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Transience of a symmetric random walk in infinite measure

Timothée Bénard

Abstract

We consider a random walk on a second countable locally compact topological space endowed with an invariant Radon measure. We show that if the walk is symmetric and if every subset which is invariant by the walk has zero or infinite measure, then one has transience in law for almost every starting point. We then deduce a converse to Eskin-Margulis recurrence theorem.

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1 Introduction

The starting point of this text is an article published by Eskin and Margulis in 2004 which studies the recurrence properties of random walks on homogeneous spaces [12]. The space in question is a quotient G/Λ , where G is a real Lie group and $\Lambda \subseteq G$ a discrete subgroup. Given a probability measure μ on G , we can define a random walk on G/Λ with transitional probabilities $(\mu \star \delta_x)_{x \in G/\Lambda}$. In more concrete terms, a random step starting at a point $x \in G/\Lambda$ is performed by choosing an element $g \in G$ randomly with law μ and letting it act on x . The two authors ask about the position of the walk at time n for large values of n . They manage to show a surprising result : if G is a simple real algebraic group, if Λ has finite covolume in G , and if the support of μ generates a Zariski-dense subgroup of G , then for every starting point $x \in G/\Lambda$, the sequence of probabilities of position $(\mu^{\star n} \star \delta_x)_{n \geq 0}$ has all its weak- \star limits of mass 1. One says there is no escape of mass. This reminds the behavior of the unipotent flow highlighted by Dani and Margulis [10, 14], who prove that the trajectories of a unipotent flow on G/Λ spend most of their time inside compact sets. Eskin-Margulis' result is actually the starting point of a fruitful analogy with Ratner theory, that led to the classification of stationary probability measures on X thanks to the work of Benoist and Quint [5, 6], followed by Eskin and Lindenstrauss [11].

This text asks the question of a converse to Eskin-Margulis Theorem :

Is the absence of escape of mass characteristic of random walks on homogeneous spaces of finite volume, or could it also happen for walks in infinite volume?

In Section 1, we give a first answer under the additional assumption that μ is symmetric, i.e. that μ is invariant under the inversion map $g \mapsto g^{-1}$. The setting is then quite general and does not rely on the algebraic frame mentionned previously.

Theorem A. *Let X be a locally compact second countable topological space equipped with a Radon measure λ , let Γ be a locally compact second countable group acting continuously on X and preserving the measure λ , let μ be a probability measure on Γ whose support generates Γ as a closed group.*

If the probability measure μ is symmetric and if every measurable Γ -invariant subset of X has zero or infinite λ -measure, then for λ -almost every starting point $x \in X$, one has the weak- \star convergence :

$$\mu^{\star n} \star \delta_x \xrightarrow[n \rightarrow +\infty]{} 0$$

To put it in a nutshell, a symmetric random walk on an infinite “quasi-ergodic” space is always transient in law : for almost every starting point,

all the mass escapes. This result can be seen as an analogue in infinite measure of equidistribution results for random walks in finite measure obtained independantly by Rota [18] and Oseledets [16].

In our statement, a measurable subset $A \subseteq X$ is considered as Γ -invariant if for every $g \in \Gamma$, $\lambda(gA\Delta A) = 0$. We will see later an equivalent characterization in terms of the Markov operator of the walk (2.3).

Note also that the proof of Theorem A yields the convergence in Cesaro-averages in the case where μ is not assumed to be symmetric.

In Section 2, we use Theorem A to address our original question of a converse to Eskin-Margulis' recurrence Theorem. This leads to the following result.

Theorem B. *Let G be a semisimple connected real Lie group with finite center, $\Lambda \subseteq G$ a discrete subgroup of infinite covolume in G , and μ a probability on G whose support generates a Zariski-dense subgroup of G .*

Then for almost every $x \in G/\Lambda$, one has the weak- \star convergence :

$$\frac{1}{n} \sum_{k=0}^{n-1} \mu^{\star k} \star \delta_x \xrightarrow[n \rightarrow +\infty]{} 0 \quad (1)$$

Moreover, if the probability measure μ is symmetric, then the convergence can be strengthened :

$$\mu^{\star n} \star \delta_x \xrightarrow[n \rightarrow +\infty]{} 0 \quad (2)$$

Observe that convergence (1) is sufficient to ensure that Eskin-Margulis' observations cannot occur when the quotient G/Λ has infinite measure. Indeed, for almost every $x \in G/\Lambda$, we obtain the existence of an extraction $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$\mu^{\star \sigma(n)} \star \delta_x \xrightarrow[n \rightarrow +\infty]{} 0$$

Section 3 is an attempt to refine Theorem B by proving convergence (2) without assumption of symmetry on μ . We manage to do so when G/Λ is an abelian cover with finite volume basis. More precisely, we show :

Theorem C. *Let G, Λ, μ as in theorem B, with μ not necessarily symmetric*

(H) : Assume that Λ is contained in lattices of G of arbitrarily large covolume.

