Towards a generalization of the logic of grounding

Francesca Poggiolesi, N. Francez

HAL Id: hal-02860310

https://hal.science/hal-02860310

Submitted on 8 Jun 2020

F. Poggiolesi, IHPST UMR8590, Paris

N. Francez, Computer Science, Technion Haifa

Towards a generalization of the logic of grounding

Abstract

The main goal of this paper is to provide a ground-analysis of two classical connectives that have so far being ignored in the literature, namely the *exclusive or*, and the *ternary or*. Such ground-analysis not only serves to extend the applicability of the logic of grounding but also leads to a generalization of [Poggiolesi, 2016]'s definition of the notion of complete and immediate grounding.

Keywords: Grounding, logic, exclusive and ternary or.

1 Introduction

In the last decade the notion of *grounding* has become prominent in philosophy. Grounding is usually taken to be a relation amongst truths or facts which is non-causal and explanatory in nature. This relation has been studied from several different perspectives: some papers retrace the history of grounding (e.g. see [Rumberg, 2013, Sebestik, 1992]), others deal with the metaphysics of grounding (e.g. see [Fine, 2012b, Schaffer, 2009]), others analyze the properties that the grounding relation might enjoy (e.g. [Krämer, 2013, de Rosset, 2013]). Yet another approach concerns the logic of grounding: several formal theories of grounding aim at clarifying this non-causal and explanatory relation with respect to logical connectives. These theories are of several types but it is possible to classify them in the following way.[1] First of all, most formal theories of grounding take the notion of grounding as primitive, namely they assume grounding to be a fundamental notion that cannot be defined in terms of others; in this type of theories, the motivation for choosing certain grounding axioms (or rules) rather than others mainly relies on appeal to our intuitions.[2] In these theories, grounding is either formalized as a connective (see [Correia, 2014, Fine, 2012a, Schnieder, 2011]) or as a predicate (see [Korbmacher, 2017]); in either ways, logical axioms or rules for the classical connectives of conjunction, disjunction and negation, as well as for universal and existential quantifiers, are proposed.

---

[1] This classification is mainly inspired by [Poggiolesi, 2020b].

[2] See [McSweeney, 2020].

There also exists an approach, which is mainly inspired by the insights of the bohemian thinker Bernard Bolzano, that takes grounding to be a notion characterizable in terms of others (see [Poggiolesi, 2016]): according to this approach grounding can be seen as a special type of derivability relation where complexity grows from the grounds to the conclusion. In this framework, grounding is formalized both as a metalinguistic relation (since it is a special type of derivability and derivability is a metalinguistic relation) but also as a connective, and grounding principles for the classical connectives of negation, conjunction and disjunction, but also for the relevant implication,[3] have been proposed.

The aim of this paper is to further develop the study of grounding from a logical perspective. In particular, we will consider the two classical connectives of *exclusive disjunction* and *trivalent disjunction* and we will elaborate an adequate logical grounding analysis of them. We will do this by employing the formal theory of grounding developed by [Poggiolesi, 2016]. In *Section 2*, we will discuss the relevance of developing a grounding analysis of the connectives of exclusive disjunction and trivalent disjunction, and of using Poggiolesi's approach as an appropriate framework for this study. After having used *Section 3* to briefly remind the reader of Poggiolesi's approach, we will dedicate *Section 4* to deal with the grounding analysis of exclusive disjunction, while *Section 5* to deal with the grounding analysis of ternary disjunction. *Section 6* will serve to evaluate the results obtained, whilst *Section 7* to draw some conclusions.

2   Motivational background behind the study of non-standard disjunctions and related methodology

In this paper we will develop a grounding analysis of two disjunctions that have not been treated in the (grounding) literature so far, namely the *exclusive or*, as well as the *trivalent or*. We will do this by using the formal theory of grounding developed by [Poggiolesi, 2016]. In this section we will discuss the relevance and interest of studying these new non-standard classical connectives from a grounding perspective, as well as the appropriateness of the chosen methodology in this context.

Let us first of all remind the reader what the exclusive disjunction and trivalent disjunction are. The exclusive disjunction is a disjunction between two options that are incompatible amongst each other. Typical examples of exclusive disjunction are:

- John is either the brother of Tessa's mother *or* he is the brother of Tessa's father,

- with four euros, John either buys a sandwich *or* a drink,

- Benjamin will either eat a cake *or* an ice-cream.

---

[3]See [Poggiolesi, 2020a].

The trivalent disjunction is a disjunction amongst three options at the same time and such that these options are not mutually exclusive. Typical examples of trivalent disjunction are:

- Fabrice will have breakfast with Anne *or* lunch *or* dinner,

- John is very talented in playing clarinet *or* he has been practicing for a long time *or* he really enjoys playing the instrument,

- Ann will watch a Tarantino movie *or* a Scorsese movie *or* a Lynch movie.

Formal theories of grounding have so far only been dealing with the classical connectives of conjunction, disjunction, negation and the quantifiers. The choice of these connectives and classical logic is typically motivated on metaphysical grounds: the relation between the classical conjunction, disjunction, negation, quantifiers and their grounds is a relation of metaphysical priority and formal theories of grounding are supposed to capture this relation at the formal level. However, quite recently, several scholars, such as [Hofweber, 2009, McSweeney, 2020, Merlo, 2020, Smithson, 2019], have been arguing against this view: according to them, two relations of grounding need to be distinguished, one metaphysical and the other logical, and the link between logical connectives and their reasons counts as an example of the latter notion rather than the former. Therefore, according to this novel perspective, formal theories of grounding deal with a logical grounding relation.

This paper focusses on formal theories of grounding, but largely remains neutral as concerns this dispute (although we will return to it in penultimate section). Whichever relation - be it metaphysical or logical - these theories are supposed to capture, since they are *formal* theories, they can themselves become the object of a study that shows their relevance, strength and utility from a logical point of view. Actually, it is quite usual to study formal theories from this angle; if, for example, we consider formal theories of modalities, they have been the object of deep mathematical analysis independently from their philosophical applications i.e. independently from whether they are looked at as capturing logical necessity or metaphysical necessity, see [Blackburn et al., 2001]. In this kind of research, standard logical questions are addressed, like: what results can be proved with these theories? Can we use them to solve paradoxes? How wide is their applicability? This paper can be seen as taking some steps in this direction, insofar as its aim is precisely to extend formal theories of grounding to cover connectives that have not so far been considered. Needless to say, the wider the applicability of the formal theories of grounding, the stronger and more attractive their logical interest is. Moreover, as we will try to show in Section 6, a wide applicability also lead to the identification of new and interesting philosophical features.

