in a neighborhood of the point  $\lambda_k$ . Comparing this expression with (4.25), we arrive to a standard formula

$$\rho(\{\lambda_k\}) = 2 \frac{f_0(\lambda_k)}{\dot{f}_{-1}(\lambda_k)}.$$

Another well known formula

$$\rho(\{\lambda\}) = \left( \sum_{n=0}^{\infty} P_n(\lambda)^2 \right)^{-1} \quad (4.26)$$

for the spectral measure at an eigenvalue  $\lambda$  of a self-adjoint Jacobi operator  $J$  can be obtained in the following way. Let  $J\psi = \lambda\psi$  where the eigenvector  $\psi = (\psi_0, \dots, \psi_n, \dots)^\top$  is normalized. Necessarily,  $\psi_n = cP_n(\lambda)$  and, by the normalization,  $c^2 \sum P_n(\lambda)^2 = 1$ . Therefore  $\rho(\{\lambda\}) = (\psi, e_0)^2 = \psi_0^2 = c^2$  is given by formula (4.26). Formula (4.26) is true without any assumptions on the coefficients  $a_n, b_n$  for an arbitrary self-adjoint extension  $J$  of the minimal Jacobi operator  $J_0$ . Note however that in the essentially self-adjoint case the condition  $\sum P_n(\lambda)^2 < \infty$  determines eigenvalues of the operator  $J = \text{clos } J_0$ . This is not true if the deficiency indices are  $(1, 1)$  when  $\sum P_n(\lambda)^2 < \infty$  for all  $\lambda$ .

## 5. DISCUSSION

**5.1. Operators with discrete spectrum.** We start with a very general result which, in particular, applies to Jacobi operators. An assertion below is essentially known, and we give it mainly for completeness of our presentation. The operator  $J$  will be defined via its quadratic form

$$(Ju, u) = \sum_{n=0}^{\infty} b_n |u_n|^2 + 2 \operatorname{Re} \sum_{n=0}^{\infty} a_n u_n \bar{u}_{n+1} \quad (5.1)$$

where  $a_n$  are complex and  $b_n$  are real numbers.

**Proposition 5.1.** *Suppose that  $b_n \rightarrow \infty$  (or  $b_n \rightarrow -\infty$ ) and that*

$$|a_{n-1}| + |a_n| \leq \epsilon |b_n| \quad (5.2)$$

*for some  $\epsilon < 1$  and all sufficiently large  $n$ . Then the spectrum of the operator  $J$  is discrete and is semi-bounded from below (resp., from above).*

*Proof.* Since finite-rank perturbations cannot change the discreteness of spectrum, we can suppose that estimate (5.2) is true for all  $n$ . Let  $B$  be the operator corresponding to the form

$$(Bu, u) = \sum_{n=0}^{\infty} b_n |u_n|^2. \quad (5.3)$$

Evidently, its spectrum is discrete and is semi-bounded from below (resp., from above). For a proof of the same result for the operator  $J$ , it suffices to check that the forms (5.1) and (5.3) are equivalent or that

$$2 \left| \operatorname{Re} \sum_{n=0}^{\infty} a_n u_n \bar{u}_{n+1} \right| \leq \epsilon \sum_{n=0}^{\infty} |b_n| |u_n|^2 \quad (5.4)$$

for some  $\epsilon < 1$ . Let us use the following obvious inequality

$$2 \left| \sum_{n=0}^{\infty} a_n u_n \bar{u}_{n+1} \right| \leq 2 \sqrt{\sum_{n=0}^{\infty} |a_n| |u_n|^2 \sum_{n=0}^{\infty} |a_n| |u_{n+1}|^2} \leq \sum_{n=0}^{\infty} (|a_{n-1}| + |a_n|) |u_n|^2.$$

Therefore estimate (5.4) is a direct consequence of the condition (5.2).  $\square$

Proposition 5.1 holds, in particular, for semi-bounded Jacobi operators when  $a_n > 0$ . It applies directly to the Friedrichs' extension of the operator  $J_0$ , but its conclusion remains true for all extensions  $J$  of  $J_0$  because the deficiency indices of  $J_0$  are finite. In particular, condition (5.2) is satisfied with any  $\epsilon \in (|\beta_\infty|^{-1}, 1)$  under the assumptions of Theorem 3.13. Thus, we recover the result of Corollary 4.16. Moreover, the corresponding operator  $J$  is semi-bounded from below (from above) if  $\beta_\infty < -1$  (if  $\beta_\infty > 1$ , respectively).

