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AN EULER-TYPE FORMULA FOR PARTITIONS OF THE MÖBIUS STRIP

PIERRE BÉRARD AND BERNARD HELFFER

Abstract. The purpose of this note is to prove an Euler-type formula for partitions of the Möbius strip. This formula was introduced in our joint paper with R. Kiwan, "Courant-sharp property for Dirichlet eigenfunctions on the Möbius strip" (arXiv:2005.01175).

1. Introduction

In [2], in collaboration with R. Kiwan, we investigated the Courant-sharp property for the eigenvalues of the Dirichlet Laplacian on the square Möbius strip, equipped with the flat metric. We pointed out that the orientability of the nodal domains (more precisely the fact that they are all orientable or not) can be detected by an Euler-type formula. The purpose of the present note is to establish this formula in the framework of partitions.

In Section 2, we fix the notation and recall (or modify) some of the definitions for partitions given in [1], and we state the Euler-type formula we are interested in, see Theorem 2.10. The proof of the theorem is given in Section 3.

2. Partitions

2.1. **Definitions and notation.** Let Σ denote a compact Riemannian surface with or without boundary. We consider Euler-type formulas in the general framework of partitions. We first recall (or modify) some of the definitions given in [1].

A *k-partition* of Σ is a collection, $\mathcal{D} = \{D_j\}_{j=1}^k$, of k pairwise disjoint, connected, open subsets of Σ. We furthermore assume that the D_j's are piecewise C^1, and that

$$\mathrm{Int}(\overline{\cup_j D_j}) = \Sigma\,. \tag{2.1}$$

The *boundary set* $\partial\mathcal{D}$ of a partition $\mathcal{D} = \{D_j\}_{j=1}^k \in \mathfrak{D}_k(\Sigma)$ is the closed set,

$$\partial\mathcal{D} = \overline{\cup_j(\partial D_j \cap \Sigma)}\,. \tag{2.2}$$

Definition 2.1. A partition $\mathcal{D} = \{D_j\}_{j=1}^k$ is called *essential* if, for all j, $1 \le j \le k$,

$$\mathrm{Int}(\overline{D_j}) = D_j\,. \tag{2.3}$$

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Definition 2.2. A *regular k-partition* is a k-partition whose boundary set $\partial\mathcal{D}$ satisfies the following properties:

(i) The boundary set $\partial\mathcal{D}$ is locally a piecewise C^1 immersed curve in Σ, except possibly at finitely many points $\{y_i \in \partial\mathcal{D} \cap \Sigma\}$ in a neighborhood of which $\partial\mathcal{D}$ is the union of $\nu(y_i)$ C^1 semi-arcs meeting at $y_i, \nu(y_i) \geq 3$.
(ii) The set $\partial\mathcal{D} \cap \partial\Sigma$ consists of finitely many points $\{z_j\}$. Near the point z_j, the set $\partial\mathcal{D}$ is the union of $\rho(z_j) \geq 1$ C^1 semi-arcs hitting $\partial\Sigma$ at z_j.
(iii) The boundary set $\partial\mathcal{D}$ has the following *transversality property*: at any *interior singular point* y_i, the semi-arcs meet transversally; at any *boundary singular point* z_j, the semi-arcs meet transversally, and they meet the boundary $\partial\Sigma$ transversally.

The subset of regular k-partitions is denoted by $\mathcal{R}_k(\Sigma) \subset \mathfrak{D}_k(\Sigma)$. When $\mathcal{D}$ is a regular partition, we denote by $\mathcal{S}(\mathcal{D})$ the set of singular points of $\partial\mathcal{D}$.

Definition 2.3. A regular k-partition $\mathcal{D} = \{D_j\}_{j=1}^k$ is called *normal*, if it satisfies the additional condition, for all $j,\ 1 \leq j \leq k$,

$$\forall x \in \partial D_j,\ \exists r > 0 \ \text{ s. t. } \ B(x,r) \cap D_j \text{ is } \textit{connected}. \tag{2.4}$$

Remark 2.4. The definition of a normal partition implies that each domain in the partition is a topological manifold with boundary (actually a piecewise C^1 surface with boundary, possibly with corners). A normal partition is essential.