If one accepts the relevance of a logical perspective on formal theories of grounding, and the interest of applicability and extendability, one still has to decide which less standard connectives to consider first. Why begin, as we done, with exclusive and trivalent disjunctions? Most importantly, because among all

classical connectives considered to date, disjunction is without doubt the most controversial one. One controversial concerns the grounds of a disjunction: indeed, it is the connective on which the several formal theories of grounding differ. On the one hand, under most approaches, the logical grounds of a disjunction are either both disjuncts or one of them (see [Correia, 2014, Fine, 2012a, Schnieder, 2011]); however, this way of treating disjunction creates the famous overdetermination phenomenon (see [Koslicki, 2015]), which basically consists in one disjunction being determined by more than one ground. By contrast, [Poggiolesi, 2016] introduces the notion of *robust condition* (see the next section for a definition) which intervenes in the grounds of disjunction. As she shows, this allows her account to overcome the overdetermination problem and settle the logical grounds of the *or* in a complete way.[4] Another controversy is the aforementioned one between metaphysical and logical grounding: indeed, the grounding-analysis of disjunction is one of the main disputed cases in this debate. In particular, according to [Hofweber, 2009, Merlo, 2020], the relation between a disjunction and its disjuncts cannot be unproblematically conceived of in terms of metaphysical priority. In sum, the case of disjunction is typically questionable and a source of debate in the grounding literature. For this reason, a study of relative disjunction connectives might shed light on the fruitfulness of different approaches, or on these debates, insofar as they give a glimpse of how they can apply beyond the case of the classical disjunctive.

Finally, let us explain why we work primarily with the theory developed by Poggiolesi, and not with the most well-known theories of [Fine, 2012a] or [Correia, 2014], to investigate the grounds of the exclusive disjunction and the trivalent disjunction. There are three main reasons for this choice. The first is *generality*: it has recently been proved by [Poggiolesi, 2020c]) that her theory is more general than the one developed by [Fine, 2012b], i.e. that there exists a formal translation according to which given Poggiolesi's approach, we can recover each Finean grounding relation between logical connectives and their grounds. The converse does not necessarily hold. Therefore, if we have a grounding analysis of, for example, trivalent disjunction according to Poggiolesi's approach, we can obtain a grounding-analysis of trivalent disjunction according to Fine's approach, whilst the converse is not necessarily true. The second concerns the previously mentioned issue of *overdetermination*: as already noted, Poggiolesi's theory does not encounter the problem of the overdetermination of disjunction and thus looks as more promising for treating other types of disjunction. The third is *structural extendability*. Poggiolesi's theory is based on a characterization of the notion of grounding as a special form of derivability; this characterization represents the formal basis to justify certain grounding rules (in particular, [Poggiolesi, 2018] puts forward grounding rules that are proved to be equivalent to the definition of the notion of complete and immediate grounding). In contrast, other formal theories of grounding defend their

---

[4]We emphasize that Poggiolesi aims to capturing the notion of *complete* grounding, that is different from the more famous notion of *full* grounding. We have the complete grounds of a certain truth A when we have a maximal set of all truths that can contribute to explain A. See [Poggiolesi, 2020a] for further details.

grounding rules or axioms on a case-by-case basis by appeal to intuitions. To this extent, Poggiolesi's framework is naturally – or structurally – extendable, by applying the general definition of grounding to the new connectives and seeing what rules result, whereas other theories are not: they give little structural indication of how they would be extended. As we will see in the next section, the grounding-analysis of the exclusive as well as ternary disjunctions will lead to a natural generalization of Poggiolesi's approach itself, but also a natural test-case for her extendable, principle-based approach. As so often, the formal principle-based framework, when applied to new cases, may give guidance where intuitions might be weaker or lacking.

3 A definition of the notion of complete and immediate formal grounding in the classical framework

We use this section to briefly recall the results of [Poggiolesi, 2016] for capturing the notion of complete and immediate grounding, which will play an important role in the sequel. Two very simple ideas motivate it. The first consists in organizing all formulas of the propositional classical language in a grounding hierarchy: each level of the hierarchy contains formulas of different complexity, with complexity increasing from bottom to top. We will call this complexity *g-complexity* to differentiate it from the standard notion of logical complexity.

Once all formulas are organized into the hierarchy, the task is to identify the formulas that stand in a dependence relation. But how is the dependence relation formally defined? By the two clauses of *positive* and *negative* derivability. Positive derivability states that the conclusion should be derivable from its grounds, while negative derivability states that the negation of the conclusion should be derivable from the negation of each ground. Whilst it is often assumed that positive derivability is a necessary condition for grounding (e.g. see [Rumberg, 2013]), negative derivability is specific to Poggiolesi's approach. Negative derivability formalizes the idea of variation: in a grounding relation not only the conclusion is derivable from its grounds, but also if something is modified in the grounds, this modification needs to affect the conclusion as well. In the negative derivability clause of Poggiolesi, the variation is conveyed by negating all the grounds en block.

As already mentioned, the account put forward in [Poggiolesi, 2016] involves a distinction between grounds and robust conditions, which can be described briefly on the example of a disjunction like $A \vee B$, in a situation where the formula A is true. In this case, A is certainly a ground for $A \vee B$; but in order for A to be the *complete* ground for $A \vee B$, it is necessary to specify that B is false (i.e. that B is not also a ground for $A \vee B$); in other terms, it is the falsity of B that ensures that, or is a *(robust) condition* for A to be the complete ground for $A \vee B$. Thus, A is the complete and immediate ground for $A \vee B$ under the robust condition that B is false. The reader is referred to [Poggiolesi, 2016]

for a detailed explanation and discussion of the idea of robust conditions in a grounding framework. Robust conditions are denoted by square brackets and will be introduced in Proposition 8.

We now present the formalism inspired by these ideas. We refer the reader to [Poggiolesi, 2016] for an even more detailed explanation of the notions introduced here.

Definition 1 *The classical language $\mathcal{L}^c$ is composed of a denumerable stock of propositional atoms (p, q, r, ...), the logical operators $\neg$, $\wedge$ and $\vee$, the parentheses (,). The connectives $\rightarrow$ and $\leftrightarrow$ are defined as usual; formulas, denoted by the letters $A, B, C, ...$, are generated in the standard way and the symbol $\bot$ is defined as $A \wedge \neg A$.*

Once the classical language $\mathcal{L}^c$ is given, we can standardly define, by means of the classical Hilbert system **C** (e.g. see [Troelstra and Schwichtenberg, 1996]), the notion of classical derivability. We will write $M \vdash_C A$ to denote the fact that the formula A is derivable in the Hilbert system for classical logic **C** from the multiset of formulas M.

We now introduce the key notion of g-complexity, which is a way of assigning a number to each formula of the language $\mathcal{L}^c$. The way that number is calculated reflects deep grounding-relevant features. As we will see, g-complexity leads to the identification of the relation of *being completely and immediately less g-complex*: if a multiset M is completely and immediately less g-complex than a formula A, then the sum of the g-complexity of its members is one less than the g-complexity of A.

Definition 2 *As it is standard, we call atoms as well as negation of atoms literals. $l, l', ...$ denote literals.*

Definition 3 *The g-complexity of a formula $A \in \mathcal{L}^c$, gcm(A), is defined in the following way:*

- *gcm(l) = 0,*

- *gcm($\neg\neg A$) = 1+ gcm(A),*

- *gcm($A \circ B$) = gcm($\neg(A \circ B)$) = 1 + gcm(A) + gcm(B).*

where the symbol $\circ$ stands for either conjunction or disjunction.