**5.2. The Carleman case.** In this subsection we still consider the case  $a_n \rightarrow \infty$  but assume that the Carleman condition (1.2) holds. Other assumptions on  $a_n$  and  $b_n$  are essentially the same as in the main part of this paper. In particular, we suppose that condition (1.5) is satisfied. Recall that under the Carleman condition the minimal Jacobi operator  $J_0$  is essentially self-adjoint. The results stated here will be published elsewhere, but some of them are close to the papers [5, 2, 14].

Now the term  $2z\alpha_n$  in the right-hand side of (2.10) cannot be neglected and the numbers  $\zeta_n$  have to be defined as approximate solutions of the equation

$$\zeta_n + \zeta_n^{-1} = 2\beta_n + 2z\alpha_n.$$

For simplicity of our discussion, we assume here that

$$\sum_{n=0}^{\infty} a_n^{-3} (1 + |b_n|) < \infty. \quad (5.5)$$

Conditions (1.2) and (5.5) admit a growth of the off-diagonal coefficients  $a_n$  as  $n^p$  for  $p \in (1/2, 1]$  and even for  $p \in (1/3, 1]$  if  $b_n = 0$ .

Suppose first that  $|\beta_\infty| < 1$ . As before, the Ansatz  $Q_n(z)$  is defined by formula (2.9), but, instead of (2.13), we now set

$$\zeta_n = (\beta_n - i\sqrt{1 - \beta_n^2}) \exp\left(iz \frac{\alpha_n}{\sqrt{1 - \beta_n^2}}\right). \quad (5.6)$$

An easy calculation shows that the corresponding remainder (2.10) is in  $\ell^1(\mathbb{Z}_+)$ . Therefore repeating the arguments of Sect. 2 and 3, we find that equation (1.7) has a solution (the Jost solution)  $f_n(z)$  distinguished by the asymptotics

$$f_n(z) = a_n^{-1/2} e^{-i\phi_n + iz\psi_n} (1 + o(1)), \quad \text{Im } z \geq 0, \quad n \rightarrow \infty, \quad (5.7)$$

where the phase  $\phi_n$  is defined by formula (1.6) and

$$\psi_n = \sum_{m=0}^{n-1} \frac{\alpha_m}{\sqrt{|1 - \beta_m^2|}}. \quad (5.8)$$

Observe that  $\psi_n \rightarrow \infty$  as  $n \rightarrow \infty$  according to (1.2), (1.5), but  $\psi_n = O(n^{2/3})$  under assumption (5.5). It now follows from (2.23) that the phase  $\psi_n$  is negligible compared to  $\phi_n$ . It is easy to see that  $f_n(z) \in \ell^2(\mathbb{Z}_+)$  for  $\text{Im } z > 0$ . The Jost functions  $f_n(z)$  are now analytic in the half-plane  $\text{Im } z > 0$  and are continuous up to the real line. We set  $f_n(z) = \overline{f_n(\bar{z})}$  for  $\text{Im } z \leq 0$ . Then the functions  $f_n(z)$  are analytic in the complex plane with the cut along  $\mathbb{R}$  and are continuous up to the cut.

If  $z = \lambda \in \mathbb{R}$ , we have two Jost solutions  $f_n(\lambda + i0)$  and  $f_n(\lambda - i0) = \overline{f_n(\lambda + i0)}$  of equation (1.7). Their Wronskian is the same as (3.25) (where  $\varkappa_\infty = 1$ ):

$$\{f(\lambda + i0), f(\lambda - i0)\} = 2i\sqrt{1 - \beta_\infty^2} \neq 0.$$

Therefore the polynomials  $P_n(\lambda)$  are linear combinations of  $f(\lambda + i0)$  and  $f(\lambda - i0)$ :

$$P_n(\lambda) = \frac{\Omega(\lambda - i0)f_n(\lambda + i0) - \Omega(\lambda + i0)f_n(\lambda - i0)}{2i\sqrt{1 - \beta_\infty^2}}, \quad n = 0, 1, 2, \dots, \quad \lambda \in \mathbb{R}, \quad (5.9)$$



