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# Defining a surreal hyperexponential

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Conway's class  $\mathbf{No}$  of surreal numbers admits a rich structure: it forms a totally ordered real closed field with an exponential functions and a derivation. The aim of this note is to construct a surreal solution  $E_\omega$  to the functional equation  $E_\omega(a+1) = \exp E_\omega(a)$  with good properties.

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## 1. INTRODUCTION

Let  $\mathbf{No}$  be Conway's class of surreal numbers. It is well known that  $\mathbf{No}$  admits a rich structure: Conway showed that  $\mathbf{No}$  forms a real closed field and Gonshor also defined an exponential function  $\exp$  on  $\mathbf{No}$  that satisfies the same first order theory as the usual exponential function on the reals. Following Conway's tradition, all *numbers* will understood to be surreal in what follows.

The aim of this note is to define a bijective function  $E_\omega: \mathbf{No}^{>>} \rightarrow \mathbf{No}^{>>}$  on the class  $\mathbf{No}^{>>}$  of positive infinitely large numbers, which is strictly increasing and satisfies the functional equation

$$E_\omega(a+1) = \exp E_\omega a \tag{1}$$

for all  $a \in \mathbf{No}^{>>}$ . Since this equation admits many solutions, the main difficulty is to single out a particular solution that will be most "natural" in a way that needs to be made precise.

The function  $E_\omega$  is said *hyperexponential* and it is the first non-trivial hyperexponential in the transfinite sequence  $(E_\alpha)_{\alpha \in \text{Ord}}$  of iterated exponentials

$$E_0 := \text{Id}, E_1 := \exp, E_2 := \exp \circ \exp, \dots, E_\omega, E_{\omega+1} := E_\omega \circ \exp, \dots, E_{2\omega} := E_\omega \circ E_\omega, \dots$$

The corresponding reciprocals are called *hyperlogarithms*:

$$L_0 := \text{Id}, L_1 := \log, L_2 := \log \circ \log, \dots, L_\omega, L_{\omega+1} := \log \circ L_\omega, \dots, L_{2\omega} := L_\omega \circ L_\omega, \dots$$

It is natural to require such more general hyperexponentials to satisfy  $E_{\omega^{\alpha+1}}(a+1) = E_{\omega^\alpha}(E_{\omega^{\alpha+1}}(a))$  for all ordinals  $\alpha$  and  $a \in \mathbf{No}^{>, >}$ . Similarly,  $L_{\omega^{\alpha+1}}(L_{\omega^\alpha}(a)) = L_{\omega^{\alpha+1}}(a) - 1$ .

There are known real analytic solutions of (1) with good properties [12, 6], even though there does not seem to exist any meaningful “most natural” solution. It is also well known that fractional iterates of  $\exp$  and  $\log$  can be defined in terms of  $E_\omega$  and  $L_\omega$ : given  $c \in \mathbb{R}$ , we take  $\exp_c(x) := E_\omega(L_\omega(x) + c)$  and  $\log_c(x) := E_\omega(L_\omega(x) - c)$ .

From a formal perspective, hyperexponentials and hyperseries were studied in detail by Schmeling and van der Hoeven [14]: they generalized transseries to include formal counterparts  $e_\alpha, \ell_\alpha$  of  $E_\alpha, L_\alpha$  for  $\alpha < \omega^\omega$  [14]. This yields in particular a natural hyperexponential on the set of positive infinitely large transseries. More recently, van den Dries, van der Hoeven and Kaplan [7] constructed the field of *logarithmic hyperseries*  $\mathbb{L}$  with  $\ell_\alpha$  for all  $\alpha$ , with corresponding natural hyperlogarithms  $L_\alpha: \mathbb{L}^{>, >} \rightarrow \mathbb{L}^{>, >}$ . The ultimate goal [11, 2] is to produce a field of hyperseries  $\mathbb{H} \supseteq \mathbb{L}$  with all hyperexponentials and to construct an isomorphism  $\mathbb{H} \cong \mathbf{No}$ . This work can be considered as another step in this direction, by constructing a natural  $E_\omega: \mathbf{No}^{>, >} \rightarrow \mathbf{No}^{>, >}$ .

**Surreal numbers and transseries** It is well known that surreal numbers  $a \in \mathbf{No}$  can be written as infinite series

$$a = \sum_{m \in \mathbf{Mo}} a_m m,$$

where  $\mathbf{Mo}$  denotes the class of surreal *monomials* and the coefficients  $a_m$  are real. In particular,  $\mathbf{No}$  is isomorphic to the Hahn field  $\mathbb{R}[[\mathbf{Mo}]]$  of formal power series. Together with the exponential function,  $\mathbf{No}$  even admits the structure of a field of *transseries* in the sense of [14]; see [4].

We will freely use notations from [1, 11] when dealing with such transseries. In particular, the *support* of  $a \in \mathbf{No}$  is defined as  $\text{supp } a := \{m \in \mathbf{Mo} : a_m \neq 0\}$  and its *infinite part* as  $a_{>} := \sum_{m \in \mathbf{Mo}, m > 1} a_m m$ . Here we used Hardy’s notation  $<$  for  $a < b \Leftrightarrow \mathbb{R}^{>} |a| < |b|$ . Similarly, we set  $a_{<} := \sum_{m \in \mathbf{Mo}, m < 1} a_m m$ . We also define  $\mathbf{No}^{>} := \{a \in \mathbf{No} : a > 1\}$ ,  $\mathbf{No}_{>} := \{a_{>} : a \in \mathbf{No}\}$ , and  $\mathbf{No}^{<} := \{a \in \mathbf{No} : a < 1\} = \{a_{<} : a \in \mathbf{No}\}$ . Finally, we write  $a \triangleleft b$  if  $\text{supp } a > a - b$ , in which case we say that  $a$  is a *truncation* of  $b$ .

**Defining exponentials** Consider a number  $a \in \mathbf{No}$  and decompose it as  $a = a_{>} + a_{=} + a_{<}$  with  $a_{>} \in \mathbf{No}_{>}$ ,  $a_{=} \in \mathbb{R}$ , and  $a_{<} \in \mathbf{No}^{<}$ . Then the functional equation of  $\exp$  yields

$$\exp a = e^{a_{>}} \exp a_{=} \exp a_{<}, \quad (2)$$

where  $\exp a_{=}$  is the usual exponential in  $\mathbb{R}$  and  $\exp a_{<} = 1 + a_{<} + \frac{1}{2} a_{<}^2 + \frac{1}{6} a_{<}^3 + \dots$ . In order to define  $\exp$  on  $\mathbf{No}^{>}$ , this relation shows that it would have sufficed to define it on  $\mathbf{No}_{>}$ . In addition, it can be shown that  $\exp$  bijectively maps the class  $\mathbf{No}_{>}$  to the class  $\mathbf{Mo}$ . Our process to define  $E_\omega$  is similar, with different subclasses  $\mathbf{Tr}$  and  $\mathbf{La}$  in the roles of  $\mathbf{No}_{>}$  and  $\mathbf{Mo}$ . The class  $\mathbf{Tr}$  is defined below and  $\mathbf{La}$  is the class of *log-atomic numbers*, i.e. numbers  $a \in \mathbf{No}^{>, >}$  such that  $\log^{\circ n} a \in \mathbf{Mo}$  for all  $n \in \mathbb{N}$ .