Then, for almost-every $x \in G/\Lambda$, we have the weak- \star convergence :

$$\mu^{\star n} \star \delta_x \xrightarrow[n \rightarrow +\infty]{} 0$$

Note that our paper focuses on the asymptotic behavior *in law* of a random walk on G/Λ . A related natural theme of study is the behavior of the walk trajectories for which analogous notions of recurrence or transience exist. Although our conclusions support the idea that walks in infinite volume are always transient in law (the mass escapes), the picture becomes mixed when it comes to considering walk trajectories. Indeed, as observed in [9] or [17], punctual recurrence or transience also depends on the nature of the ambient space.

2 A general result for transience in law

This section is dedicated to the proof of Theorem A. The proof results from a combination of Chacon-Ornstein Theorem and Akcoglu-Sucheston's pointwise convergence of alternating sequences [1]. The latter guarantees that for λ -almost every $x \in X$, the sequence of probability measures $(\mu^{*n} \star \check{\mu}^{*n} \star \delta_x)_{n \geq 0}$ weak- \star converges toward a finite measure, and is based on Rota and Osleedets' original idea to express this alternating sequence in terms of reversed martingales [18, 16]. We give a shorter proof than the one in [1]. Although our proof follows very closely the one of Rota [18] who considered walks on finite volume spaces, we use a different formalism that may be useful to illustrate the technique of "equidistribution of fibres" contained in the work of Benoist-Quint [6] (see also [7]).

2.1 Backwards martingales

We first present a convergence theorem for backwards martingales on a σ -finite measured space. It will play a crucial role in the proof of the convergence of back-and-forths (2.3).

First, let us recall the definition of conditional expectation.

Definition 2.1 (Conditional expectation). *Let (Ω, \mathcal{F}) be a measurable space, \mathcal{Q} a sub- σ -algebra of \mathcal{F} , and m a positive measure on (Ω, \mathcal{F}) whose restriction $m|_{\mathcal{Q}}$ is σ -finite. Then, for every function $f \in L^1(\Omega, \mathcal{F}, m)$, there exists a unique function $f' \in L^1(\Omega, \mathcal{Q}, m)$ such that for all \mathcal{Q} -measurable subset $A \in \mathcal{Q}$, one has $m(f 1_A) = m(f' 1_A)$. We denote this function by $\mathbb{E}_m(f|\mathcal{Q})$.*

We have the following [13, page 533] (see also [8]).

Theorem 2.2 (Convergence of backwards martingales). *Let (Ω, \mathcal{F}, m) be a measured space, $(\mathcal{Q}_n)_{n \geq 0}$ a decreasing sequence of sub- σ -algebras of \mathcal{F} such that for all $n \geq 0$, the restriction $m|_{\mathcal{Q}_n}$ is σ -finite. Then, for any function*

$f \in L^1(\Omega, \mathcal{F}, m)$, there exists $\psi \in L^1(\Omega, \mathcal{F}, m)$ such that we have the almost sure convergence :

$$\mathbb{E}_m(f|\mathcal{Q}_n) \xrightarrow[n \rightarrow +\infty]{} \psi \quad (m\text{-a.e.})$$

Remark. If the measure m is σ -finite with respect to the tail-algebra $\mathcal{Q}_\infty := \bigcap_{n \geq 0} \mathcal{Q}_n$, then Theorem 2.2 can be deduced from the probabilistic case (by restriction to \mathcal{Q}_∞ -measurable domains of finite measure), and we can certify that $\psi = \mathbb{E}_m(f|\mathcal{Q}_\infty)$. On the extreme opposite, if every \mathcal{Q}_∞ -measurable subset of Ω has m -measure 0 or $+\infty$, then, the integrability of ψ implies that $\psi = 0$. The general picture is a direct sum of these two contrasting situations as $\Omega = \Omega_\sigma \sqcup \Omega_\infty$ where Ω_σ is a countable union of \mathcal{Q}_∞ -measurable sets of finite measure, and the restricted measure $m|_{\Omega_\infty}$ takes only the values 0 or $+\infty$ on \mathcal{Q}_∞ (see [13], footnote of page 533).

2.2 Convergence of back-and-forths

We now state and show Theorem 2.3 about the convergence of back-and-forths of the μ -random walk on X . We denote by $\check{\mu} := i_*\mu$ the image of μ under the inversion map $i : \Gamma \rightarrow \Gamma, g \mapsto g^{-1}$.

Theorem 2.3 (Convergence of back-and-forths [1]). *Let X be a locally compact second countable topological space equipped with a Radon measure λ , let Γ be a locally compact second countable group acting continuously on X and preserving the measure λ , and let μ be a probability measure on Γ .*

There exists a family of finite measures $(\nu_x)_{x \in X} \in \mathcal{M}^f(X)^X$ such that for λ -almost every $x \in X$, one has the weak- \star convergence :

$$(\mu^{\star n} \star \check{\mu}^{\star n}) \star \delta_x \xrightarrow[n \rightarrow +\infty]{} \nu_x$$

Proof. The following proof is inspired by [18] and [6]. Denote

$$B := \Gamma^{\mathbb{N}^*}, \quad \beta := \mu^{\mathbb{N}^*}, \quad T : B \rightarrow B, (b_i)_{i \geq 1} \mapsto (b_{i+1})_{i \geq 1}$$

the one-sided shift. One introduces a σ -finite fibred dynamical system (B^X, β^X, T^X) setting

- $B^X := B \times X$
- $\beta^X := \beta \otimes \lambda \in \mathcal{M}^{\text{Rad}}(B \times X)$
- $T^X : B^X \rightarrow B^X, (b, x) \mapsto (Tb, b_1^{-1}x)$.

