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Transience of a symmetric random walk in infinite measure

Timothée Bénard

Abstract

We consider a random walk on a second countable locally compact topological space endowed with an invariant Radon measure. We show that if the walk is symmetric and if every subset which is invariant by the walk has zero or infinite measure, then one has transience in law for almost every starting point. We then deduce a converse to Eskin-Margulis recurrence theorem.

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Introduction

The starting point of this text is an article published by A. Eskin and G. Margulis in 2004 which studies the recurrence properties of random walks on homogenous spaces [6]. Let G be a real Lie group, $\Lambda \subseteq G$ a discrete subgroup, and denote by $X = G/\Lambda$ the quotient space. A probability measure μ on G induces a random walk on X , with transition probabilities $(\mu \star \delta_x)_{x \in X}$. The two authors ask about the position of the walk at times n for large values of n . They manage to show a surprising result : if G is a simple real algebraic group and if Λ has finite covolume in G , then for every starting point $x \in X$, the sequence of probabilities of position $(\mu^{\star n} \star \delta_x)_{n \geq 0}$ has all its weak- \star limits of mass 1. One says there is no escape of mass. This reminds the behavior of the unipotent flow highlighted by S.G. Dani and Margulis [4, 7], who prove that the trajectories of a unipotent flow on X spend most of their time inside compact sets of X . Eskin-Margulis' result is actually the starting point of a fruitful analogy leading to the classification of stationary probability measures on X , thanks to the work of Y. Benoist and J-F. Quint [2, 3], followed by Eskin and E. Lindenstrauss [5]. They show that the Cesaro averages $n^{-1} \sum_{k=0}^{n-1} \mu^{\star k} \star \delta_x$ converge for every starting point $x \in X$, with limit either the Haar probability on X or some equiprobability on a finite set. The very limited possible options reflect a phenomenon of rigidity, similar to the one noticed by M. Ratner for the unipotent flow.

This text asks the question of a converse to Eskin-Margulis theorem:

Is the absence of escape of mass characteristic of random walks on homogeneous spaces of finite volume, or could it also happen for walks in infinite volume?

We show that in finite volume and under reasonable conditions, there is always escape of (all the) mass. More precisely, for almost every starting point $x \in X$, the sequence of probabilities of position $(\mu^{\star n} \star \delta_x)_{n \geq 0}$ Cesaro-converges to the zero measure.

Plus, the average can be removed if the walk is symmetric, i.e. if μ is invariant under the inversion map : $G \rightarrow G, g \mapsto g^{-1}$.

Theorem 2.1. *Let G be a semisimple connected real Lie group with finite center, $\Lambda \subseteq G$ a discrete subgroup of infinite covolume in G , and μ a probability on G whose support generates a Zariski-dense subgroup of G .*

Then for almost every $x \in G/\Lambda$, one has the weak- \star convergence :

$$\frac{1}{n} \sum_{k=0}^{n-1} \mu^{\star k} \star \delta_x \xrightarrow{n \rightarrow +\infty} 0 \tag{1}$$

Moreover, if the probability measure μ is symmetric, then the convergence can be strengthened :

$$\mu^{\star n} \star \delta_x \xrightarrow[n \rightarrow +\infty]{} 0 \quad (2)$$

This result of transience in law contrasts strongly with the article [9], in which R. Prohaska studies the properties of punctual recurrence of a random walk on a homogeneous space G/Λ . He notably shows that if the walk has a density¹ and if the space G/Λ carries a G -invariant Radon measure with quadratic growth, then there is almost sure recurrence : for every starting point $x \in G/\Lambda$, almost every trajectory with origin x has a subsequence converging towards x .

The convergence in average (1) of our result 2.1 essentially relies on Chacon-Ornstein theorem and Howe-Moore theorem. The convergence (2) is more delicate. The difficulty comes from the fact that the Markov operator $P_\mu : f \mapsto \int_G f(g \cdot) d\mu(g)$ attached to the walk may have no spectral gap in infinite covolume : its spectral radius on $L^2(G/\Lambda)$ can be equal to 1. We circumvent this problem using a theorem of convergence for backwards martingales and obtain a very general statement, that does not rely on the algebraic frame of homogenous spaces.

Theorem 2.1 is thus a corollary of our Theorem 1.1 :

Theorem 1.1. *Let X be a locally compact second countable topological space equipped with a Radon measure λ , let Γ be a locally compact second countable group acting continuously on X and preserving the measure λ , let μ be a probability measure on Γ whose support generates Γ as a closed group.*

If the probability measure μ is symmetric and if every measurable Γ -invariant part of X has zero or infinite λ -measure, then for λ -almost every starting point $x \in X$, one has the weak- \star convergence :

$$\mu^{\star n} \star \delta_x \xrightarrow[n \rightarrow +\infty]{} 0$$

To put it in a nutshell, a symmetric random walk on an infinite quasi-ergodic space is always transient in law.

¹i.e. is given by a probability measure μ on G which is not singular with the Haar measure on G .

1 A general result for transience in law

1.1 Notations and statement of Theorem 1.1

Let X be a locally compact second countable topological space and λ a Radon measure on X , i.e. a positive measure that is finite on compact subsets of X . Let Γ be a locally compact second countable group acting continuously on X and preserving the measure λ . Let μ be a probability measure on Γ whose support generates Γ as a closed group.