where  $\Omega(z)$  is defined by formula (4.22). Note that  $\Omega(\lambda \pm i0) \neq 0$  for all  $\lambda \in \mathbb{R}$  according to (5.9). It now follows from (5.7) for  $z = \lambda \in \mathbb{R}$  that the polynomials  $P_n(\lambda)$  have asymptotics

$$P_n(\lambda) = -a_n^{-1/2} \left( |\Omega(\lambda + i0)| (1 - \beta_\infty^2)^{-1/2} \sin(\phi_n - \lambda \psi_n + \arg \Omega(\lambda + i0)) + o(1) \right), \quad n \rightarrow \infty. \quad (5.10)$$

If  $\text{Im } z \neq 0$ , then by virtue of Theorem 4.6 a solution  $g_n(z)$  of the Jacobi equation (1.7) linear independent with  $f_n(z)$  can be constructed by explicit formula (1.17). Similarly to Theorem 4.11, it can be checked that

$$g_n(z) = \frac{1}{2i\sqrt{1 - \beta_\infty^2}} \frac{1}{\sqrt{a_n}} e^{i\phi_n - iz\psi_n} (1 + o(1)), \quad \text{Im } z > 0, \quad n \rightarrow \infty,$$

so that  $g_n(z)$  grows faster than any power of  $n$  as  $n \rightarrow \infty$ . Since the Wronskian  $\{f(z), g(z)\} = 1$ , the asymptotics of the orthonormal polynomials can be easily derived from this result:

$$P_n(z) = \frac{i\Omega(z)}{2\sqrt{1 - \beta_\infty^2}} \frac{1}{\sqrt{a_n}} e^{i\phi_n - iz\psi_n} (1 + o(1)), \quad \text{Im } z > 0, \quad n \rightarrow \infty. \quad (5.11)$$

Now the Jacobi operator  $J = \text{clos } J_0$  is self-adjoint, and its resolvent  $(J - z)^{-1}$  is given by the general formula (4.24). Since the Jost solutions  $f_n(z)$  are continuous functions of  $z$  up to the real axis and  $\Omega(\lambda \pm i0) \neq 0$  for  $\lambda \in \mathbb{R}$ , the spectrum of the Jacobi operator  $J$  is absolutely continuous. Moreover, using relation (5.9), it is easy to deduce from (4.24) a representation for the spectral family  $dE(\lambda)$  of the operator  $J$ :

$$\frac{d(E(\lambda)e_n, e_m)}{d\lambda} = \pi^{-1} \sqrt{1 - \beta_\infty^2} |\Omega(\lambda + i0)|^{-2} P_n(\lambda) P_m(\lambda).$$

In particular, for the spectral measure, we have the expression

$$d\rho(\lambda) = \pi^{-1} \sqrt{1 - \beta_\infty^2} |\Omega(\lambda + i0)|^{-2} d\lambda.$$

It follows that the spectrum of  $J$  coincides with the whole real axis.

Formulas (5.10) and (5.11) are consistent with the classical asymptotic expressions for the Hermite polynomials when  $a_n = \sqrt{(n+1)/2}$  and  $b_n = 0$  (see, e.g., Theorems 8.22.6 and 8.22.7 in the G. Szegő's book [15]). Asymptotics (5.10) was obtained earlier in [5, 2, 14], but as far as (5.11) is concerned, we are aware only of the paper [11] where an asymptotics of  $|P_n(z)|$  as  $n \rightarrow \infty$  was found in terms of a behavior of the corresponding absolutely continuous spectral measure for  $|\lambda| \rightarrow \infty$ .