Let \mathcal{B} and \mathcal{X} denote the Borel σ -algebras of B and X . The Borel σ -algebra of B^X is then the product algebra $\mathcal{B} \otimes \mathcal{X}$. For all $n \geq 0$, define the *sub- σ -algebra of the n -fibres of T^X* by setting

$$\mathcal{Q}_n := (T^X)^{-n}(\mathcal{B} \otimes \mathcal{X})$$

It is a sub- σ -algebra of $\mathcal{B} \otimes \mathcal{X}$ such that for all $c \in B^X$, the smallest \mathcal{Q}_n -measurable subset of B^X containing c is the n -fibre $(T^X)^{-n}(T^X)^n(c)$. The restriction $\beta^X|_{\mathcal{Q}_n}$ is a σ -finite measure because β^X is σ -finite with respect to the σ -algebra $\mathcal{B} \otimes \mathcal{X}$ and is preserved by T^X .

As a first step, we will fix a continuous function with compact support $f \in C_c^0(X)$ and show that the sequence $(\mu^{*n} \star \check{\mu}^{*n} \star \delta_x)(f)_{n \geq 0}$ converges in \mathbb{R} for λ -almost every x . To this end, we express $(\mu^{*n} \star \check{\mu}^{*n} \star \delta_x)(f)$ using a conditional expectation and we apply Theorem 2.2. Denote

$$\tilde{f} : B^X \rightarrow \mathbb{R}, (b, x) \mapsto f(x), \quad \varphi_n := \mathbb{E}_{\beta^X}(\tilde{f} | \mathcal{Q}_n) \in L^1(B^X, \mathcal{Q}_n)$$

We first give an explicit formula for the function φ_n . Intuitively, given a point $c = (b, x) \in B^X$, the value $\varphi_n(c)$ stands for the mean value of \tilde{f} on the smallest \mathcal{Q}_n -measurable subset of B^X containing c . By definition, this subset is the n -fibre going through c and is identified with the product Γ^n under the bijection

$$h_{n,c} : \Gamma^n \rightarrow (T^X)^{-n}(T^X)^n(c), \quad a = (a_1, \dots, a_n) \rightarrow (aT^n b, a_1 \dots a_n b_n^{-1} \dots b_1^{-1} \cdot x)$$

The following lemma asserts that $\varphi_n(c)$ is nothing else than the mean value of \tilde{f} on $(T^X)^{-n}(T^X)^n(c) \equiv \Gamma^n$ with respect to the measure $\mu^{\otimes n}$.

Lemma 2.4. *Let $n \geq 0$. For β^X -almost every $(b, x) \in B^X$, one has*

$$\varphi_n(b, x) = \int_{\Gamma^n} f(a_1 \dots a_n b_n^{-1} \dots b_1^{-1} x) d\mu^{\otimes n}(a)$$

Proof of lemma 2.4. This result is extracted from [6] (lemma 3.3). We recall the proof. Up to considering separately the positive and negative parts of f , one may assume $f \geq 0$. Denote by $\varphi'_n : B^X \rightarrow [0, +\infty]$ the map defined by the right-hand side of the above equation. We show it coincides almost everywhere with φ_n by proving it also satisfies the axioms for the conditional expectation characterizing φ_n .

As the value φ'_n at a point $c \in B^X$ only depends on $(T^X)^n(c)$, the map φ'_n is \mathcal{Q}_n -measurable. It remains to show that for every $A \in \mathcal{Q}_n$, one has the equality $\beta^X(1_A \tilde{f}) = \beta^X(1_A \varphi'_n)$. Writing A as $A = (T^X)^{-n}(E)$ where $E \in \mathcal{B} \otimes \mathcal{X}$ and remembering the measure λ is preserved by Γ , one computes

that :

$$\begin{aligned}
\beta^X(1_A \varphi'_n) &= \int_{B \times X \times \Gamma^n} 1_A(b, x) f(a_1 \dots a_n b_n^{-1} \dots b_1^{-1} x) d\mu^{\otimes n}(a) d\beta(b) d\lambda(x) \\
&= \int_{B \times X \times \Gamma^n} 1_E(T^n b, b_n^{-1} \dots b_1^{-1} x) f(a_1 \dots a_n b_n^{-1} \dots b_1^{-1} x) d\mu^{\otimes n}(a) d\beta(b) d\lambda(x) \\
&= \int_{B \times X \times \Gamma^n} 1_E(T^n b, x) f(a_1 \dots a_n x) d\mu^{\otimes n}(a) d\beta(b) d\lambda(x) \\
&= \int_{B \times X} 1_E(T^n b, x) f(b_1 \dots b_n x) d\beta(b) d\lambda(x) \\
&= \int_{B \times X} 1_E(T^n b, b_n^{-1} \dots b_1^{-1} x) f(x) d\beta(b) d\lambda(x) \\
&= \beta^X(1_A \tilde{f})
\end{aligned}$$

which concludes the proof of lemma 2.4. \square

Lemma 2.4 implies that for λ -almost every $x \in X$,

$$\int_B \varphi_n(b, x) d\beta(b) = (\mu^{\star n} \star \check{\mu}^{\star n} \star \delta_x)(f) \quad (**)$$

