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Abstract

Bisexual Galton-Watson processes are discrete Markov chains where reproduction events are due to mating of males and females. Owing to this interaction, the standard branching property of Galton-Watson processes is lost. We prove tightness for conveniently rescaled bisexual Galton-Watson processes, based on recent techniques developed in [4]. We also identify the possible limits of these rescaled processes as solutions of a stochastic system, coupling two equations through singular coefficients in Poisson terms added to square roots as coefficients of Brownian motions. Under some additional integrability assumptions, pathwise uniqueness of this limiting system of stochastic differential equations and convergence of the rescaled processes are obtained. Two examples corresponding to mutual fidelity are considered.

Key words: Tightness, diffusions with jumps, scaling limits, stochastic calculus, Galton-Watson
MSC 2010: 60J27, 60J75, 60F15, 60F05, 60F10, 92D25.

Contents

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1 Introduction

Galton-Watson processes describe population dynamics for clonal populations without interactions. Biological reasons have led to generalize these processes to bisexual Galton-Watson processes modeling sexual reproduction. The number of pairing of males and females in one generation is then modeled by a mating function which can have different forms depending on different reproduction strategies: monogamous or polygamous reproduction, fidelity or not, one dominant male, etc. These bisexual Galton-Watson processes have been introduced by Daley [8] and studied in particular by Alsmeyer and Rösler [1, 2], see [3, 14] for surveys.

We are interested in the scaling limit of a bisexual Galton-Watson process. We consider a population composed of females and males. The two subpopulations have their own dynamics (clonal reproduction or intrinsic death) but can also interact through the sexual reproduction. In the latter, the mating function plays a main role. This work extends a previous paper [4], in which a general method was proposed for investigating scaling limits of finite dimensional Markov chains to diffusions with jumps. This method was applied to two one-dimensional cases in random environment. In both cases the uniqueness of the limiting one-dimensional diffusion process was based on the works of Fu and Li [12], Dawson and Li [9] and Li and Pu [20], where the authors generalized the well-known uniqueness result for Feller diffusion, with Hölder-1/2 regularity in the diffusion coefficient.

In the present situation, the two populations, females and males, are coupled by the mating, which makes the problem more difficult. We use the general result developed in [4] to prove tightness and identification of the scaling limits of the bisexual processes. The limiting values are solutions of a two-dimensional system of coupled stochastic differential equations with jumps and non regular coefficients. The main novelty concerns the uniqueness of these limiting values. Indeed the coupling of the two equations through singular coefficients in Poisson terms added to square roots as coefficients of Brownian motions raises a deep difficulty. We resolve the problem under an integrability condition on the jump measure which covers a large number of cases.

The bisexual Galton-Watson process $Z^N = (F^N, M^N)$ that we consider is defined as follows. It is a Markov process taking values in $\mathbb{N}^2$ and satisfying the following induction identity for $n \geq 0$,

$$F^N_{n+1} = F^N_n + \sum_{p=1}^{F^N_n} \mathcal{E}^{f,N}_{n,p} + \sum_{p=1}^{g_N(F^N_n, M^N_n)} L^{f,N}_{n,p}, \tag{1}$$

$$M^N_{n+1} = M^N_n + \sum_{p=1}^{M^N_n} \mathcal{E}^{m,N}_{n,p} + \sum_{p=1}^{g_N(F^N_n, M^N_n)} L^{m,N}_{n,p}, \tag{2}$$

where $N \in \mathbb{N}$ scales the population size and for each N, the family of random variables $\{M^N_0, F^N_0, \mathcal{E}^{f,N}_{n,p}, \mathcal{E}^{m,N}_{n,p}, (L^{f,N}_{n,p}, L^{m,N}_{n,p}) : n, p \geq 1\}$ is mutually independent. The random variables (M^N_0, F^N_0) are integer-valued and the random variables $(\mathcal{E}^{f,N}_{n,p}, \mathcal{E}^{m,N}_{n,p}, (L^{f,N}_{n,p}, L^{m,N}_{n,p}))$ are identically distributed for $n, p \geq 1$ and take values in $\{-1, 0, 1, 2, \ldots\} = \{-1, 0\} \cup \mathbb{N}$. We denote their distributions as follows:

$$\mathcal{E}^{\bullet,N}_{n,p} \overset{d}{=} \mathcal{E}^{\bullet,N}, \qquad (L^{f,N}_{n,p}, L^{m,N}_{n,p}) \overset{d}{=} (L^{f,N}, L^{m,N}),$$

for $\bullet \in \{f, m\}$. The terms related to the random variables $\mathcal{E}^{\bullet,N}_{n,p}$ may model either survival without offsprings ($\mathcal{E}^{\bullet,N} = 0$) or death without offsprings ($\mathcal{E}^{\bullet,N} = -1$) or more complex event including an asexual clonal reproduction with several offsprings ($\mathcal{E}^{\bullet,N} \in \mathbb{N}$). The random variables $(L^{f,N}_{n,p}, L^{m,N}_{n,p})$ model the sexual reproduction issued from mating.

The class of bisexual Galton-Watson process defined above combines the classical asexual Galton-Watson processes and the bisexual Galton-Watson processes introduced by Daley [8].

Our main result will be applied in two cases. The particular case where $\mathcal{E}^{\bullet,N}_{n,p}$ are Bernoulli random variables with values in $\{0, -1\}$, describes whether or not individuals survive in the next generation.

A second interesting example concerns the case where $\mathcal{E}^{\bullet,N}_{n,p}$ are nul and $(L^{f,N}_{n,p}, L^{m,N}_{n,p}) \overset{d}{=} -(1,1) + (L^{f,N}_{+}, L^{m,N}_{+})$, with $L^{\bullet,N}_{+} \in \{0,1,\dots\}$. This case can be interpreted as the replacement of the mating pair of female and male by a random number of females and males in the next generation, via sexual reproduction. The function g_N counts the number of mating in one generation. One of the main example of mating function is $g_N(y,z) = y \wedge z$ and we illustrate our results with this function. It counts the number of pairing of male and female when their number is given by y and z. Modeling the effective sexual interaction by such a function corresponds to monogamous mating with mutual fidelity.

In Section 2, we state our assumptions and the main results and we develop the two applications. We prove the tightness and identification of the sequence of scaled sexual Galton-Watson processes in Section 3. We then conclude the proof of the convergence theorem by using the uniqueness result proved in Section 4. In the latter, we prove a uniqueness result in a slightly more general framework which could be applied to different situations. This is is the main difficulty of the paper.

2 Main results and applications

2.1 Assumptions and statement of convergence

Let us state the assumptions under which we obtain our main result. These assumptions will be partially relaxed in the next sections for weaker results.

The two first assumptions govern the scaling of the reproduction and death events.
We introduce a truncation function $h : \mathbb{R} \to \mathbb{R}$, which is a continuous bounded function coinciding with Identity in a neighborhood of zero. For convenience, we assume in this paper that $h(u) = u$ for $u \in [-1,1]$ and as an example, one can consider $h(u) = (-1) \vee (u \wedge 1)$.

Assumption A. *We consider a non-negative sequence v_N going to $+\infty$ and we assume that*

(A1) - *For $\bullet \in \{f,m\}$, there exist $\alpha_\bullet \in \mathbb{R}$, $\sigma_\bullet \geq 0$ and a measure $\nu_\bullet$ on $(0,\infty)$ satisfying $\int_0^\infty (1 \wedge u^2)\, \nu_\bullet(du) < +\infty$,* such that

$$\lim_{N\to\infty} v_N N\, \mathbb{E}(h(\mathcal{E}^{\bullet,N}/N)) = \alpha_\bullet; \quad \lim_{N\to\infty} v_N N\, \mathbb{E}(h^2(\mathcal{E}^{\bullet,N}/N)) = \sigma^2_\bullet + \int_{(0,\infty)} h^2(u)\nu_\bullet(du);$$
$$\lim_{N\to\infty} v_N N\, \mathbb{E}(\phi(\mathcal{E}^{\bullet,N}/N)) = \int_0^\infty \phi(u)\nu_\bullet(du) \tag{3}$$

for any $\phi : \mathbb{R} \to \mathbb{R}$ continuous bounded and null in a neighborhood of 0.

(A2) - *For $\bullet, \star \in \{f,m\}$, there exist $\alpha^S_\bullet \in \mathbb{R}$ and $\sigma^S_{\bullet,\star} \in \mathbb{R}^+$ and a measure ν_S on $[0,\infty)^2$ satisfying $\int_{[0,\infty)^2} 1 \wedge (u_1^2 + u_2^2)\, \nu_S(du_1, du_2) < +\infty$, such that*

$$\lim_{N\to\infty} v_N N\, \mathbb{E}\Big(h\Big(L^{\bullet,N}/N\Big)\Big) = \alpha^S_\bullet,$$
$$\lim_{N\to\infty} v_N N\, \mathbb{E}\Big(h_\bullet h_\star\Big((L^{f,N}, L^{m,N})/N\Big)\Big) = (\sigma^S_{\bullet,\star})^2 + \int_{[0,\infty)^2} h_\bullet h_\star(u_1,u_2)\nu_S(du_1,du_2)$$

where $h_f(u_1,u_2) = h(u_1)$ and $h_m(u_1,u_2) = h(u_2)$, and

$$\lim_{N\to\infty} v_N N\, \mathbb{E}\Big(\phi\Big((L^{f,N}, L^{m,N})/N\Big)\Big) = \int_{[0,\infty)^2} \phi(u_1,u_2)\nu_S(du_1,du_2) \tag{4}$$

for any $\phi : \mathbb{R}^2 \to \mathbb{R}$ continuous bounded and null in a neighborhood of 0.

Assumption **(A1)** yields the classical necessary and sufficient condition for convergence of rescaled (asexual) Galton-Watson processes, see for instance Grimwall [13], Lamperti [18], Bansaye-Simatos [6] or [4]. Assumption **(A2)** provides a natural bisexual counterpart.

Let us now introduce the assumptions on the mating function.

Assumption B. *There exists a non-negative function g on $\mathbb{R}_+^2$ such that*
(B1)- *The sequence of mating functions g_N (defined on $\mathbb{N}^2$) uniformly converges to g, as N tends to infinity :*

$$\sup_{(y,z)\in(\mathbb{N}/N)^2} \left| \frac{g_N(Ny,Nz)}{N} - g(y,z) \right| \overset{N\to\infty}{\longrightarrow} 0. \tag{5}$$

(B2)- *The function g is dominated by $y \wedge z$: there exists $a, b \geq 0$ such that for any $y, z \geq 0$,*

$$g(y,z) \leq a(y \wedge z) + b. \tag{6}$$

(B3)- *The function g is locally Lipschitz and $g(y,0) = g(0,z) = 0$ for all y, z.*
(B4)- *The function g satisfies the ellipticity assumption: for any positive δ, n,*

$$\inf\{g(y,z) : \delta \leq y \leq n, \delta \leq z \leq n\} > 0. \tag{7}$$

As a main example, we have in mind the monogamous mating with mutual fidelity for which $g_N(y,z) = g(y,z) = y \wedge z$. We refer to [3] about mating functions and their impact on population dynamics and to [1] and [2] in the particular case of promiscuous mating. Assumption **(B2)** is restrictive regarding the behavior of g when y or z goes to infinity. But it covers our main modeling motivations. Besides it can certainly be relaxed before explosion time thanks to localization arguments to capture other mating functions.

An additional moment assumption for the jump measure will be involved for pathwise uniqueness. In this section for convenience we make the following first moment assumption and refer to the next sections for comments and extensions.

Assumption C. *We denote by ν the measure on $\mathbb{R}_+^2$ given by*

$$\nu(du_1, du_2) = \nu_f(du_1)\delta_0(du_2) + \delta_0(du_1)\nu_m(du_2) + \nu_S(du_1, du_2).$$

We assume that

$$\int_{\mathbb{R}_+^2} (u_1 + u_2)\nu(du_1, du_2) < +\infty. \tag{8}$$

Let us now state our main result.

Theorem 2.1. *Let us suppose that Assumptions* **A**, **B**, **C** *hold and that the sequence Z_0^N/N converges weakly to $Z_0 = (F_0, M_0) \in [0,\infty)^2$. Then, the sequence of processes $(Z^N_{[v_N \cdot]}/N)_N$ converges*

in law in $\mathbb{D}([0,\infty),[0,\infty)^2)$ *to the unique strong solution* $Z=(F,M)$ *of*

$$
\begin{aligned}
F_t &= F_0 + \int_0^t \alpha_f\, F_s ds + \int_0^t \sigma_f \sqrt{F_s}\, dB_s^f \\
&\quad + \int_0^t \int_{[0,\infty)^2} \mathbf{1}_{\theta\le F_{s-}} h(u) \widetilde{N}^f(ds,d\theta,du) + \int_0^t \int_{[0,\infty)^2} \mathbf{1}_{\theta\le F_{s-}} (u-h(u)) N^f(ds,d\theta,du) \\
&\quad + \int_0^t \alpha_f^S\, g(F_s,M_s)\, ds + \int_0^t \sqrt{g(F_s,M_s)} dB_s^1 \\
&\quad + \int_0^t \int_{[0,\infty)^3} \mathbf{1}_{\theta\le g(F_{s-},M_{s-})} h(u_1) \widetilde{N}^S(ds,d\theta,du_1,du_2) \\
&\quad + \int_0^t \int_{[0,\infty)^3} \mathbf{1}_{\theta\le g(F_{s-},M_{s-})} (u_1-h(u_1)) N^S(ds,d\theta,du_1,du_2), \\
M_t &= M_0 + \int_0^t \alpha_m M_s ds + \int_0^t \sigma_m \sqrt{M_s}\, dB_s^m \\
&\quad + \int_0^t \int_{[0,\infty)^2} \mathbf{1}_{\theta\le M_{s-}} h(u) \widetilde{N}^m(ds,d\theta,du) + \int_0^t \int_{[0,\infty)^2} \mathbf{1}_{\theta\le M_{s-}} (u-h(u)) N^m(ds,d\theta,du) \\
&\quad + \int_0^t \alpha_m^S\, g(F_s,M_s)\, ds + \int_0^t \sqrt{g(F_s,M_s)} dB_s^2 \\
&\quad + \int_0^t \int_{[0,\infty)^3} \mathbf{1}_{\theta\le g(F_{s-},M_{s-})} h(u_2) \widetilde{N}^S(ds,d\theta,du_1,du_2) \\
&\quad + \int_0^t \int_{[0,\infty)^3} \mathbf{1}_{\theta\le g(F_{s-},M_{s-})} (u_2-h(u_2)) N^S(ds,d\theta,du_1,du_2), \qquad (9)
\end{aligned}
$$

where B^f *and* B^m *are one-dimensional Brownian motions,*

$$
\begin{pmatrix} B^1 \\ B^2 \end{pmatrix} = \begin{pmatrix} \sqrt{(\sigma_f^S)^2 - (\sigma_{fm}^S)^4/(\sigma_m^S)^2} & (\sigma_{fm}^S)^2/\sigma_m^S \\ 0 & \sigma_m^S \end{pmatrix} .B,
$$

B is a two-dimensional Brownian motion, N^S, N^f and N^m are Poisson point measures on $[0,\infty)^4$ and $[0,\infty)^3$, respectively with intensity measures $dsd\theta\nu_S(du_1,du_2)$, $dsd\theta\nu_1(du)$ and $dsd\theta\nu_m(du)$, $\widetilde{N}$ being the compensated measure of N and all these processes are independent.

Note that in the previous statement, $\sigma^S_{\bullet,\bullet}$ is denoted by $\sigma^S_\bullet$ for convenience. Besides Assumption **A** ensures that the quantity $(\sigma_f^S)^2-(\sigma_{fm}^S)^4/(\sigma_m^S)^2$ is positive. It can be deduced from Cauchy-Schwarz inequality applied to $\mathbb{E}\Big(\chi_\varepsilon(L^{f,N})\chi_\varepsilon(L^{m,N})\Big)$ by choosing $\chi_\varepsilon = h(1-\phi_\varepsilon)$ with ϕ_ε an even continuous bounded function on $\mathbb{R}$ null in $[0,\varepsilon]$ and equal to 1 in $[2\varepsilon,\infty)$.