To understand the notion of g-complexity, it must be kept in mind that grounding is concerned entirely with truths. Accordingly, the appropriate notion of complexity should track relationships among the truths expressed by the formulas if they were true. If A and B express truths, then the truth expressed by $A \wedge B$ or $A \vee B$ is obtained from the previous truths using a single operation, just as the formulas $A \wedge B$ and $A \vee B$ are constructed from the formulas A and B using a single connective. Counting the connective in this case is faithful to the relationship of interest among truths and indeed $gcm(A \circ B) = gcm(A) + gcm(B)) +1$.

By contrast, the negation is different, because there is no sense in which if a formula of the form $\neg A$ expresses a truth, then that truth is constructed from A itself. Consider for instance the formulas p and $\neg p$ (namely the literals). p is atomic thus has g-complexity 0, but does that mean that $\neg p$ should count as having g-complexity 1? That would be justified if the truth $\neg p$ (when it is a truth) was constructed from the truth p; but this is not the case in general, not least because when one of the formulas is a truth, the other is not. From the point of view of grounding, which deals solely in truths, there is no truth from which $\neg p$ can be formally constructed, so, like p, it is atomic. Similar points hold for formulas of the form A, $\neg A$, where A is either a conjunction or a disjunction: the complexity of the latter cannot be counted as one more than the complexity of the former, since it is not reducible to it. Therefore in the formula $\neg A$ (where A does not itself start with a negation), the only g-complexity to count is that of A. This is precisely what Definition 3 does, by setting the complexity of $A \circ B$ and $\neg(A \circ B)$ on the same level.

The case of the double negation, however, is different. A formula like $\neg\neg A$, if true, can be reduced to another, simpler truth, namely A. Moreover, such reduction is direct: there is no "intermediate" truth that one passes through to obtain the former from the latter. Thus, it makes sense to count the g-complexity of $\neg\neg A$ as equal to that of A plus one.

Let us now move to the key notion of *being completely and immediately less g-complex*. In order to define this notion, we first need to introduce other notions, namely that of *converse* of a formula, and the relations of *a-c* equivalence and $\cong$. (The notion of converse of a formula and the relation $\cong$ will be directly used to define the relation of "being completely and immediately less g-complex"; the relation of *a-c* equivalence serves to define the relation $\cong$).

Definition 4 *Let D be a formula. The* converse *of D, written D^*, is defined in the following way*

$$D^* = \begin{cases} \neg^{n-1}E, & \text{if } D = \neg^n E \text{ and } n \text{ is odd} \\ \neg^{n+1}E, & \text{if } D = \neg^n E \text{ and } n \text{ is even} \end{cases}$$

where the principal connective of E is not a negation, $n \geqslant 0$ and 0 is taken to be an even number.[5]

Note that the advantage of working with the notion of *converse of a formula A* rather than with the *negation of a formula A* is that, while negation might increase the g-complexity of A, the converse of A is a formula B which has the same g-complexity as A, whilst having the opposite truth value than A just as the negation.

Definition 5 *Consider a formula A. We will say that A is* a-c equiv *to B, if, and only if, A can be obtained from B by applications of associativity and commutativity of conjunction and disjunction.*

---

[5]Note that $\neg^0 E$ is just E.

Definition 6 *For any two formulas A, B, $A \cong B$ if, and only if:*

> A *is* a-c equiv *to* B *or* A *is* a-c equiv *to* B^*

As extensively discussed in [Poggiolesi, 2016], two formulas A and B stand in the relation denoted by $\cong$ when they *are about*, or *pertain to*, or *concern* the same issue. The relation $\cong$ is thus analogous (though not equivalent) to the notion of factual equivalence discussed in [Correia, 2014, Correia, 2016].

Definition 7 *Given a multiset of formulas M and a formula C of the classical language $\mathcal{L}^c$, we say that M is* completely and immediately less g-complex *than C, if, and only if:*

- *$C \cong \neg\neg B$ and $M = \{B\}$ or $M = \{B^*\}$, or*

- *$C \cong (B \circ D)$ and $M = \{B, D\}$, or $M = \{B^*, D\}$, or $M = \{B, D^*\}$, or $M = \{B^*, D^*\}$.*

The multiset[6] M is completely and immediately less g-complex than the formula C since it contains *all* those 'subformulas'[7] of C which are such that the sum of their g-complexity is *one less than* that of C.

Definition 8 *For any consistent multiset of formulas $C \cup M$ such that C and M are formulated in the classical language $\mathcal{L}^c$, we say that, under the robust condition C (that may be empty), M completely and immediately formally grounds A, in symbols $[C] \, M \, \mid\!\sim \, A$, if and only if:*

- *$M \vdash_C A$*

- *$C, \neg(M) \vdash_C \neg A$*

- *$C \cup M$ is completely and immediately less g-complex than A in the sense of Definition 7.*

where $\neg(M) := \{\neg B | B \in M\}$.

Under the robust condition C, the multiset M completely and immediately grounds A if, and only if, (i) A is derivable from M – positive derivability; (ii) $\neg A$ is derivable from $\neg(M)$ plus C – negative derivability; (iii) $C \cup M$ is completely and immediately less g-complex than A.

As said in the Introduction, the goal of this paper is to use the approach developed by of Poggiolesi in order to give an adequate ground-analysis of the two connectives of *exclusive disjunction* and *trivalent disjunction*. But such an approach has been explicitly formulated for the classical connectives of negation, conjunction and disjunction. As an example, the relation of *being completely*

---

[6] We work with multisets rather than with sets since we need to keep track of the number of occurrences of each formula.

[7] For the rigorous definition of subformula in a grounding framework see [Poggiolesi, 2016].

and immediately less g-complex only concerns formulas built up from these connectives. Hence, in order for it to work for other connectives and still keep the spirit of the original one, we need to conservatively extend it, i.e. we need to generalize it in such a way that it works as it used to for the standard classical connectives but it also covers the new ones. Here it is the list of the reasonable moves to obtain a reasonable generalization.

1. We need to replace, in the main definition, classical derivability $\vdash_C$ by derivability in the logic where the new connectives are introduced.[8] This is the easy step. We recall that the original definition in both [Poggiolesi, 2016] and [Poggiolesi, 2018] considers derivability (both positive and negative) to be formulated in axiomatic (Hilbert-style) proof-systems. Here, for the sake of simplicity, we will mainly employ (the deductively equivalent) natural deduction (ND) proof systems for derivability.

2. We need to find a suitable notion of variation, expressing the expected connection between the complete immediate grounds and the grounded in the logic where the new connectives are introduced.

3. We need to find a suitable g-complexity measure and related notion of *being completely and immediately less g-complex* which involve the new connectives under consideration.

4 A logic with exclusive disjunction

The first connective we analyse is the exclusive disjunction, also said *xor* and denoted by $\oplus$, and its negation. We will study this connective in the logic $L_\oplus^c$ obtained by adding $\oplus$ to classical logic.