Example 2.5. The partition of a compact surface Σ (with or without boundary) associated with an eigenfunction is called a *nodal partition*. The domains of the partition are the nodal domains of Φ, the boundary set $\partial\mathcal{D}$ is the nodal set $\mathcal{Z}(\Phi)$, the singular set $\mathcal{S}(\mathcal{D})$ is the set of critical zeros of Φ. This is an example of an essential, regular partition. Nodal partitions are not necessarily normal due to the singular set, see Figure 3.1 (middle) and Figure 3.3 (A).

For a partition $\mathcal{D} \in \mathfrak{D}(\Sigma)$, we introduce the following numbers.

(a) $\kappa(\mathcal{D}, \Sigma)$ denotes the number of domains of the partition;
(b) $\beta(\mathcal{D}, \Sigma)$ is defined as $b_0(\partial\mathcal{D} \cup \partial\Sigma) - b_0(\partial\Sigma)$, the difference between the number of connected components of $\partial\mathcal{D} \cup \partial\Sigma$, and the number of connected components of $\partial\Sigma$;
(c) $\omega(\mathcal{D}, \Sigma)$ is the *orientability character* of the partition,

$$\begin{cases} \omega(\mathcal{D}, \Sigma) = 0, & \text{if all the domains of the partition are orientable;} \\ \omega(\mathcal{D}, \Sigma) = 1, & \text{if at least one domain of the partition is non-orientable.} \end{cases}$$

Obviously, $\omega(\mathcal{D}, \Sigma) = 0$ whenever the surface Σ is orientable.

Remarks 2.6.

(i) We use the definition of orientability given in [3, Chap. 5.3] (via differential forms of degree 2), or the similar form given in [5, Chap. 4.5] in the setting of topological manifolds (via the degree). A topological surface is orientable if one can choose an atlas whose changes of chart are homeomorphisms with degree 1.

(ii) A compact surface (with boundary) is non-orientable if and only if it contains the homeomorphic image of a Möbius strip. One direction is clear since the Möbius strip is not orientable. For the other direction, one can use the classification of compact surfaces (with boundary).

For a regular partition $\mathcal{D}$, we define the *index* of a point $x \in \mathcal{S}(\mathcal{D})$ to be,

$$\iota(x) = \begin{cases} \nu(x) - 2\,, & \text{if } x \text{ is an interior singular point,} \\ \rho(x)\,, & \text{if } x \text{ is a boundary singular point.} \end{cases} \tag{2.5}$$

For a regular partition, define the number $\sigma(\mathcal{D}, \Sigma)$ to be,

$$\sigma(\mathcal{D}, \Sigma) = \frac{1}{2} \sum_{x \in \mathcal{S}(\mathcal{D})} \iota(x)\,. \tag{2.6}$$

Finally, we introduce the number

$$\delta(\mathcal{D}, \Sigma) = \omega(\mathcal{D}, \Sigma) + \beta(\mathcal{D}, \Sigma) + \sigma(\mathcal{D}, \Sigma) - \kappa(\mathcal{D}, \Sigma)\,. \tag{2.7}$$

Lemma 2.7 (Normalization)**.** *Let $\mathcal{D}$ be an essential, regular partition of Σ. Then, one can construct a normal partition $\widetilde{\mathcal{D}}$ of Σ such that $\delta(\widetilde{\mathcal{D}}, \Sigma) = \delta(\mathcal{D}, \Sigma)$, and $\omega(\widetilde{\mathcal{D}}, \Sigma) = \omega(\mathcal{D}, \Sigma)$.*

Proof. Using condition (2.3), we see that an essential, regular partition $\mathcal{D} = \{D_j\}_{j=1}^k$ is normal except possibly at points in $\mathcal{S}(\mathcal{D})$, with index $\iota > 1$, and for which there exists some domain D_j such that $B(x, \varepsilon) \cap D_j$ has at least two connected components. Here, $B(x, \varepsilon)$ denotes the disk with center x and radius ε in Σ. Let x be such a point. For ε small enough, introduce the partition $\mathcal{D}_x$ whose elements are the $D_j \setminus \overline{B(x, \epsilon)}$, and the extra domain $B(x, \varepsilon) \cap \Sigma$. In this procedure, we have $\kappa(\mathcal{D}_x) = \kappa(\mathcal{D}) + 1$; an interior singular point x, with $\nu(x) \geq 4$ is replaced by $\nu(x)$ singular points of index 3 for which condition (2.4) is satisfied. Hence, $\sigma(\mathcal{D}_x) = \sigma(\mathcal{D}) + 1$ and $\delta(\mathcal{D}_x) = \delta(\mathcal{D})$. A similar procedure is applied at a boundary singular point. By recursion, we can in this way eliminate all the singular points at which condition (2.4) is not satisfied. Choosing ε small enough, this procedure does not change the orientability of the modified domains, while the added disks are orientable. □