This setting allows to define a random walk on X whose transition probabilities are $(\mu \star \delta_x)_{x \in X}$. In other words, given a starting point $x \in X$ and a measurable subset $A \subseteq X$, the probability to move from x to A in one step is $\mu\{g \in \Gamma, g.x \in A\}$.

We now introduce a few terminologies :

- The probability measure μ is *symmetric* if $\mu = i_*\mu$ where $i : \Gamma \rightarrow \Gamma, g \mapsto g^{-1}$ denotes the inverse map.
- The *Markov operator* P_μ attached to μ acts on the set of non-negative measurable functions on X via the formula

$$P_\mu\varphi(x) := \int_G \varphi(gx)d\mu(g)$$

It extends into a contraction on the spaces $L^p(X, \lambda)$ for every $p \in [1, \infty]$.

- A measurable subset $A \subseteq X$ is Γ -*invariant* if for all $g \in \Gamma$, one has $\lambda(A\Delta gA) = 0$. This is equivalent to say that $P_\mu 1_A = 1_A$ λ -a.e. on X .

Let us check the equivalence stated in the definition of a Γ -invariant subset. The point is to show the converse direction, so we consider a measurable subset $A \subseteq X$ such that $P_\mu 1_A = 1_A$ λ -a.e. on X . This means that for λ -almost every $x \in X$, μ -almost every $g \in G$, one has $1_A(gx) = 1_A(x)$. Fubini theorem then implies that for μ -almost every $g \in \Gamma$, one has $\lambda(A\Delta gA) = 0$. The subgroup $D \subseteq \Gamma$ generated by such elements g is dense in Γ and leaves the set A λ -a.e.-invariant. So we just need to check that the λ -a.e.-invariance is preserved by taking limits. Let $g \in \Gamma$, $(g_n) \in D^{\mathbb{N}}$ such that $g_n \rightarrow g$, let $\varphi \in C_c^0(X)$. By dominated convergence,

$$\int_{g_n A} \varphi d\lambda - \int_{g A} \varphi d\lambda = \int_A \varphi(g_n \cdot) - \varphi(g \cdot) d\lambda \xrightarrow{n \rightarrow +\infty} 0$$

We deduce that $\int_A \varphi d\lambda = \int_{g A} \varphi d\lambda$. As this is true for every $\varphi \in C_c^0(X)$, one concludes that $\lambda(A\Delta gA) = 0$.

We now state the main result of this text, expressing that a symmetric random walk on an infinite “quasi-ergodic” space is transient in law. The proof is given in section 1.3.

Theorem 1.1. *Let X be a locally compact second countable topological space equipped with a Radon measure λ , let Γ be a locally compact second countable group acting continuously on X and preserving the measure λ , let μ be a probability measure on Γ whose support generates Γ as a closed group.*

If the probability measure μ is symmetric and if every measurable Γ -invariant part of X has zero or infinite λ -measure, then for λ -almost every starting point $x \in X$, one has the weak- \star convergence :

$$\mu^{\star n} \star \delta_x \xrightarrow[n \rightarrow +\infty]{} 0$$

Remarks.

1) The assumption on the measure λ implies that the Γ -orbit of λ -almost every point of X is unbounded in X . There is equivalence if the action of Γ preserves a distance inducing the topology on X .

2) Without the assumption of symmetry, the proof gives the convergence in average

$$\frac{1}{n} \sum_{k=0}^{n-1} \mu^{\star k} \star \delta_x \xrightarrow[n \rightarrow +\infty]{} 0$$

We can not hope for the convergence of probabilities $(\mu^{\star n} \star \delta_x)_{n \geq 0}$ if we remove the hypothesis of symmetry. For example, let us consider $S_{\mathbb{Z}}$ a \mathbb{Z} -cover of a hyperbolic compact surface. One can realize its unitary bundle $T^1 S_{\mathbb{Z}}$ as a homogeneous space G/Λ where $G = SL_2(\mathbb{R})$ and $\Lambda \subseteq G$ is a discrete subgroup.

Set $\mu = \delta_{u_1}$ where $u_1 := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The μ -walk on G/Λ is now a deterministic process that corresponds to a discretized horocycle flow on $T^1 S_{\mathbb{Z}}$. One can check that every subset of $T^1 S_{\mathbb{Z}}$ which is invariant under the walk has zero or infinite measure (direct consequence of Howe-Moore theorem, see proof of Theorem 2.1). However, the walk is almost everywhere recurrent [1], so we can not have the convergence $\mu^{\star n} \star \delta_x \rightarrow 0$ for almost every x .

We now summarize the proof of Theorem 1.1. The key point is to show that for λ -almost every starting point $x \in X$, the sequence of position probabilities $(\mu^{\star 2n} \star \delta_x)_{n \geq 0}$ has a weak- \star limit (which is a measure on X of mass less or equal than 1). We obtain this using the symmetry of μ and a theorem of convergence for backwards martingales, extended beforehand to a context where measures are σ -finite. Once the convergence of $(\mu^{\star 2n} \star \delta_x)_{n \geq 0}$ is established, we just need to check that the sequence of Cesaro averages $(\frac{1}{n} \sum_{k=0}^{n-1} \mu^{\star k} \star \delta_x)_{n \geq 1}$ weakly converges to the zero measure. This comes from Chacon-Ornstein theorem and does not use the symmetry assumption on the probability measure μ .