The cases  $|\beta_\infty| > 1$  and  $|\beta_\infty| < 1$  are technically rather similar, but the asymptotic behavior of orthogonal polynomials and spectral properties of the Jacobi operators are quite different in these cases. If  $|\beta_\infty| > 1$ , we set (cf. (5.6))

$$\zeta_n = \text{sgn } \beta_n (|\beta_n| - \sqrt{\beta_n^2 - 1}) \exp \left( -z \frac{\alpha_n}{\sqrt{\beta_n^2 - 1}} \right).$$

Then again the corresponding remainder (2.10) is in  $\ell^1(\mathbb{Z}_+)$ . Therefore repeating the arguments of Sect. 2 and 3, we find that equation (1.7) has a solution (the Jost solution)  $f_n(z)$  distinguished by the asymptotics

$$f_n(z) = a_n^{-1/2} (\text{sgn } \beta_\infty)^n e^{-\varphi_n - z\psi_n} (1 + o(1)), \quad n \rightarrow \infty,$$

where  $\varphi_n$  and  $\psi_n$  are defined by formulas (1.9) and (5.8), respectively. Now the functions  $f_n(z)$  are analytic in the whole complex plane. The second solution  $g_n(z)$  of the Jacobi equation (1.7) is again given by equality (1.17) which leads to the asymptotic formula

$$P_n(z) = -\Omega(z) \frac{(\text{sgn } \beta_\infty)^{n+1}}{2\sqrt{\beta_\infty^2 - 1}} \frac{e^{\varphi_n + z\psi_n}}{\sqrt{a_n}} (1 + o(1)), \quad n \rightarrow \infty. \quad (5.12)$$

As in Sect. 4.4, the resolvent of  $J$  is determined by formula (4.24), but, in contrast to the case  $|\beta_\infty| < 1$ , its singularities are due to zeros of the denominator  $\Omega(z)$  only. Therefore the spectrum of  $J$  is discrete which is consistent with Proposition 5.1.

**5.3. The non-Carleman case versus Carleman one.** Let us now compare the results of this paper for the non-Carleman case (1.4) with those described in the previous subsection.

Suppose first that  $|\beta_\infty| < 1$ . According to Theorem 3.9 all solutions of equation (1.7) for every  $z \in \mathbb{C}$  belong to  $\ell^2(\mathbb{Z}_+)$  so that the Jacobi operator  $J_0$  has a one parameter family of self-adjoint extensions  $J$ . In this case formula (1.7) for the resolvents of the operators  $J$  makes no sense. Nevertheless the scalar products  $((J - z)^{-1}e_0, e_0)$ , that is, the Cauchy-Stieltjes transforms of the corresponding spectral measures  $d\rho_J(\lambda)$ , can be expressed via the orthogonal polynomials of the first  $P_n(z)$  and of the second  $\tilde{P}_n(z)$  kinds by the Nevanlinna formula obtained by him in [9] (see, e.g., formulas (2.31), (2.32) in the book [1]). This remarkable formula implies, in particular, that the spectra of all self-adjoint extensions  $J$  of  $J_0$  are discrete. The spectral measure at eigenvalues of  $J$  is given by formula (4.26). Our construction is quite independent of the Nevanlinna theory, but if some link with this theory exists, it would be desirable to find it.

Asymptotic formulas (4.7) and (5.10) look rather similar although the phase in (5.10) contains an additional term  $\lambda\psi_n$ . The amplitude factor  $a_n^{-1/2}$  in (4.7) and (5.10) is the same. However, under assumption (1.2) the sequence (5.10) never belongs to  $\ell^2(\mathbb{Z}_+)$  while under assumption (1.4) the sequence (4.7) belongs to this space for all  $\lambda \in \mathbb{R}$ .

The difference between the Carleman and non-Carleman cases is even more obvious for  $\text{Im } z \neq 0$ . According to Theorem 4.1 in the non-Carleman case, all solutions of the Jacobi equation oscillate as  $n \rightarrow \infty$ , but they are in  $\ell^2(\mathbb{Z}_+)$  due to the amplitude factor  $a_n^{-1/2}$ . In the Carleman case, the solution  $f_n(z)$  exponentially decays while the solution  $g_n(z)$  exponentially grows as  $n \rightarrow \infty$ .

In the case  $|\beta_\infty| > 1$ , the asymptotic formulas (4.23) and (5.12) for orthogonal polynomials are close to each other, but the first of them contains an additional factor  $\varkappa_\infty$  while (5.12) contains a factor  $e^{z\psi_n}$ . We emphasize that in the Carleman case (1.2) the Jacobi operators  $J_0$  are essentially self-adjoint for all sequences  $b_n$  while in the non-Carleman case (1.4) the Jacobi operators  $J_0$  are essentially self-adjoint if and only if condition (1.19) is satisfied (see Corollary 4.12).

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