**Surreal substructures** It turns out that each of the classes  $\mathbf{No}_{>}$ ,  $\mathbf{Mo}$ ,  $\mathbf{Tr}$ , and  $\mathbf{La}$  are examples of so-called surreal substructures, which were extensively studied in [3]. Let us quickly recall a few basic facts; see also section 2.

Besides the usual ordering, the class  $\mathbf{No}$  of surreal numbers admits a well-founded partial order  $\sqsubseteq$  called the *simplicity relation*. A *surreal substructure* is a subclass  $\mathbf{S}$  of  $\mathbf{No}$  that is isomorphic to  $(\mathbf{No}, \leq, \sqsubseteq)$  for the induced relations by  $\leq$  and  $\sqsubseteq$  on  $\mathbf{S}$ . Equivalently, this means that for any subsets  $L, R$  of  $\mathbf{S}$  with  $L < R$ , the *cut*

$$(L | R)_{\mathbf{S}} := \{a \in \mathbf{S} : L < a < R\}$$

in  $\mathbf{S}$  admits a  $\sqsubseteq$ -minimum which is then denoted  $\{L | R\}_{\mathbf{S}}$ . We call  $(L, R)$  a *cut representation* in  $\mathbf{S}$ . We extend this notation to the case when  $\mathbf{L}, \mathbf{R}$  are classes, provided  $\{\mathbf{L} | \mathbf{R}\}_{\mathbf{S}}$  indeed exists. If  $a \in \mathbf{S}$ , then we let

$$a_L^{\mathbf{S}} := \{a' \in \mathbf{S} : a' \sqsubseteq a \wedge a' < a\}, \quad a_R^{\mathbf{S}} := \{a'' \in \mathbf{S} : a'' \sqsubseteq a \wedge a'' > a\}, \quad a_{\mathbf{C}}^{\mathbf{S}} := a_L^{\mathbf{S}} \sqcup a_R^{\mathbf{S}}.$$

Then  $a = \{a_L^{\mathbf{S}} | a_R^{\mathbf{S}}\}_{\mathbf{S}}$ . Moreover, for any cut  $(L | R)_{\mathbf{S}}$  in  $\mathbf{S}$  containing  $a$  (such as  $a = \{L | R\}_{\mathbf{S}}$ ), the set  $L$  (resp.  $R$ ) is cofinal (resp. coinital) with respect to  $a_L^{\mathbf{S}}$  (resp.  $a_R^{\mathbf{S}}$ ).

Important examples of surreal substructures include  $\mathbf{No}$ , the class  $\mathbf{No}^>$  of strictly positive numbers, the class  $\mathbf{No}^{>>}$  of positive infinitely large numbers, the classes  $\mathbf{Mo}$  and  $\mathbf{Mo}^>$  of monomials and infinite monomials, the class  $\mathbf{No}_{>} := \mathbb{R}[[\mathbf{Mo}^>]]$  of purely infinite numbers, and the class  $\mathbf{No}^< := \mathbb{R}[[\mathbf{Mo}^>]]$  of infinitesimal numbers.

**Truncated numbers** Assume that  $E_{\omega} a$  has been defined for some positive purely infinite number  $a \in \mathbf{No}_{>}^> := \mathbf{No}_{>} \cap \mathbf{No}^>$ . For sufficiently small  $\varepsilon$ , we wish to define  $E_{\omega}(a + \varepsilon)$  using Taylor series expansion:

$$E_{\omega}(a + \varepsilon) := E_{\omega} a + (E'_{\omega} a) \varepsilon + \frac{1}{2} (E''_{\omega} a) \varepsilon^2 + \dots \quad (3)$$

The successive derivatives  $E'_{\omega} a, E''_{\omega} a, \dots$  can be defined in  $\mathbf{No}$  as ordinary (and so-called logarithmic) transseries applied to  $b := E_{\omega} a$ :

$$\begin{aligned} E'_{\omega} a &:= b(L_1 b)(L_2 b) \dots \\ E''_{\omega} a &= b(L_1 b)^2(L_2 b)^2 \dots + b(L_1 b)(L_2 b)^2(L_3 b)^2 + \dots \\ &\vdots \end{aligned}$$

It can be shown that (3) converges formally, provided that  $\varepsilon < 1/E_{\omega} a$ . More generally, consider  $\delta \in \mathbf{No}$  with  $\delta < 1/L_k b$  for a certain  $k \in \mathbb{N}$ . Assuming (1), this means that  $\delta < 1/E_{\omega}(a - k)$ , which allows us to define

$$E_{\omega}(a - k + \delta) := E_{\omega}(a - k) + (E'_{\omega}(a - k)) \delta + \frac{1}{2} (E''_{\omega}(a - k)) \delta^2 + \dots \quad (4)$$

and set

$$E_{\omega}(a + \delta) := E_k(E_{\omega}(a - k + \delta)). \quad (5)$$

It follows that it is sufficient to define  $E_{\omega}$  at numbers  $\varphi \in \mathbf{No}^{>>}$  with

$$\text{supp } \varphi \succ \frac{1}{L_{\mathbb{N}} E_{\omega} \varphi},$$

where  $L_{\mathbb{N}} = \{L_0, L_1, \dots\}$ . Those numbers are said to be *truncated* and we will write  $\mathbf{Tr}$  for the class of all truncated numbers. It turns out that  $\mathbf{Tr}$  is a surreal substructure.

**Defining hyperexponentials** Our definition of  $E_{\omega}$  proceeds in three stages:

1. We first define  $E_{\omega}$  on  $\mathbf{No}_{>}^>$ . For any two positive purely infinite numbers  $\theta_1, \theta_2$  with  $\theta_1 < \theta_2$ , we have  $\theta_1 + \mathbb{N} < \theta_2 - \mathbb{N}$ . By the functional equation, we should have  $E_{\omega}(\theta_1 + \mathbb{N}) = E_{\mathbb{N}} E_{\omega} \theta_1 < L_{\mathbb{N}} E_{\omega} \theta_2 = E_{\omega}(\theta_2 + \mathbb{N})$ . We deduce that for  $\theta \in \mathbf{No}_{>}^>$ , the number  $E_{\omega} \theta$  should lie in the cut  $(L | R)$  in  $\mathbf{No}$  where

$$L := E_{\mathbb{N}} \theta \cup E_{\mathbb{N}} E_{\omega} \theta_L^{\mathbf{No}_{>}^>}, \quad R := L_{\mathbb{N}} E_{\omega} \theta_R^{\mathbf{No}_{>}^>}$$

The simplest way to ensure this is to define

$$E_\omega(\theta) := \{E_{\mathbb{N}} \theta, E_{\mathbb{N}} E_\omega \theta_L^{\mathbf{No}^>} \mid L_{\mathbb{N}} E_\omega \theta_R^{\mathbf{No}^>}\}$$

for all  $\theta \in \mathbf{No}^>$ .

2. We next extend  $E_\omega$  to  $\mathbf{Tr}$ . Similar arguments and the simplicity heuristic impose

$$E_\omega \varphi := \{E_{\mathbb{N}} \varphi, \mathcal{E} E_\omega \varphi_L^{\mathbf{Tr}} \mid \mathcal{E} E_\omega \varphi_R^{\mathbf{Tr}}\} \in \mathbf{La},$$

where  $\mathcal{E}$  is a function group to be defined in Section 2. Here  $\mathbf{Tr}$ ,  $\mathbf{La}$ , and (3) play a similar role as  $\mathbf{No}^>$ ,  $\mathbf{Mo}$ , and (2) when extending the definition of  $\exp$ .