1.2 Backwards martingales

This section extends the convergence theorem for backwards martingales, well known for probability spaces (“Lévy’s downward theorem”, [11], section 14.4), to the case of σ -finite measure spaces. We will use it to prove Theorem 1.6.

First, let us recall the definition of conditional expectation.

Definition 1.2 (Conditional expectation). *Let (Ω, \mathcal{F}) be a measurable space, \mathcal{Q} a sub- σ -algebra of \mathcal{F} , and m a positive measure on (Ω, \mathcal{F}) whose restriction $m|_{\mathcal{Q}}$ is σ -finite. Then, for every function $f \in L^1(\Omega, \mathcal{F}, m)$, there exists a unique function $f' \in L^1(\Omega, \mathcal{Q}, m)$ such that for all \mathcal{Q} -measurable subset $A \in \mathcal{Q}$, one has $m(f 1_A) = m(f' 1_A)$. We denote this function by $\mathbb{E}_m(f|\mathcal{Q})$.*

We show the following.

Theorem 1.3 (Convergence a.s. of backwards martingales). *Let (Ω, \mathcal{F}, m) be a measured space, $(\mathcal{Q}_n)_{n \geq 0}$ a decreasing sequence of sub- σ -algebras of \mathcal{F} such that for all $n \geq 0$, the restriction $m|_{\mathcal{Q}_n}$ is σ -finite. Then, for any function $f \in L^1(\Omega, \mathcal{F}, m)$, there exists $\psi \in L^1(\Omega, \mathcal{F}, m)$ such that we have the almost sure convergence :*

$$\mathbb{E}_m(f|\mathcal{Q}_n) \xrightarrow{n \rightarrow +\infty} \psi \quad (m\text{-a.e.})$$

Remark. If the measure m is σ -finite with respect to the tail-algebra $\mathcal{Q}_\infty := \bigcap_{n \geq 0} \mathcal{Q}_n$, then Theorem 1.3 can be deduced from the probabilistic case (by restriction to \mathcal{Q}_∞ -measurable domains of finite measure), and we can precise that $\psi = \mathbb{E}_m(f|\mathcal{Q}_\infty)$. On the extreme opposite, if the tail algebra \mathcal{Q}_∞ does not contain any subset of Ω with m -measure in $]0, +\infty[$, then, the integrability of ψ implies that $\psi = 0$.

The key point is lemma 1.4, that allows to control the ratio of two conditional expectations in infinite measure by some conditional expectation in finite measure.

Lemma 1.4. *Let (Ω, \mathcal{F}, m) be a measured space and $\mathcal{Q} \subseteq \mathcal{F}$ be a sub- σ -algebra such that $m|_{\mathcal{Q}}$ is σ -finite. Let $f, h \in L^1(\Omega, \mathcal{F}, m)$ be integrable functions with $h > 0$. Then one has almost sure equality :*

$$\mathbb{E}_m(f|\mathcal{Q}) = \mathbb{E}_m(h|\mathcal{Q}) \mathbb{E}_{hm}\left(\frac{f}{h}|\mathcal{Q}\right) \quad (m\text{-a.e.})$$

Proof. One can assume $f \geq 0$. Denote

$$\varphi = \mathbb{E}_m(f|\mathcal{Q}), \quad \psi = \mathbb{E}_m(h|\mathcal{Q}) \mathbb{E}_{hm}\left(\frac{f}{h}|\mathcal{Q}\right)$$

These are two non-negative \mathcal{Q} -measurable functions on Ω and we want to show they coincide almost everywhere. As the measure $m|_{\mathcal{Q}}$ is σ -finite, it is enough to check they have the same integral on every \mathcal{Q} -measurable subset. Let $A \in \mathcal{Q}$.

$$\int_A \psi \, dm = \int_A h \mathbb{E}_{hm}\left(\frac{f}{h} \mid \mathcal{Q}\right) \, dm = \int_A \frac{f}{h} \, dhm = \int_A \varphi \, dm$$

□

Lemma 1.5. *Keep the setting of Theorem 1.3 and assume moreover that $m(\Omega) = +\infty$. Let $f \in L^1(\Omega, \mathcal{F}, m)$ be an integrable function with $f \geq 0$. Then for every $\varepsilon > 0$,*

$$m\{\limsup \mathbb{E}_m(f \mid \mathcal{Q}_n) < \varepsilon\} > 0$$

Proof. We fix a function $h : \Omega \rightarrow]0, +\infty[$ which is measurable, positive, bounded and satisfies $m(h) = 1$ (and will be adjusted later). Lemma 1.4 allows to write for $n \geq 0$:

$$\mathbb{E}_m(f \mid \mathcal{Q}_n) = \mathbb{E}_m(h \mid \mathcal{Q}_n) \mathbb{E}_{hm}\left(\frac{f}{h} \mid \mathcal{Q}_n\right) \quad (m\text{-a.e.})$$

On the one hand, $\mathbb{E}_m(h \mid \mathcal{Q}_n) \leq \|h\|_{\infty}$ m -a.e.