Remark 2.8. By the same process, one could also remove all singular points with index $\iota > 1$. We do not need to do that for our purposes.

2.2. **Partitions and Euler-type formulas.** In the case of partitions of the sphere $\mathbb{S}^2$, or of a planar domain Ω, we have the following Euler-type formula, which appears in [9, 10] (sphere) or [7] (planar domain).

Proposition 2.9. *Let Σ be $\mathbb{S}^2$, or a bounded open set Ω in $\mathbb{R}^2$, with piecewise C^1 boundary, and let $\mathcal{D}$ be a regular partition with $\partial\mathcal{D}$ the boundary set. Then,*

$$\kappa(\mathcal{D}, \Sigma) = 1 + \beta(\mathcal{D}, \Sigma) + \sigma(\mathcal{D}, \Sigma)\,. \tag{2.8}$$

Theorem 2.10. *Let $\mathcal{D}$ be an essential, regular partition of the Möbius strip M_1. With the previous notation, we have,*

$$\kappa(\mathcal{D}, M_1) = \omega(\mathcal{D}, M_1) + \beta(\mathcal{D}, M_1) + \sigma(\mathcal{D}, M_1)\,. \tag{2.9}$$

3. Proof of Theorem 2.10

The idea to prove Theorem 2.10 is to examine how $\delta(\mathcal{D},\Sigma)$ changes when the partition $\mathcal{D}$ and the surface Σ are modified, starting out from a partition $\mathcal{D}$ of the Möbius strip, and arriving at a partition $\widetilde{\mathcal{D}}$ of a domain Ω in $\mathbb{R}^2$, on which we can apply (2.8).

Lemma 3.1. *Let $\mathcal{D}$ be a regular partition of M_1. Assume that there exists a simple piecewise C^1 curve $\ell : [0,1] \to M_1$, such that $\ell \subset \partial\mathcal{D}$, $\ell(0), \ell(1) \in \partial M_1$, and ℓ transversal to ∂M_1. If $\Omega := M_1 \setminus \ell$ is simply-connected [1], then $\delta(\mathcal{D}, M_1) = 0$.*

Proof. Observing that $\mathcal{D}$ can also be viewed as a partition of Ω, it is enough to prove that

$$\delta(\mathcal{D},\Omega) = \delta(\mathcal{D}, M_1)\,.$$

For this, we make the following observations.

- Since $1 = b_0(\partial\mathcal{D} \cup \partial\Omega) = b_0(\partial\mathcal{D} \cup \partial M_1)$ and $b_0(\partial\Omega) = b_0(\partial M_1)$, we have
$$\beta(\mathcal{D},\Omega) = \beta(\mathcal{D}, M_1)\,.$$
- $\omega(\mathcal{D}, M_1) = \omega(\mathcal{D},\Omega) = 0$. Indeed, each domain in $\mathcal{D}$ is orientable because Ω is simply-connected.
- $\kappa(\mathcal{D}, M_1) = \kappa(\mathcal{D},\Omega)$.
- Let x be a singular point of $\mathcal{D}$ in M_1 belonging to ℓ, with index $\nu(x)$. After scissoring, we obtain two boundary points $x_+ \in \ell_+$ and $x_- \in \ell_-$, with indices ρ_+ and ρ_- such that $\rho_+ + \rho_- = \nu(x) - 2\,.$
- After scissoring, the boundary singular point $\ell(0)$ of ℓ yields two boundary singular points $y_{0,+}$ and $y_{0,-}$ such that $\rho_{0,+}+\rho_{0,-} = \rho(\ell(0))-1$, with a similar property for $\ell(1)$.
- As a consequence of the two previous items, we have
$$\sigma(\mathcal{D}, M_1) = \sigma(\mathcal{D},\Omega) + 1\,.$$