3. We finally extend the definition of  $E_\omega$  to  $\mathbf{No}^{>,>}$  by relying on (4) and (5).

## 2. EQUATIONS AND CONVEX PARTITIONS

Before we define  $E_\omega$ , let us briefly recall a general method to define surreal substructures using convex partitions. For more details, see [3, Section 6].

**Uniform equations** Let  $\mathbf{T}$  be a surreal substructure and  $F: \mathbf{S} \rightarrow \mathbf{T}$  be a function. Let  $\lambda, \rho$  be functions defined for cut representations in  $\mathbf{S}$  and such that  $(\lambda(L, R), \rho(L, R))$  is a cut representation in  $\mathbf{T}$  whenever  $(L, R)$  is a cut representation in  $\mathbf{S}$ . We say that  $(\lambda, \rho)$  is an *equation* of  $F$  if, for all  $a \in \mathbf{S}$ , we have

$$F(a) = \{\lambda(a_L^{\mathbf{S}}, a_R^{\mathbf{S}}) \mid \rho(a_L^{\mathbf{S}}, a_R^{\mathbf{S}})\}_{\mathbf{T}}.$$

We say that the equation is *uniform* if we have  $F(\{L \mid R\}_{\mathbf{S}}) = \{\lambda(L, R) \mid \rho(L, R)\}_{\mathbf{T}}$  whenever  $(L, R)$  is a cut representation in  $\mathbf{S}$ . For instance, by [9, Theorem 3.2], for  $r \in \mathbb{R}$ , the following equation for the translation  $a \in \mathbf{No} \mapsto a + r$  by  $r$  is uniform:

$$a + r = \{a_L + r, a + r_L \mid a + r_R, a_R + r\}. \quad (6)$$

**Remark 1.** Assume that  $F$  has an equation  $(\lambda, \rho)$  with

$$\forall a \in \mathbf{S}, \forall a' \in a_L^{\mathbf{S}}, \forall a'' \in a_R^{\mathbf{S}}, \exists b' \in \lambda(a_L, a_R), \exists b'' \in \rho(a_L, a_R), \quad F(a') \leq b' \wedge F(a'') \geq b''.$$

This is in particular the case if  $F(a_L^{\mathbf{S}}) \subseteq \lambda(a_L^{\mathbf{S}}, a_R^{\mathbf{S}})$  and  $F(a_R^{\mathbf{S}}) \subseteq \rho(a_L^{\mathbf{S}}, a_R^{\mathbf{S}})$  for all  $a \in \mathbf{S}$ . Then we claim that  $F$  is strictly increasing. To see this, consider  $a_0, a_1 \in \mathbf{S}$  with  $a_0 < a_1$ . By [3, Proposition 4.6], there is a  $\sqsubseteq$ -maximal element  $c$  of  $\mathbf{S}$  with  $c \sqsubseteq a, b$ , and we have  $a < c \leq b$  or  $a \leq c < b$ . We treat the first case, the other one being symmetric. Since  $a < c$  and  $c \sqsubseteq a$ , we have  $c \in a_R^{\mathbf{S}}$  so there is  $b'' \in \rho(a_L^{\mathbf{S}}, a_R^{\mathbf{S}})$  with  $b'' \leq F(c)$ . We have  $F(a) < b'' \leq F(c)$ . A similar argument yields  $F(c) \leq F(b)$ , so  $F(a) < F(b)$ .

**Convex partitions** Let  $\mathbf{S}$  be a surreal substructure and let  $\mathbf{\Pi}$  be a partition of  $\mathbf{S}$  into convex subclasses, each of which admits a cofinal and coinital subset. We refer to  $\mathbf{\Pi}$  as a *thin convex partition* of  $\mathbf{S}$ . For  $a \in \mathbf{S}$ , we let  $\mathbf{\Pi}[a]$  denote the unique member of  $\mathbf{\Pi}$  containing  $a$ . We also write  $\mathbf{\Pi}[\mathbf{X}] := \bigcup_{x \in \mathbf{X}} \mathbf{\Pi}[x]$  for any subclass  $\mathbf{X}$  of  $\mathbf{S}$ . We say that an element  $a \in \mathbf{S}$  is  $\mathbf{\Pi}$ -*simple* if it is the  $\sqsubseteq$ -minimum of its class  $\mathbf{\Pi}[a]$ . This is equivalent to the existence of a cut  $(L \mid R)_{\mathbf{S}}$  in  $\mathbf{S}$  with  $a = \{L \mid R\}_{\mathbf{S}}$  and  $\mathbf{\Pi}[L] < a < \mathbf{\Pi}[R]$ .

Then the class  $\mathbf{Smp}_{\mathbf{\Pi}}$  of  $\mathbf{\Pi}$ -simple elements forms a surreal substructure which is contained in  $\mathbf{S}$ . For  $a \in \mathbf{Smp}_{\mathbf{\Pi}}$ , we have  $a = \{\mathbf{\Pi}[a_L^{\mathbf{Smp}_{\mathbf{\Pi}}}] \mid \mathbf{\Pi}[a_R^{\mathbf{Smp}_{\mathbf{\Pi}}}]\}_{\mathbf{Smp}_{\mathbf{\Pi}}}$ .

**Function groups** A *function group*  $\mathcal{G}$  on a surreal substructure  $\mathbf{S}$  is a group of strictly increasing bijections  $\mathbf{S} \rightarrow \mathbf{S}$  under functional composition. We regard elements  $f, g$  of  $\mathcal{G}$  as actions on  $\mathbf{S}$  and sometimes write  $fg$  and  $fa$  for  $a \in \mathbf{S}$  rather than  $f \circ g$  and  $f(a)$ . For such a function group  $\mathcal{G}$ , the collection  $\mathbf{\Pi}_{\mathcal{G}} := (\mathcal{G}[a])_{a \in \mathbf{S}}$  of classes

$$\mathcal{G}[a] := \{b \in \mathbf{S} : \exists f, g \in \mathcal{G}, fa \leq b \leq ga\}$$

is a thin convex partition of  $\mathbf{S}$  and we define  $\mathbf{Smp}_{\mathcal{G}} := \mathbf{Smp}_{\Pi_{\mathcal{G}}}$ .

For  $f, g \in \mathcal{G}$ , the relation  $f < g \iff \forall a \in \mathbf{S}, (fa < ga)$  is a partial order on  $\mathcal{G}$ . We will frequently rely on the elementary fact that  $(\mathcal{G}, <)$  is *partially bi-ordered*, i.e. that we have

$$\forall f, g, h \in \mathcal{G}, \quad g > \text{id}_{\mathbf{S}} \iff fgh > fh.$$

**Common function groups** As an example, we can obtain the previous structures as the classes  $\mathbf{Smp}_{\mathcal{G}}$  for actions of the following function groups  $\mathcal{G}$  acting on  $\mathbf{No}$ ,  $\mathbf{No}^>$  or  $\mathbf{No}^{>,>}$ .