On the other hand, the convergence theorem for backwards martingales in finite measure ([11], 14.4) asserts that

$$\mathbb{E}_{hm}\left(\frac{f}{h} \mid \mathcal{Q}_n\right) \rightarrow \mathbb{E}_{hm}\left(\frac{f}{h} \mid \mathcal{Q}_{\infty}\right) \quad (m\text{-a.e.})$$

This last function is non-negative and satisfies $\int_{\Omega} \mathbb{E}_{hm}\left(\frac{f}{h} \mid \mathcal{Q}_{\infty}\right) \, dhm = \|f\|_{L^1(m)}$, giving the inequality

$$hm\{\mathbb{E}_{hm}\left(\frac{f}{h} \mid \mathcal{Q}_{\infty}\right) \leq 2\|f\|_{L^1(m)}\} \geq \frac{1}{2}$$

Combining the above observations, we proved the existence of some non-zero measure subset of Ω on which

$$\limsup \mathbb{E}_m(f \mid \mathcal{Q}_n) \leq 2\|h\|_{\infty}\|f\|_{L^1(m)} \quad (*)$$

Let $\varepsilon > 0$. As $m(\Omega) = +\infty$, one can choose h from the start such that $\|h\|_{\infty} < \frac{\varepsilon}{2\|f\|_{L^1(m)}}$ and inequality (*) then gives the expected result.

□

We conclude section 1.2 proving the convergence theorem for backwards martingales in infinite measure.

Proof of Theorem 1.3. One can assume $f \geq 0$. Fix $\varepsilon > 0$ and denote

$$\Omega' := \{\omega \in \Omega, \limsup_{n \rightarrow +\infty} \mathbb{E}_m(f | \mathcal{Q}_n)(\omega) - \liminf_{n \rightarrow +\infty} \mathbb{E}_m(f | \mathcal{Q}_n)(\omega) > \varepsilon\}$$

We need to show that $m(\Omega') = 0$. Argue by contradiction assuming $m(\Omega') > 0$. As Ω' is \mathcal{Q}_∞ -measurable, one has for all $n \geq 0$ the equality

$$\mathbb{E}_m(f | \mathcal{Q}_n)_{|\Omega'} = \mathbb{E}_{m_{|\Omega'}}(f_{|\Omega'} | \mathcal{Q}_{n|\Omega'}) \quad (m_{|\Omega'}\text{-a.e.})$$

The convergence theorem for backwards martingales in finite measure and the definition of Ω' imply that $m(\Omega') = +\infty$. Applying lemma 1.5 to the restrictions $\Omega', m_{|\Omega'}, (\mathcal{Q}_{n|\Omega'})_{n \geq 0}, f_{|\Omega'}$, one deduces that

$$m_{|\Omega'}\{\limsup \mathbb{E}_m(f | \mathcal{Q}_n) < \varepsilon\} > 0$$

This contradicts the definition of Ω' or the non-negativeness of f , which together lead to $\limsup \mathbb{E}_m(f | \mathcal{Q}_n) > \varepsilon$ $m_{|\Omega'}$ -almost everywhere. \square

1.3 Convergence of back-and-forths and proof of Theorem 1.1

The goal of this section is to prove Theorem 1.1 on the transience in law of a random walk in infinite measure. The proof will use an extension of Rota's "alternierende verfahren" theorem (see [10]) to the case of random walks in infinite measure (Theorem 1.6).

Given a probability measure μ on a group Γ , we shall denote by $\check{\mu} := i_*\mu$ the image of μ under the inverse map $i : \Gamma \rightarrow \Gamma, g \mapsto g^{-1}$.

Theorem 1.6 (Convergence of back-and-forths). *Let X be a locally compact second countable topological space equipped with a Radon measure λ , let Γ be a locally compact second countable group acting continuously on X and preserving the measure λ , and let μ be a probability measure on Γ .*

There exists a family of finite measures $(\nu_x)_{x \in X} \in \mathcal{M}^f(X)^X$ such that for λ -almost every $x \in X$, one has the weak- \star convergence :

$$(\mu^{*n} \star \check{\mu}^{*n}) \star \delta_x \xrightarrow[n \rightarrow +\infty]{} \nu_x$$

Proof. The proof is inspired by the phenomenon of equidistribution of fibres observed by Benoist-Quint in [3]. Denote

$$B := \Gamma^{\mathbb{N}^*}, \quad \beta := \mu^{\mathbb{N}^*}, \quad T : B \rightarrow B, (b_i)_{i \geq 1} \mapsto (b_{i+1})_{i \geq 1}$$

the one-sided shift. One introduces a σ -finite fibred dynamical system (B^X, β^X, T^X) setting

- $B^X := B \times X$
- $\beta^X := \beta \otimes \lambda \in \mathcal{M}^{\text{Rad}}(B \times X)$

- $T^X : B^X \rightarrow B^X, (b, x) \mapsto (Tb, b_1^{-1}x)$.