Since Ω is homeomorphic to a simply-connected domain in $\mathbb{R}^2$, we have $\kappa(\mathcal{D},\Omega) = 1 + \beta(\mathcal{D},\Omega) + \sigma(\mathcal{D},\Omega)$. Taking into account the preceding identities, it follows that $\delta(\mathcal{D},\Omega) = 0$. □

Lemma 3.2. *Let $\mathcal{D}$ be a regular partition of M_1. Then, there exists a simple piecewise C^1 path $\ell : [0,1] \to M$ such that:*

- *$M_1 \setminus \ell$ is simply connected;*
- *ℓ crosses $\partial\mathcal{D}$ and hits ∂M_1 transversally;*
- *$\mathcal{S}(\mathcal{D}) \cap \ell = \emptyset$.*

Proof. Starting from any line ℓ_0 such that $M_1 \setminus \ell_0$ is simply connected, it is easy to deform ℓ_0 in order to get the two other properties. □

Lemma 3.3. *Let $\mathcal{D}$ be a regular partition of M_1 whose elements D_i are all simply-connected. Let $\ell : [0,1] \to M_1$ be a path as given by Lemma 3.2. Let $\widetilde{\mathcal{D}}$ be the partition of M_1 whole elements are the connected components of the $D_i \setminus \ell$. Then, $\partial\widetilde{\mathcal{D}} = \partial\mathcal{D} \cup \ell$, and $\delta(\mathcal{D}, M_1) = \delta(\widetilde{\mathcal{D}}, M_1)$. In particular $\widetilde{\mathcal{D}}$ satisfies the assumptions of Lemma 3.1, and*

$$\delta(\mathcal{D}, M_1) = 0\,.$$

[1] More precisely, after scissoring M_1 along ℓ, and unfolding, one can view Ω as a subset $\widetilde{\Omega}$ of $\mathbb{R}^2$. The former ℓ is now split into two lines ℓ_+ and ℓ_- in the boundary of $\widetilde{\Omega}$. For simplicity, we denote by ℓ, both the curve, and the image $\ell([0,1])$.

Proof. Denote by $y_i, x_1, \dots, x_N, y_f$ the points of $\ell \cap \partial\mathcal{D}$, where

- ⋄ y_i and y_f belong to ∂M_1;
- ⋄ each open interval J on the path ℓ, delimited by two consecutive points in the sequence, is contained in some element D_J of the partition, and the end points of J belong to ∂D_J.

Using the assumption that the elements of $\mathcal{D}$ are simply-connected, and comparing $\widetilde{\mathcal{D}}$ with $\mathcal{D}$, we observe that:

- ⋄ $\kappa(\widetilde{\mathcal{D}}) = \kappa(\mathcal{D}) + N + 1$;
- ⋄ the partition $\widetilde{\mathcal{D}}$ has two extra simple singular boundary points, and N interior singular points with index $\iota = 4$, so that
$$\sigma(\widetilde{\mathcal{D}}) = \sigma(\mathcal{D}) + N + 1\,;$$
- ⋄ $\beta(\widetilde{\mathcal{D}}) = \beta(\mathcal{D})$ and $\omega(\widetilde{\mathcal{D}}) = \omega(\mathcal{D}) = 0$.

The lemma follows. □

Lemma 3.4. *Let S be (the interior of) a surface with or without boundary. Let D be an open subset of S, and $K \subset D$ a compact subset. Assume that $D\backslash K$ is connected. Then, $S\backslash K$ is connected.*

Proof. It suffices to prove the following claim.

Claim. Given any $x_0, x_1 \in S\backslash K$, there exists a path $\gamma : [0,1] \to S\backslash K$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

⋄ Since S is connected, there exists a path $\ell : [0,1] \to S$, with $\ell(0) = x_0$ and $\ell(1) = x_1$. If x_0 and x_1 both belong to D, the claim is clear. Without loss of generality, we may now assume that $x_0 \notin D$.