For  $r \in \mathbb{R}$  and  $s \in \mathbb{R}^>$ , we define

$$\begin{aligned} T_r &:= a \mapsto a + r && \text{acting on } \mathbf{No} \text{ or } \mathbf{No}^{>,>} \\ H_s &:= a \mapsto sa && \text{acting on } \mathbf{No}^> \text{ or } \mathbf{No}^{>,>} \\ P_s &:= a \mapsto a^s = \exp(s \log a) && \text{acting on } \mathbf{No}^> \text{ or } \mathbf{No}^{>,>} \end{aligned}$$

We define

$$\begin{aligned} \mathcal{T} &:= \{T_r : r \in \mathbb{R}\} \\ \mathcal{H} &:= \{H_s : s \in \mathbb{R}^>\} \\ \mathcal{D} &:= \{P_s : s \in \mathbb{R}^>\} \\ \mathcal{E} &:= \langle E_n H_s L_n : n \in \mathbb{N}, s \in \mathbb{R}^> \rangle \\ \mathcal{E}^* &:= \{E_n, L_n : n \in \mathbb{N}\}. \end{aligned}$$

We then have the following list of identities [3, Section 7.1]:

- The action of  $\mathcal{T}$  on  $\mathbf{No}$  (resp.  $\mathbf{No}^{>,>}$ ) yields  $\mathbf{No}_>$  (resp.  $\mathbf{No}_>^>$ ).
- The action of  $\mathcal{H}$  on  $\mathbf{No}^>$  (resp.  $\mathbf{No}^{>,>}$ ) yields  $\mathbf{Mo}$  (resp.  $\mathbf{Mo}^>$ ).
- The action of  $\mathcal{D}$  on  $\mathbf{No}^{>,>}$  yields  $\exp \mathbf{Mo}^>$ .
- The action of  $\mathcal{E}$  on  $\mathbf{No}^{>,>}$  yields  $\mathbf{La}$  [4, Corollary 5.17].
- The action of  $\mathcal{E}^*$  on  $\mathbf{No}^{>,>}$  yields the class  $\mathbf{K}$  of [13].

For  $a \in \mathbf{No}^{>,>}$ , we will denote  $\mathfrak{d}_{\omega}(a)$  the unique log-atomic element of  $\mathcal{E}[a]$  and we will denote  $\mathfrak{d}_{\omega}^*(a)$  the unique element of  $\mathcal{E}^*[a]$  lying in  $\mathbf{K}$ . We have  $\mathfrak{d}_{\omega}^*(a) \sqsubseteq \mathfrak{d}_{\omega}(a) \sqsubseteq a$ .

### 3. LOGARITHMIC TRANSSERIES AND HYPERSERIES

Let  $\mathbb{L}_{<\omega} = \mathbb{R}[[\mathcal{L}_{<\omega}]]$  denote the field of logarithmic hyperseries from [7]. For each  $n \in \mathbb{N}$ , we define  $\ell_n := L_n x \in \mathbb{L}_{<\omega}$ . Recall that  $\mathbb{R}[[\mathcal{L}_{<\omega} \times \varepsilon^{\mathbb{N}}]]$  stands for the set of series with Noetherian support in the partially ordered set  $\mathcal{L}_{<\omega} \times \varepsilon^{\mathbb{N}}$ . We may consider elements of  $\mathbb{R}[[\mathcal{L}_{<\omega} \times \varepsilon^{\mathbb{N}}]]$  as bivariate series that are logarithmic transseries with respect to  $x$  and ordinary series with respect to  $\varepsilon$ .

**LEMMA 2.** *Consider a series  $f \in \mathbb{R}[[\mathcal{L}_{<\omega} \times \varepsilon^{\mathbb{N}}]]$  such that the substitution  $f(\bar{\varepsilon})$  of  $\varepsilon$  by  $\bar{\varepsilon}$  in  $f$  vanishes for all  $\bar{\varepsilon} < 1$  in  $\mathbb{L}_{<\omega}$ . Then  $f = 0$ .*

**Proof.** Assume for contradiction that  $f \neq 0$ . Write  $f = f_0 + f_1 \varepsilon + f_2 \varepsilon^2 + \dots$  and let  $k \in \mathbb{N}$  be minimal with  $f_k \neq 0$ . Since  $f$  is Noetherian as a series in  $\mathbb{R}[[\mathcal{L}_{<\omega} \times \varepsilon^{\mathbb{N}}]]$ , the set  $\bigcup_{i \in \mathbb{N}} \text{supp } f_i$  is well based and admits a largest element  $m$ . Taking  $\bar{\varepsilon} \in x^{\mathbb{R}}$  sufficiently small such that  $m \bar{\varepsilon} < f_k$ , it follows that  $f_{k+1} \bar{\varepsilon}^{k+1} + f_{k+2} \bar{\varepsilon}^{k+2} + \dots < f_k \bar{\varepsilon}^k$ , whence  $f(\bar{\varepsilon}) = f_k \bar{\varepsilon}^k + f_{k+1} \bar{\varepsilon}^{k+1} + f_{k+2} \bar{\varepsilon}^{k+2} + \dots \sim f_k \bar{\varepsilon}^k \neq 0$ .  $\square$

LEMMA 3. Given integers  $p \geq k \geq 0$  and  $\delta(\varepsilon) = L_{p-k}(L_k x + \varepsilon) - L_p x$ , we have

$$(L'_\omega L_k x) \varepsilon + \frac{1}{2} (L''_\omega L_k x) \varepsilon^2 + \cdots = (L'_\omega L_p x) \delta(\varepsilon) + \frac{1}{2} (L''_\omega L_p x) \delta(\varepsilon)^2 + \cdots \quad (7)$$

as an identity in  $\mathbb{R}[[\mathfrak{L}_{<\omega} \times \varepsilon^{\mathbb{N}}]]$ .

**Proof.** The left and right hand sides of (7) are clearly Noetherian series in  $\mathbb{R}[[\mathfrak{L}_{<\omega} \times \varepsilon^{\mathbb{N}}]]$ . For any  $\bar{\varepsilon} < L_k x$  in  $\mathbb{L}_{<\omega}$ , the following Taylor series expansions hold in  $\mathbb{L}_{\leq\omega}$ :

$$\begin{aligned} L_\omega(L_k x + \bar{\varepsilon}) &= L_\omega L_k x + (L'_\omega L_k x) \bar{\varepsilon} + \frac{1}{2} (L''_\omega L_k x) \bar{\varepsilon}^2 + \cdots \\ L_\omega(L_p x + \delta(\bar{\varepsilon})) &= L_\omega L_p x + (L'_\omega L_p x) \delta(\bar{\varepsilon}) + \frac{1}{2} (L''_\omega L_p x) \delta(\bar{\varepsilon})^2 + \cdots \end{aligned}$$

Subtracting both expansions, the identity (7) holds for  $\varepsilon$  substituted by  $\bar{\varepsilon} < 1$  in  $\mathbb{L}_{<\omega}$ . We conclude by Lemma 2.  $\square$

In [7], we defined the field  $\mathbb{L}_{\leq\omega} = \mathbb{R}[[\mathfrak{L}_{\leq\omega}]]$ , as well as a hyperlogarithmic function  $L_\omega$  on  $\mathbb{L}_{\leq\omega}^{\succ}$  for which  $\mathbb{L}_{\leq\omega} \cong \mathbb{R}[[\ell_\omega^{\mathbb{R}}]][[\mathfrak{L}_{<\omega}]]$  with  $\ell_\omega = L_\omega x$ . Let  $u := L_\omega x \in \mathbb{L}_{\leq\omega}$ . Then logarithmic transseries in  $u$  can be considered as elements in  $L_{<\omega} \circ L_\omega \subseteq \mathbb{L}_{\leq\omega}$  and the successive derivatives of  $E_\omega(u)$  with respect to  $u$  are given by

$$E_\omega^{(k)}(u) = \vartheta^k x \in \mathbb{L}_{<\omega}, \quad \vartheta := \gamma^{-1} \frac{\partial}{\partial x'}, \quad \gamma := \ell'_\omega = \prod_{k < \omega} \frac{1}{\ell_k}.$$