Let \mathcal{B} and \mathcal{X} denote the borel σ -algebras of B and X . The borel σ -algebra of B^X is then the product algebra $\mathcal{B} \otimes \mathcal{X}$. For all $n \geq 0$, define the *sub- σ -algebra of the n -fibres* of T^X by setting

$$\mathcal{Q}_n := (T^X)^{-n}(\mathcal{B} \otimes \mathcal{X})$$

It is a sub- σ -algebra of $\mathcal{B} \otimes \mathcal{X}$ such that for all $c \in B^X$, the smallest \mathcal{Q}_n -measurable subset of B^X containing c is the n -fibre $(T^X)^{-n}(T^X)^n(c)$. The restriction $\beta^X|_{\mathcal{Q}_n}$ is a σ -finite measure because β^X is σ -finite with respect to the σ -algebra $\mathcal{B} \otimes \mathcal{X}$ and is preserved by T^X .

As a first step, we will fix a continuous function with compact support $f \in C_c^0(X)$ and show that the sequence $(\mu^{*n} \star \check{\mu}^{*n} \star \delta_x)(f)_{n \geq 0}$ converges in \mathbb{R} for λ -almost every x . To this end, we express $(\mu^{*n} \star \check{\mu}^{*n} \star \delta_x)(f)$ using a conditional expectation and we apply Theorem 1.3. Denote

$$\tilde{f} : B^X \rightarrow \mathbb{R}, (b, x) \mapsto f(x), \quad \varphi_n := \mathbb{E}_{\beta^X}(\tilde{f} | \mathcal{Q}_n) \in L^1(B^X, \mathcal{Q}_n)$$

We first give an explicit formula for the function φ_n . Intuitively, given a point $c = (b, x) \in B^X$, the value $\varphi_n(c)$ stands for the mean value of \tilde{f} on the smallest \mathcal{Q}_n -measurable subset of B^X containing c . By definition, this subset is the n -fibre going through c and is identified with the product Γ^n under the bijection

$$h_{n,c} : \Gamma^n \rightarrow (T^X)^{-n}(T^X)^n(c), \quad a = (a_1, \dots, a_n) \rightarrow (aT^n b, a_1 \dots a_n b_n^{-1} \dots b_1^{-1} x)$$

The following lemma asserts that $\varphi_n(c)$ is nothing else than the mean value of \tilde{f} on $(T^X)^{-n}(T^X)^n(c) \equiv \Gamma^n$ with respect to the measure $\mu^{\otimes n}$.

Lemma 1.7. *Let $n \geq 0$. For β^X -almost every $(b, x) \in B^X$, one has*

$$\varphi_n(b, x) = \int_{\Gamma^n} f(a_1 \dots a_n b_n^{-1} \dots b_1^{-1} x) d\mu^{\otimes n}(a)$$

Proof of lemma 1.7. This result is extracted from [3] (lemma 3.3). We recall the proof. Up to considering separately the positive and negative parts of f , one may assume $f \geq 0$. Denote by $\varphi'_n : B^X \rightarrow [0, +\infty]$ the map defined by the right-hand side of the above equation. We show it coincides almost everywhere with φ_n by proving it also satisfies the axioms for the conditional expectation characterizing φ_n .

As the value φ'_n at a point $c \in B^X$ only depends on $(T^X)^n(c)$, the map φ'_n is \mathcal{Q}_n -measurable. It remains to show that for every $A \in \mathcal{Q}_n$, one has the equality $\beta^X(1_A \tilde{f}) = \beta^X(1_A \varphi'_n)$. Writing A as $A = (T^X)^{-n}(E)$ where $E \in \mathcal{B} \otimes \mathcal{X}$ and remembering the measure λ is preserved by Γ , one computes that :

$$\begin{aligned}
\beta^X(1_A \varphi'_n) &= \int_{B \times X \times \Gamma^n} 1_A(b, x) f(a_1 \dots a_n b_n^{-1} \dots b_1^{-1} x) d\mu^{\otimes n}(a) d\beta(b) d\lambda(x) \\
&= \int_{B \times X \times \Gamma^n} 1_E(T^n b, b_n^{-1} \dots b_1^{-1} x) f(a_1 \dots a_n b_n^{-1} \dots b_1^{-1} x) d\mu^{\otimes n}(a) d\beta(b) d\lambda(x) \\
&= \int_{B \times X \times \Gamma^n} 1_E(T^n b, x) f(a_1 \dots a_n x) d\mu^{\otimes n}(a) d\beta(b) d\lambda(x) \\
&= \int_{B \times X} 1_E(T^n b, x) f(b_1 \dots b_n x) d\beta(b) d\lambda(x) \\
&= \int_{B \times X} 1_E(T^n b, b_n^{-1} \dots b_1^{-1} x) f(x) d\beta(b) d\lambda(x) \\
&= \beta^X(1_A \tilde{f})
\end{aligned}$$

which concludes the proof of lemma 1.7. \square

Lemma 1.7 implies that for λ -almost every $x \in X$,

$$\int_B \varphi_n(b, x) d\beta(b) = (\mu^{\star n} \star \check{\mu}^{\star n} \star \delta_x)(f) \quad (**)$$

But our Theorem 1.3 on convergence of backwards martingales asserts the sequence of conditional expectations $(\varphi_n)_{n \geq 0}$ converges β^X -almost-surely. Noticing that $\|\varphi_n\|_\infty \leq \|f\|_\infty$, the dominated convergence theorem and equation (**) imply that for λ -almost every $x \in X$, the sequence

$$((\mu^{\star n} \star \check{\mu}^{\star n} \star \delta_x)(f))_{n \geq 0}$$

has a limit in \mathbb{R} .