But Theorem 2.2 on convergence of backwards martingales asserts the sequence of conditional expectations $(\varphi_n)_{n \geq 0}$ converges β^X -almost-surely. Noticing that $\|\varphi_n\|_\infty \leq \|f\|_\infty$, the dominated convergence Theorem and equation (**) imply that for λ -almost every $x \in X$, the sequence

$$((\mu^{\star n} \star \check{\mu}^{\star n} \star \delta_x)(f))_{n \geq 0}$$

has a limit in \mathbb{R} .

We deduce from the previous paragraph that for λ -almost every $x \in X$, the sequence of probability measures $(\mu^{\star n} \star \check{\mu}^{\star n} \star \delta_x)_{n \geq 0}$ has a weak- \star limit (which is a measure on X whose mass is less or equal to one, and possibly null). It is indeed a standard argument, that uses the separability of the space of continuous functions with compact support on X equipped with the supremum norm $(C_c^0(X), \|\cdot\|_\infty)$, and the representation of non negative linear forms on $C_c^0(X)$ by Radon measures (Riesz Theorem). This concludes the proof of Theorem 2.3. \square

2.3 Proof of Theorem A

We now prove that a symmetric random walk on an infinite “quasi-ergodic” space is always transient in law. Recall first the precise statement.

Theorem A. *Let X be a locally compact second countable topological space equipped with a Radon measure λ , let Γ be a locally compact second countable group acting continuously on X and preserving the measure λ , let μ be a probability measure on Γ whose support generates Γ as a closed group.*

If the probability measure μ is symmetric and if every measurable Γ -invariant subset of X has zero or infinite λ -measure, then for λ -almost every starting point $x \in X$, one has the weak- \star convergence :

$$\mu^{\star n} \star \delta_x \xrightarrow{n \rightarrow +\infty} 0$$

Remark. Without the assumption of symmetry, the proof gives the convergence in average

$$\frac{1}{n} \sum_{k=0}^{n-1} \mu^{\star k} \star \delta_x \xrightarrow{n \rightarrow +\infty} 0$$

We cannot hope for the convergence of probabilities $(\mu^{\star n} \star \delta_x)_{n \geq 0}$ if we remove the hypothesis of symmetry. For example, let us consider $S_{\mathbb{Z}}$ a \mathbb{Z} -cover of a hyperbolic compact surface. One can realize its unitary bundle $T^1 S_{\mathbb{Z}}$ as a homogeneous space G/Λ where $G = SL_2(\mathbb{R})$ and $\Lambda \subseteq G$ is a discrete subgroup.

Set $\mu = \delta_{u_1}$ where $u_1 := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The μ -walk on G/Λ is now a deterministic process that corresponds to a discretized horocycle flow on $T^1 S_{\mathbb{Z}}$. One can check that every subset of $T^1 S_{\mathbb{Z}}$ which is invariant under the walk has zero or infinite measure (direct consequence of Howe-Moore Theorem, see proof of Theorem B). However, the walk is almost everywhere recurrent [2], so we cannot have the convergence $\mu^{\star n} \star \delta_x \rightarrow 0$ for almost every x .

The proof will use the *Markov operator* P_μ attached to μ . It acts on the set of non-negative measurable functions on X via the formula

$$P_\mu \varphi(x) := \int_G \varphi(gx) d\mu(g)$$

and can be extended to a contraction on the spaces $L^p(X, \lambda)$ for $p \in [1, \infty]$.

Recall from the introduction that a measurable subset $A \subseteq X$ is Γ -invariant if for all $g \in \Gamma$, one has $\lambda(A \Delta gA) = 0$. This can be rephrased¹ in terms of the Markov operator as the equality $P_\mu 1_A = 1_A$ λ -a.e. on X .

¹The point is to show that a measurable subset $A \subseteq X$ such that $P_\mu 1_A = 1_A$ λ -a.e. on X is actually Γ -invariant. The assumption on A means that for λ -almost every $x \in X$, μ -almost every $g \in G$, one has $1_A(gx) = 1_A(x)$. Fubini Theorem then implies that for μ -almost every $g \in \Gamma$, one has $\lambda(A \Delta gA) = 0$. The subgroup $D \subseteq \Gamma$ generated by such elements g is dense in Γ and leaves the set A λ -a.e.-invariant. So we just need to check that the λ -a.e.-invariance is preserved by taking limits. Let $g \in \Gamma$, $(g_n) \in D^{\mathbb{N}}$ such that $g_n \rightarrow g$, let $\varphi \in C_c^0(X)$. By dominated convergence,

$$\int_{g_n A} \varphi d\lambda - \int_{gA} \varphi d\lambda = \int_A \varphi(g_n \cdot) - \varphi(g \cdot) d\lambda \xrightarrow{n \rightarrow +\infty} 0$$

We deduce that $\int_A \varphi d\lambda = \int_{gA} \varphi d\lambda$. As this is true for every $\varphi \in C_c^0(X)$, one concludes that $\lambda(A \Delta gA) = 0$.