For any  $f \in \mathbb{L}_{<\omega}$ , we have

$$\text{supp } \vartheta f \subseteq \text{supp } \vartheta + \text{supp } f, \quad \text{supp } \vartheta = \gamma^{-1} \text{supp } \frac{\partial}{\partial x} = \gamma^{-1} \left\{ \frac{1}{\ell_0}, \frac{1}{\ell_0 \ell_1}, \dots \right\},$$

whence

$$\text{supp } E_\omega^{(k)}(u) = \text{supp } \vartheta^k x \subseteq \left( \frac{\gamma}{\ell_0} \right)^{-k} \left\{ 1, \frac{1}{\ell_1}, \frac{1}{\ell_1 \ell_2}, \dots \right\}^\infty \ell_0.$$

For  $\delta < 1/x$  in  $\mathbb{R}[[\mathfrak{L}_{<\omega} \times \varepsilon^{\mathbb{N}}]]$ , this allows us to define  $E_\omega(u + \delta)$  using the Taylor series expansion

$$E_\omega(u + \delta) := E_\omega(u) + E'_\omega(u) \delta + \frac{1}{2} E''_\omega(u) \delta^2 + \cdots. \quad (8)$$

LEMMA 4. The following identity holds in  $\mathbb{R}[[\mathfrak{L}_{<\omega} \times \varepsilon^{\mathbb{N}}]]$ :

$$E_\omega L_\omega(x + \varepsilon) = x + \varepsilon. \quad (9)$$

**Proof.** The left hand side is well defined by (8) for  $\delta = L_\omega(x + \varepsilon) - u = L_\omega(x + \varepsilon) - L_\omega x = \gamma \varepsilon + \frac{1}{2} \gamma' \varepsilon^2 + \cdots < 1/x$  in  $\mathbb{R}[[\mathfrak{L}_{<\omega} \times \varepsilon^{\mathbb{N}}]]$ . The fact that (9) holds for  $\varepsilon$  substituted by  $\bar{\varepsilon}$  in  $\mathbb{L}_{<\omega}^{\succ}$  follows from the usual rules of iterated derivatives of inverse functions. For a detailed proof, we refer to [14, section 6.4]. We conclude by Lemma 2.  $\square$

#### 4. HYPEREXPONENTIALS OF TRUNCATED NUMBERS

We recursively define  $E_\omega$  for positive purely infinite numbers  $\theta \in \mathbf{No}_{>}^{\succ}$  by

$$E_\omega \theta := \left\{ \vartheta_\omega^*(\theta), E_\omega \theta_L^{\mathbf{No}_{>}^{\succ}} \mid E_\omega \theta_R^{\mathbf{No}_{>}^{\succ}} \right\}_{\mathbf{K}}. \quad (10)$$

PROPOSITION 5. The function  $E_\omega$  defines a strictly increasing bijection  $\mathbf{No}_{>}^{\succ} \rightarrow \mathbf{K}$ . Moreover, the previous equation is uniform.

**Proof.** The function  $E_\omega$  is well-defined and strictly increasing by Remark 1. The uniformity of the equation follows immediately.

Let  $L_\omega$  denote the partial inverse function of  $E_\omega$  and prove that  $L_\omega$  is defined on  $\mathbf{K}$  by induction on  $\sqsubseteq$ . Let  $\kappa \in \mathbf{K}$  such that  $\kappa_{\sqsubseteq}^{\mathbf{K}}$  is contained  $E_\omega \mathbf{No}_{\succ}^{\mathbf{K}}$ . Since  $E_\omega$  is injective, its inverse is defined on  $\kappa_{\sqsubseteq}^{\mathbf{K}}$ . Let

$$\theta := \{L_\omega \kappa_L^{\mathbf{K}} \mid L_\omega \kappa_R^{\mathbf{K}}, \kappa\}_{\mathbf{No}_{\succ}^{\mathbf{K}}}.$$

This number is well defined since  $L_\omega: \kappa_{\sqsubseteq}^{\mathbf{K}} \rightarrow \mathbf{No}_{\succ}^{\mathbf{K}}$  is strictly increasing and for  $\kappa' \in \kappa_L^{\mathbf{K}}$ , we have  $L_\omega \kappa' < \kappa' < \kappa$ . By uniformity, we have  $E_\omega \theta = \{\mathfrak{d}_\omega^*(\theta), \kappa_L^{\mathbf{K}} \mid \kappa_R^{\mathbf{K}}, E_\omega \kappa\}_{\mathbf{K}}$  where  $\kappa = \{\kappa_L^{\mathbf{K}} \mid \kappa_R^{\mathbf{K}}\}_{\mathbf{K}}$ . In order to conclude that  $E_\omega \theta = \kappa$ , it therefore suffices to show that  $\kappa$  lies in the cut  $(\mathfrak{d}_\omega^*(\theta) \mid E_\omega \kappa)$ . We have  $E_\omega \kappa > \kappa$  by (10) and  $\theta < \kappa$  by definition of  $\theta$ , whence  $\mathfrak{d}_\omega^*(\theta) < \kappa$  since  $\kappa \in \mathbf{K}$ . We conclude by induction that  $E_\omega: \mathbf{No}_{\succ}^{\mathbf{K}} \rightarrow \mathbf{K}$  is surjective.  $\square$

We next identify the class of truncated numbers. For  $a \in \mathbf{No}^{>, >}$ , we consider the following convex class

$$\mathbf{\Pi}[a] := \left\{ b \in \mathbf{No}^{>, >} : \exists n \in \mathbb{N}, a - b < \frac{1}{L_n E_\omega a_{>}} \right\}.$$

PROPOSITION 6. *The classes  $\mathbf{\Pi}[a]$  for  $a \in \mathbf{No}^{>, >}$  form a thin convex partition of  $\mathbf{No}^{>, >}$ .*

**Proof.** Given  $a \in \mathbf{No}^{>, >}$ , it is clear that the class  $\mathbf{\Pi}[a]$  is convex and that it contains  $a$ . Note that for  $a \in \mathbf{No}^{>, >}$ , we have  $\mathbf{\Pi}[a] \subseteq a + \mathbf{No}^{<}$ . Let  $a, b \in \mathbf{No}^{>, >}$  with  $\mathbf{\Pi}[a] \neq \mathbf{\Pi}[b]$ . We claim that  $\mathbf{\Pi}[a] \cap \mathbf{\Pi}[b] = \emptyset$ . If  $a_{>} \neq b_{>}$ , then we have  $a + \mathbf{No}^{<} \cap b + \mathbf{No}^{<} = \emptyset$ , which yields the result. Assume that  $a_{>} = b_{>}$ . Assume for contradiction that there are  $c \in \mathbf{No}^{>, >}$  and  $m, n \in \mathbb{N}$  with  $a - c < \frac{1}{L_m E_\omega a_{>}}$  and  $b - c < \frac{1}{L_n E_\omega a_{>}}$ . Given  $d \in \mathbf{\Pi}[a]$ , there is a number  $p \geq m$ ,  $n$  with  $a - d < \frac{1}{L_p E_\omega a_{>}}$ . Therefore  $a - c, b - c, a - d$  are dominated by  $\frac{1}{L_p E_\omega a_{>}}$ , whence  $b - d = b - c - (a - c) + (a - d) < \frac{1}{L_p E_\omega a_{>}}$ . This proves that  $\mathbf{\Pi}[a] \subseteq \mathbf{\Pi}[b]$  and symmetric arguments yield  $\mathbf{\Pi}[a] \supseteq \mathbf{\Pi}[b]$ : a contradiction. This proves our claim. It only remains to see that the class  $\mathbf{\Pi}[a]$  admits a cofinal and cointial subset for any  $a \in \mathbf{No}^{>, >}$ . Indeed, we can take  $a \pm \frac{1}{L_n E_\omega a_{>}}$  as examples of such sets.  $\square$