4.1 Defining the logic $L_\oplus^c$

Definition 9 *The language $\mathcal{L}_\oplus^c$ extends the language $\mathcal{L}^c$ thanks to the addition of the binary connective $\oplus$. Formulas are generated as usual by the operation*

$$p \mid \neg A \mid A \wedge B \mid A \vee B \mid A \oplus B$$

Truth is defined by classical-like bivalent valuations v, assigning arbitrarily truth-values to atomic formulas and extended to standard compound formulas as usual, and to formulas such as $A \oplus B$ by

$$v[\![A \oplus B]\!] = t \text{ iff } v[\![A]\!] \neq v[\![B]\!]$$

That is, there are two possibilities for $v[\![A \oplus B]\!] = t$:

1. $v[\![A]\!] = t$ and $v[\![B]\!] = f$

---

[8]See also [Poggiolesi, 2020a].

$$\frac{A \quad \neg B}{A \oplus B} \; (\oplus I_1) \qquad \frac{\neg A \quad B}{A \oplus B} \; (\oplus I_2)$$

$$\frac{A \oplus B \quad A}{\neg B} \; (\oplus E_1) \qquad \frac{A \oplus B \quad B}{\neg A} \; (\oplus E_2) \qquad \frac{A \oplus B \quad \neg A}{B} \; (\oplus E_3) \qquad \frac{A \oplus B \quad \neg B}{A} \; (\oplus E_4)$$

Figure 1: ND for $L_\oplus$

2. $v[\![A]\!] = f$ and $v[\![B]\!] = t$

For derivability, we have the ND-system $\mathcal{N}_\oplus^c$ obtained by adding to the natural deduction calculus for classical logic the rules in Figure 1. We use $M \vdash_{\mathcal{N}_\oplus^c} A$ to indicate derivability in $L_\oplus^c$.

4.2 Complete immediate grounds in $L_\oplus^c$

Let us start by considering the notion of g-complexity, but also the related relation of *being completely and immediately less g-complex*, in the framework of the logic $L_\oplus^c$. Actually these notions are easily obtained by adapting the insights involved in the analysis of the standard connectives illustrated in the previous section to the new connective xor. Hence we have what follows.

Definition 10 *The g-complexity of a formula $A \in \mathcal{L}_\oplus^c$, $gcm'(A)$, is defined in the following way:*

- $gcm'(l) = 0$,

- $gcm'(\neg\neg A) = 1 + gcm'(A)$,

- $gcm'(A \circ B) = gcm'(\neg(A \circ B)) = 1 + gcm'(A) + gcm'(B)$.

where $\circ = \{\wedge, \vee, \oplus\}$.

Definition 11 *Let D be a formula. The* converse *of D, written D^*, is defined as in Definition 4 but covers the language $\mathcal{L}_\oplus^c$; this implies that we have the converse of $A \oplus B$, namely $(A \oplus B)^*$, which corresponds to $\neg(A \oplus B)$.*

Definition 12 *Consider a formula A. We will say that A is* a-c$'$ equiv *to B, if, and only if, A can be obtained from B by applications of associativity and commutativity of conjunction, disjunction and exclusive disjunction.*

Let us provide some examples of formulas that are *a-c$'$ equiv*. If A is of the form $E \oplus F$, then the formula $F \oplus E$ is *a-c$'$ equiv* to it. To take another example, if A is of the form $(B \oplus C) \oplus D$ the formulas $D \oplus (B \oplus C)$ and $(D \oplus C) \oplus B$ are formulas *a-c$'$ equiv* to it.

Definition 13 *For any two formulas A, B, $A \cong' B$ if, and only if:*

A is a-c' *equiv to B or A is* a-c' *equiv to B^**

Definition 14 *Given a multiset of formulas M and a formula C of the language $\mathcal{L}_\oplus^c$, we say that M is* completely and immediately less g-complex *than C, if, and only if:*

- *$C \cong' \neg\neg B$ and $M = \{B\}$ or $M = \{B^*\}$, or*

- *$C \cong' (B \circ D)$ and $M = \{B, D\}$, or $M = \{B^*, D\}$, or $M = \{B, D^*\}$, or $M = \{B^*, D^*\}$.*

The relation of being *completely and immediately less g-complex* is very useful for the task of identifying the grounds of a certain formula, in this case the formulas of the form $A \oplus B$ which are the centre of our interest. Since, according the approach of Poggiolesi, the complete and immediate grounds of $A \oplus B$ need to be completely and immediately less complex than $A \oplus B$, we know that the grounds for $A \oplus B$ necessarily consist of one of these four multisets $\{A, B\}$ or $\{A^*, B\}$ or $\{A, B^*\}$ or $\{A^*, B^*\}$. This widely restricts the possibilities. Let us now analyse each of these multisets. Still according to Poggiolesi's approach, we know that the grounds and their conclusion need to be such that the latter is derivable from the former. Of course now derivability is to be understood relative to the logic $L_\oplus^c$. But then of the four multisets $\{A, B\}$, $\{A^*, B\}$, $\{A, B^*\}$, $\{A^*, B^*\}$, only two can serve as grounds, since neither from $\{A, B\}$ nor from $\{A^*, B^*\}$ the formula $A \oplus B$ is derivable according to the rules of Figure 1. So we are left with $\{A^*, B\}$ and $\{A, B^*\}$: from $\{A^*, B\}$, thanks to the rule $\oplus I_1$ and possibly some applications of the negation rule, and from $\{A, B^*\}$, thanks to the rule $\oplus I_2$ and possibly some applications of the negation rule, the formula $A \oplus B$ is derivable. Even at the intuitive level these multisets seem to be two good candidates. The exclusive disjunction is indeed true if, and only if, its components do not have the same truth value; hence, if we wonder why a formula like $A \oplus B$ is true, the natural answer is either because A is true and B^* is true, or because B is true and A^* is true. Moreover each of these grounds is complete because each gathers all truths that serve to explain $A \oplus B$, and finally even the immediacy condition is satisfied since nothing can possibly lie in between $\{A^*, B\}$ or $\{A, B^*\}$ and $A \oplus B$.

So far so god. But here is where the problem comes. According to Poggiolesi's definition, for $\{A^*, B\}$ and $\{A, B^*\}$ to be the complete and immediate grounds of $A \oplus B$, not only the relation of being completely and immediately less g-complex and positive derivability need to be satisfied, but negative derivability as well. In particular, by negating *each* ground and having an empty robust condition, the negative derivability clause would require:

1.
$$\neg(A^*), \neg B \vdash_{\mathcal{N}_\oplus^c} \neg(A \oplus B)$$

2.
$$\neg A, \neg(B^*) \vdash_{\mathcal{N}_\oplus^c} \neg(A \oplus B)$$

The problem is that neither 1 nor 2 hold and that therefore $\{A^*, B\}$ and $\{A, B^*\}$ cannot be considered as the complete and immediate grounds of $A \oplus B$, according to Poggiolesi's approach. In front of this situation we have two options: either we stick with Poggiolesi's original notion of negative derivability, we reject $\{A^*, B\}$ and $\{A, B^*\}$ as possible grounds of $A \oplus B$ and we try to change something in g-complexity and positive derivability in the attempt of finding new grounds. Or we believe that the arguments and definitions provided so far for g-complexity and positive derivability are sound and that moreover even at the intuitive level $\{A^*, B\}$ and $\{A, B^*\}$ seem to be ideal candidates for the complete and immediate grounds of $A \oplus B$, and that hence something needs to be changed in the original Poggiolesi's notion of negative derivability. This second option looks more plausible and thus we will follow it in the rest of the section.