Note also that if instead of Assumption **C**, we only require $\int_{\mathbb{R}_+^2} 1\wedge(u_1+u_2)\,\nu(\mathrm{d}u_1,\mathrm{d}u_2)<+\infty$, then the convergence hold before the explosion time $\mathbb{T}_e=\lim_{n\to\infty}\inf\{t\ge 0: F_t\ge n \text{ or } M_t\ge n\}$. Moreover the tightness and identification can be achieved under the optimal two-order moment condition : $\int_{\mathbb{R}_+^2} 1\wedge(u_1^2+u_2^2)\,\nu(du_1,du_2)<+\infty$. The proof of the uniqueness requires a stronger moment assumption close to 0, which slightly extends the first moment condition.

Lastly, note that because of **(B3)**, $(0,0)$ is an absorbing point and any solution issued from $\mathbb{R}_+^2$ stays in $\mathbb{R}_+^2$.

The proof of Theorem 2.1 will be given in Section 3. It consists in first proving the tightness and the identification of the limit under Assumptions **A** and **(B1)**, **(B2)**, using a suitable functional space which exploits the independence of random variables. The uniqueness of the limit will be proved with additional Assumptions **(B3)**–**(B4)** and **C**. This uniqueness result is the main point

of the paper. Indeed, the stochastic system given in (9) is a true coupled system (the coupling being due to the mating), with radical diffusion coefficients and accumulating jumps. At the best of our knowledge, it is the first result of this type in the case of non polynomial coefficients. For the polynomial case, we mention the general approach for the study of the martingale problem developed in e.g. [7, 11]. Our uniqueness result is stated and proved in a general framework in Section 4.

2.2 Applications

We consider now two examples for which we apply the previous result. For sake of clarity, we focus on the classical mating function

$$g(y,z) = y \wedge z.$$

2.2.1 Survival and sexual reproduction

In this first application, the probability for a given mating to leave one offspring or more in the next time step (generation) is low. But a large number of offsprings may be produced in a single mating. This random integer number is denoted by D^N. The sex is determined independently for each offspring: each new born is a female (resp. a male) with probability $q \in (0,1)$ (resp. $1-q$). Besides, we fix $p_f, p_m \geq 0$ to determine the death probability of males and females in each generation.

We make the following assumption.

Assumption D. *Let us consider $\alpha \in [0,\infty)$ and a measure μ on $[0,\infty)$ such that*

$$\int_0^\infty u\mu(du) < \infty. \tag{10}$$

We also consider a sequence $(v_N)_N$ of positive real numbers tending to infinity and a truncation function h and assume that

$$\lim_{N\to\infty} v_N N\,\mathbb{E}\left(h\left(\frac{D^N}{N}\right)\right) = \alpha + \int_0^\infty h(u)\,\mu(du); \quad \lim_{N\to\infty} v_N N\,\mathbb{E}\left(\phi\left(\frac{D^N}{N}\right)\right) = \int_0^\infty \phi(u)\,\mu(du).$$

for any ϕ continuous bounded and null in a neighborhood of 0.

We observe that the choice of the truncation function h does not impact (α,μ). Moreover, given a triplet $(\mu,\alpha,(v_N)_N)$ as previously, the sequence of random variables $(D^N)_N$ satisfying

$$P(D^N=1) = \frac{\alpha}{v_N}, \qquad \forall u \in [2/N,\infty), \quad \mathbb{P}(D^N \geq Nu) = \frac{\mu[u,\infty)}{Nv_N}$$

satisfies Assumption **D**.

We now give by induction a formal definition of the bisexual Galton-Watson $Z^N = (F^N, M^N)$ with sexual reproduction D^N, sex ratio q and death rates (p_f, p_m). Given $(F_0^N, M_0^N) \in \mathbb{N}^2$, we define for $n \geq 0$,

$$F_{n+1}^N = F_n^N + \sum_{p=1}^{F_n^N} \mathcal{E}_{n,p}^{f,N} + \sum_{p=1}^{M_n^N \wedge F_n^N} L_{n,p}^{f,N}, \tag{11}$$

$$M_{n+1}^N = M_n^N + \sum_{p=1}^{M_n^N} \mathcal{E}_{n,p}^{m,N} + \sum_{p=1}^{M_n^N \wedge F_n^N} L_{n,p}^{m,N}, \tag{12}$$

where for each N, the family of random variables $\{M_0^N, F_0^N, \mathcal{E}_{n,p}^{f,N}, \mathcal{E}_{n,p}^{m,N}, (L_{n,p}^{f,N}, L_{n,p}^{m,N}) : n,p \geq 1\}$ is mutually independent. Moreover for each $n,p \geq 1$ and $\bullet \in \{f,m\}$,

$$\mathbb{P}(\mathcal{E}_{n,p}^{\bullet,N} = -1) = 1 - \mathbb{P}(\mathcal{E}_{n,p}^{\bullet,N} = 0) = p_\bullet / v_N$$

corresponds to the probability of death of each female and each male, while

$$(L_{n,p}^{f,N}, L_{n,p}^{m,N}) \stackrel{d}{=} (L^{f,N}, L^{m,N}) = \sum_{j=1}^{D^N} (\mathcal{B}_j, 1 - \mathcal{B}_j)$$

describes the sex repartition of offsprings, where $(\mathcal{B}_j)_{j\geq 1}$ are independent Bernoulli random variables with parameter q independent of D^N.

Theorem 2.2. *Under the weak convergence of Z_0^N/N to $Z_0 = (F_0, M_0) \in [0,\infty)^2$ and Assumption* **D**, *the sequence of processes $(Z_{[v_N \cdot]}^N/N)_N$ converges in law in $\mathbb{D}([0,\infty),[0,\infty)^2)$ to the unique strong solution $Z = (F,M)$ of the following SDE :*

$$\begin{aligned} F_t &= F_0 - \int_0^t p_f F_s ds + \alpha q \int_0^t (F_s \wedge M_s)\, ds + \int_0^t \int_{[0,\infty)^2} \mathbf{1}_{\theta \leq F_{s-} \wedge M_{s-}} q u N(ds, d\theta, du), \qquad (13) \\ M_t &= M_0 - \int_0^t p_m M_s ds + \alpha(1-q) \int_0^t (F_s \wedge M_s)\, ds + \int_0^t \int_{[0,\infty)^2} \mathbf{1}_{\theta \leq F_{s-} \wedge M_{s-}} (1-q) u N(ds, d\theta, du), \end{aligned}$$

where N is a Poisson point measure on $[0,\infty)^3$ with intensity measure $ds d\theta \mu(du)$.

To apply Theorem 2.1, the technical point to check is Assumption **(A2)**. It is deduced from the next lemma.

Lemma 2.3. *For any integers $(k,\ell) \neq (0,0)$,*

$$N v_N \mathbb{E}\left(1 - \exp\left(-k\frac{L^{f,N}}{N} - \ell \frac{L^{m,N}}{N}\right)\right) \stackrel{N\to\infty}{\longrightarrow} a_{k,\ell}\, \alpha + \int_0^\infty (1 - e^{-a_{k,\ell} u}) \mu(du),$$

where $a_{k,\ell} = kq + \ell(1-q)$.

Proof. By independence of the random variables $\mathcal{B}_j$ and conditioning by D^N,

$$\mathbb{E}\left(1 - e^{-kL^{f,N}/N - \ell L^{m,N}/N}\right) = 1 - \mathbb{E}\left(\left[qe^{-k/N} + (1-q)e^{-\ell/N}\right]^{D^N}\right) = \mathbb{E}\left(f_{a_{k,\ell}^N}(D^N/N)\right),$$

where $f_a(x) = 1 - \exp(-ax)$ and

$$a_{k,\ell}^N = -N \log\left(qe^{-k/N} + (1-q)e^{-\ell/N}\right).$$

Letting $N \to \infty$ and noticing that $a_{k,\ell}^N \to a_{k,\ell} > 0$, we prove that $N v_N \mathbb{E}\left(f_{a_{k,\ell}}(D^N/N)\right) \to a_{k,\ell}\alpha + \int_0^\infty f_{a_{k,\ell}}\, d\mu$ by Assumption **D** and conclude. More precisely, let us use a family of non-negative continuous bounded functions $\varphi_\varepsilon : [0,\infty) \to [0,1]$, which are equal to zero in $[0,\varepsilon]$ and equal to 1 in $[2\varepsilon,\infty)$. The decomposition $f_a = ah + (\varphi_\varepsilon + 1 - \varphi_\varepsilon)(f_a - ah)$ yields

$$\begin{aligned} N v_N \mathbb{E}\left(f_{a_{k,\ell}^N}(D^N/N)\right) &= a_{k,\ell}^N . N v_N \mathbb{E}(h(D^N/N)) + N v_N \mathbb{E}\left(\varphi_\varepsilon (f_{a_{k,\ell}^N} - a_{k,\ell}^N h)(D^N/N)\right) \\ &\quad + N v_N \mathbb{E}\left((1-\varphi_\varepsilon)(f_{a_{k,\ell}^N} - a_{k,\ell}^N h)(D^N/N)\right). \end{aligned}$$

By Assumption **D**, the first term converges to $a_{k,\ell}(\alpha + \int h\, d\mu)$ as N tends to infinity.
The last term vanishes as ε tends to 0. To see that, we use that there exists $C > 0$ such that for ε small enough and a in a bounded set, $|(1-\varphi_\varepsilon)(f_a - ah)|(x) \leq C\varepsilon h(x)$ for any $x \geq 0$ and

$$v_N N \left|\mathbb{E}\left((1-\varphi_\varepsilon)(f_a - ah)(D^N/N)\right)\right| \leq C\varepsilon v_N N \mathbb{E}(h(D^N/N)),$$

while $v_N N\mathbb{E}(h(D^N/N))$ is bounded by Assumption **D**.
The facts that the sequence of functions $(f_{a^N_{k,\ell}} - a^N_{k,\ell}h)(x) - (f_{a_{k,\ell}} - a_{k,\ell}h)(x)$ tends uniformly to 0 on the interval $[\varepsilon, \infty)$ as N tends to infinity and that $\left(Nv_N\mathbb{E}\left(\varphi_\varepsilon(D^N/N)\right)\right)_N$ is bounded by the last part of Assumption **D** ensures that

$$Nv_N\left\{\mathbb{E}\left(\varphi_\varepsilon(f_{a^N_{k,\ell}} - a^N_{k,\ell}h)(D^N/N)\right) - \mathbb{E}\left(\varphi_\varepsilon(f_{a_{k,\ell}} - a_{k,\ell}h)(D^N/N)\right)\right\} \overset{N\to\infty}{\longrightarrow} 0,$$

for any $\varepsilon > 0$. We conclude using the convergence of $Nv_N\mathbb{E}\left(\varphi_\varepsilon(f_{a_{k,\ell}} - a_{k,\ell}h)(D^N/N)\right)$ to $\int \varphi_\varepsilon(f_{a_{k,\ell}} - a_{k,\ell}h)\, d\mu$ which also comes from Assumption **D**. □

Proof of Theorem 2.2. The previous lemma ensures via an approximation argument relying on Stone-Weierstrass local theorem (see [4] for details), that

$$\begin{aligned}
&\lim_{N\to\infty} v_N N\,\mathbb{E}\Big(h\Big(L^{\bullet,N}/N\Big)\Big) = \alpha^S_\bullet;\\
&\lim_{N\to\infty} v_N N\,\mathbb{E}\Big(h_\bullet h_\star\Big((L^{f,N}, L^{m,N})/N\Big)\Big) = (\sigma^S_{\bullet,\star})^2 + \int_{\mathbb{R}^2_+} h_\bullet h_\star(u_1,u_2)\nu_S(du_1,du_2)\\
&\lim_{N\to\infty} v_N N\,\mathbb{E}\Big(\phi\Big((L^{f,N}, L^{m,N})/N\Big)\Big) = \int_{[0,\infty)^2} \phi(u_1,u_2)\nu_S(du_1,du_2),
\end{aligned}$$

where we recall that $h_f(u_1,u_2) = h(u_1)$ and $h_m(u_1,u_2) = h(u_2)$ and ϕ is continuous bounded and null in neighborhood of 0 and where we set

$$\begin{aligned}
\alpha^S_f &= \alpha q + \int_0^\infty h(qu)\mu(du), \quad \alpha^S_m = \alpha(1-q) + \int_0^\infty h((1-q)u)\mu(du),\\
\nu^S(A) &= \int_0^\infty 1_{(qu,(1-q)u)\in A}\,\mu(du).
\end{aligned}$$

Assumption **(A1)** is obviously satisfied :

$$\begin{aligned}
&\lim_{N\to\infty} v_N N\,\mathbb{E}(h(\mathcal{E}^{\bullet,N}/N)) = \lim_{N\to\infty} -v_N N\mathbb{P}(\mathcal{E}^{\bullet,N} = -1)/N = -p_\bullet;\\
&\lim_{N\to\infty} v_N N\,\mathbb{E}(h^2(\mathcal{E}^{\bullet,N}/N)) = \lim_{N\to\infty} v_N N\,\mathbb{P}(\mathcal{E}^{\bullet,N} = -1)/N^2 = 0;\\
&\lim_{N\to\infty} v_N N\,\mathbb{E}(\phi(\mathcal{E}^{\bullet,N}/N)) = 0.
\end{aligned}$$

Assumption **B** is guaranteed by our choice of mating function $x \wedge y$ and Assumption **C** comes from (10). We can apply Theorem 2.1 and conclude. □

2.2.2 Replacement of couples

We assume that for each N, $\mathcal{E}^{\bullet,N} = 0$. Besides, the reproduction random variables $L^{f,N}$ and $L^{m,N}$ are independent random variables taking values in $\{-1, 0, 1, \ldots\}$ and the marginal laws satisfy the following scaling assumption.

Assumption E. *We consider two triplets $(\alpha_\bullet, \sigma_\bullet, \nu_\bullet)$ for $\bullet \in \{f, m\}$ with the conditions*

$$\alpha_\bullet \in \mathbb{R}; \qquad \sigma_\bullet \geq 0; \qquad \int_0^\infty u\nu_\bullet(du) < \infty.$$

We consider also a truncation function h and a non-negative sequence v_N going to $+\infty$. Let us assume that for $\bullet \in \{f, m\}$,

$$\lim_{N\to\infty} v_N N\,\mathbb{E}(h(L^{\bullet,N}/N)) = \alpha_\bullet; \quad \lim_{N\to\infty} v_N N\,\mathbb{E}(h^2(L^{\bullet,N}/N)) = \sigma_\bullet + \int_0^\infty h^2(u)\,\nu_\bullet(du);$$

$$\lim_{N\to\infty} v_N N\,\mathbb{E}(\varphi(L^{\bullet,N}/N)) = \int_0^\infty \varphi(u)\,\nu_\bullet(du).$$

for any φ continuous bounded and null in a neighborhood of 0.

We know from the historical study of Galton-Watson processes that for any such triplet (α, σ, ν), there exist $(v_N)_N$ and $(L^N)_N$ satisfying Assumption **E**, see [16, 15, 6].

We consider for each $N \geq 1$ the following Markov chain where every pair dies after reproduction and leaves independently a random number of males and females, independent from each other and distributed as $(L^{f,N}, L^{m,N})$. It is defined by

$$\begin{aligned} F^N_{n+1} &= F^N_n + \sum_{p=1}^{M^N_n \wedge F^N_n} L^{f,N}_{n,p}, \\ M^N_{n+1} &= M^N_n + \sum_{p=1}^{M^N_n \wedge F^N_n} L^{m,N}_{n,p}, \end{aligned}$$

where $(L^{f,N}_{n,p}, L^{m,N}_{n,p} : n \geq 0, p \geq 1)$ are independent and distributed as $(L^{f,N}, L^{m,N})$. Writing $(L^{f,N}, L^{m,N}) = -(1,1) + (L^{f,N}_+, L^{m,N}_+)$, it means that the pairs disappear in the next generation and are replaced by a number of males and females given by $L^{\bullet,N}_+ \in \{0, 1, \ldots\}$.
Assumption **E** and the independence of $L^{f,N}$ and $L^{m,N}$ make Assumptions **A** and **C** easy to check, while Assumption **B** is again a direct consequence of the choice of g_N. We obtain

Theorem 2.4. *Under the weak convergence of $(Z^N_0/N)_N$ to $Z_0 = (F_0, M_0) \in [0,\infty)^2$ and Assumption* **E**, *the sequence of processes $(Z^N_{[v_N \cdot]}/N)_N$ converges in law in $\mathbb{D}([0,\infty), [0,\infty)^2)$ to the unique strong solution $Z = (F, M)$ of the stochastic differential equation*

$$\begin{aligned} F_t &= F_0 + \alpha_f \int_0^t F_s \wedge M_s \, ds + \sigma_f \int_0^t \sqrt{F_s \wedge M_s} dB^f_s \\ &\quad + \int_0^t \int_{[0,\infty)^2} \mathbf{1}_{\theta \leq F_{s-} \wedge M_{s-}} h(u) \widetilde{N}^f(ds, du, d\theta) + \int_0^t \int_{[0,\infty)^2} \mathbf{1}_{\theta \leq F_{s-} \wedge M_{s-}} (u - h(u)) N^f(ds, du, d\theta), \\ M_t &= M_0 + \alpha_m \int_0^t F_s \wedge M_s \, ds + \sigma_m \int_0^t \sqrt{F_s \wedge M_s} dB^m_s \\ &\quad + \int_0^t \int_{[0,\infty)^2} \mathbf{1}_{\theta \leq F_{s-} \wedge M_{s-}} h(u) \widetilde{N}^m(ds, du, d\theta) + \int_0^t \int_{[0,\infty)^2} \mathbf{1}_{\theta \leq F_{s-} \wedge M_{s-}} (u - h(u)) N^m(ds, du, d\theta), \end{aligned}$$

where B^f, B^m, N^f, N^m are independent, B^f and B^m are Brownian motions, N^f and N^m are Poisson point measures on $\mathbb{R}^3_+$, respectively with intensity measures $ds du \nu_f(d\theta)$ and $ds du \nu_m(d\theta)$.