COROLLARY 7. *The class  $\mathbf{Tr} := \mathbf{Smp}_{\mathbf{\Pi}}$  is a surreal substructure.*

Let  $a \in \mathbf{No}^{>, >}$  and let  $\varphi$  denote the  $\leq$ -supremum of truncations  $\psi$  of  $a$  (i.e. series with  $\psi \leq a$ ) with  $\text{supp } \psi > \frac{1}{L_n E_\omega a_{>}}$ . In particular, we have  $a_{>} \leq \varphi$  since  $\text{supp } a_{>} > 1$ . We see that  $\varphi$  satisfies  $\varphi \leq a$  and  $\text{supp } \varphi > \frac{1}{L_n E_\omega a_{>}}$ . Write  $\varphi = a_{>} + \delta$  and  $a = \varphi + \varepsilon = a_{>} + \delta + \varepsilon$ . By  $\leq$ -maximality of  $\varphi$ , we have  $\varepsilon < \frac{1}{L_n E_\omega a_{>}} = \frac{1}{L_n E_\omega \varphi_{>}}$  so  $a \in \mathbf{\Pi}[\varphi]$ , or equivalently  $\varphi \in \mathbf{\Pi}[a]$ . We deduce that  $\varphi$  is the  $\leq$ -minimum, hence  $\sqsubseteq$ -minimum of  $\mathbf{\Pi}[a]$ , so  $\varphi \in \mathbf{Tr}$ . We also see that for  $\theta \in \mathbf{No}_{\succ}^{\mathbf{K}}$  and  $r \in \mathbb{R}$ , we have  $\theta + r \subseteq \mathbf{Tr}$ . Since  $\mathbf{Tr}$  is a surreal substructure, we may use recursion on  $\varphi \in \mathbf{Tr}$  to define

$$\hat{E}_\omega(\varphi) := \{E_{\mathbb{N}} \varphi, \mathcal{E} \hat{E}_\omega(\varphi_{\mathbf{L}}^{\mathbf{Tr}}) \mid \mathcal{E} \hat{E}_\omega(\varphi_{\mathbf{R}}^{\mathbf{Tr}})\}. \quad (11)$$

PROPOSITION 8. *The equation (11) is uniform and  $\hat{E}_\omega$  is a strictly increasing function  $\mathbf{Tr} \rightarrow \mathbf{La}$ .*

**Proof.** Since  $\mathbf{Tr}$  is a surreal substructure, the definition, strict monotonicity and uniformity follow by Remark 1. For  $\varphi \in \mathbf{Tr}$ , we have  $\hat{E}_\omega(\varphi) > \mathcal{E} E_{\mathbb{N}} \varphi$  since  $\mathcal{E} < \text{exp}$  on  $\mathbf{No}^{>, >}$ . We deduce that  $\hat{E}_\omega(\varphi)$  is  $\mathcal{E}$ -simple, hence log-atomic.  $\square$

By [4, Lemma 2.4], for every infinite monomial  $\mathbf{m} \in \mathbf{Mo}_{\succ}^{\mathbf{K}}$ , we have

$$\text{exp } \mathbf{m} = \{\mathcal{D} \mathbf{m}, \mathcal{D} \text{exp } \mathbf{m}_L^{\mathbf{Mo}_{\succ}^{\mathbf{K}}} \mid \mathcal{D} \text{exp } \mathbf{m}_R^{\mathbf{Mo}_{\succ}^{\mathbf{K}}}\}. \quad (12)$$



PROPOSITION 9. We have  $\forall \varphi \in \mathbf{Tr}, \hat{E}_\omega(\varphi + 1) = e^{\hat{E}_\omega(\varphi)}$ .

**Proof.** We prove this by induction on  $(\mathbf{Tr}, \sqsubseteq)$ . Let  $\varphi \in \mathbf{Tr}$  such that this holds on  $\varphi_{\mathbf{L}}^{\mathbf{Tr}}$ . Note that  $\mathbf{\Pi}[a + r] = \mathbf{\Pi}[a] + r$  for all  $a \in \mathbf{No}^{>, >}$  and  $r \in \mathbb{R}$ . We have  $\varphi = \{\mathbb{R}, \mathbf{\Pi}[\varphi_{\mathbf{L}}^{\mathbf{Tr}}] \mid \mathbf{\Pi}[\varphi_{\mathbf{R}}^{\mathbf{Tr}}]\}$ , so

$$\begin{aligned} \varphi + 1 &= \{\varphi, \mathbf{\Pi}[\varphi_{\mathbf{L}}^{\mathbf{Tr}}] + 1 \mid \mathbf{\Pi}[\varphi_{\mathbf{R}}^{\mathbf{Tr}}] + 1\} \\ &= \{\mathbf{\Pi}[\varphi], \mathbf{\Pi}[\varphi_{\mathbf{L}}^{\mathbf{Tr}} + 1] \mid \mathbf{\Pi}[\varphi_{\mathbf{R}}^{\mathbf{Tr}} + 1]\} \\ &= \{\varphi, \varphi_{\mathbf{L}}^{\mathbf{Tr}} + 1 \mid \varphi_{\mathbf{R}}^{\mathbf{Tr}} + 1\}_{\mathbf{Tr}}. \end{aligned}$$

We deduce that

$$\begin{aligned} \hat{E}_\omega(\varphi + 1) &= \{E_{\mathbb{N}}(\varphi + 1), \mathcal{E} \hat{E}_\omega(\varphi_{\mathbf{L}}^{\mathbf{Tr}} + 1) \mid \mathcal{E} \hat{E}_\omega(\varphi_{\mathbf{R}}^{\mathbf{Tr}} + 1)\} \\ &= \{E_{\mathbb{N}} \varphi, \mathcal{E} \exp \hat{E}_\omega(\varphi_{\mathbf{L}}^{\mathbf{Tr}}) \mid \mathcal{E} \exp \hat{E}_\omega(\varphi_{\mathbf{R}}^{\mathbf{Tr}})\}. \end{aligned}$$

Since  $\hat{E}_\omega(\varphi) \in \mathbf{La} \subseteq \mathbf{Mo}^{>, >}$ , we may apply (12). We also note that  $\exp \mathcal{E} a$  and  $\mathcal{E} \exp a$  are mutually cofinal and cointial for all  $a \in \mathbf{No}^{>, >}$  to obtain

$$\begin{aligned} \exp \hat{E}_\omega(\varphi) &= \{\mathcal{D} \hat{E}_\omega(\varphi), \mathcal{D} \exp E_{\mathbb{N}} \varphi, \mathcal{D} \exp \mathcal{E} \hat{E}_\omega(\varphi_{\mathbf{L}}^{\mathbf{Tr}}) \mid \mathcal{D} \exp \mathcal{E} \hat{E}_\omega(\varphi_{\mathbf{R}}^{\mathbf{Tr}})\} \\ &= \{\mathcal{D} \hat{E}_\omega(\varphi), E_{\mathbb{N}} \varphi, \mathcal{D} \mathcal{E} \exp \hat{E}_\omega(\varphi_{\mathbf{L}}^{\mathbf{Tr}}) \mid \mathcal{D} \mathcal{E} \exp \hat{E}_\omega(\varphi_{\mathbf{R}}^{\mathbf{Tr}})\} \\ &= \{\mathcal{D} \hat{E}_\omega(\varphi), E_{\mathbb{N}} \varphi, \mathcal{E} \exp \hat{E}_\omega(\varphi_{\mathbf{L}}^{\mathbf{Tr}}) \mid \mathcal{E} \exp \hat{E}_\omega(\varphi_{\mathbf{R}}^{\mathbf{Tr}})\}. \end{aligned}$$