Let us remind the reader that the notion of negative derivability serves, in Poggiolesi's intentions, to formalize a notion of variation. However, the choice made in [Poggiolesi, 2016], to negate all of the grounds *in block* fits classical conjunction and disjunction, but it is a quite strong request. Indeed if we take M as a multiset of complete and immediate grounds, in order to vary this multiset and check whether this variation affects the conclusion, it seems enough to vary at least one element of M by replacing it with its negation, or even better with its converse that it is an even more general requirement, and check that this variation passes through the conclusion. This small change already gives a modification of our original multiset of grounds and it might already affect the truth-value of the grounded formula. So negative derivability can become a much more general request *to find at least a variation of the multiset of grounds that affects the truth-value of the conclusion*, i.e. it is such that from that variation we can derive that the converse of the grounded formula is true. This leaves the choice in [Poggiolesi, 2016] correct by being a *special case* of the generalization here, the special case where one chooses to vary all of the grounds.

Let us see the effect of such a change in our case-study. Let us focus on $\{A, B^*\}$ as being the complete and immediate grounds of $A \oplus B$. There are two natural choices for varying this multiset of grounds.

1. Turn B^* to $(B^*)^*$, namely to B. We then have that

$$\{A, B\} \vdash_{\mathcal{N}_\oplus^c} \neg(A \oplus B)$$

The derivation is the following:

$$\cfrac{\cfrac{\cfrac{[A \oplus B]_1 \quad A}{\neg B} \oplus E_1 \quad B}{\bot} \wedge I}{\neg(A \oplus B)} \neg I^1$$

2. Turn A to A^*. We then have that

$$\{A^*, B^*\} \vdash_{\mathcal{N}_\oplus^c} \neg(A \oplus B)$$

The derivation is the following[9]

$$\frac{\dfrac{[A \oplus B]_1 \quad A^*}{B} \oplus E_3 \qquad B^*}{\dfrac{\bot}{\neg(A \oplus B)} \neg I^1} \wedge I$$

So, indeed, there are two ways for the negative derivability to be satisfied and hence we can conclude that according to this new analysis $\{A, B^*\}$ is the complete and immediate ground of $A \oplus B$. Note that an analogous analysis can be carried out for $\{A^*, B\}$ but is omitted.[10]

These observations not only have helped us to provide a grounding-analysis of the exclusive disjunction, but have also led to a natural extension of Poggiolesi's definition which we rigorously reformulate in the following way.

Definition 15 *For any consistent multiset of formulas $C \cup M$ such that C and M are formulated in the language $\mathcal{L}_\oplus^c$, we say that, under the robust condition C (that may be empty), M completely and immediately formally grounds A, in symbols $[C] \ M \mid\!\sim \ A$, if and only if:*

- *$M \vdash_{\mathcal{N}_\oplus^c} A$,*

- *for some non-empty (possibly non-proper) sub-multiset M' of M we have that $C, (M')^*, M^- \vdash_{\mathcal{N}_\oplus^c} (A)^*$, where $(M')^* := \{B^*|B \in M'\}$ and M^- is the complement of M',*

- *$C \cup M$ is completely and immediately less g-complex than A in the sense of Definition 21.*

This new definition of complete and immediate grounding conservatively extends the one of [Poggiolesi, 2016]: not only it is straightforward to check that it provides the same complete and immediate grounds for double negation, conjunction and disjunction (as well as negation of conjunction and negation of disjunction), moreover it treats in an adequate way the new connective $\oplus$. The definition also correctly yields the complete and immediate grounds for $\neg(A \oplus B)$. The complete and immediate grounds of $\neg(A \oplus B)$ are indeed either $\{A, B\}$ or $\{A^*, B^*\}$. Let us analyse the relation between $\{A, B\}$ and $\neg(A \oplus B)$, the one between $\{A^*, B^*\}$ and $\neg(A \oplus B)$ is analogous. We have that from $\{A, B\}$, the formula $\neg(A \oplus B)$ can be derived:

---

[9]For the sake of simplicity, in the application of the rules E_3 and $\neg I$ we use A^* instead of $\neg A$ and B^* instead of $\neg B$. The eventual difference disappears thanks to the applications of the classical negation rules.

[10]This analysis cannot be carried out for the multisets $\{A, B\}$ and $\{A^*, B^*\}$ since none of them satisfies positive derivability with $A \oplus B$, although both multisets are completely and immediately less g-complex than $A \oplus B$.

$$\frac{\dfrac{[A \oplus B]_1 \quad A}{\neg B} \oplus E \quad B}{\dfrac{\bot}{\neg(A \oplus B)} \neg I^1} \wedge I$$

hence positive derivability is satisfied. We also have that from $\{A^*, B\}$ - which is a possible variation of the multiset of grounds - the formula $(\neg(A \oplus B))^*$, namely $(A \oplus B)$, can be derived. This is so by means of a simple application of the rule $\oplus I_1$ or by means of an application of the rule $\oplus I_1$ and the application of the classical rules of negation. Therefore also negative derivability is satisfied. Finally, it is easy to check that the multiset $\{A, B\}$ is completely and immediately less g-complex than $\neg(A \oplus B)$ according to Definition 21.

Let us end the section with the following important observation. We have analyzed the grounds of the exclusive disjunction, namely of the formulas of the form $A \oplus B$ and we have reached the conclusion that the complete and immediate grounds of such formulas are either the multiset $\{A, B^*\}$ or the multiset $\{A^*, B\}$. This is supported by the generalized version of Poggiolesi's approach, but also by our intuitions. Let us for example consider the sentence "with four euros John either buys a sandwich or a drink" and suppose one asks for the reasons why this sentence is true. The natural answer is because "with four euros John buys a sandwich and John does not buy a drink", or because "with four euros John buys a drink and John does not buy a sandwich", which precisely coincide with the conclusion that we have drawn in this section, that therefore lay on a solid basis. However one could formulate the following perplexity in front of our conclusions. Consider the logic $L_\oplus^c$ in which we have been working in this section. In $L_\oplus^c$ it is possible to formulate the equivalence between the formula $A \oplus B$ and the formula $(A \wedge \neg B) \vee (\neg A \wedge B)$. However, according to Definition 15, we have that:

- the complete and immediate grounds of $A \oplus B$ are $\{A, B^*\}$ or $\{A^*, B\}$

- the complete and immediate grounds of $(A \wedge \neg B) \vee (\neg A \wedge B)$ are either $\{A \wedge \neg B\}$ under the robust condition $(\neg A \wedge B)^*$, or $\neg A \wedge B$ under the robust condition $(A \wedge \neg B)^*$.