We deduce from the previous paragraph that for λ -almost every $x \in X$, the sequence of probability measures $(\mu^{\star n} \star \check{\mu}^{\star n} \star \delta_x)_{n \geq 0}$ has a weak- \star limit (which is a measure on X whose mass is less or equal to one, and possibly null). It is indeed a standard argument, that uses the separability of the space of continuous functions with compact support on X equipped with the supremum norm $(C_c^0(X), \|\cdot\|_\infty)$, and the representation of non negative linear forms on $C_c^0(X)$ by Radon measures (Riesz theorem). This concludes the proof of Theorem 1.6. \square

We finish the section with the proof that a symmetric random walk on an infinite quasi-ergodic space is always transient in law. Recall first the precise statement.

Theorem 1.1. *Let X be a locally compact second countable topological space equipped with a Radon measure λ , let Γ be a locally compact second countable group acting continuously on X and preserving the measure λ , let μ be a probability measure on Γ whose support generates Γ as a closed group.*

If the probability measure μ is symmetric and if every measurable Γ -invariant part of X has zero or infinite λ -measure, then for λ -almost every starting point $x \in X$, one has the weak- \star convergence :

$$\mu^{\star n} \star \delta_x \xrightarrow{n \rightarrow +\infty} 0$$

Proof of Theorem 1.1. We just need to prove that for λ -almost every $x \in X$, one has the convergence $\mu^{\star 2n} \star \delta_x \rightarrow 0$. According to Theorem 1.6 and the symmetry of μ , the sequence $(\mu^{\star 2n} \star \delta_x)_{n \geq 0}$ converges to a finite measure, so it is enough to check the following convergence in average : for λ -almost every $x \in X$,

$$\frac{1}{n} \sum_{k=0}^{n-1} \mu^{\star k} \star \delta_x \rightarrow 0$$

As announced in section 1.1, we show this last convergence *without using the assumption of symmetry on μ* . We need to check that for every non-negative continuous function with compact support $\varphi \in C_c^0(X)^+$,

$$\frac{1}{n} \sum_{k=0}^{n-1} P_\mu^k \varphi \rightarrow 0 \quad (\lambda\text{-a.e.})$$

where P_μ denotes the Markov operator of the walk (see section 1.1).

Chacon-Ornstein theorem (cf [8]) implies that the sequence of functions $(\frac{1}{n} \sum_{k=0}^{n-1} P_\mu^k \varphi)_{n \geq 1}$ converges almost-surely to some function $\psi : X \rightarrow \mathbb{R}_+$. As the functions $P_\mu^k \varphi$ are uniformly bounded in $L^2(X, \lambda)$, Fatou lemma implies that ψ is λ -integrable. Furthermore, the function φ being bounded, the dominated convergence theorem applied to the probability space (Γ, μ) gives the P_μ -invariance

$$P_\mu \psi = \psi \quad (\lambda\text{-a.e.})$$

We now infer that ψ is Γ -invariant, meaning that for $g \in \Gamma$, one has the equality $\psi \circ g = \psi$ λ -a.e. on X . To this end, observe that the P_μ -invariance of ψ expresses ψ as a barycenter of translates $\psi \circ g$:

$$\int_{\Gamma} \psi \circ g \, d\mu(g) = \psi \quad (\lambda\text{-a.e.})$$

But the functions $\psi \circ g$ all are in $L^2(X, \lambda)$ and have the same norm as ψ . The strict convexity of balls in a Hilbert space then gives for μ -almost every $g \in \Gamma$, the equality $\psi \circ g = \psi$ λ -almost everywhere. As the support of μ generates Γ

as a closed subgroup, we infer by a method already discussed in section 1.1, that for all $g \in \Gamma$, one has $\psi \circ g = \psi$ λ -a.e., which is the Γ -invariance announced above.

The Γ -invariance of ψ implies that for every constant $c > 0$, the set $\{\psi > c\}$ is Γ -invariant, so has zero or infinite λ -measure by hypothesis. As ψ^2 is integrable, we must have $\lambda\{\psi > c\} = 0$. Finally, we get that $\psi = 0$ λ -almost everywhere, which finishes the proof. □

2 Application : A converse to Eskin-Margulis recurrence theorem

2.1 Statement of Theorem 2.1

We use Theorem 1.1 to show that a Zariski-dense symmetric random walk on a homogeneous space G/Λ of infinite volume is transient in law (Theorem 2.1). The difficulty lies in the fact that the Markov operator P_μ of the walk may not have a spectral gap, meaning its action on $L^2(G/\Lambda)$ may have a spectral radius equal to 1. Our result can be seen as a converse to Eskin-Margulis recurrence theorem stating that a “Zariski-dense random walk on a homogeneous space with finite volume is uniformly recurrent in law”.

Theorem (Eskin-Margulis, 2004, [6]). *Let G be a semisimple connected real Lie group with finite center, $\Lambda \subseteq G$ a lattice, μ a probability measure on G whose support is compact and generates a Zariski-dense subgroup of G .*

Then the random walk on G/Λ induced by μ is uniformly recurrent in law.