We have  $\hat{E}_\omega(\varphi + 1) > \hat{E}_\omega(\varphi)$ , so  $\hat{E}_\omega(\varphi + 1) > \mathcal{E} \hat{E}_\omega(\varphi)$  and  $\hat{E}_\omega(\varphi + 1) > \mathcal{D} \hat{E}_\omega(\varphi)$ . We clearly have  $\exp \hat{E}_\omega(\varphi) > E_{\mathbb{N}} \varphi$ . We deduce that  $\exp \hat{E}_\omega(\varphi) = \hat{E}_\omega(\varphi + 1)$ . By induction, the relation is valid on  $\mathbf{Tr}$ .  $\square$

PROPOSITION 10. For  $\theta \in \mathbf{No}^{>, >}$ , we have  $\hat{E}_\omega(\theta) = E_\omega \theta$ .

**Proof.** We prove this by induction on  $(\mathbf{No}^{>, >}, \sqsubseteq)$ . Let  $\theta \in \mathbf{No}^{>, >}$  be such that this holds on  $\theta_{\mathbf{L}}^{\mathbf{No}^{>, >}}$ . For  $\varphi \in \theta_{\mathbf{L}}^{\mathbf{Tr}}$ , we have  $\varphi_{>} \in \theta_{\mathbf{L}}^{\mathbf{No}^{>, >}}$ , and there is  $n \in \mathbb{N}$  with  $\varphi \leq \varphi_{>} + n$ . We deduce that  $\hat{E}_\omega(\varphi) \leq E_n \hat{E}_\omega(\varphi_{>}) = E_n E_\omega \varphi_{>}$ . In particular, we have  $\hat{E}_\omega(\varphi) < E_{n+1} E_\omega \varphi_{>}$  so  $\mathcal{E} \hat{E}_\omega(\varphi) < E_{n+2} E_\omega \varphi_{>}$ . This proves that  $E_{\mathbb{N}} \hat{E}_\omega \theta_{\mathbf{L}}^{\mathbf{No}^{>, >}}$  is cofinal with respect to  $\mathcal{E} \hat{E}_\omega(\theta_{\mathbf{L}}^{\mathbf{Tr}})$ . For  $\theta' \in \theta_{\mathbf{L}}^{\mathbf{No}^{>, >}}$  and  $n \in \mathbb{N}$ , we have  $E_n E_\omega \theta' = E_n \hat{E}_\omega(\theta') = \hat{E}_\omega(\theta' + n)$  where  $\theta' + n \in \theta_{\mathbf{L}}^{\mathbf{Tr}}$ , so  $\mathcal{E} \hat{E}_\omega(\theta_{\mathbf{L}}^{\mathbf{Tr}})$  is cofinal with respect to  $E_{\mathbb{N}} E_\omega \theta_{\mathbf{L}}^{\mathbf{No}^{>, >}}$ . Symmetric arguments yield that  $L_{\mathbb{N}} E_\omega \theta_{\mathbf{R}}^{\mathbf{No}^{>, >}}$  and  $\mathcal{E} \hat{E}_\omega(\theta_{\mathbf{R}}^{\mathbf{Tr}})$  are mutually cointial. We conclude that  $\hat{E}_\omega(\theta) = \{E_{\mathbb{N}} \theta, E_{\mathbb{N}} E_\omega \theta_{\mathbf{L}}^{\mathbf{No}^{>, >}} \mid L_{\mathbb{N}} E_\omega \theta_{\mathbf{R}}^{\mathbf{No}^{>, >}}\} = E_\omega \theta$ .  $\square$

PROPOSITION 11. The function  $E_\omega: \mathbf{Tr} \rightarrow \mathbf{La}$  is bijective. Its reciprocal  $L_\omega$  admits the following uniform equation on  $\mathbf{La}$ :

$$L_\omega \mathbf{a} = \{L_\omega \mathbf{a}_{\mathbf{L}}^{\mathbf{La}} \mid L_\omega \mathbf{a}_{\mathbf{R}}^{\mathbf{La}}, L_{\mathbb{N}} \mathbf{a}\}_{\mathbf{Tr}}.$$

**Proof.** Noticing that  $E_\omega \varphi = \hat{E}_\omega(\varphi) = \{E_{\mathbb{N}} \mathfrak{d}_\omega(\varphi), E_\omega \varphi_{\mathbf{L}}^{\mathbf{Tr}} \mid E_\omega \varphi_{\mathbf{R}}^{\mathbf{Tr}}\}_{\mathbf{La}}$  for all  $\varphi \in \mathbf{Tr}$ , this follows from the same arguments as in Proposition 5.  $\square$

## 5. HYPEREXPONENTIALS OF ARBITRARY NUMBERS

The field  $\mathbb{L}_{<\omega} = \mathbb{R}[[\mathcal{L}_{<\omega}]]$  of logarithmic hyperseries of [7] is a subfield of the class of all well-based transseries in an infinitely large variable  $x$ . Both  $\mathbb{L}_{<\omega}$  and the class of all transseries are closed under derivation and under composition [8, 10, 14]. For every positive infinite number  $a \in \mathbf{No}^{>, >}$ , there also exists an evaluation embedding  $\mathbb{L}_{<\omega} \rightarrow \mathbf{No}; f \mapsto f(a)$  such that  $f(g(a)) = (f \circ g)(a)$  for all  $f, g \in \mathbb{L}_{<\omega}$ : see [5].

Given  $a \in \mathbf{No}^{>, >}$ , let  $\varphi = \varphi_a \in \mathbf{Tr}$  be the unique truncated series with  $a \in \mathbf{\Pi}[\varphi]$ . If  $a \neq \varphi$ , then there is a smallest number  $n = n_a \in \mathbb{Z}$  with

$$a - \varphi < \frac{1}{L_n E_\omega \varphi} = \frac{1}{E_\omega(\varphi - n)}.$$

Write  $\varepsilon := a - \varphi$ . With  $\vartheta$  as in section 3, we define for every  $k \in \mathbb{N}$ :

$$E_\omega^{(k)}(\varphi - n) := (\vartheta^k x)(E_\omega(\varphi - n)).$$

Substitution of  $E_\omega(\varphi - n)$  for  $x$  in (8) allows us to extend the definition of  $E_\omega$  by

$$E_\omega(a - n) := \sum_{k \in \mathbb{N}} \frac{1}{k!} E_\omega^{(k)}(\varphi - n) \varepsilon^k,$$

and

$$E_\omega(a) := E_n(E_\omega(a - n)).$$

PROPOSITION 12. For all  $a \in \mathbf{No}^{>, >}$ , we have

$$E_\omega(a + 1) = E_1 E_\omega a.$$

**Proof.** If  $\varphi_a = a$ , then this is Proposition 9. Otherwise, we have  $\varphi_{a+1} = \varphi_a + 1 \neq a + 1$  and  $n_{a+1} = n_a + 1$ , whence  $E_\omega(a + 1 - n_{a+1}) = E_\omega(a - n_a)$  and

$$E_\omega(a + 1) = E_{n_a+1}(E_\omega(a - n_a)) = E_1 E_{n_a}(E_\omega(a - n_a)) = E_1 E_\omega a. \quad \square$$