⋄ Define the set $J := \{t \in [0,1] \mid \ell(t) \notin K\}$. Since $x_0 \notin K$, the set J in not empty, and we define $t_0 := \inf J$. Clearly, $\ell(t_0) \in K$, and since $K \subset D$, $\ell(t_0) \in D$. Since D is open, there exists some $\varepsilon > 0$ such that $\ell([t_0 - \varepsilon, t_0 + \varepsilon]) \subset D$, and hence $x_\varepsilon = \ell(t_0 - \varepsilon) \in D$. If $x_1 \in D$, then there exists a path γ_ε from x_ε to x_1, contained in $D\backslash K$. The path $\gamma_\varepsilon \cdot \ell|_{[0,t_0-\varepsilon]}$ is contained in $S\backslash K$ and links x_0 and x_1.

⋄ If $x_1 \notin D$, we can consider the path ℓ^{-1} and apply the preceding argument. Define $t_1 = \sup J$. Then t_1 exists and $\ell(t_1) \in K$, and there exists $\delta > 0$ such that $x_\delta = \ell(t_1 + \delta) \in D$. There exists a path $\gamma_{\varepsilon,\delta}$ linking x_ε and x_δ in $D\backslash K$. The path $\ell|_{[t1+\delta,1]} \cdot \gamma_{\varepsilon,\delta} \cdot \ell|_{[0,t_0-\varepsilon]}$ is contained in $S\backslash K$ and links x_0 and x_1.

□

Lemma 3.5. *Let $\mathcal{D}$ be a normal partition of M_1, and let D be an element of $\mathcal{D}$. Assume that D is orientable. Then, D is homeomorphic to a sphere with q discs removed, and one can find q piecewise C^1 cuts ℓ_j (i.e. disjoint simple curves joining two points of ∂D and hitting the boundary transversally) such that $D\backslash \cup\ell_j$ is simply connected.*

Proof. Since $\mathcal{D}$ is normal, the domain D is an orientable surface with boundary. According to the classification theorem [5, Chap. 6], D is homeomorphic to some $\Sigma_{0,g,q}$, a sphere with g handles attached, and q disks removed. We claim that $g = 0$. Indeed, if $g \neq 0$, there exists g disjoint simple closed curves whose union does not disconnect D and hence, by Lemma 3.4 does not disconnect M_1. Since M_1 has genus 1, we must have $g \leq 1$. If $g = 1$, we have a simple closed curve γ which disconnects D, and preserves orientation since D is orientable. A simple closed curve which

disconnects M_1 reverses orientation, a contradiction, see [8]. One can draw cuts on the model surface $\Sigma_{0,0,q}$ and pull them back to D.

Claim 3.6. *One can find piecewise C^1 cuts*

Indeed, we can start from a continuous cut $\gamma : [0,1] \to M_1$, with $\gamma(0,1) \subset D$, joining two points $\gamma(0) \in \Gamma_0$ and $\gamma(1) \in \Gamma_1$, two components of ∂D. In order to finish the proof, we need to approximate γ by a piecewise C^1 path γ_1 which is transversal to ∂D at $\gamma(0)$ and $\gamma(1)$. For this purpose, we choose $B(\gamma(0), 2r_0) \cap D$, $B(\gamma(1), 2r_1) \cap D$, with r_0, r_1 small enough so that these sets do not intersect $\partial D \backslash \Gamma_i$. We choose $t_0, t_1 \in (0,1)$ such that $t_0 \in B(\gamma(0), r_0)$, $\gamma(t_1) \in B(\gamma(1), r_1)$. We cover $\gamma([t_0, t_1])$ with small disks $B(\gamma(t), r) \subset D$, with r small enough so that the disks $B(\gamma(t), r)$ do not meet ∂D. By compactness, we can extract a finite covering of $\gamma([t_0, t_1])$. We can now easily construct the desired path γ_1, close to γ by using geodesics segments contained in the disks. ✓