The assumption $\int_0^\infty u \, \nu_\bullet(du) < \infty$ guarantees both non-explosion and pathwise uniqueness. For tightness and identification, we just need $\int_0^\infty (u^2 \wedge 1) \, \nu_\bullet(du) < \infty$ for $\bullet \in \{f, m\}$, while $\int_0^\infty (u \wedge 1) \, \nu_\bullet(du) < \infty$ is sufficient for pathwise uniqueness before the explosion time.

Proof. We have

$$\mathbb{E}\Big(h\Big(L^{f,N}/N\Big) h\Big(L^{m,N}/N\Big)\Big) = \mathbb{E}\Big(h\Big(L^{f,N}/N\Big)\Big) \mathbb{E}\Big(h\Big(L^{m,N}/N\Big)\Big)$$

is of order of magnitude of $(1/v_N N)^2$ so that

$$\lim_{N \to \infty} v_N N \, \mathbb{E}\Big(h\Big(L^{f,N}/N\Big) h\Big(L^{m,N}/N\Big)\Big) = 0.$$

Similarly for $\phi(u_1, u_2) = \phi_{f,1}(u_1) + \phi_{m,1}(u_2) + \sum_{k=2}^K \phi_{f,k}(u_1) \phi_{m,k}(u_2)$ and $\phi_{\bullet,k}$ continuous bounded and equal to zero in a neighborhood of zero, we have

$$\lim_{N \to \infty} v_N N \, \mathbb{E}\Big(\phi\Big((L^{f,N}, L^{m,N})/N\Big)\Big) = \int_0^{+\infty} \phi_{f,1}(u_1) \nu_f(du_1) + \int_0^{+\infty} \phi_{m,1}(u_2) \nu_m(du_2).$$

Thus, Assumption **A** holds with $\sigma^S_{fm} = 0$ and $\nu_S(du_1, du_2) = \delta_0(du_1) \nu_m(du_2) + \nu_f(du_1) \delta_0(du_2)$ and applying Theorem 2.1 yields the result. □

3 Proof of the convergence

The proof is organized as follows. First, using [4] applied to a compactified version of the bisexual process $Z^N = (F^N, M^N)$, we prove tightness and that the limiting points of Z^N are weak solution of SDE (9). Second, we prove that pathwise uniqueness holds for (9). This point is new and is the main difficulty of the paper. It is the object of forthcoming Proposition 3.5, whose proof is a direct adaptation of the uniqueness result stated and proved in a more convenient setting in Section 4.

3.1 Tightness and identification

Tightness and identification are proved under more general assumptions. We only need Assumptions **A** and **(B1)**, **(B2)**.

Proposition 3.1. *Suppose Assumptions* **A** *and* **(B1)**, **(B2)** *hold and suppose the sequence* $(Z_0^N/N)_N$ *converges weakly to* $Z_0 = (F_0, M_0) \in [0,\infty)^2$. *Then, the sequence of processes* $(Z^N_{[v_N\cdot]}/N)_N$ *is tight in* $\mathbb{D}([0,\infty),[0,\infty]^2)$ *and the limiting values* $Z = (F, M)$ *are weak solutions of* (9) *before the explosion time* $\mathbb{T}_e = \lim_{n\to\infty} \inf\{t \geq 0 : F_t \geq n \text{ or } M_t \geq n\}$.

The proof below provides an identification of the limiting points before the explosion time. Assumption **(B2)** on the domination of the mating function could also be relaxed before explosion using localization argument.

Let us apply the approach developed in [4] for the asexual case. The method is based on the convergence of the characteristics of the associated semi-martingales developed in Jacod-Shiryaev [15], with the use of a specific functional space. This latter exploits the population recurrence-type structure and the independence of the random variables $\{M_0^N, F_0^N, \mathcal{E}_{n,p}^{f,N}, \mathcal{E}_{n,p}^{m,N}, (L_{n,p}^{f,N}, L_{n,p}^{m,N}), n, p \geq 1\}$. This method allows us to prove tightness and identification under the optimal moment assumption on the jump measure, see Assumption **A**.

Let us quickly summarize what we will do. We first remark that depending on the reproduction laws, we can have explosion of the process under Assumption **A**. To deal with this problem and to guarantee the boundedness of the characteristics, we compactify the process as in [4] by considering the new process X^N defined as follows :

$$X_n^N = \left(\exp(-F_n^N/N), \exp(-M_n^N/N)\right).$$

This exponential transform combined with a functional space $\mathcal{H}$ formed by polynomials allow to exploit independence and positivity of the reproduction random variables.

(I) In our setting, the characteristics of the exponential transform of the process are given by formulas (14) and (15) below. It has been proved in [4] (see also Appendix A) that their uniform convergence, in the sense of Lemma 3.3 below guarantees the tightness of the sequence $(X^N_{[v_N\cdot]})_N$ and yields the characteristics of limiting semimartingales.

(II) To identify the limiting values as solutions of a stochastic differential equation, we need to exploit the explicit form given in Lemma 3.3. This representation is obtained in Lemma 3.4.

(III) We come back to the initial process Z^N using Itô's formula, up to the explosion time and prove that the limiting values of the sequence $(Z^N)_N$ are solutions of the stochastic differential system (9). This will complete the proof of Proposition 3.1.

Let us now develop this program.

(I) The first part consists in introducing functional space $\mathcal{H}$ and in proving Assumption (H1) recalled in Appendix A. This assumption ensures the convergence of the characteristics of the

rescaled Markov chain $(X^N_{[v_N.]})_N$ for test functions belonging to $\mathcal{H}$ and provides their limiting form. Note that since $(X^N_{[v_N.]})_N$ is bounded, Assumption (H0) of [4] is obvious.
We consider the space $\mathcal{U} = [-1,1]^2$ and the space of monomial functions (on $\mathcal{U}$) defined by

$$\mathcal{H} = \left\{(u_1,u_2) \in \mathcal{U} \to H_{i,j}(u_1,u_2) = (u_1)^i(u_2)^j \ ; \ i \geq 0, j \geq 0, i, j \neq 0\right\}.$$

Following [15, 4], we consider the following family of linear operators characterizing the law of the increments of the scaled Markov chain. It is defined for H measurable and bounded and for $x = (\exp(-y), \exp(-z)) \in \mathcal{X} = (0,1]^2$ by

$$\mathcal{G}^N_x(H) = v_N \mathbb{E}\left(H(X^N_1 - x) | X^N_0 = x\right),$$

where

$$\begin{aligned} X^N_1 - x \quad = \quad & \left(e^{-y} \left(\exp\left(-\frac{1}{N} \sum_{p=1}^{[Ny]} \mathcal{E}^{f,N}_p - \frac{1}{N} \sum_{p=1}^{g_N([Ny],[Nz])} L^{f,N}_p \right) - 1 \right), \right. \\ & \left. e^{-z} \left(\exp\left(-\frac{1}{N} \sum_{p=1}^{[Nz]} \mathcal{E}^{m,N}_p - \frac{1}{N} \sum_{p=1}^{g_N([Ny],[Nz])} L^{m,N}_p \right) - 1 \right) \right) \end{aligned}$$

Assumption (H1.1,2) is a direct consequence of Stone-Weierstrass theorem and the convergence needed in (H1.3) is proved in forthcoming Lemmas 3.2 and 3.3. For that purpose, we set

$$A^N_{k,\ell}(x) \quad = \quad \mathbb{E}\left(\exp\left(-\frac{k}{N} \sum_{p=1}^{[Ny]} \mathcal{E}^{f,N}_p - \frac{\ell}{N} \sum_{p=1}^{[Nz]} \mathcal{E}^{m,N}_p - \frac{1}{N} \sum_{p=1}^{g_N([Ny],[Nz])} L^{k,\ell,N}_p \right) \right), \tag{14}$$

where $L^{k,\ell,N}_p = kL^{f,N}_p + \ell L^{m,N}_p$ and using that $\sum_{k=0}^i \sum_{\ell=0}^j (-1)^{i-k+j-\ell} \binom{i}{k}\binom{j}{\ell} = 0$, we get by expansion

$$\mathcal{G}^N_x(H_{i,j}) \quad = \quad e^{-iy-jz} \sum_{k=0}^{i} \sum_{\ell=0}^{j} (-1)^{i-k+j-\ell} \binom{i}{k}\binom{j}{\ell} v_N\left(A^N_{k,\ell}(x) - 1\right). \tag{15}$$

Furthermore, we set for $u, v \in \mathbb{R}$,

$$f_k(u) = 1 - e^{-ku}, \qquad f_{k,\ell}(u,v) = 1 - e^{-ku-\ell v}$$

for $k, \ell \in \mathbb{N}$, and by independence of the reproduction events, we have

$$A^N_{k,\ell} = a^N_1 a^N_2 a^N_3, \tag{16}$$

for any $x = (\exp(-y), \exp(-z)) \in (0,1]^2$, where

$$\begin{aligned} a^N_1(x) \quad &= \quad \exp\left([Ny] \log\left(1 - \epsilon^N_1\right)\right), \qquad && \epsilon^N_1 = \mathbb{E}\left(f_k\left(\mathcal{E}^{f,N}/N\right)\right) \\ a^N_2(x) \quad &= \quad \exp\left([Nz] \log\left(1 - \epsilon^N_2\right)\right), \qquad && \epsilon^N_2 = \mathbb{E}\left(f_\ell\left(\mathcal{E}^{m,N}/N\right)\right) \\ a^N_3(x) \quad &= \quad \exp\left(g_N([Ny],[Nz]) \log\left(1 - \epsilon^N_3\right)\right), \quad && \epsilon^N_3 = \mathbb{E}\left(f_{k,\ell}\left((L^{f,N}, L^{m,N})/N\right)\right). \end{aligned}$$

We use the following functions f_k and $f_{k,\ell}$ and their decompositions

$$\begin{aligned} f_k(u) \quad &= \quad kh(u) - \frac{k^2}{2} h^2(u) + R_k(u), \\ f_{k,\ell}(u,v) \quad &= \quad kh(u) + \ell h(v) - \frac{k^2}{2} h^2(u) - \frac{\ell^2}{2} h^2(v) - k\ell h(u)h(v) + R_{k,\ell}(u,v), \end{aligned}$$

where R_k (resp. $R_{k,\ell}$) is continuous bounded and $o(u^2)$ (resp. $o(\|(u,v)\|^2)$) in a neighborhood of 0 (resp. $(0,0)$). These decompositions allow us to derive the asymptotic behavior of ϵ_i^N from Assumption **A**, by summing the three components. Indeed, R_k (resp. $R_{k,\ell}$) are not null in a neighborhood of zero but small enough and a simple approximation argument, which follows e.g. [4, Section 4], yields $v_N N\mathbb{E}(R_k(L^{\bullet,N})) \to \int R_k d\nu_\bullet$ and $v_N N\mathbb{E}(R_{k,\ell}((L^{f,N}, L^{m,N})/N)) \to \int R_{k,\ell} d\nu_S$ as $N \to \infty$. We get

$$v_N N\,\epsilon_1^N \overset{N\to\infty}{\longrightarrow} \gamma_k^f = \alpha_f\, k - \frac{1}{2}\,\sigma_f^2\, k^2 + \int_0^\infty \big(f_k(u) - kh(u)\big)\nu_f(du), \tag{17}$$

$$v_N N\,\epsilon_2^N \overset{N\to\infty}{\longrightarrow} \gamma_\ell^m = \alpha_m\, \ell - \frac{1}{2}\,\sigma_m^2\, \ell^2 + \int_0^\infty \big(f_\ell(u) - \ell h(u)\big)\nu_m(du), \tag{18}$$

$$\begin{aligned} v_N N\,\epsilon_3^N \overset{N\to\infty}{\longrightarrow} \gamma_{k,\ell}^S = {} & \alpha_f^S\, k + \alpha_m^S\, \ell - \frac{1}{2}\,(\sigma_f^S)^2\, k^2 - \frac{1}{2}\,(\sigma_m^S)^2\, \ell^2 - (\sigma_{fm}^S)^2 k\ell \\ & + \int_{\mathbb{R}_+^2} \big(f_{k,\ell}(u_1,u_2) - kh(u_1) - \ell h(u_2)\big)\nu_S(du_1 du_2), \end{aligned} \tag{19}$$

where we recall that $\sigma_{\bullet\bullet}^S$ is denoted by $\sigma_\bullet^S$.

Letting $N \to \infty$, we obtain the following uniform convergence:

Lemma 3.2. *For any $(i,j) \in \mathbb{N}^2 \setminus \{(0,0)\}$, for any $k, \ell \in \mathbb{N}^2$,*

$$\sup_{x\in(0,1]^2} e^{-iy-jz} \left| v_N(A_{k,\ell}^N(x) - 1) + \gamma_k^f y + \gamma_\ell^m z + \gamma_{k,\ell}^S g(y,z) \right| \overset{N\to\infty}{\longrightarrow} 0,$$

where $x = (e^{-y}, e^{-z})$.

Proof. We use the expression (16) which is rewritten :

$$\begin{aligned} A_{k,\ell}^N = {} & 1 + (a_1^N - 1) + (a_2^N - 1) + (a_3^N - 1) + (a_1^N - 1)(a_2^N - 1) + (a_1^N - 1)(a_3^N - 1) \\ & + (a_2^N - 1)(a_3^N - 1) + (a_1^N - 1)(a_2^N - 1)(a_3^N - 1) \end{aligned} \tag{20}$$

for a convenient Taylor expansion. We show now the uniform convergences

$$\sup_{x\in(0,1]^2} e^{-(iy+jz)/3} \left| v_N(a_p^N(x) - 1) - \gamma_p(x) \right| \overset{N\to\infty}{\longrightarrow} 0, \tag{21}$$

for $p = 1, 2, 3$, where

$$\gamma_1(x) = \gamma_k^f\, y \;;\; \gamma_2(x) = \gamma_\ell^m\, z \;;\; \gamma_3(x) = \gamma_{k,\ell}^S\, g(y,z).$$

The terms for $p = 1, 2$ correspond to the scaling of a Galton-Watson process and have already been considered in [4]. Hence, we focus on the third term, which is more delicate. Using (19) and Assumption **(B.1)**, we first expand

$$a_3^N(x) = e^{g_N([Ny],[Nz])\log\left(1-\epsilon_3^N\right)} = 1 + \frac{1}{v_N}\left(\gamma_{k,\ell}^S\, g(y,z) + (g(y,z)+1)o_N(1)\right) + O\left(\frac{g(y,z)^2+1}{v_N^2}\right),$$

as $N \to \infty$, uniformly for x such that $g_N(Ny, Nz)/N \le v_N$. Combining this estimate and Assumption **(B.2)** yields

$$\sup_{g_N(Ny,Nz)/N\le v_N} e^{-(iy+jz)/3} \left| v_N(a_3^N(x) - 1) - \gamma_{k,\ell}^S\, g(y,z) \right| \overset{N\to\infty}{\longrightarrow} 0.$$

Besides, $\gamma^* = \sup_N \left\{ N v_N \log(1-\epsilon_3^N) \right\}$ is finite since $N v_N \epsilon_3^N$ has a finite limit. For x such that $g_N(Ny,Nz)/N \geq v_N$, we have

$$\begin{aligned} e^{-(iy+jz)/3} v_N(a_3^N(x)+1) &\leq e^{-(iy+jz)/3} \frac{g_N(Ny,Nz)}{N} (e^{(\gamma^*/v_N) g_N(Ny,Nz)/N} + 1) \\ &\leq e^{-(iy+jz)/3} (g(y,z)+o(1)) (e^{(\gamma^*/v_N)(g(y,z)+o(1))} + 1), \end{aligned}$$

where $o(1)$ is uniform with respect to x using (5). Recalling (6), we get that both $e^{-(iy+jz)/3} v_N(a_3^N - 1)$ and $e^{-(iy+jz)/3}\gamma_3(x)$ converge to 0 as N tends to infinity, uniformly for $g_N(Ny,Nz)/N \geq v_N$. This ends the proof of (21).