Given that the two formulas are logically equivalent, shouldn't the grounds of $A \oplus B$ be the same as the grounds of $(A \wedge \neg B) \vee (\neg A \wedge B)$ and thus our conclusion be wrong? In order to dissipate this doubt, let us make the following remark. It is by now a widespread opinion (e.g. see [Correia, 2016, Krämer, 2019]) that ground-theoretic equivalence, i.e. the equivalence required to claim that the two formulas $A \oplus B$ and $(A \wedge \neg B) \vee (\neg A \wedge B)$ have the same complete and immediate grounds, is more fine-grained than logical equivalence. This is so because of the negation connective – A and $\neg\neg A$ are logically equivalent but it is not the case that they have the same complete and immediate grounds rather the former grounds the latter – and this is so also because of the number of occurrences of a connective or a subformula in a given formula to which grounding is sensitive,

for example $A \wedge (B \vee C)$ is logically equivalent to $(A \wedge B) \vee (A \wedge C)$ but they do not have the same grounds (e.g. see [Krämer and Roski, 2015]). In other words, in view of these examples that are commonly accepted in the literature, and because of the different structure displayed by the two formulas $A \oplus B$ and $(A \wedge \neg B) \vee (\neg A \wedge B)$, it should not come as a surprise, rather it seems to confirm the granularity of the notion of ground-theoretic equivalence, the fact that $A \oplus B$ and $(A \wedge \neg B) \vee (\neg A \wedge B)$, although logically equivalent, have different complete and immediate grounds.

5 A logic with ternary disjunction

In this section we consider a classical-like ternary connective $+(A, B, C)$, to be interpreted as a ternary disjunction. Namely, classically equivalent to $(A \vee B) \vee C$ (or any of its $ac - equiv$ formula), but considered primitive. We will study this connective in the logic $L^c_{\oplus+}$ obtained by adding to the logic $L^c_{\oplus}$ the ternary connective $+$.

5.1 Defining the logic $L^c_{\oplus+}$

Definition 16 *The language $\mathcal{L}^c_{\oplus+}$ extends the language $\mathcal{L}^c_{\oplus}$ thanks to the addition of the trivalent connective $+$. Formulas are generated as usual by the operation*

$$p \mid \neg A \mid A \wedge B \mid A \vee B \mid A \oplus B \mid +(A, B, C)$$

Truth is defined by classical-like bivalent valuations v, assigning arbitrarily truth-values to atomic formulas and extended to standard compound formulas as usual, and to formulas such as $+(A, B, C)$ by

$$v[\![+(A, B, C)]\!] = t \text{ iff } v[\![A]\!] = t \text{ or } v[\![B]\!] = t \text{ or } v[\![C]\!] = t$$

For derivability, we have the ND-system $\mathcal{N}^c_{\oplus+}$ obtained by adding to the natural deduction calculus $\mathcal{N}^c_{\oplus}$ the rules in Figure 2. We use $M \vdash_{\mathcal{N}^c_{\oplus+}} A$ to indicate derivability in $L^c_{\oplus+}$.

5.2 Complete immediate grounds in $L^c_{\oplus+}$

As before, we start by considering the notion of g-complexity, but also the related relation of *being completely and immediately less g-complex*, in the framework of the logic $L^c_{\oplus+}$. Once more these notions are easily obtained by adapting the insights involved in the analysis of the standard connectives to the trivalent disjunction. Hence we have what follows.

Definition 17 *The g-complexity of a formula $A \in \mathcal{L}^c_{\oplus+}$, $gcm''(A)$, is defined in the following way:*

- $gcm''(l) = 0$,

$$\frac{A}{+(A,B,C)} \ (+I_1) \qquad \frac{B}{+(A,B,C)} \ (+I_2) \qquad \frac{C}{+(A,B,C)} \ (+I_3) \tag{1}$$

$$\frac{+(A,B,C) \quad \begin{matrix} [A]_i \\ \vdots \\ D \end{matrix} \quad \begin{matrix} [B]_j \\ \vdots \\ D \end{matrix} \quad \begin{matrix} [C]_k \\ \vdots \\ D \end{matrix}}{D} \ (+E^{i,j,k}) \tag{2}$$

Figure 2: ND for L_+

- $gcm''(\neg\neg A) = 1 + gcm''(A)$,

- $gcm''(A \circ B) = gcm''(\neg(A \circ B)) = 1 + gcm''(A) + gcm''(B)$.

- $gcm'' + (A,B,C) = gcm''(\neg + (A,B,C)) = 1 + gcm''(A) + gcm''(B) + gcm''(C)$.

where $\circ = \{\wedge, \vee, \oplus\}$.

Definition 18 *Let D be a formula. The* converse *of D, written D^*, is defined as in Definition 4 but covers the language $\mathcal{L}^c_{\oplus+}$; this implies that we have the converse of $+(A,B,C)$, namely $(+(A,B,C))^*$, which corresponds to $\neg + (A,B,C)$.*

Definition 19 *Consider a formula A. We will say that A is* a-c-extend equiv *to B, if, and only if, A can be obtained from B by applications of associativity and commutativity of conjunction, disjunction and exclusive disjunction, or it can be obtained by applications of the following laws:*

- $+(A,B,C) \leftrightarrow +(A,C,B)$

- $+(A,B,C) \leftrightarrow +(B,A,C)$

- $+(A,B,C) \leftrightarrow +(C,A,B)$

- $+(A,B,C) \leftrightarrow +(B,C,A)$

- $+(A,B,C) \leftrightarrow +(C,B,A)$

- $+(+(A,B,C),D,E) \leftrightarrow +(A,+(B,C,D),E)$

- $+(+(A,B,C),D,E) \leftrightarrow +(A,B,+(C,D,E))$

- $+(A,+(B,C,D),E) \leftrightarrow +(A,B,+(C,D,E))$

Definition 20 *For any two formulas $A, B \in \mathcal{L}^c_{\oplus+}$, $A \cong'' B$ if, and only if:*

> *A is* a-c-extend equiv *to B or A is* a-c-extend equiv *to B^**

Definition 21 *Given a multiset of formulas M and a formula C of the language $\mathcal{L}^c_{\oplus+}$, we say that M is* completely and immediately less g-complex *than C, if, and only if:*

- *$C \cong'' \neg\neg B$ and $M = \{B\}$ or $M = \{B^*\}$, or*

- *$C \cong'' (B \circ D)$ and $M = \{B, D\}$, or $M = \{B^*, D\}$, or $M = \{B, D^*\}$, or $M = \{B^*, D^*\}$.*

- *$C \cong'' +(B, D, E)$ and $M = \{B, D, E\}$, or $M = \{B^*, D, E\}$, or $M = \{B, D^*, E\}$, or $M = \{B, D, E^*\}$, or $M = \{B^*, D^*, E\}$, or $M = \{B, D^*, E^*\}$, or $M = \{B^*, D, E^*\}$, or $M = \{B^*, D^*, E^*\}$.*