□

Lemma 3.7. *Let $\mathcal{D}$ be a normal partition of M_1 such that all the domains $D_i \in \mathcal{D}$ are orientable. Then, there exists a new partition $\widetilde{\mathcal{D}}$ of M_1 all of whose domains are simply-connected, and such that $\delta(\mathcal{D}, M_1) = \delta(\widetilde{\mathcal{D}}, M_1)$.*
In particular,

$$\delta(\mathcal{D}, M_1) = 0\,.$$

Proof. This is essentially an application of Lemma 3.5, with Lemma 2.7 in mind. Each time, we apply one step in the proof of Lemma 3.5 to one of the domains $D_i \in \mathcal{D}$, β decreases by 1, and we create two critical points x_1 and x_2 with $\iota(x_1) = \iota(x_2) = 1$. □

Lemma 3.8. *Let $\mathcal{D}$ be a normal partition of M_1. Assume that some domain $D \in \mathcal{D}$ is non-orientable. Then, there exists a connected component Γ of ∂D, and a piecewise C^1 path $\ell : [0,1] \to M_1$, such that $\ell(0), \ell(1) \in \Gamma$, with ℓ transversal to Γ at its end points, $\ell((0,1)) \subset D$, and $D \setminus \ell$ is connected and orientable.*

Proof. The partition $\mathcal{D}$ being normal, the domain D is a non-orientable surface with boundary and hence, there exists a homeomorphism $f : D \to \Sigma_{1,c,q}$, one of the standard non-orientable surfaces, a sphere with $c \geq 1$ cross-caps attached, and q discs removed (c is the genus[2] of $\Sigma_{1,c}$, and q is the number of boundary components of D). The surface $\Sigma_{1,c,q}$ contains c pairwise disjoint simple closed curves whose union does not disconnect the surface, each cross-cap contributes for one such curve. Since the genus of M_1 is 1 (M_1 is a sphere with one cross-cap attached, and one disk removed), Lemma 3.4 implies that $c = 1$. It follows that $f : D \to \Sigma_{1,1,q}$, a Möbius strip $\Sigma_{1,1}$ with q disks removed. Denote by $\Gamma = f^{-1}(\partial\Sigma_{1,1})$, the inverse image of the boundary of the Möbius strip. It is easy to cut $\Sigma_{1,1,q}$ by some path ℓ_0, with end points on $\partial\Sigma_{1,1}$, in such a way that $\Sigma_{1,1,q} \backslash \ell_0$ is simply-connected. The path ℓ_1 is given by $\ell_1 = f^{-1}(\ell_0)$. In order to finish the proof, it suffices to approximate ℓ_1 by a piecewise C^1 path ℓ transversal to ∂D. For this purpose, we can use the same arguments as in the proof of Claim 3.6. □

[2] For an orientable surface without boundary, the genus is defined as the number of handles attached to the sphere. For a non-orientable surface without boundary, the genus is defined as the number of cross-caps attached to the sphere.

Proof of Theorem 2.10

Let $\mathcal{D}$ be a regular partition of the Möbius strip. Let $\delta(\mathcal{D}, M_1)$ be defined in (2.7). By Lemma 2.7, we may assume that $\mathcal{D}$ is a normal partition, without changing the value of $\delta(\mathcal{D}, M_1)$.

$\diamond$ Assume that all the domains in $\mathcal{D}$ are orientable. Applying Lemma 3.7, we conclude that $\delta(\mathcal{D}, M_1) = 0$, and the theorem is proved in this case.

$\diamond$ Assume that (at least) one of the domains in $\mathcal{D}$, call it D_1 is non-orientable. We claim that D_1 is actually the only non-orientable domain in $\mathcal{D}$. Indeed, assume that there is another non-orientable domain D_2. By Lemma 2.7, both domains are surfaces with boundary, with genus 1. Each $D_i, i = 1, 2$, contains a simple closed curve γ_i which does not disconnect D_i. Since $D_1 \cap D_2 = \emptyset$, we would have two disjoint simple closed curves γ_1, γ_2. Applying Lemma 3.4 twice, we obtain that $\gamma_1 \cup \gamma_2$ does not disconnect M_1. This is a contradiction since M_1 has genus 1.