Combining the three uniform convergences in (20) yields the conclusion. □

We can now compute the limit of (15), as N tends to infinity, which is achieved in the following lemma.

Lemma 3.3. *For any $(i,j) \in \mathbb{N}^2 \setminus \{(0,0)\}$, we have*

$$\sup_{x \in (0,1]^2} \left| \mathcal{G}_x^N(H_{i,j}) - \mathcal{G}_x(H_{i,j}) \right| \overset{N \to \infty}{\longrightarrow} 0,$$

where, writing $x = (e^{-y}, e^{-z})$ and denoting by $\delta_{i,j}$ the Kronecker symbol:

$$\begin{aligned} -e^{iy+jz}\mathcal{G}_x(H_{i,j}) &= y\,\delta_{j,0}\left(\delta_{i,1}\alpha_f - (2\delta_{i,2}+\delta_{i,1})\sigma_f^2/2 + \int_0^\infty \left((-1)^{i+1} f_1(u)^i - \delta_{i,1} h(u)\right) \nu_f(du)\right) \\ &\quad + z\,\delta_{i,0}\left(\delta_{j,1}\alpha_m - (2\delta_{j,2}+\delta_{j,1})\sigma_m^2/2 + \int_0^\infty \left((-1)^{j+1} f_1(u)^j - \delta_{j,1} h(u)\right) \nu_m(du)\right) \\ &\quad + g(y,z)\Big(\delta_{j,0}\left[\delta_{i,1}\alpha_f^S - (2\delta_{i,2}+\delta_{i,1})(\sigma_f^S)^2/2\right] + \delta_{i,0}\left[\delta_{j,1}\alpha_m^S - (2\delta_{j,2}+\delta_{j,1})(\sigma_m^S)^2/2\right] \\ &\qquad + \delta_{i,1}\delta_{j,1}(\sigma_{fm}^S)^2 + \int_{[0,\infty)^2} g_{i,j}(u_1,u_2)\nu_S(du_1,du_2)\Big), \end{aligned}$$

and

$$\begin{aligned} g_{i,j}(u) &= \delta_{j,0}\big((-1)^{i+1} f_1(u_1)^i - \delta_{i,1} h(u_1)\big) \\ &\quad + \delta_{i,0}\big((-1)^{j+1} f_1(u_2)^j - \delta_{j,1} h(u_2)\big) - (-1)^{i+j} f_1(u_1)^i f_2(u_2)^j 1_{i\neq 0} 1_{j\neq 0}. \end{aligned}$$

Proof. Combining (15) and the uniform convergence of the previous lemma, we obtain that $\mathcal{G}_x^N(H_{i,j})$ converges uniformly, as n tends to infinity, to $\mathcal{G}_x(H_{i,j})$ which satisfies

$$-e^{iy+jz}\mathcal{G}_x(H_{i,j}) = \sum_{k=0}^{i}\sum_{\ell=0}^{j} (-1)^{i-k+j-\ell} \binom{i}{k}\binom{j}{\ell} (y\gamma_k^f + z\gamma_\ell^m + g(y,z)\gamma_{k,\ell}^S).$$

Plugging the expressions of the constants γ given in (17) and (18) and (19), the sum above can be simplified using $f_{k,\ell}(u_1,u_2) = f_k(u_1) + f_\ell(u_2) - f_k(u_1)f_\ell(u_2)$ and

$$\sum_{k=0}^{i} \binom{i}{k} (-1)^{i-k} = \delta_{0,i} \quad ; \quad \sum_{k=0}^{i} \binom{i}{k} (-1)^{i-k} k = \delta_{1,i},$$

$$\sum_{k=0}^{i} \binom{i}{k} (-1)^{i-k} k^2 = 2\delta_{2,i} + \delta_{1,i} \quad ; \quad \sum_{k=0}^{i} \binom{i}{k} (-1)^{i-k} f_k(u) = (-1)^{i+1} f_1(u)^i 1_{i>0}.$$

We obtain the expected result. □

(II) We now proceed with the representation of the limiting points. For that purpose, we proceed with the successive identification of the coefficients of the stochastic differential equation associated with the limiting characteristics obtained above. Firstly, we gather the jump terms in a common Poisson representation. Indeed, considering first $\mathcal{G}_x(H_{i,j})$ for $i+j \geq 3$ leads us to define the measure μ on $V = \{1,2,3\} \times [0,+\infty) \times [0,\infty)^2$ by

$$\begin{aligned}\mu(dk, d\theta, du_1, du_2) &= \delta_1(dk)\, d\theta\, \nu_f(du_1)\delta_0(du_2) + \delta_2(dk)\, d\theta\, \delta_0(du_1)\, \nu_m(du_2) \\ &\quad + \delta_3(dk)\, d\theta\, \nu_S(du_1, du_2), \end{aligned} \tag{22}$$

where δ_k is the Dirac mass in k. The jump image function $K = (K_1, K_2)$ is the measurable function $K : (x,v) \in [0,1]^2 \times V \to K(x,v) \in \mathbb{R}^2$ given by

$$K_1(x,v) = K_1(x,k,\theta,u_1,u_2) = -e^{-y}.\Big(f_1(u_1)\, \mathbb{1}_{k=1,\, \theta \leq y} + f_1(u_1)\, \mathbb{1}_{k=3,\, \theta \leq g(y,z)}\Big), \tag{23}$$

$$K_2(x,v) = K_2(x,k,\theta,u_1,u_2) = -e^{-z}.\Big(f_1(u_2)\, \mathbb{1}_{k=2,\, \theta \leq z} + f_1(u_2)\, \mathbb{1}_{k=3,\, \theta \leq g(y,z)}\Big), \tag{24}$$

where we recall that $x = (\exp(-y), \exp(-z))$. Let us observe that $\int_V 1 \wedge |K(.,v)|^2 \mu(dv) < +\infty$. Secondly, using $\mathcal{G}_x(H_{i,j})$ for $i+j=2$, we define the diffusion coefficients $\sigma(.) \in \mathcal{M}_{2,4}(\mathbb{R})$ as follows

$$\sigma_{11}(x) = e^{-y}\sqrt{y}\sigma_f, \quad \sigma_{12}(x) = 0, \quad \sigma_{21}(x) = 0, \quad \sigma_{22}(x) = e^{-z}\sqrt{z}\sigma_m,$$

and

$$\sigma_{13}(x) = e^{-y}\sqrt{g(y,z)}\sqrt{(\sigma_f^S)^2 - (\sigma_{fm}^S)^4/(\sigma_m^S)^2}, \quad \sigma_{14}(x) = e^{-y}\sqrt{g(y,z)}(\sigma_{fm}^S)^2/\sigma_m^S$$

$$\sigma_{23}(x) = 0, \quad \sigma_{24}(x) = e^{-z}\sqrt{g(y,z)}\sigma_m^S.$$

Finally we set the drift term $b(.) = (b_1(.), b_2(.)) \in \mathbb{R}^2$:

$$\begin{aligned} b_1(x) = \mathcal{G}_x(H_{1,0}) &= e^{-y}y\left(-\alpha_f + \frac{\sigma_f^2}{2} - \int_0^\infty (f_1(u) - h(u))\, \nu_f(du)\right) \\ &\quad + e^{-y}g(y,z)\left(-\alpha_f^S + \frac{(\sigma_f^S)^2}{2} - \int_0^\infty (f_1(u_1) - h(u_1))\, \nu_S(du_1, du_2)\right); \\ b_2(x) = \mathcal{G}_x(H_{0,1}) &= e^{-z}z\left(-\alpha_m + \frac{\sigma_m^2}{2} - \int_0^\infty (f_2(u) - h(u))\, \nu_m(du)\right) \\ &\quad + e^{-z}g(y,z)\left(-\alpha_f^S + \frac{(\sigma_f^S)^2}{2} - \int_0^\infty (f_1(u_2) - h(u_2))\, \nu_S(du_1, du_2)\right). \end{aligned}$$

These parameters yield the following representation of the limiting points of $(X^N_{[v_N .]})_N$.

Lemma 3.4. *Any limiting value in $\mathbb{D}([0,\infty),[0,1]^2)$ of the sequences of processes $\left(X^N_{[v_N .]}\right)_N$ is a semimartingale solution of the stochastic differential system*

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s + \int_0^t \int_V K(X_{s-}, v)\tilde{N}(ds, dv), \tag{25}$$

where B is a 4-dimensional Brownian motion and N is a Poisson point measure on $\mathbb{R}_+ \times V$ with intensity $ds\mu(dv)$, X_0, B, N are independent and $\tilde{N}$ is the compensated martingale measure of N.

Proof. We need to prove that (H2) in [4] (cf. Appendix A) is satisfied. The continuity of $x \in \mathcal{X} \to \mathcal{G}_x(H)$ for $H \in \mathcal{H}$ is a direct consequence of the continuity of g, which is guaranteed by (**B3**). The continuous extension to $\overline{\mathcal{X}}$ is due to (6). Using our definition of parameters b, σ, K, μ, let us now check that for any $H \in \mathcal{H}$,

$$\mathcal{G}_x(H) = \sum_{a\in\{1,2\}} \alpha_a(H) b_a(x) + \sum_{a,b\in\{1,2\}} \beta_{a,b}(H) c_{a,b}(x) + \int_V \overline{H}(K(x,v))\mu(dv), \tag{26}$$

where for any $a, b \in \{1,2\}$,

$$c_{a,b}(x) = \sum_{i=1}^4 \sigma_{a,i}(x)\sigma_{b,i}(x) + \int_V K_a K_b(x,v)\mu(dv)$$

and $\alpha_a(H), \beta_{a,b}(H)$ are the first and second order coefficients of H in its Taylor expansion and $\overline{H} = H - \sum_{a\in\{1,2\}} \alpha_a(H) - \sum_{a,b\in\{1,2\}} \beta_{a,b}(H)$ is the remaining term. We first observe that for $H \in \mathcal{H}$, these coefficients are trivial. There is a unique coefficient which is non zero for $H_{i,j}$ when $i + j \leq 2$ and it is equal to 1. Besides dor $i + j \geq 3$, $H_{i,j} = \overline{H_{i,j}}$ and $\alpha_\cdot(H_{i,j}) = \beta_{\cdot,\cdot}(H_{i,j}) = 0$. Then using the triplet (V, μ, K) introduced above, we directly check that

$$\mathcal{G}_x(H_{i,j}) = \int_V H_{i,j}(K(x,v))\mu(dv) = \int_V \overline{H}_{i,j}(K(x,v))\mu(dv)$$

and $H_{i,j}$ satisfies (26) for $i + j \geq 3$. Then we can check that (26) is satisfied for $H_{2,0}$:

$$\begin{aligned}\mathcal{G}_x(H_{2,0}) &= e^{-2y} y \left(\sigma_f^2 + \textstyle\int_0^\infty f_1(u)^2 \nu_f(du)\right) + e^{-2y} g(y,z) \left((\sigma_f^S)^2 + \textstyle\int_{[0,\infty)^2} f_1(u_1)^2 \nu_S(du_1, du_2)\right) \\ &= \textstyle\sum_{i=1}^4 \sigma_{1i}^2(x) + \int_V K_1^2(x,v)\mu(dv) = c_{1,1}(x).\end{aligned}$$

Similarly

$$\begin{aligned}\mathcal{G}_x(H_{0,2}) &= e^{-2z} z \left(\sigma_m^2 + \textstyle\int_0^\infty f_1(u)^2 \nu_m(du)\right) + e^{-2z} g(y,z) \left((\sigma_m^S)^2 + \textstyle\int_{[0,\infty)^2} f_1(u_2)^2 \nu_S(du_1, du_2)\right) \\ &= \textstyle\sum_{i=1}^4 \sigma_{2i}^2(x) + \int_V K_2^2(x,v)\mu(dv) = c_{2,2}(x),\end{aligned}$$

and (26) is satisfied for $H_{0,2}$. Finally, the crossed term writes

$$\begin{aligned}\mathcal{G}_x(H_{1,1}) &= e^{y+z} g(y,z) \left((\sigma_{fm}^S)^2 + \int_{[0,\infty)^2} f_1(u_1) f_1(u_2) \nu_S(du_1, du_2)\right) \\ &= \sum_{i=1}^4 \sigma_{1i}(x)\sigma_{2i}(x) + \int_V K_1(x,v) K_2(x,v)\mu(dv) = c_{1,2}(x) = c_{2,1}(x)\end{aligned}$$

and (26) is proved for any $H \in \mathcal{H}$, recalling that the definition of b guarantees the identity for $i + j = 1$. This proves that (H2) is satisfied and recalling that (H1) is already proved, we can apply Theorem 2.4 in [4], see also Theorem A.2 in Appendix . It ends the proof. □

(III) Let us now come back to the initial processes.

We write $V = V_1 \cup V_2$, where $V_1 = \{1,2,3\} \times [0,+\infty) \times (0,1]^2$ and $V_2 = \{1,2,3\} \times [0,+\infty) \times (1,\infty)^2$, to split small and large jumps.

We have seen in Lemma 3.4 that

$$\begin{aligned}X_t^1 = \exp(-F_t) &= X_0^1 + \int_0^t \overline{b_1}(X_s)ds + \sum_{i=1}^4 \int_0^t \sigma_{1,i}(X_s)dB_s^i \\ &\quad + \int_0^t \int_{V_1} K_1(X_{s-}, v)\tilde{N}(ds, dv) + \int_0^t \int_{V_2} K_1(X_{s-}, v)N(ds, dv),\end{aligned}$$

where

$$\overline{b_1}(x) = e^{-y}y\left(-\alpha_f + \frac{\sigma_f^2}{2} - \int_{(0,1]}(f_1(u)-h(u))\,\nu_f(du) + \int_{(1,\infty)} h(u)\nu_f(du)\right) \tag{27}$$
$$+e^{-y}g(y,z)\left(-\alpha_f^S + \frac{(\sigma_f^S)^2}{2} + \int_{(0,1]}(h(u_1)-f_1(u_1))\,\nu_S(du_1,du_2) + \int_{(1,\infty)} h(u_1)\nu_S(du_1,du_2)\right).$$

Using Itô's formula we get before the explosion time $\mathbb{T}_e$:

$$\log X_t^1 = -F_t = -F_0 + \int_0^t \frac{1}{X_s^1}\overline{b_1}(X_s)ds - \frac{1}{2}\sum_{i=1}^4 \int_0^t \frac{\sigma_{1,i}^2(X_s)}{(X_s^1)^2}ds + \sum_{i=1}^4 \int_0^t \frac{\sigma_{1,i}(X_s)}{X_s^1}dB_s^i$$
$$+\int_0^t\int_{V_1}\Big\{\log\big(X_{s-}^1 + K_1(X_{s-},v)\big) - \log(X_{s-}^1)\Big\}\tilde{N}(ds,dv)$$
$$+\int_0^t\int_{V_1}\Big\{\log\big(X_{s-}^1 + K_1(X_{s-},v)\big) - \log(X_{s-}^1) + K_1(X_{s-},v)\frac{1}{X_{s-}^1}\Big\}\mu(dv)ds.$$
$$+\int_0^t\int_{V_2}\Big\{\log\big(X_{s-}^1 + K_1(X_{s-},v)\big) - \log(X_{s-}^1)\Big\}N(ds,dv).$$

By definition of the coefficients introduced previously and by identification of the Brownian terms, we obtain

$$\frac{\sigma_{1,1}(X_s)}{X_s^1} = \sqrt{F_s}\sigma_f \;;\; \sigma_{1,2}(X_s) = 0\;;$$

$$\frac{\sigma_{1,3}(X_s)}{X_s^1} = \sqrt{g(F_s,M_s)}\sqrt{(\sigma_f^S)^2 - (\sigma_{fm}^S)^4/(\sigma_m^S)^2}\;;\; \frac{\sigma_{1,4}(X_s)}{X_s^1} = \sqrt{g(F_s,M_s)}\frac{(\sigma_{fm}^S)^2}{\sigma_m^S}.$$

We also recall from (23) that for any positive two-dimensional x, for $v=(k,\theta,u_1,u_2)$,

$$\frac{K_1(x,v)}{x^1} = -\Big(f_1(u_1)\mathbf{1}_{k=1,\theta\le y} + f_1(u_1)\mathbf{1}_{k=3,\theta\le g(y,z)}\Big).$$

Writing $N^f(ds,d\theta,du) = N(ds,\{1\},d\theta,du,\{0\})$ and $N^S(ds,d\theta,du_1,du_2) = N(ds,\{3\},d\theta,du_1,du_2)$, computation gives that

$$\int_{V_1}\Big\{\log\big(X_{s-}^1 + K_1(X_{s-},v)\big) - \log(X_{s-}^1)\Big\}\tilde{N}(ds,dv)$$
$$= \int_{[0,\infty)\times[0,1)} 1_{\theta\le F_{s-}}u\tilde{N}^f(ds,du) + \int_{[0,\infty)\times[0,1)} 1_{\theta\le g(F_{s-},M_{s-})}u_1\tilde{N}^S(ds,du_1,du_2).$$

We obtain similarly the last jumps terms, without compensation. Finally, the drift term of F is given by the remaining terms. Recall that μ is defined in (22) and replacing $\overline{b_1}(x)$ by its value given in (27), it is equal to

$$-\frac{1}{x^1}\overline{b_1}(x) + \frac{1}{2}\sum_{i=1}^4\frac{\sigma_{1,i}^2(x)}{(x^1)^2}ds + \int_{(0,1]}(h(u)-f_1(u))\,\nu_f(du) + \int_{(1,\infty)}h(u)\nu_f(du)$$
$$+\int_{(0,1]}(h(u_1)-f_1(u_1))\,\nu_S(du_1,du_2) + \int_{(1,\infty)}h(u_1)\nu_S(du_1,du_2) = \alpha_f y + \alpha_f^S g(y,z).$$

This yields the expected equation for F_t. Following the same lines for $M_t = -\log(X_t^2)$ ends the proof of Proposition 3.1.