The considerations involved in the determination of the complete and immediate grounds of formulas of the form $+(A, B, C)$ are quite similar to those involved in the complete and immediate grounds of the binary classical disjunction. But precisely because of this similarity, a new possibility naturally emerges, namely that more than just one robust condition is needed for getting the complete immediate grounds of classical formulas. Consider the sentence "Ann will watch a Tarantino's movie or a Scorsese's movie or a Lynch's movie." In case Ann will watch the three, then the three sentences "Ann will watch a Tarantino's movie," "Ann will watch a Scorsese's movie" and "Ann will watch a Lynch's movie" will be the complete and immediate grounds. The second possibility is that Ann watches two movies out of three, say Tarantino and Scorsese, and then we have that under the robust condition that "Ann will not watch a Lynch's movie," "Ann will watch a Tarantino's movie" and "Ann will watch a Scorsese's movie" are the complete and immediate grounds of the sentence "Ann will watch a Tarantino's movie or a Scorsese's movie or a Lynch's movie." There exists a third possibility according to which Ann watches only one movie out of three, say Tarantino, and in this case we need to have that under the two robust conditions that "Ann will not watch a Lynch's movie" and "Ann will not watch a Scorsese's movie," "Ann will watch a Tarantino's movie" is the complete and immediate grounding of "Ann will watch a Tarantino's movie or a Scorsese's movie or a Lynch's movie." This corresponds to the fact that formulas of the form $+(A, B, C)$ not only will have as complete and immediate grounds the multiset $\{A, B\}$ under the robust condition C^*, but also the set $\{A\}$ under the robust condition B^* and C^*.

These observations not only have helped us to provide a grounding-analysis of the ternary disjunction, but have also led to a natural extension of Poggiolesi's definition which we rigorously reformulate in the following way.

Definition 22 *For any consistent multiset of formulas $N \cup M$ such that N and M are formulated in the language $\mathcal{L}^c_{\oplus+}$, we say that, under the robust conditions N (that may be empty), M completely and immediately formally grounds A, in symbols $[N] \; M \mathrel{\vdash\!\sim} A$, if and only if:*

- $M \vdash_{\mathcal{N}^c_{\oplus +}} A$,

- *for some non-empty (possibly non-proper) sub-multiset M' of M we have that $N, (M')^*, M^- \vdash_{\mathcal{N}^c_{\oplus +}} (A)^*$, where $(M')^* := \{B^* | B \in M'\}$ and M^- is the complement of M',*

- *$N \cup M$ is completely and immediately less g-complex than A in the sense of Definition 21.*

This new definition of complete and immediate grounding conservatively extends Definition 15: not only it is straightforward to check that it provides the same complete and immediate grounds for double negation, conjunction, disjunction and exclusive disjunction (as well as the negation of each of these last three connectives), moreover it treats in an adequate way the new connective $+$. The complete and immediate grounds of $+(A, B, C)$ that emerge from Definition 15 are:

- $\{A, B, C\}$

- $\{A, B\}$ under the robust condition C^*

- $\{B, C\}$ under the robust condition A^*

- $\{A, C\}$ under the robust condition B^*

- $\{A\}$ under the robust conditions B^*, C^*

- $\{B\}$ under the robust conditions A^*, C^*

- $\{C\}$ under the robust conditions A^*, B^*

Let us consider in detail one of these cases, namely the one where A is the complete and immediate ground under the robust conditions $\{B^*, C^*\}$. We have that from A one can derive $+(A, B, C)$ by means of one application of the rule $+I_1$ so that positive derivability is satisfied. Negative derivability is also satisfied since we have

$$
\cfrac{[+(A,B,C)]_1 \qquad \cfrac{\cfrac{[A]_2, A^*}{\bot}\wedge I \quad \cfrac{[B]_3, B^*}{\bot}\wedge I \quad \cfrac{[C]_4, C^*}{\bot}\wedge I}{\bot}+E_{2,3,4}}{\neg + (A,B,C)}\neg I_1
$$

Finally it is easy to verify that the multiset $\{A, B^*, C^*\}$ is completely and immediately less g-complex than $+(A, B, C)$ according to Definition 21.

Note that Definition 15 helps us to correctly identify the complete and immediate grounds of formulas of the form $\neg + (A, B, C)$; it is the multiset $\{A^*, B^*, C^*\}$ which satisfies together with formulas of the form $\neg + (A, B, C)$ positive and negative derivability plus g-complexity. Note also that in case $+(A, B, C)$ has the form $+(p, q, +(r, s, t))$ and its grounds is the multiset, say,

$\{p, q, +(r, s, t)\}$, then the multiset $\{p, q, +(r, s, t)\}$ is also the complete and immediate ground of the following formulas $+(+(p, q, r), s, t)$, $+(p, +(q, r, s), t)$, but also $+(q, +(p, r, s), t)$, $+(q, +(p, t, s), r)$, $+(p, +(r, t, s), q)$ and so on.

It is interesting to underline that in natural language not only do we have ternary disjunctions, but also n-ary disjunctions (e.g. see [Francez, 2019]). The complete and immediate grounds of n-ary disjunction will be analogous to those of the ternary disjunction, and they will be obtained by straightforwardly generalizing the key-notions of this section - namely g-complexity, converse, a-c equivalence, completely and immediately less g-complex - to the n-ary case.

Finally a brief observation on the links between logical equivalence and ground-theoretic equivalence. Note that although the two formulas $+(A, B, C)$ and $A \vee (B \vee C)$ are logically equivalent and have the same complete and mediate grounds, they are not ground-theoretically-equivalent, namely they do not have the same complete and immediate grounds. This is so because the connective $+$ is assumed as primitive in the language and thus it represents a basic ternary choice that does not need to pass through a double disjunction to be explained.

6 Discussion

Motivated by the belief that the formal theories of grounding are of intrinsic interest and can and should be studied from a logical perspective, the aim of this paper has been to enrich their applicability, and use them to get a grounding analysis for two non-standard connectives, namely the *exclusive or* and the *ternary or*. We have developed this analysis drawing on Poggiolesi's approach, which is based on the notions of derivability and complexity. Our first result is the rigorous identification of the complete and immediate grounds for the connectives $\oplus$ and $+$. Let us emphasize the importance of this result from a conceptual perspective.[11] First of all, one could object against the utility of our study: since both the connectives $\oplus$ and $+$ are definable in terms of the other classical connectives, one could attempt to obtain the grounds of $\oplus$ and $+$ simply from the grounds of these latter. For example, given that $A \oplus B$ is logically equivalent to, and hence definable as, the formula $(A \wedge \neg B) \vee (\neg A \wedge B)$, one proposal could be to take its complete and immediate grounds to just be those of $(A \wedge \neg B) \vee (\neg A \wedge B)$. This kind of reasoning, although common, depends on the assumption that ground-theoretic equivalence is the same as logical equivalence. However, as our intuitions suggest, and as has emerged from other recent analysis, e.g. see [Correia, 2016], ground-theoretic equivalence is *not* the same as logical equivalence. Rather, it is a much more fine-grained hyperintensional notion (e.g. see [Leitgeb, 2019]) that is sensitive to the structure of the formula, i.e. to the number as well as the order of the connectives that occur in a formula. Our study is fully coherent with these findings, insofar as it treats the two previously unstudied connectives as primitive, rather than as defined from others, and brings out important ground-theoretic differences.