Proof of Theorem A. We just need to show that for λ -almost every $x \in X$, one has the convergence $\mu^{\star 2n} \star \delta_x \rightarrow 0$. According to Theorem 2.3 and the symmetry of μ , the sequence $(\mu^{\star 2n} \star \delta_x)_{n \geq 0}$ converges to a finite measure, so it is enough to check the following convergence in average : for λ -almost every $x \in X$,

$$\frac{1}{n} \sum_{k=0}^{n-1} \mu^{\star k} \star \delta_x \rightarrow 0$$

As announced in the preceding remark, we show this last convergence *without using the assumption of symmetry on μ* . We need to check that for every non-negative continuous function with compact support $\varphi \in C_c^0(X)^+$,

$$\frac{1}{n} \sum_{k=0}^{n-1} P_\mu^k \varphi \rightarrow 0 \quad (\lambda\text{-a.e.})$$

where P_μ denotes the Markov operator of the walk.

Chacon-Ornstein Theorem (cf [15]) implies that the sequence of functions $(\frac{1}{n} \sum_{k=0}^{n-1} P_\mu^k \varphi)_{n \geq 1}$ converges almost-surely to some function $\psi : X \rightarrow \mathbb{R}_+$. As the functions $P_\mu^k \varphi$ are uniformly bounded in $L^2(X, \lambda)$, Fatou lemma implies that $\psi \in L^2(X, \lambda)$. Furthermore, the function φ being bounded, the dominated convergence Theorem applied to the probability space (Γ, μ) gives the P_μ -invariance

$$P_\mu \psi = \psi \quad (\lambda\text{-a.e.})$$

We now infer that ψ is Γ -invariant, meaning that for $g \in \Gamma$, one has the equality $\psi \circ g = \psi$ λ -a.e. on X . To this end, observe that the P_μ -invariance of ψ expresses ψ as a barycenter of translates $\psi \circ g$:

$$\int_{\Gamma} \psi \circ g \, d\mu(g) = \psi \quad (\lambda\text{-a.e.})$$

But the functions $\psi \circ g$ all are in $L^2(X, \lambda)$ and have the same norm as ψ . The strict convexity of balls in a Hilbert space then gives for μ -almost every $g \in \Gamma$, the equality $\psi \circ g = \psi$ λ -almost everywhere. As the support of μ generates Γ as a closed subgroup, we infer by a method already discussed in the footnote 2 that for all $g \in \Gamma$, one has $\psi \circ g = \psi$ λ -a.e., which is the Γ -invariance announced above.

The Γ -invariance of ψ implies that for every constant $c > 0$, the set $\{\psi > c\}$ is Γ -invariant, so has zero or infinite λ -measure by hypothesis. As ψ^2 is integrable, we must have $\lambda\{\psi > c\} = 0$. Finally, we get that $\psi = 0$ λ -almost everywhere, which finishes the proof. □

3 Application : A converse to Eskin-Margulis recurrence theorem

3.1 Context

We use Theorem A to show that a Zariski-dense symmetric random walk on a homogeneous space G/Λ of infinite volume is transient in law (Theorem B). The difficulty lies in the fact that the Markov operator P_μ of the walk may not have a spectral gap, meaning its action on $L^2(G/\Lambda)$ may have a spectral radius equal to 1. Our result can be seen as a converse to Eskin-Margulis recurrence theorem stating that a “Zariski-dense random walk on a homogeneous space with finite volume is uniformly recurrent in law”.

Theorem (Eskin-Margulis, 2004, [12]). *Let G be a semisimple connected real Lie group with finite center, $\Lambda \subseteq G$ a lattice, μ a probability measure on G whose support is compact and generates a Zariski-dense subgroup of G .*

Then the random walk on G/Λ induced by μ is uniformly recurrent in law.

We explain the above terminology :

A subgroup of G is *Zariski-dense* if it is dense from an algebraic point of view. More formally, denote by \mathfrak{g} the Lie algebra of G , and $\text{Ad} : G \rightarrow \text{Aut}(\mathfrak{g}), g \mapsto T_e(g.g^{-1})$ its adjoint representation. A subgroup $\Gamma \subseteq G$ is Zariski-dense if every polynomial function on the space of endomorphisms of \mathfrak{g} that vanishes on $\text{Ad}(\Gamma)$ also vanishes on $\text{Ad}(G)$.

The *uniform recurrence in law* means that if K is some big enough compact subset of G/Λ , then for every starting point $x \in G/\Lambda$, the μ -walk starting from x will have a very strong probability to be in K at time n , as long as n is large enough. More formally, we ask that for every $\varepsilon > 0$, there exists a compact subset $K \subseteq G/\Lambda$ such that for every $x \in G/\Lambda$, there exists an integer $N_x \geq 0$ such that if $n \geq N_x$ then $\mu^{\star n} \star \delta_x(K) \geq 1 - \varepsilon$.