3.2 Uniqueness and convergence

Using the results of forthcoming Section 4, we are able to prove the uniqueness needed for Theorem 2.1 in a slightly more general framework. Recall that the measure

$$\nu(du_1, du_2) = \nu_f(du_1)\delta_0(u_2) + \nu_m(du_2)\delta_0(u_1) + \nu_S(du_1, du_2)$$

has been introduced in Assumption **C**.

Proposition 3.5. *Let us assume that Hypotheses* **(B2)**–**(B4)** *are satisfied and*

$$\int_{[0,\infty)^2} (u_1^2 + u_2^2) \wedge 1 \, \nu(du_1, du_2) < \infty$$

Let us moreover assume that that there exists $\varepsilon_0 > 0$ *such that*

$$\liminf_{a \to 0} \; e^{\varepsilon_0 \left(\int_{A(a)} (u_1+u_2) \; \nu(du_1, du_2) \right)} \int_{B(a)} (u_1^2 + u_2^2) \, \nu(du_1, du_2) = 0, \tag{28}$$

with $A(a) = \{(u_1, u_2) : \, a < u_1 \leq 1, a < u_2 \leq 1\}, B(a) = \{(u_1, u_2) : \, 0 < u_1 \leq a, 0 < u_2 \leq a\}$.

Then, the stochastic differential system (9) *has a unique strong (positive) solution up to the explosion time*

$$\mathbb{T}_{\mathbf{e}} = \lim_{n \to \infty} \inf\{t \geq 0 : F_t \geq n \text{ or } M_t \geq n\}.$$

If ν *satisfies the extra assumption* $\int_{[0,\infty)^2} (u_1^2 + u_2^2) \wedge (u_1 + u_2) \, \nu(du_1, du_2) < \infty$, *then* $\mathbb{T}_{\mathbf{e}} = \infty$ *a.s.*

Note that Assumption **C** obviously implies (28). Observe also that under Assumptions **(B2)**–**(B4)**, g is locally Lipschitz with linear growth and satisfies the ellipticity assumption.

The proof of Proposition 3.5 is a simple adaptation of the proof of uniqueness of the next section. The measure that plays the role of λ in Section 4, is ν. The representation of jumps in (9) relies on the three Poisson point measures N^f, N^m, N^S. These measures can be gathered in a single Poisson point measure for convenience.

Finally, combining Propositions 3.1 and 3.5, we have proved the convergence stated in Theorem 2.1.

4 Pathwise uniqueness

We have seen previously that the main technical problems to prove uniqueness for the system (9), come from the presence of the square root as coefficient on the Brownian terms, the presence of singular coefficients for the compensated Poisson terms and the fact that this is a two-dimensional system. In this section, we present a simpler version of this system by focusing on the sexual coupling term. This system contains all the difficulties mentioned, improving the known results in the literature. We do this to keep notation as simple as possible. Without additional complexity, we actually consider here a more general diffusion and jump terms.

4.1 The system of equations

We study the uniqueness problem for the following system of stochastic differential equations. This system has a form similar to the one obtained in (9) and contains all its difficulties. It is given by

$$
\begin{aligned}
X_t &= x_0 + \int_0^t b_1(X_s, Y_s)ds + \int_0^t \sqrt{\ell_1(X_s, Y_s)}\, dB_s^1 \\
&+ \int_0^t \int_{\mathbb{R}_+^2} \mathbf{1}_{\theta \le \kappa_1(X_{s-}, Y_{s-})} p_1(X_{s-}, Y_{s-}) h(z)\, \widetilde{N}^1(ds, d\theta, dz) \\
&+ \int_0^t \int_{\mathbb{R}_+^2} \mathbf{1}_{\theta \le \kappa_1(X_{s-}, Y_{s-})} p_1(X_{s-}, Y_{s-}) (z - h(z))\, N^1(ds, d\theta, dz); \\
Y_t &= y_0 + \int_0^t b_2(X_s, Y_s)ds + \int_0^t \sqrt{\ell_2(X_s, Y_s)}\, dB_s^2 \\
&+ \int_0^t \int_{\mathbb{R}_+^2} \mathbf{1}_{\theta \le \kappa_2(X_{s-}, Y_{s-})} p_2(X_{s-}, Y_{s-}) h(z)\, \widetilde{N}^2(ds, d\theta, dz) \\
&+ \int_0^t \int_{\mathbb{R}_+^2} \mathbf{1}_{\theta \le \kappa_2(X_{s-}, Y_{s-})} p_2(X_{s-}, Y_{s-}) (z - h(z))\, N^2(ds, d\theta, dz). \qquad (29)
\end{aligned}
$$

The processes B^1 and B^2 are Brownian motions and N^1 and N^2 are Poisson point measures on $(\mathbb{R}_+)^3$ with intensities $ds\, d\theta\, \lambda_1(dz)$ and $ds\, d\theta\, \lambda_2(dz)$, not necessarily independent.

In what follows, we will denote

$$\lambda(dz) = \lambda_1(dz) + \lambda_2(dz),$$

and throughout this section we assume that λ satisfies the hypothesis

(F0) $\int_0^\infty (z^2 \wedge 1)\, \lambda(dz) < \infty$.

The coefficient are defined on $\mathbb{R}_+^2$ and for $i = 1, 2$ the hypotheses about these coefficients are

(F1) $b_i, \ell_i, \kappa_i, p_i$ are locally Lipschitz on $\mathbb{R}_+ \times \mathbb{R}_+$. We also assume that for all $z \in \mathbb{R}_+$ it holds $b_i(0, z) = \ell_i(0, z) = \kappa_i(0, z) = b_i(z, 0) = \ell_i(z, 0) = \kappa_i(z, 0) = 0$.

(F2) ℓ_i, κ_i, p_i are nonnegative, and p_i are strictly positive in every compact set of $[0, \infty)^2$.

(F3) b_i, ℓ_i, κ_i have linear growth and p, q are bounded. We denote by $\mathbf{L}, \mathbf{A}$ two constants such that

$$|b_1(x, y)| + |b_2(x, y)| + \ell_1(x, y) + \ell_2(x, y) + \kappa_1(x, y) + \kappa_2(x, y) \le \mathbf{L}(x + y) + \mathbf{A}.$$

We assume without loss of generality that p_i are bounded by 1.

(F4) The function $h \in C_b(\mathbb{R}_+, \mathbb{R}_+)$ and it satisfies $h(z) = z$ in a neighborhood of 0.

We point out the following facts that are direct consequences of **(F0)** and **(F4)**.

1. $\int_0^1 z^2\, \lambda(dz) < \infty$, $\int_0^1 h^2(z)\, \lambda(dz) < \infty$, $\int_0^1 |z - h(z)|\, \lambda(dz) < \infty$ and $\lambda([1, \infty)) < \infty$.
2. $\int_1^\infty z\, \lambda(dz) < \infty$ if and only if $\int_0^\infty |z - h(z)|\, \lambda(dz) < \infty$.
3. $\int_0^\infty z^2 \wedge z\, \lambda(dz) < \infty$ if and only if $\int_0^\infty h^2(z)\, \lambda(dz) < \infty$ and $\int_0^\infty |z - h(z)|\, \lambda(dz) < \infty$.

Note also that because of **(F1)**, $(0, 0)$ is an absorbing point and any solution issued from $\mathbb{R}_+^2$ stays in $\mathbb{R}_+^2$.

In some of the computations below, we shall use Burkhölder-Davis-Gundy inequality with $p = 1$, which provides a finite constant $\mathbf{C}_1$, such that

$$\mathbb{E}\left(\sup_{s \le \tau} |M_s|\right) \le \mathbf{C}_1\, \mathbb{E}([M, M]_\tau^{1/2}) \qquad (30)$$

for any local martingale M, and any stopping time τ (cf. Dellacherie-Meyer [10] VII.92). We also need to use similar inequalities relating the supremum of a local martingale and its predictable quadratic variation. Namely, there exists a constant $\overline{\mathbf{c}}_1 > 0$, such that if the jumps of M are bounded in absolute value by $\mathbf{\Delta}$ then (see Lenglart-Lépingle-Pratelli [19])

$$\begin{aligned} \overline{\mathbf{c}}_1 \mathbb{E}(\langle M, M\rangle_\tau^{1/2}) &\leq \mathbb{E}\left(\sup_{s\leq\tau} |M_s|\right) + \mathbf{\Delta}; \\ \mathbb{E}([M, M]_\tau^{1/2}) &\leq 3\,\mathbb{E}(\langle M, M\rangle_\tau^{1/2}). \end{aligned} \tag{31}$$

Note that if $M_t = \int_0^t \int_0^\infty H_{s-}(z)\tilde{N}(ds, dz)$ where N is a Poisson point measure with intensity ν, then $[M, M]_\tau = \int_0^t \int_0^\infty H_{s-}^2(z)N(ds, dz)$ and $\langle M, M\rangle_\tau = \int_0^\tau \int_0^\infty H_{s-}^2(z)\nu(ds, dz)$.

Our first result is an a priori bound for system (29) and we set

$$X_t^* = \sup_{s\leq t} |X_s|.$$

Proposition 4.1. *Assume that $(x_0, y_0) \in \mathbb{R}_+^2$. Assume that $\int_0^\infty (z^2 \wedge z)\,\lambda(dz) < \infty$ and* **(F1)**–**(F4)** *hold. If (X, Y) is a nonnegative solution of (29) then, the following a priori estimates hold for all $t > 0$*

$$\mathbb{E}(X_t + Y_t) \leq (x_0 + y_0 + a\mathbf{A}t)e^{a\mathbf{L}\,t}$$

and

$$\mathbb{E}(X_t^* + Y_t^*) \leq \big(x_0 + y_0 + D + (D + a)\mathbf{A}\,t\big)e^{(D+a)\mathbf{L}\,t},$$

where $\mathbf{L}$ *and* $\mathbf{A}$ *are given in* **(F3)**, $a = 2 + \int_0^\infty |z - h(z)|\,\lambda(dz)$ *and*
$D = \mathbf{C}_1 \left(2 + \sqrt{\int_0^\infty h^2(z)\,\lambda_1(dz)} + \sqrt{\int_0^\infty h^2(z)\,\lambda_2(dz)}\right)$.

Proof. We consider $S_n^X = \inf\{t > 0 : X_t \geq n\}, S_n^Y = \inf\{t > 0 : Y_t \geq n\}$ and $S_n = S_n^X \wedge S_n^Y$. Then, we have

$$\begin{aligned} \mathbb{E}(X_{t\wedge S_n}) &= x_0 + \int_0^t \mathbb{E}(b_1(X_s, Y_s), s < S_n)\,ds \qquad (32) \\ &\quad + \int_0^\infty (z - h(z))\,\lambda_1(dz) \int_0^t \mathbb{E}(\kappa_1(X_s, Y_s)p_1(X_s, Y_s), s < S_n)\,ds \\ &\leq x_0 + \big(1 + \int_0^\infty |z - h(z)|\,\lambda_1(dz)\big)\mathbf{A}t + \qquad (33) \\ &\qquad \big(1 + \int_0^\infty |z - h(z)|\,\lambda_1(dz)\big)\mathbf{L} \int_0^t \mathbb{E}(X_{s\wedge S_n} + Y_{s\wedge S_n})\,ds. \end{aligned}$$

Proceeding similarly for Y, this implies that

$$\mathbb{E}(X_{t\wedge S_n} + Y_{t\wedge S_n}) \leq x_0 + y_0 + a\mathbf{A}t + a\mathbf{L}\int_0^t \mathbb{E}(X_{s\wedge S_n} + Y_{s\wedge S_n})\,ds.$$

To apply Gronwall's inequality, we need to bound $\mathbb{E}(X_{t\wedge S_n} + Y_{t\wedge S_n})$. This is not direct because the processes may jump at S_n.

The first lines of (32) show that $\mathbb{E}(X_{t\wedge S_n}) \leq x_0 + t(2\mathbf{L}\,n + \mathbf{A})(1 + \int_0^\infty |z - h(z)|\,\lambda_1(dz))$, proving that for all t we have $\mathbb{E}(X_{t\wedge S_n}) < \infty$. A similar conclusion holds for Y.