---

[11]We take it to be evident its interest from a technical perspective.

Secondly, this paper extends Poggiolesi's approach to previously unconsidered connectives, and generalizes her characterization of the notion of complete and immediate grounding – on the one hand, by introducing a wider notion of variation, on the other hand, by allowing several robust conditions. It thus shows that Poggiolesi's method copes well with two new connectives, as well as suggesting that the method is well-positioned for accurate analysis of other grounding cases not considered to date.

Extendability is an important property of any formal model of a philosophic notion, on at least two fronts. One centers on future formal development. In the current case, it can be an argument for convincing logicians to take an interest in grounding as a new and fertile area of research, for instance. The analysis of the notion of grounding is currently central in philosophy, but attracts little interest from a logical perspective. However, as the case of modality has shown - a metaphysical notion that has given rise to one of the most central contemporary domains of (philosophical) logic - interest and input from logicians has both enriched the study of modal logics and provided a fruitful feedback into philosophical debate on modality. The other is purely philosophical: an extendable account of grounding has a strong claim to capturing the structure of the notion itself. For instance, we mentioned previously the purported contrast between metaphysical and logical grounding, and the debates about where the boundary between them lies, or whether one or the other even exists (see for instance [Hofweber, 2009] or [Merlo, 2020]), some of which specifically discuss classic disjunction. The development in this paper of a single framework that covers a range of connectives suggests that it may seize characteristics of grounding that are independent of the specific type of grounding involved, and in this sense is perhaps "deeper" than the contrast between metaphysical and logical grounding, for instance. It may be that the connectives used here have impacts in the context of this debate - we leave this as a topic for future research.

7 Conclusions

In this article we have considered two different types of disjunction, namely exclusive disjunction as well as trivalent disjunction, and we have applied Poggiolesi's method to obtain a grounding analysis of both of them. We have underlined the technical as well as the conceptual advantages of this operation. We thus hope that this paper can be taken as a first step towards a systematic grounding analysis of several new - classical but also non-classical - connectives that may result in a wide grounding framework that has an interest not only for its philosophical roots but also for its formal features.

References

[Blackburn et al., 2001] Blackburn, P., de Rijke, M., and Venema, Y. (2001). *Modal Logic*. Cambridge University Press, Cambridge.

[Correia, 2014] Correia, F. (2014). Logical grounds. *Review of Symbolic Logic*, 7(1):31–59.

[Correia, 2016] Correia, F. (2016). On the logic of factual equivalence. *Review of Symbolic Logic*, 9:103–122.

[de Rosset, 2013] de Rosset, L. (2013). What is weak ground? *Essays in Philosophy*, 14(1):7–18.

[Fine, 2012a] Fine, K. (2012a). Guide to ground. In Correia, F. and Schnieder, B., editors, *Metaphysical grounding*, pages 37–80. Cambridge University Press, Cambridge.

[Fine, 2012b] Fine, K. (2012b). The pure logic of ground. *Review of Symbolic Logic*, 25(1):1–25.

[Francez, 2019] Francez, N. (2019). A proof-theoretic semantics for parametric logical constants. *Logique et Analyse*, 62:225–244.

[Hofweber, 2009] Hofweber, T. (2009). Ambitious, yet modest, metaphysics. In Chalmers, D., Manley, D., and Wasserman, R., editors, *Metamteaphysics: New Essays on the Foundations of Ontology*, pages 260–289. Clarendon Press, Oxford.

[Korbmacher, 2017] Korbmacher, J. (2017). Axiomatic theories of partial ground i. the base theory. *Journal of Philosophical Logic*, 47:161–191.

[Koslicki, 2015] Koslicki, K. (2015). The coarse-grainedness of grounding. *Oxford Studies in Metaphysics*, 9(2):306–344.

[Krämer, 2013] Krämer, S. (2013). A simpler puzzle of ground. *Thought*, 2(2):85–89.

[Krämer, 2019] Krämer, S. (2019). Ground-theoretic equivalence. *Synthese*, forthcoming:1–41.

[Krämer and Roski, 2015] Krämer, S. and Roski, S. (2015). A note on the logic of worldly ground. *Thought*, 4:59–68.

[Leitgeb, 2019] Leitgeb, H. (2019). Hype: A system of hyperintensional logic (with an application to semantic paradoxes). *Journal of Philosophical Logic*, 48:305–405.

[McSweeney, 2020] McSweeney, M. (2020). Grounding logically complex facts. In Raven, M., editor, *Routledge Handbook of metaphysical grounding*, pages 228–242. Routledge.

[Merlo, 2020] Merlo, G. (2020). Disjunction and the logic of grounding. *Erkenntnis*, pages 1–23.

[Poggiolesi, 2016] Poggiolesi, F. (2016). On defining the notion of complete and immediate formal grounding. *Synthese*, 193:3147–3167.

[Poggiolesi, 2018] Poggiolesi, F. (2018). On constructing a logic for the notion of complete and immediate formal grounding. *Synthese*, 195:1231–1254.

[Poggiolesi, 2020a] Poggiolesi, F. (2020a). Grounding principles for relevant implication. *Synthese*, 22:1–28.

[Poggiolesi, 2020b] Poggiolesi, F. (2020b). Logics of grounding. In Raven, M., editor, *Routledge Handbook of Metaphysical Grounding*, pages 213–227. Routledge.

[Poggiolesi, 2020c] Poggiolesi, F. (2020c). A proof-based framework for several types of grounding. *HAL, hal-02498116*, 3:1–34.

[Rumberg, 2013] Rumberg, A. (2013). Bolzano's theory of grounding against the background of normal proofs. *Review of Symbolic Logic*, 6(3):424–459.

[Schaffer, 2009] Schaffer, J. (2009). On what grounds what. In Chalmers, D., Manley, D., and Wasserman, R., editors, *Metametaphysics: New Essays on the Foundations of Ontology*, pages 291–309. Oxford University Press.

[Schnieder, 2011] Schnieder, B. (2011). A logic for 'Because'. *The Review of Symbolic Logic*, 4(03):445–465.

[Sebestik, 1992] Sebestik, J. (1992). *Logique et mathematique chez Bernard Bolzano*. J. Vrin.

[Smithson, 2019] Smithson, R. (2019). Metaphysical and conceptual grounding. *Erkenntnis*, forthcoming:1–25.

[Troelstra and Schwichtenberg, 1996] Troelstra, A. S. and Schwichtenberg, H. (1996). *Basic Proof Theory*. Cambridge University Press.