We show the following converse to Eskin-Margulis Theorem :

Theorem B. *Let G be a semisimple connected real Lie group with finite center, $\Lambda \subseteq G$ a discrete subgroup of infinite covolume in G , and μ a probability on G whose support generates a Zariski-dense subgroup of G .*

Then for almost every $x \in G/\Lambda$, one has the weak- \star convergence :

$$\frac{1}{n} \sum_{k=0}^{n-1} \mu^{\star k} \star \delta_x \xrightarrow[n \rightarrow +\infty]{} 0 \quad (1)$$

Plus, if the probability measure μ is symmetric, then the convergence can be strengthened :

$$\mu^{\star n} \star \delta_x \xrightarrow[n \rightarrow +\infty]{} 0 \quad (2)$$

Remarks. 1) Theorem B is still true under the more general assumption that the subgroup generated by the support of μ has unbounded projections in the non-compact factors of G (see the proof).

2) Whether or not μ is symmetric, convergence (1) implies that for almost every $x \in G/\Lambda$, there exists an extraction $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$\mu^{\star\sigma(n)} \star \delta_x \xrightarrow{n \rightarrow +\infty} 0$$

Hence, the phenomenon observed by Eskin and Margulis cannot happen in a context of infinite measure.

3) Theorem B describes the asymptotic behavior of probabilities of position for *almost every* starting point $x \in G/\Lambda$. One may not hope for transience in law for every starting point as it is possible that the orbit $\Gamma.x$ is finite.

4) It is reasonable to believe that convergence (2) might hold without assumption of symmetry on μ (see 3.3)

3.2 Proof of Theorem B

Theorem B will be a consequence of Theorem A and the remark of Section 2.3. To apply these, we need to check the assumption of “quasi-ergodicity” : every subset of G/Λ invariant by the walk has zero or infinite mass for the Haar measure.

Remark. In infinite volume, the action of Γ on the homogeneous space G/Λ has *no reason to be ergodic* for the Haar measure. This is obvious if Λ is the trivial subgroup, but we can also construct examples where Λ is Zariski-dense. To this end, denote by \mathbb{D} the Poincaré disk, set $G = SL_2(\mathbb{R}) \cong \text{Isom}^+(\mathbb{D}) \cong T^1\mathbb{D}$, and consider a Schottky subgroup $S_0 \subseteq G$ whose limit set \mathcal{L}_0 on the boundary of \mathbb{D} is contained under four geodesic arcs, which are disjoint and small enough. Set $\Gamma = \Lambda = S_0$. For some non-zero measure subset of unitary vectors $x \in T^1\mathbb{D}$, the set $\overline{x\Lambda} \cap \partial\mathbb{D} = x\mathcal{L}_0$ does not intersect the limit set \mathcal{L}_0 of Γ . Given such an x and looking in the quotient space, the orbital map $\Lambda \rightarrow \Gamma \backslash G, g \mapsto \Gamma xg$ is proper, so its image cannot be dense. Thus, the right action of Λ on $\Gamma \backslash G$ is not ergodic, or equivalently, the left action of Γ on G/Λ is not ergodic.

Quasi-ergodicity will be a consequence of Howe-Moore Theorem. First recall its statement [20, Theorem 2.2.20] :

Theorem (Howe-Moore). *Let G be a semisimple connected real Lie group with finite center, and π a continuous morphism from G to the unitary group of a separable Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. Assume that every factor G_i of G has a trivial set of fixed points, i.e. $\mathcal{H}^{G_i} := \{x \in \mathcal{H}, G_i.x = x\}$ is $\{0\}$.*

Then for every $v, w \in \mathcal{H}$, one has

$$\langle \pi(g).v, w \rangle \xrightarrow{g \rightarrow \infty} 0$$

In the above statement, the unitary group $U(\mathcal{H})$ is endowed with the strong operator topology, and the notation $g \rightarrow \infty$ means that g leaves every compact subset of G . Also recall the definition of *factors* of G we will use in the sequel. Denote by \mathfrak{g} the Lie algebra of G . It can be uniquely decomposed as a direct sum of simple ideals : $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$. The factors of G are the immersed connected subgroups G_1, \dots, G_s of G whose Lie algebras are $\mathfrak{g}_1, \dots, \mathfrak{g}_s$. They are closed in G and commute mutually : for $i \neq j \in \{1, \dots, s\}$ and $g_i \in G_i, g_j \in G_j$ one has $g_i g_j = g_j g_i$. Lastly, the product map $\pi : G_1 \times \cdots \times G_s \rightarrow G, (g_1, \dots, g_s) \mapsto g_1 \cdots g_s$ is a morphism of groups which is onto and has finite kernel. In the sequel, we will say that a subgroup $\Gamma \subseteq G$ has unbounded projections in the factors of G if for every $i \in \{1, \dots, s\}$, the projection of $\pi^{-1}(\Gamma) \subseteq G_1 \times \cdots \times G_s$ in G_i is unbounded

Howe-Moore Theorem implies a lemma of rigidity.