From Gronwall's inequality, using that X, Y are nonnegative and they have only upward jumps, we obtain

$$n\,\mathbb{P}(S_n < t) \leq \mathbb{E}(X_{t\wedge S_n} + Y_{t\wedge S_n}) \leq (x_0 + y_0 + a\mathbf{A}t)e^{a\mathbf{L}t},$$

proving that $S_n \to \infty$ a.s., as $n \to \infty$. Now, Fatou's lemma shows that

$$\mathbb{E}(X_t + Y_t) \leq \liminf_{n\to\infty} \mathbb{E}(X_{t\wedge S_n} + Y_{t\wedge S_n}) \leq (x_0 + y_0 + a\mathbf{A}t)e^{a\mathbf{L}t} \tag{34}$$

which proves the first part of the lemma. Besides,

$$\begin{aligned}\mathbb{E}(X^*_{t\wedge S_n}) \quad \leq \quad & x_0 + \int_0^t \mathbb{E}(|b_1(X_s,Y_s)|, s<S_n)\,ds \\ & + \int_0^\infty |z-h(z)|\,\lambda_1(dz) \int_0^t \mathbb{E}(\kappa_1(X_s,Y_s)p_1(X_s,Y_s), s<S_n)\,ds \\ & + \mathbb{E}\left(\sup_{s\leq t\wedge S_n}\left|\int_0^s \sqrt{\ell_1(X_s,Y_s)}\,dB^1_s\right|\right) \\ & + \mathbb{E}\left(\sup_{s\leq t\wedge S_n}\left|\int_0^s \int_{[0,\infty)^2} \mathbf{1}_{\theta\leq\kappa_1(X_{s-},Y_{s-})}p_1(X_{s-},Y_{s-})h(z)\,\widetilde{N}^1(ds,d\theta,dz)\right|\right)\end{aligned}$$

Using inequality (34) for the two first terms of the right hand side above and (30) for the two last terms together with Cauchy Schwarz for the jump term, we obtain

$$\begin{aligned}\mathbb{E}(X^*_{t\wedge S_n}) \quad \leq \quad & x_0 + \left(1+\int_0^\infty |z-h(z)|\,\lambda_1(dz)\right)\left[\mathbf{A}t + \mathbf{L}\int_0^t \mathbb{E}(X_{s\wedge S_n}+Y_{s\wedge S_n})\,ds\right] \\ & + \mathbf{C}_1\left(\mathbb{E}\left(\int_0^{t\wedge S_n}\ell_1(X_s,Y_s)\,ds\right)\right)^{1/2} \\ & + \mathbf{C}_1\left(\mathbb{E}\left(\int_0^{t\wedge S_n}\int_{[0,\infty)^2}\mathbf{1}_{\theta\leq\kappa_1(X_{s-},Y_{s-})}p_1^2(X_{s-},Y_{s-})h^2(z)\,N^1(ds,d\theta,dz)\right)\right)^{1/2},\end{aligned}$$

where we have used that the square root is a concave function. Therefore,

$$\begin{aligned}\mathbb{E}(X^*_{t\wedge S_n}) &\leq x_0 + \left(1+\textstyle\int_0^\infty |z-h(z)|\,\lambda_1(dz)\right)\left[\mathbf{A}t + \mathbf{L}\int_0^t \mathbb{E}(X_{s\wedge S_n}+Y_{s\wedge S_n})\,ds\right] \\ &\quad + \mathbf{C}_1\left(1+\sqrt{\textstyle\int_0^\infty h^2(z)\,\lambda_1(dz)}\right)\left(\mathbf{A}t + \mathbf{L}\textstyle\int_0^t \mathbb{E}(X_{s\wedge S_n}+Y_{s\wedge S_n})\,ds\right)^{1/2} \\ &\leq x_0 + d_1 + d_2\mathbf{A}t + d_2\,\mathbf{L}\textstyle\int_0^t \mathbb{E}(X_{s\wedge S_n}+Y_{s\wedge S_n})\,ds,\end{aligned}$$

wiith $d_1 = \mathbf{C}_1\left(1+\sqrt{\int_0^\infty h^2(z)\,\lambda_1(dz)}\right)$, $d_2 = d_1 + 1 + \int_0^\infty |z-h(z)|\,\lambda_1(dz)$ (here we have used that $\sqrt{a}\leq 1+a$). The result follows from Gronwall's inequality. □

Proposition 4.2. *Let $(x_0,y_0)\in\mathbb{R}^2_+$. We assume that (X,Y) is a nonnegative solution of the system (29), such that $x_0=0$, and $\int_0^\infty (z^2\wedge z)\,\lambda(dz)<\infty$. If (**F1**) – (**F4**) hold, then for all $t\geq 0$, we have $X_t=0, Y_t=y_0$. A similar conclusion holds if $y_0=0$.*

Proof. Consider $U_\epsilon = \inf\{t>0 : X_t\geq\epsilon \text{ or } Y_t\geq y_0+\epsilon\}$ where $\epsilon>0$. Note that $U_\epsilon>0$ a.s. since the process (X,Y) is right-continuous. As in the proof of the previous proposition and using

$b_1(0,.) = \kappa_1(0,.) = 0$, we have the following estimate

$$\begin{aligned}
\mathbb{E}(X_{t\wedge U_\epsilon}) &\leq \int_0^t \mathbb{E}(|b_1(X_s,Y_s)|, s < U_\epsilon)\, ds \\
&\qquad + \int_0^\infty |z - h(z)|\, \lambda_1(dz) \int_0^t \mathbb{E}(\kappa_1(X_s,Y_s)p(X_s,Y_s), s < U_\epsilon)\, ds \\
&= \int_0^t \mathbb{E}(|b_1(X_s,Y_s) - b_1(0,Y_s)|, s < U_\epsilon)\, ds \\
&\qquad + \int_0^\infty |z - h(z)|\, \lambda_1(dz) \int_0^t \mathbb{E}(\kappa_1(X_s,Y_s)p_1(X_s,Y_s) - \kappa_1(0,Y_s)p_1(0,Y_s), s < U_\epsilon)\, ds \\
&\leq R\big(1 + \int_0^\infty |z - h(z)|\, \lambda_1(dz)\big) \int_0^t \mathbb{E}(X_s, s < U_\epsilon)\, ds \\
&\leq R\big(1 + \int_0^\infty |z - h(z)|\, \lambda_1(dz)\big) \int_0^t \mathbb{E}(X_{s\wedge U_\epsilon})\, ds,
\end{aligned}$$

where R is a Lipschitz constant for $b_1, \kappa_1 p_1$ on $[0,\epsilon] \times [0, y_0 + \epsilon]$. Gronwall's inequality gives that $\mathbb{E}(X_{t\wedge U_\epsilon}) = 0$, which implies that $X_{t\wedge U_\epsilon} = 0$ a.s. and then $Y_{t\wedge U_\epsilon} = y_0$ a.s.. In particular, on the trajectories where $U_\epsilon < \infty$, there is a small time $s_0 > 0$ such that for all $0 \leq s \leq s_0$, $X_{s+U_\epsilon} < \varepsilon$ and $Y_{s+U_\epsilon} < y_0 + \epsilon$ (by right continuity), which gives a contradiction. Therefore, the only possible conclusion is that $U_\epsilon = \infty$ and we conclude that $X_t = 0$ and $Y_t = y_0$, for all t. □

4.2 Uniqueness

In this section we shall prove pathwise uniqueness for the system (29). We need the ellipticity assumption for the coefficients $\ell_i, i = 1, 2$, given in Assumption **(B4)**,

(F5) For $i = 1, 2$ and for every $0 < \delta \leq n < \infty$, there exists $\zeta = \zeta(\delta, n) > 0$ such that

$$\zeta \leq \inf\{\ell_i(x,y) : (x,y) \in [\delta, n]^2\}.$$

We also need to have a control on the small jumps and this is done through the following hypothesis on λ, which is the analog of (28) in Proposition 3.5.

(F6) There exists $\varepsilon_0 > 0$ such that

$$\liminf_{a \downarrow 0} \left[e^{\varepsilon_0 \int_a^1 z\, \lambda(dz)} \int_0^a z^2 \lambda(dz) \right] = 0. \tag{35}$$

We notice that if $\int_0^\infty (z \wedge 1)\, \lambda(dz) < +\infty$ then λ satisfies hypotheses **(F0)** and **(F6)**. Also it is quite direct to show that if $\mu \leq \lambda$ and λ satisfies **(F0)** and **(F6)**, then μ fulfils **(F0)** and **(F6)**.

For a solution (X, Y) of system (29), we denote by $\mathbb{T}_\mathbf{e}$ the explosion time of (X, Y), which is given by

$$\mathbb{T}_\mathbf{e} = \lim_{n\to\infty} S_n^X \wedge S_n^Y.$$

Now, we are ready to state a uniqueness result.

Theorem 4.3. *Assume that $(x_0, y_0) \in \mathbb{R}^2_+$. Assume that the coefficients of the system (29) satisfy* **(F1)**–**(F5)**, *and $\lambda = \lambda_1 + \lambda_2$ satisfies* **(F0)** *and* **(F6)**. *Then, pathwise uniqueness holds for this system, that is, if (X, Y) and $(\widetilde{X}, \widetilde{Y})$ are two solutions up to their respective explosion times $\mathbb{T}_\mathbf{e}$ and $\widetilde{\mathbb{T}}_\mathbf{e}$, then $\mathbb{T}_\mathbf{e} = \widetilde{\mathbb{T}}_\mathbf{e}$ a.s. and for all $t < \mathbb{T}_\mathbf{e}$ we have $(X_t, Y_t) = (\widetilde{X}_t, \widetilde{Y}_t)$ a.s..*

Under the extra hypothesis $\int_0^\infty (z^2 \wedge z)\, \lambda(dz) < \infty$, we have $\mathbb{T}_\mathbf{e} = \infty$ a.s.

Proof. (i) In the first part of the proof, we assume the extra condition

$$\int_0^\infty (z^2 \wedge z)\, \lambda(dz) < \infty \tag{36}$$

and Proposition 4.1 guarantees non-explosion of solutions. Since λ satisfies **(F6)**, there exists $\varepsilon_0 > 0$ such that

$$\liminf_{a \downarrow 0} e^{\varepsilon_0 \int_a^\infty h(z)\, \lambda(dz)} \int_0^a h(z)^2 \lambda(dz) = 0,$$

because h agrees with the identity on a neighborhood of 0 and h is bounded then $\int_1^\infty h(z)\lambda(dz) < \infty$. In what follows, we denote by $\mathbf{\Xi}$ a bound for h.

We consider (X, Y) and $(\widetilde{X}, \widetilde{Y})$ two strong solutions of the system (29). Let us fix $0 < \delta < x_0 \wedge y_0$ and $n \in \mathbb{N}^*$ and let us take $T_\delta = \inf\{t > 0 : X_t \wedge Y_t \wedge \widetilde{X}_t \wedge \widetilde{Y}_t < \delta\}$, $S_n = S_n^X \wedge S_n^Y \wedge S_n^{\widetilde{X}} \wedge S_n^{\widetilde{Y}}$, where we recall notation $S_n^Z = \inf\{t \geq 0 : Z_t \geq n\}$, and

$$T_{n,\delta} = T_\delta \wedge S_n. \tag{37}$$

We will prove that there exists $t_0 > 0$ and a constant $A > 0$ such that for all $t \leq t_0$

$$\mathbb{E}((X - \widetilde{X})^*_{t \wedge T_{n,\delta}} + (Y - \widetilde{Y})^*_{t \wedge T_{n,\delta}}) \leq A \liminf_{a \to 0} \left(e^{\epsilon_0 \int_a^\infty h(z)\, \lambda(dz)} \int_0^a h(z)^2\, \lambda(dz) \right)^{\frac{1}{2}} = 0, \tag{38}$$

where we recall that we write $Z_t^* = \sup\{Z_s : s \leq t\}$. Uniqueness will be shown on the interval $[0, t_0 \wedge T_{n,\delta}]$. Similarly, it will extend to the interval $[t_0 \wedge T_{n,\delta}, 2t_0 \wedge T_{n,\delta}]$ and by iterating this argument, uniqueness will be shown in $[0, T_{n,\delta}]$ (when $T_{n,\delta} = \infty$ we take this interval to be $[0, \infty)$).

Then, since the processes do not explode, we can take the limit as $n \to \infty$, to conclude uniqueness on $[0, T_\delta]$. Finally, we deduce that $X = \widetilde{X}, Y = \widetilde{Y}$ on the interval $[0, T_0)$, where $T_0 = \lim \uparrow T_\delta$. Notice that one of the coordinates has to be 0 on the left of T_0, when T_0 is finite. Say that $X_{T_0-} = \widetilde{X}_{T_0-} = 0$. Since T_0 is a predictable stopping time the Poisson processes cannot jump at this time, which implies that $X_{T_0} = \widetilde{X}_{T_0} = 0$ and therefore from the uniqueness starting from 0 we conclude $X_t = \widetilde{X}_t = 0$ for all $t \geq T_0$, which also implies that $Y_t = \widetilde{Y}_t = Y_{T_0-}$ for all $t \geq T_0$, showing the desired uniqueness.

Let us now prove (38).

In what follows we denote by

$$\begin{array}{l}
\Delta X_s = X_s - \widetilde{X}_s \ ; \ \Delta Y_s = Y_s - \widetilde{Y}_s; \\
\Delta b_s = b_1(X_s, Y_s) - b_1(\widetilde{X}_s, \widetilde{Y}_s) \ ; \ \Delta \ell_s^{1/2} = (\ell_1(X_s, Y_s))^{1/2} - (\ell_1(\widetilde{X}_s, \widetilde{Y}_s))^{1/2}; \\
\Delta \kappa_s = \kappa_1(X_{s-}, Y_{s-}) - \kappa_1(\widetilde{X}_{s-}, \widetilde{Y}_{s-}) \ ; \ \Delta p_s = p_1(X_{s-}, Y_{s-}) - p_1(\widetilde{X}_{s-}, \widetilde{Y}_{s-}) \ ; \\
\Delta u_s(\theta) = \mathbf{1}_{\theta \leq \kappa_1(X_{s-}, Y_{s-})} p_1(X_{s-}, Y_{s-}) - \mathbf{1}_{\theta \leq \kappa_1(\widetilde{X}_{s-}, \widetilde{Y}_{s-})} p_1(\widetilde{X}_{s-}, \widetilde{Y}_{s-}),
\end{array}$$

and

$$\Gamma_t = \int_0^t \int_{[0,\infty)^2} \Delta u_s(\theta)\, h(z)\, \widetilde{N}^1(ds, d\theta, dz).$$

We observe that

$$\Delta X_t = \int_0^t \Delta b_s ds + \int_0^t \Delta \ell_s^{1/2} dB_s^1 + \Gamma_t + \int_0^{t \wedge T_{n,\delta}} \int_{[0,\infty)^2} \Delta u_s(\theta)(z - h(z)) N^1(ds, d\theta, dz) \tag{39}$$

and get

$$\mathbb{E}((\Delta X)^*_{t\wedge T_{n,\delta}}) \le \mathbb{E}\left(\int_0^{t\wedge T_{n,\delta}} |\Delta b_s|\, ds\right) + \mathbb{E}\left(\sup_{0\le r\le t\wedge T_{n,\delta}}\left|\int_0^r \Delta\ell_s^{1/2} dB_s^1\right|\right) \tag{40}$$
$$+ \mathbb{E}\left(\sup_{0\le r\le t\wedge T_{n,\delta}} |\Gamma_r|\right) + \mathbb{E}\left(\int_0^{t\wedge T_{n,\delta}}\int_{[0,\infty)^2} |\Delta u_s(\theta)||z-h(z)| N^1(ds,d\theta,dz)\right).$$

Consider $r(n)$ a common Lipschitz constant for all the functions b, ℓ, κ, p, q on the interval $[0,n]$ and denote by $\mathbf{K}(n) = (2\mathbf{L}\, n + \mathbf{A} + 1)(r(n)+1)$, where $\mathbf{L}, \mathbf{A}$ are given by the linear growth condition on **(F3)**. In particular, $\mathbf{K}(n)$ serves as a Lipschitz constant for all the coefficients of the system in the interval $[0,n]$, as well as a bound for these functions on $[0,n]^2$.