Inversely, consider an arbitrary positive infinite number  $b \in \mathbf{No}^{>, >}$ . Then there exists a  $k \in \mathbb{N}$  such that  $L_k b = L_k a + \varepsilon$  for some log-atomic  $a \in \mathbf{La}$  and  $\varepsilon < L_k a$ . We extend the definition of  $L_\omega$  to any such number  $b$  by

$$L_\omega b := L_\omega a + (\ell'_\omega \circ \ell_k(a)) \varepsilon + \frac{1}{2} (\ell''_\omega \circ \ell_k(a)) \varepsilon^2 + \dots$$

In view of Lemma 3, the value of  $L_\omega b$  does not depend on the choice of  $k$ . Note also that this definition indeed extends our previous definition of  $L_\omega$  on  $\mathbf{La}$ .

PROPOSITION 13. For all  $b \in \mathbf{No}^{>, >}$ , we have

$$L_\omega L_1 b = L_\omega b - 1.$$

**Proof.** With  $L_k b = L_k a + \varepsilon$  as above (while taking  $k > 0$ ), we have

$$\begin{aligned} L_\omega L_1 b &= L_\omega L_1 a + (L'_\omega L_{k-1} L_1 a) \varepsilon + \frac{1}{2} (L''_\omega L_{k-1} L_1 a) \varepsilon^2 + \dots \\ &= L_\omega a - 1 + (L'_\omega L_k a) \varepsilon + \frac{1}{2} (L''_\omega L_k a) \varepsilon^2 + \dots \\ &= L_\omega b - 1, \end{aligned}$$

where  $L_\omega L_1 a = L_\omega a - 1$  because of Proposition 9. □

PROPOSITION 14. For any  $b \in \mathbf{No}^{>, >}$ , we have  $E_\omega L_\omega b = b$ .

**Proof.** Let  $k \in \mathbb{N}$  be such that  $L_k b = L_k a + \bar{\varepsilon}$ , where  $a \in \mathbf{La}$  and  $\bar{\varepsilon} < L_k a$ . Let us first consider the special case when  $k = 0$ . Since  $a$  is log-atomic, we have  $\mathbb{L}_{<\omega} \cong \mathbb{L}_{<\omega}(a)$ . From Lemma 4, it therefore follows that  $E_\omega L_\omega(a + \varepsilon) = a + \varepsilon$  inside  $\mathbb{R}[[\mathcal{L}_{<\omega}(a) \times \varepsilon^{\mathbb{N}}]]$ . The result follows by specializing this relation at  $\bar{\varepsilon}$ . If  $k > 0$ , then  $L_\omega b = L_\omega L_k b + k = L_\omega(L_k a + \bar{\varepsilon}) + k$  by Proposition 13. Applying the result for the special case when  $k = 0$ , we have  $E_\omega(L_\omega b - k) = L_k a + \bar{\varepsilon} = L_k b$ . We conclude by Proposition 12. □

In particular, the function  $E_\omega: \mathbf{No}^{>, >} \rightarrow \mathbf{No}^{>, >}$  is surjective. We next prove that it is strictly increasing, concluding our proof that  $E_\omega: \mathbf{No}^{>, >} \rightarrow \mathbf{No}^{>, >}$  is a strictly increasing bijection with reciprocal  $L_\omega$ .

LEMMA 15. For  $\varphi, \psi \in \mathbf{Tr}$  with  $\varphi < \psi$ , we have  $\mathcal{E}E_\omega(\mathbf{\Pi}[\varphi]) < \mathcal{E}E_\omega(\mathbf{\Pi}[\psi])$ .

**Proof.** Note that  $\mathbf{La} \ni E_\omega(\varphi) < E_\omega(\psi) \in \mathbf{La}$ , so it is enough to prove that  $E_\omega(a) \in \mathcal{E}[E_\omega(\varphi)]$  for all  $a \in \mathbf{\Pi}[\varphi]$ . For such  $a$ , there is  $n \in \mathbb{N}$  with  $\varepsilon := a - \varphi < E_\omega(\varphi - n)^{-1}$ , and

$$\begin{aligned} L_n E_\omega(a) &= \sum_{k \in \mathbb{N}} \frac{E_\omega^{(k)}(\varphi - n)}{k!} \varepsilon^k \\ &= E_\omega(\varphi - n) + \delta, \end{aligned}$$

where  $\delta := \sum_{k > 0} \frac{E_\omega^{(k)}(\varphi - n)}{k!} \varepsilon^k$  is infinitesimal. So  $L_n E_\omega(a) \sim L_n E_\omega(\varphi)$ , whence  $E_\omega(a) \in \mathcal{E}[E_\omega(\varphi)]$ .  $\square$

LEMMA 16. For  $\varphi \in \mathbf{Tr}$  and  $a, b \in \mathbf{No}^{>, >}$  with  $a, b \in \mathbf{\Pi}[\varphi]$ , there is  $n \in \mathbb{N}$  with

$$L_n E_\omega(b) - L_n E_\omega(a) \sim E'_\omega(\varphi - n) (b - a).$$

**Proof.** Write  $a = \varphi + \varepsilon_a$  and  $b = \varphi + \varepsilon_b$  where  $\varepsilon_a, \varepsilon_b < 1$  and let  $n \in \mathbb{N}$  with  $\varepsilon_a, \varepsilon_b < E_\omega(\varphi - n)^{-1}$ . Writing  $\varepsilon_k := \varepsilon_b^k - \varepsilon_a^k$  for  $k \in \mathbb{N}^>$ , we have  $\varepsilon_k < E_\omega(\varphi - n)^{-k}$ . We deduce that

$$\begin{aligned} L_n E_\omega(b) - L_n E_\omega(a) &= \sum_{k > 0} \frac{E_\omega^{(k)}(\varphi - n)}{k!} \varepsilon_k \\ &\sim E'_\omega(\varphi - n) (\varepsilon_b - \varepsilon_a) \\ &\sim E'_\omega(\varphi - n) (b - a). \end{aligned} \quad \square$$

PROPOSITION 17. The function  $E_\omega$  is strictly increasing on  $\mathbf{No}^{>, >}$ .

**Proof.** Let  $a, b \in \mathbf{No}^{>, >}$  with  $a < b$ . If  $a < \mathbf{\Pi}[b]$ , then we get  $E_\omega(a) < E_\omega(b)$  by Lemma 15. Otherwise, we have  $a \in \mathbf{\Pi}[b]$  so by Lemma 16, there are  $\varphi \in \mathbf{Tr}$  and  $n \in \mathbb{N}$  with

$$L_n E_\omega(b) - L_n E_\omega(a) \sim E'_\omega(\varphi - n) (b - a).$$

Since  $E'_\omega(\varphi - n) > 0$ , we conclude that  $L_n E_\omega(b) > L_n E_\omega(a)$ , whence  $E_\omega(b) > E_\omega(a)$ .  $\square$

COROLLARY 18. The function  $E_\omega$  is bijective, with reciprocal  $L_\omega$ .

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