Lemma 3.1. *Let G be a semisimple connected real Lie group with finite center, $\Gamma \subseteq G$ a subgroup with unbounded projections in the factors of G . Let (\mathcal{H}, ρ) be a unitary representation of G on a separable Hilbert space.*

$$\text{If } \mathcal{H}^G = \{0\} \text{ then } \mathcal{H}^\Gamma = \{0\}$$

Proof of lemma 3.1. Denote by G_1, \dots, G_s the factors of G . Up to pulling back the representation of G by the product map $\pi : G_1 \times \cdots \times G_s \rightarrow G, (g_1, \dots, g_s) \mapsto g_1 \cdots g_s$, one may assume that G is a direct product of quasi-simple² connected real Lie groups with finite center $G = G_1 \times \cdots \times G_s$. Assume $s = 2$. The hypothesis $\mathcal{H}^G = \{0\}$ implies that $\mathcal{H}^{G_1} \cap \mathcal{H}^{G_2} = \{0\}$. Thus, we can decompose

$$\mathcal{H} = \mathcal{H}^{G_1} \oplus \mathcal{H}^{G_2} \oplus \mathcal{H}'$$

where \mathcal{H}' is the orthogonal of $\mathcal{H}^{G_1} \oplus \mathcal{H}^{G_2}$ in \mathcal{H} . Moreover, each subspace is invariant by G . Let $v \in \mathcal{H}$ be a Γ -invariant vector. Decompose v as $v = v_1 + v_2 + v'$ with $v_i \in \mathcal{H}^{G_i}, v' \in \mathcal{H}'$. The representation of G on \mathcal{H} leads to a unitary representation of G_2 on \mathcal{H}^{G_1} and the Γ invariance of v implies that v_1 is invariant under $p_2(\Gamma)$, projection of Γ on the factor G_2 . As $p_2(\Gamma)$ is unbounded in G_2 , one can apply Howe-Moore Theorem to obtain $v_1 = 0$. In the same way $v_2 = 0$. Thus $v = v' \in \mathcal{H}'$. The representations of G_1 and G_2 induced by G on \mathcal{H}' have no non-trivial fix point. Hence, we can apply Howe-Moore Theorem one more time to infer that $v' = 0$. Finally, $\mathcal{H}^\Gamma = \{0\}$.

For the general case where $s \geq 1$, argue by induction on s using the previous method and the decomposition of \mathcal{H} as $\mathcal{H}^{G_1 \times \cdots \times G_{s-1}} \oplus \mathcal{H}^{G_s} \oplus \mathcal{H}'$. \square

We infer from the last lemma that for a group G with no compact factor, the Haar measure on G/Λ is “quasi-ergodic”.

²A real Lie group is said to be quasi-simple if its Lie algebra is simple.

Lemma 3.2. *Keep the setting of Theorem B and assume the group G has no compact factor. Let Γ be the smallest closed subgroup of G that contains the support of μ , and λ a Haar measure on G/Λ .*

Then every Γ -invariant subset of G/Λ has zero or infinite λ -measure.

Proof of lemma 3.2. Argue by contradiction assuming the existence of some Γ -invariant measurable subset $A \subseteq G/\Lambda$ such that $\lambda(A) \in]0, +\infty[$. Consider the regular unitary representation of G on $L^2(G/\Lambda)$, given by the formula $g.f = f(g^{-1}\cdot)$. The characteristic function $1_A \in L^2(G/\Lambda)$ is a non-zero fix point for the action of Γ . As G has no compact factor, lemma 3.1 and the Zariski-density of μ imply there exists a non-zero fix point $\varphi \in L^2(G/\Lambda)$ for the action of G . Such a function is λ -a.e. constant, implying that λ has finite mass. Absurd. □

We now prove Theorem B.

Proof of Theorem B. Assume first that the group G has no compact factor. If the probability measure μ is symmetric, then convergence (2) comes from lemma 3.2 and Theorem A. If there is no assumption of symmetry, we still get the convergence in Cesaro average (1) via the remark of Section 2.3.

We now explain how to reduce Theorem B to the case where G has no compact factor. Denote by G_1, \dots, G_s the factors of G , and π the induced finite cover of G , i.e. $\pi : G_1 \times \dots \times G_s \rightarrow G, (g_1, \dots, g_s) \mapsto g_1 \dots g_s$. There exists a probability measure $\tilde{\mu}$ on $\prod_{i=1}^s G_i$ whose support is $\pi^{-1}(\text{supp } \mu)$ and such that the $\tilde{\mu}$ -walk on $\prod_{i=1}^s G_i/\pi^{-1}(\Lambda)$ lifts the μ -walk on G/Λ . It is enough to show our result of transience for this $\tilde{\mu}$ -walk. Denote by G_1, \dots, G_k the non compact factors of G and $p : \prod_{i=1}^s G_i \rightarrow \prod_{i=1}^k G_i, (g_i)_{i \leq s} \mapsto (g_i)_{i \leq k}$ the projection on their product (notice that $k \geq 1$ otherwise G would not have a discrete subgroup of infinite covolume). Then the projection $p(\pi^{-1}(\Lambda))$ is a discrete subgroup of infinite covolume in $\prod_{i=1}^k G_i$. It is enough to prove our result of transience for the image $p_*\tilde{\mu}$ on $\prod_{i=1}^k G_i$. But this probability measure is Zariski-dense. Hence, we have reduced Theorem B to the case of a group with no compact factor, which finishes the proof. □

3.3 Removing the assumption of symmetry

We conclude this paper with a refinement of theorem B stating that convergence (2) holds without assumption of symmetry if the discrete subgroup Λ is contained in lattices of G of arbitrary large covolume. This assumption is for instance satisfied if G/Λ is an *abelian cover with finite volume basis*, i.e. if there exists a lattice $\Lambda_0 \subseteq G$ containing Λ as a normal subgroup and such that the quotient Λ_0/Λ is abelian.