We introduce

$$R_t = \mathbb{E}((X-\widetilde{X})^*_{t\wedge T_{n,\delta}} + (Y-\widetilde{Y})^*_{t\wedge T_{n,\delta}}) = \mathbb{E}((\Delta X)^*_{t\wedge T_{n,\delta}} + (\Delta Y)^*_{t\wedge T_{n,\delta}}),$$

and the first term in the RHS of (40) is clearly bounded by

$$\mathbb{E}\left(\int_0^{t\wedge T_{n,\delta}} |\Delta b_s|\, ds\right) \le \mathbf{K}(n)\, t\, R_t. \tag{41}$$

Let us bound the second term (Brownian term) in (40). By definition for $s < t\wedge T_{n,\delta}$, we have $X_s, Y_s, \widetilde{X}_s, \widetilde{Y}_s \in [\delta, n]$ and therefore $a = \ell_1(X_s,Y_s) \ge \zeta, b = \ell_1(\widetilde{X}_s,\widetilde{Y}_s) \ge \zeta$, where $\zeta = \zeta(\delta,n) > 0$ is given by the ellipticity assumption **(F5)**. Now, for $a, b \ge \zeta$ we have $|\sqrt{a} - \sqrt{b}| \le \frac{1}{2\sqrt{\zeta}}|a-b|$ and we get from (30) that

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\le r\le t\wedge T_{n,\delta}}\left|\int_0^r \Delta\ell_s^{1/2} dB_s^1\right|\right) &\le \mathbf{C}_1\mathbb{E}\left(\left(\int_0^{t\wedge T_{n,\delta}}\left|\left(\ell_1(X_s,Y_s)\right)^{1/2} - \left(\ell_1(\widetilde{X}_s,\widetilde{Y}_s)\right)^{1/2}\right|^2 ds\right)^{1/2}\right)\\
&\le \frac{\mathbf{C}_1}{2\sqrt{\zeta}}\mathbb{E}\left(\left(\int_0^{t\wedge T_{n,\delta}}\left|\ell_1(X_s,Y_s) - \ell_1(\widetilde{X}_s,\widetilde{Y}_s)\right|^2 ds\right)^{1/2}\right)\\
&\le \frac{\mathbf{K}(n)\mathbf{C}_1}{2\sqrt{\zeta}}\sqrt{t}\, R_t
\end{aligned} \tag{42}$$

For the last term in (40), we use that $0 \le p \le 1$ and the triangular inequality

$$|\Delta u_s(\theta)| \le |\mathbf{1}_{\theta\le\kappa_1(X_{s-},Y_{s-})} - \mathbf{1}_{\theta\le\kappa_1(\widetilde{X}_{s-},\widetilde{Y}_{s-})}| + \mathbf{1}_{\theta\le\kappa_1(\widetilde{X}_{s-},\widetilde{Y}_{s-})}|p_1(X_{s-},Y_{s-}) - p_1(\widetilde{X}_{s-},\widetilde{Y}_{s-})|.$$

This implies that

$$\begin{aligned}
\int_0^\infty |\Delta u_s(\theta)|\, d\theta &\le |\kappa_1(X_{s-},Y_{s-}) - \kappa_1(\widetilde{X}_{s-},\widetilde{Y}_{s-})| + \kappa_1(\widetilde{X}_{s-},\widetilde{Y}_{s-})|p(X_{s-},Y_{s-}) - p(\widetilde{X}_{s-},\widetilde{Y}_{s-})|\\
&\le \mathbf{K}(n)\left(|X_{s-} - \widetilde{X}_{s-}| + |Y_{s-} - \widetilde{Y}_{s-}|\right)
\end{aligned}$$

and therefore

$$\begin{aligned}
\mathbb{E}\left(\int_0^{t\wedge T_{n,\delta}}\int_0^\infty |\Delta u_s(\theta)||z-h(z)| N^1(ds,d\theta,dz)\right) &= \int_0^\infty |z-h(z)|\,\lambda_1(dz)\,\mathbb{E}\left(\int_0^{t\wedge T_{n,\delta}}\int_0^\infty |\Delta u_s(\theta)|\, ds d\theta\right)\\
&\le \mathbf{K}(n)\, t\int_0^\infty |z-h(z)|\,\lambda_1(dz)\, R_t.
\end{aligned}$$

Let us now concentrate on the third term in (40). We write

$$\Gamma_r^a = \int_0^r\int_{[0,\infty)^2} \Delta u_s(\theta)\, h(z)\mathbf{1}_{0\le z\le a}\, \widetilde{N}^1(ds,d\theta,dz), \quad \Gamma_r^{a\to} = \int_0^r\int_{[0,\infty)^2} \Delta u_s(\theta)\, h(z)\mathbf{1}_{a<z}\, \widetilde{N}^1(ds,d\theta,dz).$$

for $a \geq 0$ and $\sup_{0\leq s\leq t\wedge T_{n,\delta}} |\Gamma_s| \leq \sup_{0\leq s\leq t\wedge T_{n,\delta}} |\Gamma^a_s| + \sup_{0\leq s\leq t\wedge T_{n,\delta}} |\Gamma^{a\rightarrow}_s|$. Using now (30) and (31) and $\mathbf{1}_{a<z}|d\widetilde{N}^1| \leq \mathbf{1}_{a<z} d(N^1+\nu^1)$, we get

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leq s\leq t\wedge T_{n,\delta}} |\Gamma_s|\right)\\
&\quad\leq \mathbb{E}\left(\sup_{0\leq s\leq t\wedge T_{n,\delta}} |\Gamma^a_s|\right) + \mathbb{E}\left(\int_0^{t\wedge T_{n,\delta}}\int_0^\infty |\Delta u_s(\theta)|\, h(z)\mathbf{1}_{a<z}\, d(N^1+\nu^1)\right)\\
&\quad\leq \mathbf{C}_1\mathbb{E}\left([\Gamma^a,\Gamma^a]^{1/2}_{t\wedge T_{n,\delta}}\right) + 2\int_a^\infty h(z)\,\lambda_1(dz)\,\mathbb{E}\left(\int_0^{t\wedge T_{n,\delta}}\int_0^\infty |\Delta u_s(\theta)|\, ds d\theta\right) \qquad (43)\\
&\quad\leq 3\mathbf{C}_1\mathbb{E}\left(\langle\Gamma^a,\Gamma^a\rangle^{1/2}_{t\wedge T_{n,\delta}}\right) + 2\mathbf{K}(n)\int_a^\infty h(z)\,\lambda_1(dz)\; R_t. \qquad (44)
\end{aligned}$$

It remains to estimate $\mathbb{E}\left(\langle\Gamma^a,\Gamma^a\rangle^{1/2}_{t\wedge T_{n,\delta}}\right)$. If we denote by $W_1(a) = \int_{[0,a]} h^2(z)\lambda_1(dz)$ and $W_1 = W_1(\infty)$, then for $0 < a \leq 1$, to be fixed later on, we get

$$\begin{aligned}
\mathbb{E}\left(\langle\Gamma^a,\Gamma^a\rangle^{1/2}_{t\wedge T_{n,\delta}}\right) &= \sqrt{W_1(a)}\;\mathbb{E}\left(\left(\int_0^{t\wedge T_{n,\delta}}\int_0^\infty |\Delta u_s|\, d\theta ds\right)^{1/2}\right)\\
&= \frac{\sqrt{W_1(a)}}{\sqrt{W_1}}\;\sqrt{W_1}\;\mathbb{E}\left(\left(\int_0^{t\wedge T_\delta}\int_0^\infty |\Delta u_s|\, d\theta ds\right)^{1/2}\right)\\
&= \frac{\sqrt{W_1(a)}}{\sqrt{W_1}}\;\mathbb{E}\left(\langle\Gamma,\Gamma\rangle^{1/2}_{t\wedge T_{n,\delta}}\right)\\
&\leq \frac{\sqrt{W_1(a)}}{\sqrt{W_1}}\;\overline{\mathbf{c}}_1^{-1}\left(\mathbb{E}\left(\sup_{0\leq s\leq t\wedge T_{n,\delta}} |\Gamma_s|\right) + \Xi\right),
\end{aligned}$$

where we applied (31) to $(\Gamma_s)_s$. Here $\overline{\mathbf{c}}_1^{-1}$ is a finite constant, and obviously since Ξ is a bound for h, then Ξ is a bound for the jumps of Γ.
It remains to remark from (39) that

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\leq r\leq t\wedge T_{n,\delta}} |\Gamma_r|\right) &\leq \mathbb{E}\left((\Delta X)^*_{T_{n,\delta}}\right) + \mathbb{E}\left(\int_0^{T_{n,\delta}} |\Delta b_s|\, ds\right) + \mathbb{E}\left(\sup_{0\leq r\leq T_{n,\delta}}\left|\int_0^r \Delta\ell_s^{1/2} dB^1_s\right|\right)\\
&\quad + \mathbb{E}\left(\int_0^{T_{n,\delta}}\int_{[0,\infty)^2} |\Delta u_s(\theta)||z-h(z)| N^1(ds,d\theta,dz)\right) = \mathscr{K} < +\infty. \qquad (45)
\end{aligned}$$

Coming back to (43) we get

$$\mathbb{E}\left(\sup_{0\leq s\leq t\wedge T_{n,\delta}} |\Gamma_s|\right) \leq \frac{\sqrt{W_1(a)}}{\sqrt{W_1}}\, 3\mathbf{C}_1\overline{\mathbf{c}}_1^{-1}(\mathscr{K}+\Xi) + 2\mathbf{K}(n)\int_a^\infty h(z)\,\lambda_1(dz)\;\int_0^t R_s ds.$$

Finally, adding all the estimates in (40), we obtain the following inequality

$$\mathbb{E}((\Delta X)^*_{t\wedge T_{n,\delta}}) \leq \beta_1(a)\mathbb{E}\left((\Delta X)^*_{t\wedge T_{n,\delta}}\right) + \beta_1(a)\Xi + \gamma_1(a)\int_0^t R_s\, ds + \rho_1(a,t)R_t,$$

with

$$\begin{aligned}
&\beta_1(a) = \frac{\sqrt{W_1(a)}}{\sqrt{W_1}}\, 3\mathbf{C}_1\overline{\mathbf{c}}_1^{-1};\; \gamma_1(a) = 2\mathbf{K}(n)\int_a^\infty h(z)\,\lambda_1(dz),\\
&\rho_1(a,t) = \mathbf{K}(n)\left(t + t\int_0^\infty |z-h(z)|\,\lambda_1(dz) + \frac{\mathbf{C}_1}{2\sqrt{\zeta}}\sqrt{t}\right)(1+\beta_1(a)).
\end{aligned}$$

In a similar way, we get the upper bound for $\mathbb{E}((\Delta Y)^*_{t\wedge T_{n,\delta}})$. We call $\beta_2, \gamma_2, \rho_2$ the corresponding quantities. Then, summing up these upper bounds gives the following upper bound for R_t.

$$R_t \leq (\beta_1(a) \vee \beta_2(a))\, R_t + \Xi\beta(a) + \gamma(a) \int_0^t R_s\, ds + \rho(a,t) R_t,$$

with $\beta = \beta_1 + \beta_2, \gamma = \gamma_1 + \gamma_2, \rho = \rho_1 + \rho_2$.

We first choose $0 < a_0 < 1$ such that for all $a \leq a_0$, we have $\beta(a) \leq 1/4$, and we choose $0 < t_0 = t_0(n)$ such that

$$\begin{aligned} \rho(1,t_0) \quad &= \mathbf{K}(n) \left(t_0 + 2t_0 \textstyle\int_0^\infty |z - h(z)|\, \lambda_1(dz) + \frac{\mathbf{C}_1}{2\sqrt{\zeta}} \sqrt{t_0} \right) (1 + \beta_1(1)) \\ &\quad + \mathbf{K}(n) \left(t_0 + 2t_0 \textstyle\int_0^\infty |z - h(z)|\, \lambda_2(dz) + \frac{\mathbf{C}_1}{2\sqrt{\zeta}} \sqrt{t_0} \right) (1 + \beta_2(1)) \;\leq\; 1/4. \end{aligned}$$

Hence, for all $a \leq a_0, t \leq t_0$, we get $R_t \leq \frac{1}{2} R_t + \Xi\beta(a) + \gamma(a) \int_0^t R_s\, ds$, and a fortiori it holds

$$R_t \leq 2\Xi\beta(a) + 2\gamma(a) \int_0^t R_s\, ds. \tag{46}$$

Gronwall's inequality shows that, for all $0 < a \leq a_0, t \leq t_0$

$$\begin{aligned} R_t \quad &= \quad 6\,\Xi\, \mathbf{C}_1 \bar{\mathbf{c}}_1^{-1} \left(\sqrt{\frac{W_1(a)}{W_1}} + \sqrt{\frac{W_2(a)}{W_2}} \right) e^{2t\gamma(a)} \\ &\leq \quad 12\,\Xi \frac{\mathbf{C}_1 \bar{\mathbf{c}}_1^{-1}}{\sqrt{W_1} \wedge \sqrt{W_2}} \left(\int_0^a h(z)^2\, \lambda(dz)\, e^{8\mathbf{K}(n)t \int_a^\infty h(z)\, \lambda(dz)} \right)^{\frac{1}{2}} \\ &\leq \quad A \left(\int_0^a h^2(z)\, \lambda(dz)\, e^{8\mathbf{K}(n)t_0 \int_a^\infty z\, \lambda(dz)} \right)^{\frac{1}{2}}, \end{aligned}$$

where $A = 12\,\Xi \frac{\mathbf{C}_1 \bar{\mathbf{c}}_1^{-1}}{\sqrt{W_1} \wedge \sqrt{W_2}}$. Hence, if we also assume that $8\mathbf{K}(n)t_0 \leq \epsilon_0$, we have for all $t \leq t_0$

$$\mathbb{E}((X - \widetilde{X})^*_{t\wedge T_{n,\delta}} + (Y - \widetilde{Y})^*_{t\wedge T_{n,\delta}}) \leq A \liminf_{a \to 0} \left(\int_0^a h(z)^2\, \lambda(dz)\, e^{\epsilon_0 \int_a^\infty h(z)\, \lambda(dz)} \right)^{\frac{1}{2}} = 0.$$

This result was our aim and as previously detailed, uniqueness is then proved under (36).

(ii) Now, we relax the extra integrability condition (36). We truncate the Poisson processes as follows

$$N^{i,D}(dt, d\theta, dz) = \mathbf{1}_{z \leq D}\, N^i(dt, d\theta, dz),\ i = 1, 2,$$

where now the intensities are $d\widehat{\nu}_i^D(dt, d\theta, dz) = \mathbf{1}_{z \leq D}\, dt\, d\theta\, \lambda_i(dz)$. In particular, we have $\widehat{\lambda}_i^D(dz) = \mathbf{1}_{z \leq D}\, \lambda_i(dz)$, which satisfy **(F6)** and the extra condition of part (i) is satisfied:

$$\int_0^\infty (z^2 \wedge z) \widehat{\lambda}_i^D(dz) = \int_0^D (z^2 \wedge z)\, \lambda(dz) < \infty.$$

We consider the associated drift term, where compensation has been truncated :

$$b_i^D(x,y) = b_i(x,y) - \kappa_i(x,y) p_i(x,y) \int_{(D,\infty)} h(z) \lambda(dz)$$

With these truncated Poisson processes, consider the analogue of (29)

$$\begin{aligned}
\widehat{X}_t &= x_0 + \int_0^t b_1^D(\widehat{X}_s,\widehat{Y}_s)ds + \int_0^t \sqrt{\ell_1(\widehat{X}_s,\widehat{Y}_s)}\,dB_s^1 \\
&\quad + \int_0^t \int_{[0,\infty)^2} \mathbf{1}_{\theta\le\kappa_1(\widehat{X}_{s-},\widehat{Y}_{s-})}\, p_1(\widehat{X}_{s-},\widehat{Y}_{s-})\, h(z)\, \widetilde{N}^{1,D}(ds,d\theta,dz) \\
&\quad + \int_0^t \int_{[0,\infty)^2} \mathbf{1}_{\theta\le\kappa_1(\widehat{X}_{s-},\widehat{Y}_{s-})}\, p_1(\widehat{X}_{s-},\widehat{Y}_{s-})\, (z-h(z))\, N^{1,D}(ds,d\theta,dz); \\
\widehat{Y}_t &= y_0 + \int_0^t b_2^D(\widehat{X}_s,\widehat{Y}_s)ds + \int_0^t \sqrt{\ell_1(\widehat{X}_s,\widehat{Y}_s)}\,dB_s^2 \\
&\quad + \int_0^t \int_{[0,\infty)^2} \mathbf{1}_{\theta\le\kappa_2(\widehat{X}_{s-},\widehat{Y}_{s-})}\, p_2(\widehat{X}_{s-},\widehat{Y}_{s-})\, h(z)\, \widetilde{N}^{2,D}(ds,d\theta,dz) \\
&\quad + \int_0^t \int_{[0,\infty)^2} \mathbf{1}_{\theta\le\kappa_2(\widehat{X}_{s-},\widehat{Y}_{s-})}\, p_2(\widehat{X}_{s-},\widehat{Y}_{s-})\, (z-h(z))\, N^{2,D}(ds,d\theta,dz). \qquad (47)
\end{aligned}$$

We claim that if (X,Y), $(\widetilde{X},\widetilde{Y})$ are two solutions of (29), then they are also solutions of (47), on the interval $[0,\tau^D)$, where τ^D is the first time when the point measure induces a jump z larger than D:

$$\tau^D = \inf\left\{ t>0 : \int_0^t \int_{[0,\infty)^2} \mathbf{1}_{\theta\le\kappa_1(X_{s-},Y_{s-}),\, z>D} N^1(ds,d\theta,dz) + \mathbf{1}_{\theta\le\kappa_2(X_{s-},Y_{s-}),\, z>D} N^2(ds,d\theta,dz) \;>\; 0 \right\}$$

Indeed, $\mathbf{1}_{s<\tau^D,\,\theta\le\kappa_i(X_{s-},Y_{s-})}\mathbf{1}_{z>D}N^i(ds,d\theta,dz) = 0$ and

$$\mathbf{1}_{s<\tau^D,\,\theta\le\kappa_i(X_{s-},Y_{s-})}\, N^i(ds,d\theta,dz) = \mathbf{1}_{s<\tau^D,\,\theta\le\kappa_i(X_{s-},Y_{s-})}\, N^{i,D}(ds,d\theta,dz),$$

while $b_i^D - b_i$ is the correction of the drift coming from the compensation of $N^{1,D} - N^1$.