Apply Lemma 3.8: D_1 is homeomorphic to a Möbius trip with q disks removed, there is a path ℓ which does not disconnect D_1, and such that $D_1 \setminus \ell$ is orientable. In doing so, using the fact that ℓ is piecewise C^1 and transversal to Γ, we obtain a regular partition $\mathcal{D}' = \left\{D_1 \setminus \ell, D_2, ..., D_{\kappa(\mathcal{D})}\right\}$ of M_1. Since all the domains of $\mathcal{D}'$ are orientable, by the preceding argument, we have $\delta(\mathcal{D}', M_1) = 0$. On the other-hand, we have $\omega(\mathcal{D}', M_1) = 0$, $\beta(\mathcal{D}', M_1) = \beta(\mathcal{D}, M_1)$, $\kappa(\mathcal{D}', M_1) = \kappa(\mathcal{D}, M_1)$, and since we have the extra arc ℓ in $\partial\mathcal{D}'$, whose end points are singular points of index $\iota = 1$, we have $\sigma(\mathcal{D}', M_1) = \sigma(\mathcal{D}, M_1) + 1$. It follows that,

$$0 = \beta(\mathcal{D}', M_1) + \sigma(\mathcal{D}', M_1) - \kappa(\mathcal{D}', M_1) = \beta(\mathcal{D}, M_1) + \sigma(\mathcal{D}, M_1) + 1 - \kappa(\mathcal{D}, M_1)\,,$$

and hence $\delta(\mathcal{D}, M_1) = 0$. The proof of Theorem 2.10 is now complete. $\square$

Figure 3.1 displays the typical nodal patterns of the Dirichlet eigenfunction

$$\Phi_{\beta,\theta}(x, y) = \cos\theta \sin(2x) \sin(3y) + \sin\theta \sin(3x) \sin(2y + \beta)\,, \tag{3.1}$$

when $\beta \in (0, \frac{\pi}{3})$ is fixed and $\theta \in (0, \frac{\pi}{2})$. As explained in [2, Section 5.4], there is a dramatic change in the nodal pattern when θ passes some value $\theta(\beta)$ (this value is precisely defined in [2, Eq. (5.26)]). For $0 < \theta \leq \theta(\beta)$ the nodal domains are all orientable; for $\theta(\beta) < \theta < \frac{\pi}{2}$, there is one non-orientable nodal domain, homeomorphic to a Möbius strip (the domain in green). When $\theta = \theta(\beta)$, the nodal domain in green is not a surface with boundary due to the singular point at the boundary.

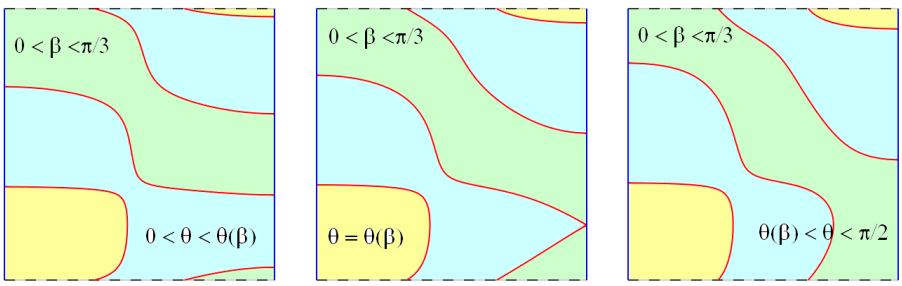


FIGURE 3.1. Example 1

Figure 3.2 displays the nodal patterns of the eigenfunctions $\sin(3x)$ (left) and $\sin(5x)$ (right), on a 3D representation[3] of the Möbius strip. The nodal domains are colored according to sign. In both cases, there is one nodal domain which is homeomorphic to a Möbius strip. The other nodal domains are cylinders, and hence orientable though not simply-connected.