We explain the above terminology :

A subgroup of G is *Zariski-dense* if it is dense from an algebraic point of view. More formally, denote by \mathfrak{g} the Lie algebra of G , and $\text{Ad} : G \rightarrow \text{Aut}(\mathfrak{g}), g \mapsto T_e(g.g^{-1})$ its adjoint representation. A subgroup $\Gamma \subseteq G$ is Zariski-dense if every polynomial function on the space of endomorphisms of \mathfrak{g} that vanishes on $\text{Ad}(\Gamma)$ also vanishes on $\text{Ad}(G)$.

The *uniform recurrence in law* means that if K is some big enough compact subset of G/Λ , then for every starting point $x \in G/\Lambda$, the μ -walk starting from x will have a very strong probability to be in K at time n , as long as n is large enough. More formally, we ask that for every $\varepsilon > 0$, there exists a compact subset $K \subseteq G/\Lambda$ such that for every $x \in G/\Lambda$, there exists a rank $N_x \geq 0$ such that if $n \geq N_x$ then $\mu^{*n} \star \delta_x(K) \geq 1 - \varepsilon$.

We show the following converse to Eskin-Margulis theorem :

Theorem 2.1. *Let G be a semisimple connected real Lie group with finite center, $\Lambda \subseteq G$ a discrete subgroup of infinite covolume in G , and μ a probability on G whose support generates a Zariski-dense subgroup of G .*

Then for almost every $x \in G/\Lambda$, one has the weak- \star convergence :

$$\frac{1}{n} \sum_{k=0}^{n-1} \mu^{\star k} \star \delta_x \xrightarrow{n \rightarrow +\infty} 0 \quad (1)$$

Plus, if the probability measure μ is symmetric, then the convergence can be strengthened :

$$\mu^{\star n} \star \delta_x \xrightarrow{n \rightarrow +\infty} 0 \quad (2)$$

Remarks. 1) Theorem 2.1 is still true under the more general assumption that the subgroup generated by the support of μ has unbounded projections in the non-compact factors of G (see the proof).

2) Whether or not μ is symmetric, we get for almost every $x \in G/\Lambda$ the existence of an extraction $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$\mu^{\star \sigma(n)} \star \delta_x \xrightarrow{n \rightarrow +\infty} 0$$

Hence, the phenomenon observed by Eskin and Margulis can not happen in a context of infinite measure.

3) Theorem 2.1 describes the asymptotic behavior of probabilities of position for *almost every* starting point $x \in G/\Lambda$. One may not hope for transience in law for every starting point as it is possible that the orbit $\Gamma.x$ is finite.

4) It is reasonable to believe that convergence (2) could hold without assumption of symmetry on μ .

2.2 Proof of Theorem 2.1

Theorem 2.1 is a consequence of Theorem 1.1 and the remark 2) that follows. To apply these, we need to check that the Haar measure on G/Λ gives zero or infinite mass to any subset invariant by the walk. This will be a consequence of Howe-Moore theorem. First recall its statement (cf. [12], Theorem 2.2.20) :

Theorem (Howe-Moore). *Let G be a semisimple connected real Lie group with finite center, and π a continuous morphism from G to the unitary group of a separable Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. Assume that every factor G_i of G has a trivial set of fixed points, i.e. $\mathcal{H}^{G_i} := \{x \in \mathcal{H}, G_i.x = x\}$ is $\{0\}$.*

Then for every $v, w \in \mathcal{H}$, one has

$$\langle \pi(g).v, w \rangle \xrightarrow{g \rightarrow \infty} 0$$

In the above statement, the notation $g \rightarrow \infty$ means that g leaves every compact subset of G . Also recall the definition of *factors* of G we will use in the sequel. Denote by \mathfrak{g} the Lie algebra of G . It can be uniquely decomposed as a direct sum of simple ideals : $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$. The factors of G are the immersed connected subgroups G_1, \dots, G_s of G whose Lie algebras are $\mathfrak{g}_1, \dots, \mathfrak{g}_s$. They are closed in G and commute mutually : for $i \neq j \in \{1, \dots, s\}$ and $g_i \in G_i, g_j \in G_j$ one has $g_i g_j = g_j g_i$. Lastly, the product map $\pi : G_1 \times \cdots \times G_s \rightarrow G, (g_1, \dots, g_s) \mapsto g_1 \dots g_s$ is a morphism of groups which is onto and has finite kernel. In the sequel, we will say that a subgroup $\Gamma \subseteq G$ has unbounded projections in the factors of G if for every $i \in \{1, \dots, s\}$, the projection of $\pi^{-1}(\Gamma) \subseteq G_1 \times \cdots \times G_s$ in G_i is unbounded

Howe-Moore theorem implies a lemma of rigidity.