Theorem C. *Let G, Λ, μ as in theorem B, with μ not necessarily symmetric (H) : Assume that Λ is contained in lattices of G of arbitrarily large covolume.*

Then, for almost-every $x \in G/\Lambda$, we have the weak- \star convergence :

$$\mu^{\star n} \star \delta_x \xrightarrow{n \rightarrow +\infty} 0$$

Remark. The hypothesis (H) is not satisfied in general. For instance, set $G := SL_3(\mathbb{R})$, $\lambda_0 > 0$, $g_0 := \text{diag}(\lambda_0, \lambda_0^{-1}, 1) \in G$ and $\Lambda := \{g_0^k, k \in \mathbb{Z}\}$. If the coefficient λ_0 is not algebraic over \mathbb{Q} , then Λ is a discrete subgroup of G that is not included in any lattice of G (consequence of Margulis' superrigidity Theorem, cf. [4], section 11.3).

The key input behind theorem C is that a Zariski-dense random walk on a finite volume homogeneous space admits a spectral gap (modulo a few assumptions). This is the content of the following proposition :

Proposition 3.3. *Let G be a semi-simple connected real Lie group with finite center and no compact factor. Let Λ_0 be a lattice in G , and μ a probability measure on G whose support generates a Zariski-dense subgroup of G . Let λ_0 be the Haar probability measure on G/Λ_0 and*

$$L_0^2(G/\Lambda_0) := \left\{ f \in L^2(G/\Lambda_0), \int_{G/\Lambda_0} f d\lambda_0 = 0 \right\}$$

Then the Markov operator P_μ acting on $L_0^2(G/\Lambda_0)$ has a spectral radius strictly less than 1.

The conclusion of proposition 3.3 means that a certain power of the operator $P_\mu : L_0^2(G/\Lambda_0) \rightarrow L_0^2(G/\Lambda_0)$ has a norm strictly less than 1.

Proof. It is shown in [3, lemma 3] that the regular representation of G on $L_0^2(G/\Lambda_0)$ does not weakly contain the trivial representation. Moreover, as G has no compact factor, the assumption of Zariski-density on μ implies that its projections in the various factors of G are not supported by amenable subgroups. The announced result is then a consequence of [19, Theorem C]. \square

We deduce from the spectral gap the equidistribution of the probabilities of position for almost-every starting point.

Corollary 3.4. *Under the assumptions of proposition 3.3, for almost every $x \in G/\Lambda_0$, the sequence of probabilities of position $(\mu^{\star n} \star \delta_x)_{n \geq 0}$ weak- \star converges toward the Haar probability measure on G/Λ_0 .*

Proof. Let $f \in C_c^0(G/\Lambda_0)$ be a continuous function with compact support on G/Λ_0 , set $f_0 = f - \int_{G/\Lambda_0} f d\lambda_0$. According to proposition 3.3, the sequence $(\|P_\mu^n f_0\|_{L^2})_{n \geq 0}$ converges to 0 at exponential speed, hence the sequence of functions $(\sum_{k=0}^n P_\mu^k f_0)_{n \geq 0}$ is λ -a.e. convergent. We infer that $P_\mu^n f_0 \rightarrow 0$ λ -a.e., in other words, for λ -almost every $x \in G/\Lambda_0$,

$$\mu^{\star n} \star \delta_x(f) \longrightarrow \int_{G/\Lambda_0} f d\lambda_0$$

The corollary follows, using the separability of the space of continuous functions on G/Λ_0 with compact support for the norm $\|\cdot\|_\infty$. □

We can now conclude.

Proof of theorem C. Arguing as in theorem B, we may suppose the group G has no compact factor. We need to show that for every compact subset $K \subseteq G/\Lambda$, every $\varepsilon > 0$, almost-every $x \in G/\Lambda$, one has

$$\limsup \mu^{\star n} \star \delta_x(K) \leq \varepsilon$$

Fix a compact set K in G/Λ , a constant $\varepsilon > 0$. According to the hypothesis (H), there exists a lattice $\Lambda_0 \subseteq G$ such that, denoting by $\pi_0 : G/\Lambda \rightarrow G/\Lambda_0$ the projection, we have

$$\lambda_0(\pi_0(K)) < \varepsilon$$

Corollary 3.4 entails that for almost every $y \in G/\Lambda_0$,

$$\limsup \mu^{\star n} \star \delta_y(\pi_0(K)) \leq \lambda_0(\pi_0(K)) < \varepsilon$$

However, given $x \in G/\Lambda$, we have $\mu^{\star n} \star \delta_x(K) \leq \mu^{\star n} \star \delta_{p_0(x)}(p_0(K))$. We infer that for almost every $x \in G/\Lambda$, we have $\limsup \mu^{\star n} \star \delta_x(K) \leq \varepsilon$, which concludes the proof. □

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