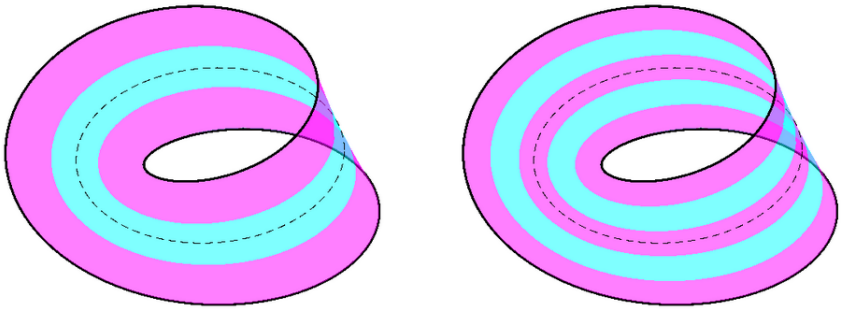


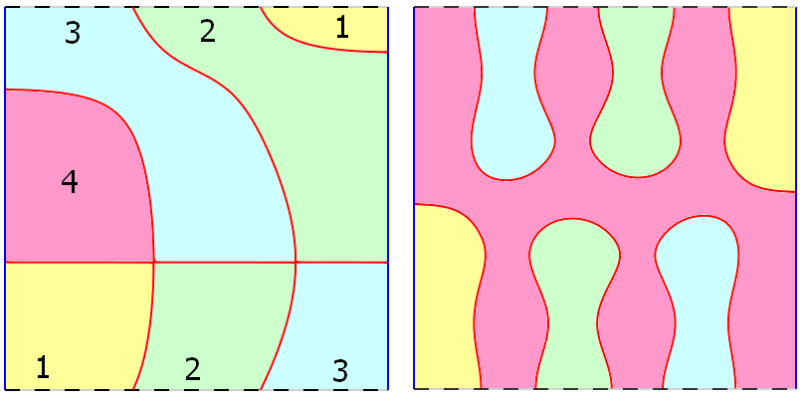
FIGURE 3.2. Example 2

Figure 3.3 (A) displays the nodal patterns of the eigenfunction (3.1), with $\beta = \frac{\pi}{3}$ and $\frac{\pi}{4} < \theta\frac{\pi}{2}$. This is explained in [2, Section 5.5]. Notice that the nodal domains labeled (2) and (3) are not surfaces with boundary due to one of the singular points. There are two interior singular points (with $\nu = 4$), and four boundary singular points (with $\rho = 1$).

Figure 3.3 (B) displays the nodal pattern of the function

$$\cos\theta\sin(x)\cos(6y) + \sin\theta\sin(6x)\cos(y),$$

with $\theta = 0.4\,\pi$. There is one non-orientable domain (colored in pink), homeomorphic to a Möbius strip with two holes (nodal domains colored in blue or in green). There is another disk-like nodal domain (colored in yellow).



(A) (B)

FIGURE 3.3. Example 3

[3]We work with the flat metric on the Möbius strip, and use an embedding into $\mathbb{R}^3$ which is not isometric.

References

[1] P. Bérard and B. Helffer. Remarks on the boundary set of spectral equipartitions. Philosophical Transactions of the Royal Society A 2014 372, 20120492, published 16 December 2013. 1

[2] P. Bérard, B. Helffer and R. Kiwan. Courant-sharp property for Dirichlet eigenfunctions on the Möbius strip. arXiv:2005.01175. 1, 7, 8

[3] M. Berger and B. Gostiaux. Differential geometry: manifolds, curves and surfaces. Springer, 1988. 2

[4] V. Bonnaillie-Noël, B. Helffer. Nodal and spectral minimal partitions –The state of the art in 2016–. In Shape Optimization and Spectral Theory (pp. 353-397) (A. Henrot Editor). De Gruyter Open.

[5] Gallier Jean, Xu Dianna (2013). A guide to the classification theorem for compact surfaces, collection "Geometry and Computing", Springer, Heidelberg. doi: 10.1007/978-3-642-34364-3 2, 5

[6] B. Helffer, T. Hoffmann-Ostenhof, and S. Terracini. Nodal domains and spectral minimal partitions. *Ann. Inst. H. Poincaré AN* 26, 101–138, 2009.

[7] T. Hoffmann-Ostenhof, P. Michor, and N. Nadirashvili. Bounds on the multiplicity of eigenvalues for fixed membranes. Geom. Funct. Anal. 9 (1999), no. 6, 1169–1188. 3

[8] M. Kreck See the entries "Orientation covering" and "Orientation of manifolds" in the Bulletin of Manifold Atlas. 6

[9] J. Leydold. Nodal properties of spherical harmonics. PhD Thesis, Universität Wien, 1993. 3

[10] J. Leydold. On the number of nodal domains of spherical harmonics. Topology 35 (1996) 301–321. 3

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