The first part (i) then ensures that (X,Y) and $(\widetilde{X},\widetilde{Y})$ coincide up to time τ^D. Writing $S_n = S_n(X,Y,\widetilde{X},\widetilde{Y})$ the first time when either X or Y or $\widetilde{X}$ or $\widetilde{Y}$ goes beyond n, we observe that

$$\{\tau^D < t\wedge S_n\} \subset \cup_{i\in\{1,2\}}\{N^i([0,t]\times[0,\sup\kappa_i([0,n]^2)]\times[D,\infty)) > 0\}.$$

Besides, for each $n\in\mathbb{N}$ and $t>0$ the probability of the event of the right hand side goes to 0 as $D\to\infty$. Letting n and then D go to infinity ensures uniqueness up to explosion time $\mathbb{T}_e$. The proof is completed. □

A Hypotheses (H1) and (H2)

In this appendix, we recall the framework introduced in [4] Section 2, adapted to our setting.

Let $\mathcal{X}$ be the bounded subset $(0,1]^2$ of $\mathbb{R}^2$ and $\mathcal{U} = [-1,1]^2$.

For any $N\ge 1$, we consider a discrete time $\mathcal{X}$-valued Markov chain $(X_k^N : k\in\mathbb{N})$ with increments $X_{k+1}^N - X_k^N$ taking values in $\mathcal{U}$. Let $(v_N)_N$ be a given sequence of positive real numbers going to infinity when N tends to infinity. For $x\in\mathcal{X}$, we define

$$\mathcal{G}_x^N(H) = v_N\,\mathbb{E}\big(H(X_{k+1}^N - X_k^N)\,|\,X_k^N = x\big) = v_N\,\mathbb{E}\big(H(X_1^N - X_0^N)\,|\,X_0^N = x\big), \qquad (48)$$

for real valued bounded measurable functions H defined on $\mathcal{U}$.

We first observe that Hypothesis (H0) in [4] is obviously satisfied since the state space is bounded.

We introduce the functional space

$$C^2_{b,0} = C^2_{b,0}(\mathcal{U}, \mathbb{R}) = \left\{ H \in C_b(\mathcal{U}, \mathbb{R}) \,:\, H(u) = \sum_{i=1}^{2} \alpha_i u_i + \sum_{i,j=1}^{2} \beta_{i,j} u_i u_j + o(|u|^2),\ \alpha_i, \beta_{i,j} \in \mathbb{R} \right\}$$

and here the **specific function** h is the two dimensional identity function

$$h = (h^1, h^2) \in (C^2_{b,0})^2 \ ;\ h^i(u) = u_i \quad (i = 1, 2). \tag{49}$$

Hypotheses (H1) There exists a functional space $\mathcal{H}$ such that

1. *$\mathcal{H}$ is a subset of $C^2_{b,0}$ and $h^i, h^i h^j \in Vect(\mathcal{H})$ for $i, j = 1, \ldots, 2$.*
2. *For any $g \in C_c(\mathcal{U}, \mathbb{R})$ with $g(0) = 0$, there exists a sequence $(g_n)_n \in C^2_{b,0}$ such that $\lim_{n\to\infty} \|g - g_n\|_{\infty,\mathcal{U}} = 0$ and $|h|^2 g_n \in Vect(\mathcal{H})$.*
3. *There exists a family of real numbers $(\mathcal{G}_x(H); x \in \mathcal{X}, H \in \mathcal{H})$ such that for any $H \in \mathcal{H}$,*

$$(i) \qquad \lim_{N\to\infty} \sup_{x\in\mathcal{X}} \left| \mathcal{G}^N_x(H) - \mathcal{G}_x(H) \right| = 0.$$
$$(ii) \qquad \sup_{x\in\mathcal{X}} |\mathcal{G}_x(H)| < +\infty.$$

Theorem A.1. *Assume that the sequence $(X^N_0)_N$ is tight in $\overline{\mathcal{X}}$ and $(H1)$ hold. Then the sequence of processes $(X^N_{[v_N \cdot]}, N \in \mathbb{N})$ is tight in $\mathbb{D}([0,\infty), \overline{\mathcal{X}})$.*

We observe that that for any $H \in C^2_{b,0}$, there exists a unique decomposition of the form

$$H = \sum_{i=1}^{2} \alpha_i(H) h^i + \sum_{i,j=1}^{2} \beta_{i,j}(H) h^i\, h^j + \overline{H}, \tag{50}$$

where $\overline{H}(u_1, u_2) = o(|(u_1, u_2)|^2)$ is a continuous and bounded function and $\alpha_i(H)$, $\beta_{i,j}(H)$, $i, j = 1 \cdots 2$ are real coefficients and β is a symmetric matrix.

The next hypothesis (H2) in addition to (H1) is sufficient to get the identification of the limiting values by their semimartingale characteristics, and then their representation as solutions of a stochastic differential equation.

Hypotheses (H2)

1. *For any $H \in \mathcal{H}$, the map $x \in \mathcal{X} \to \mathcal{G}_x(H)$ is continuous and extendable by continuity to $\overline{\mathcal{X}}$.*
2. *For any $x \in \overline{\mathcal{X}}$ and any $H \in \mathcal{H}$,*

$$\mathcal{G}_x(H) = \sum_{i=1}^{2} \alpha_i(H) b_i(x) + \sum_{i,j=1}^{2} \beta_{i,j}(H) c_{i,j}(x) + \int_V \overline{H}(K(x,v))\mu(dv), \tag{51}$$

where

i) α_i, $\beta_{i,j}$ and $\overline{H}$ have been defined in (50),

ii) b_i and $c_{i,j}$ are measurable functions defined on $\overline{\mathcal{X}}$,

iii) *V is a Polish space, μ is a σ-finite positive measure on V, K is a measurable function from $\overline{\mathcal{X}} \times V$ with values in $\mathcal{U}$, $\int_V 1 \wedge |K(.,v)|^2 \mu(dv) < +\infty$ and*

$$c_{i,j}(x) = \sum_{k=1}^{4} \sigma_{i,k}(x)\sigma_{j,k}(x) + \int_V K_i(x,v)K_j(x,v)\mu(dv),$$

where $\sigma_{i,k}(x)$ are measurable functions defined on $\overline{\mathcal{X}}$ for $1 \leq i \leq 2$ and $1 \leq k \leq 4$.

The main general result in [4] yields the following statement here. A slight adaptation is needed since here the dimension 2 of the process X differs from the dimension 4 of the brownian motion involved in the representation (one can also consider a 4-dimensional process by adding two coordinates identically null to match the precise framework of [4]).

Theorem A.2. *If the sequence $(X_0^N)_N$ is tight in $\overline{\mathcal{X}}$ and (H1) and (H2) hold then any limiting value of $(X^N_{[v_N .]}, N \in \mathbb{N})$ is a semimartingale solution of the stochastic differential system*

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s + \int_0^t \int_V K(X_{s-}, v)\tilde{N}(ds, dv), \tag{52}$$

where $X_0 \in \overline{\mathcal{X}}$ and B is a 4-dimensional Brownian motion and N is a Poisson Point measure on $\mathbb{R}_+ \times V$ with intensity $ds\mu(dv)$. Moreover X_0, B, N are independent and $\tilde{N}$ is the compensated martingale measure of N.

B Hypothesis (F6)

In this appendix, we shall study more closely hypothesis (**F6**).

Lemma B.1. *Assume λ_1 satisfies* (**F6**)*, and λ_2 satisfies $\int (z \wedge 1)\, \lambda_2(dz) < \infty$. Then $\lambda = \lambda_1 + \lambda_2$ satisfies* (**F6**).

Proof. If λ_1 also satisfies $\int (z \wedge 1)\, \lambda_1(dz) < \infty$, we have for $\lambda = \lambda_1 + \lambda_2$ and any $a \leq 1, \epsilon > 0$

$$e^{\epsilon \int_{(a,1]} z\, \lambda(dz)} \int_0^a z^2\, \lambda(dz) \leq e^{\epsilon \int_{[0,1]} z\, \lambda(dz)} \int_0^a z^2\, \lambda(dz),$$

which converges to 0 as a converges to 0. Hence, λ satisfies (**F6**).

So, for the rest of the proof we shall assume that $\int_0^1 z\, \lambda_1(dz) = +\infty$. In what follows we denote by $K = \int_{[0,1]} z\, \lambda_2(dz) < \infty$. We define inductively $c_0 = 1$ and given c_n we consider $0 < c_{n+1} < c_n$ characterized by

$$c_{n+1} = \sup \left\{ 0 \leq c < c_n : \int_{[c, c_n)} z\, \lambda_1(dz) \geq 1 \right\}.$$

Now, since λ_1 satisfies (**F6**) there exists a sequence $(a_k)_k \subset [0,1]$ such that $a_k \downarrow 0$ and

$$r(a_k) = e^{\epsilon_0 \int_{(a_k,1]} z\lambda_1(dz)} \int_0^{a_k} z^2\, \lambda_1(dz) \to 0.$$

Consider for every k the unique c_{n_k} such that $a_k \in [c_{1+n_k}, c_{n_k})$. We consider two possible situations:

(i) $\int_{[c_{1+n_k}, a_k]} z\lambda_1(dz) \leq 1/2$;

(ii) $\int_{[c_{1+n_k}, a_k]} z\lambda_1(dz) > 1/2$.

In the first case we have $\int_{(c_{2+n_k},a_k]} z\,\lambda_1(dz) \le 3/2$ and $\int_{[c_{2+n_k},a_k]} z\,\lambda_1(dz) \ge 1$. On the one hand

$$\begin{aligned} r(a_k) &= e^{\epsilon_0 \int_{(a_k,1]} z\lambda_1(dz)} \int_0^{a_k} z^2\,\lambda_1(dz) \ge e^{\epsilon_0 \int_{(a_k,1]} z\lambda_1(dz)} \int_{[c_{2+n_k},a_k]} z^2\,\lambda_1(dz) \\ &\ge c_{2+n_k} e^{\epsilon_0 \int_{(a_k,1]} z\lambda_1(dz)} \int_{[c_{2+n_k},a_k]} z\,\lambda_1(dz) \ge c_{2+n_k} e^{\epsilon_0 \int_{(a_k,1]} z\lambda_1(dz)} \\ &\ge e^{\epsilon_0 \int_{(c_{2+n_k},1]} z\lambda_1(dz) - \frac{3}{2}\epsilon_0}\, c_{2+n_k}. \end{aligned}$$

With this estimation we obtain for $d_k = c_{2+n_k}$

$$\begin{aligned} e^{\epsilon_0 \int_{(d_k,1]} z\lambda_1(dz)} \int_0^{d_k} z^2\lambda_2(dz) &\le c_{2+n_k} e^{\epsilon_0 \int_{(c_{2+n_k},1]} z\lambda_1(dz)} \int_0^{d_k} z\lambda_2(dz) \\ &\le K e^{\frac{3}{2}\epsilon_0}\, r(a_k). \end{aligned}$$

On the other hand

$$\begin{aligned} e^{\epsilon_0 \int_{(d_k,1]} z\lambda_1(dz)} \int_0^{d_k} z^2\lambda_1(dz) &\le e^{\epsilon_0 \int_{(a_k,1]} z\lambda_1(dz) + \frac{3}{2}\epsilon_0} \int_0^{a_k} z^2\lambda_1(dz) \\ &\le e^{\frac{3}{2}\epsilon_0}\, r(a_k). \end{aligned}$$

This gives the bound

$$e^{\epsilon_0 \int_{(d_k,1]} z\lambda(dz)} \int_0^{d_k} z^2\lambda(dz) \le e^{\epsilon_0(K+\frac{3}{2})}(K+1)\, r(a_k).$$

In the second case we have $\int_{(c_{1+n_k},a_k]} z\lambda_1(dz) < 1$, by the definition of c_{1+n_k}, and therefore

$$\begin{aligned} r(a_k) &= e^{\epsilon_0 \int_{(a_k,1]} z\lambda_1(dz)} \int_0^{a_k} z^2\,\lambda_1(dz) \ge e^{\epsilon_0 \int_{(a_k,1]} z\lambda_1(dz)} \int_{[c_{1+n_k},a_k]} z^2\,\lambda_1(dz) \\ &\ge c_{1+n_k} e^{\epsilon_0 \int_{(a_k,1]} z\lambda_1(dz)} \int_{[c_{1+n_k},a_k]} z\,\lambda_1(dz) \ge \tfrac{1}{2} c_{1+n_k} e^{\epsilon_0 \int_{(a_k,1]} z\lambda_1(dz)} \\ &\ge \tfrac{1}{2} c_{1+n_k} e^{\epsilon_0 \int_{(c_{1+n_k},1]} z\lambda_1(dz) - \epsilon_0}. \end{aligned}$$

Similarly as before, we take $d_k = c_{1+n_k}$ which gives

$$\begin{aligned} e^{\epsilon_0 \int_{(d_k,1]} z\lambda_1(dz)} \int_0^{d_k} z^2\lambda_2(dz) &\le c_{1+n_k} e^{\epsilon_0 \int_{(c_{1+n_k},1]} z\lambda_1(dz)} \int_0^{d_k} z\lambda_2(dz) \\ &\le 2K e^{\epsilon_0}\, r(a_k). \end{aligned}$$

Again, we have

$$\begin{aligned} e^{\epsilon_0 \int_{(d_k,1]} z\lambda_1(dz)} \int_0^{d_k} z^2\lambda_1(dz) &\le e^{\epsilon_0 \int_{(a_k,1]} z\lambda_1(dz) + \epsilon_0} \int_0^{a_k} z^2\lambda_1(dz) \\ &\le e^{\epsilon_0}\, r(a_k). \end{aligned}$$

which allows us to show

$$e^{\epsilon_0 \int_{(d_k,1]} z\lambda(dz)} \int_0^{d_k} z^2\lambda(dz) \le e^{\epsilon_0(K+1)}(2K+1)\, r(a_k).$$

We summarize these estimations in both cases as

$$e^{\epsilon_0 \int_{(d_k,1]} z\lambda(dz)} \int_0^{d_k} z^2\lambda(dz) \le e^{\epsilon_0(K+\frac{3}{2})}(2K+1)\, r(a_k)$$

The result follows by noticing that $(d_k)_k$ converges to 0. □

Remark B.2. *Notice that if* $\int (z\wedge 1)\,\lambda(dz) < \infty$ *then both* $\int (z\wedge 1)\,\lambda_1(dz) < \infty, \int (z\wedge 1)\,\lambda_2(dz) < \infty$ *and a fortiori both* λ_1, λ_2 *fullfill* (**F6**). *Moreover,* $\lambda = \lambda_1 + \lambda_2$ *satisfies* (**F6**).

It is also direct to show that if λ *satisfies* (**F6**)*, then both* λ_1, λ_2 *fulfill* (**F6**). *Nevertheless, it is not true that if both* λ_1, λ_2 *satisfies* (**F6**) *then* λ *satisfies* (**F6**). *This makes the previous Lemma more interesting.*

We now give sufficient conditions to have hypothesis (**F6**). Assume that $\lambda(dz) = \frac{f(z)}{z^2}dz$, with $f \geq 0$ and $\int_0^1 f(z)dz < \infty$, $\int_{0+} \frac{f(z)}{z}\,dz = \infty$. After taking logarithm, condition (**F6**) holds if

$$\sup_{0<a<a_0} \frac{\int_a^{a_0} \frac{f(z)}{z}dz}{-\log(\int_0^a f(z)\,dz)} < \infty, \tag{53}$$

for some small a_0. This condition is satisfied if

$$r = \sup_{0<a<a_0} \frac{\int_0^a f(z)dz}{a} < \infty. \tag{54}$$

Indeed, for all small z we have $\int_0^z f(u)du \leq rz$ and therefore

$$\int_a^{a_0} \frac{f(z)}{z}dz \leq r\int_a^{a_0} \frac{f(z)}{\int_0^z f(u)du}dz = r\left(\log\left(\int_0^{a_0} f(u)du\right) - \log\left(\int_0^a f(u)du\right)\right).$$

Hence, if a_0 is small, we have $\int_0^{a_0} f(u)du < 1$ and then

$$0 \leq \frac{\int_a^{a_0} \frac{f(z)}{z}dz}{-\log(\int_0^a f(z)\,dz)} \leq r\left(\frac{\log\left(\int_0^{a_0} f(u)du\right)}{-\log(\int_0^a f(z)\,dz)} + 1\right) \leq r.$$

which shows (53).

Notice that (54) is satisfied if f is bounded near 0. For example $f = 1$, which gives $\lambda(dz) = \frac{1}{z^2}dz$. Clearly $f(z) = -\log(z)$, for z small, does not satisfies (54). It is quite direct to show that it does not satisfies (53) nor (35).

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