Lemma 2.2. *Let G be a semisimple connected real Lie group with finite center, $\Gamma \subseteq G$ a subgroup with unbounded projections in the factors of G . Let (\mathcal{H}, ρ) be a unitary representation of G on a separable Hilbert space.*

$$\text{If } \mathcal{H}^G = \{0\} \text{ then } \mathcal{H}^\Gamma = \{0\}$$

Proof of lemma 2.2. Denote by G_1, \dots, G_s the factors of G . Up to pulling back the representation of G by the product map $\pi : G_1 \times \cdots \times G_s \rightarrow G, (g_1, \dots, g_s) \mapsto g_1 \dots g_s$, one may assume that G is a direct product of quasi-simple² connected real Lie groups with finite center $G = G_1 \times \cdots \times G_s$. Assume $s = 2$. The hypothesis $\mathcal{H}^G = \{0\}$ implies that $\mathcal{H}^{G_1} \cap \mathcal{H}^{G_2} = \{0\}$. Thus, we can decompose

$$\mathcal{H} = \mathcal{H}^{G_1} \oplus \mathcal{H}^{G_2} \oplus \mathcal{H}'$$

where \mathcal{H}' is the orthogonal of $\mathcal{H}^{G_1} \oplus \mathcal{H}^{G_2}$ in \mathcal{H} . Moreover, each subspace is invariant by G . Let $v \in \mathcal{H}$ be a Γ -invariant vector. Decompose v as $v = v_1 + v_2 + v'$ with $v_i \in \mathcal{H}^{G_i}, v' \in \mathcal{H}'$. The representation of G on \mathcal{H} leads to a unitary representation of G_2 on \mathcal{H}^{G_1} and the Γ invariance of v implies that v_1 is invariant under $p_2(\Gamma)$, projection of Γ on the factor G_2 . As $p_2(\Gamma)$ is unbounded in G_2 , one can apply Howe-Moore theorem to obtain $v_1 = 0$. In the same way $v_2 = 0$. Thus $v = v' \in \mathcal{H}'$. The representations of G_1 and G_2 induced by G on \mathcal{H}' have no non-trivial fix point. Hence, we can apply Howe-Moore theorem one more time to infer that $v' = 0$. Finally, $\mathcal{H}^\Gamma = \{0\}$.

For the general case where $s \geq 1$, argue by induction on s using the previous method and the decomposition of \mathcal{H} as $\mathcal{H}^{G_1 \times \cdots \times G_{s-1}} \oplus \mathcal{H}^{G_s} \oplus \mathcal{H}'$. \square

We infer from the last lemma that for a group G with no compact factor, the Haar measure on G/Λ is “quasi-ergodic”.

²A real Lie group is said to be quasi-simple if its Lie algebra is simple.

Lemma 2.3. *Keep the setting of Theorem 2.1 and assume the group G has no compact factor. Let Γ be the smallest closed subgroup of G that contains the support of μ , and λ a Haar measure on G/Λ .*

Then every Γ -invariant subset of G/Λ has zero or infinite λ -measure.

Remark. In finite volume, the action of Γ on the homogeneous space G/Λ has *no reason to be ergodic* for the Haar measure. This is obvious if Λ is the trivial subgroup, but we can also construct examples where Λ is Zariski-dense. To this end, denote by \mathbb{D} the Poincaré disk, set $G = SL_2(\mathbb{R}) \equiv \text{Isom}^+(\mathbb{D}) \equiv T^1\mathbb{D}$, and consider a Schottky subgroup $S_0 \subseteq G$ whose limit set \mathcal{L}_0 on the boundary of \mathbb{D} is contained under four geodesic arcs, which are disjoint and small enough. Set $\Gamma = \Lambda = S_0$. For some non-zero measure subset of unitary vectors $x \in T^1\mathbb{D}$, the set $x\Lambda \cap \partial\mathbb{D} = x\mathcal{L}_0$ does not intersect the limit set \mathcal{L}_0 of Γ . Given such an x and looking in the quotient space, the orbital map $\Lambda \rightarrow \Gamma \backslash G, g \mapsto \Gamma xg$ is proper, so its image can not be dense. Thus, the right action of Λ on $\Gamma \backslash G$ is not ergodic, or equivalently, the left action of Γ on G/Λ is not ergodic.

Proof of lemma 2.3. Argue by contradiction assuming the existence of some Γ -invariant measurable subset $A \subseteq G/\Lambda$ such that $\lambda(A) \in]0, +\infty[$. Consider the regular unitary representation of G on $L^2(G/\Lambda)$, given by the formula $g.f = f(g^{-1}\cdot)$. The characteristic function $1_A \in L^2(G/\Lambda)$ is a non-zero fix point for the action of Γ . As G has no compact factor, lemma 2.2 and the Zariski-density of μ imply there exists a non-zero fix point $\varphi \in L^2(G/\Lambda)$ for the action of G . Such a function is λ -a.e. constant, implying that λ has finite mass. Absurd. □

We conclude by the proof of Theorem 2.1, converse to Eskin-Margulis recurrence theorem.

Proof of Theorem 2.1. Assume first that the group G has no compact factor. If the probability measure μ is symmetric, then convergence (2) comes from lemma 2.3 and Theorem 1.1. If there is no assumption of symmetry, we still get the convergence in Cesaro average (1) via the remark 2) following Theorem 1.1.

We now explain how to reduce Theorem 2.1 to the case where G has no compact factor. Denote by G_1, \dots, G_s the factors of G , and π the induced finite cover of G , i.e. $\pi : G_1 \times \dots \times G_s \rightarrow G, (g_1, \dots, g_s) \mapsto g_1 \dots g_s$. There exists a probability measure $\tilde{\mu}$ on $\prod_{i=1}^s G_i$ whose support is $\pi^{-1}(\text{supp } \mu)$ and such that the $\tilde{\mu}$ -walk on $\prod_{i=1}^s G_i / \pi^{-1}(\Lambda)$ lifts the μ -walk on G/Λ . It is enough to show our result of transience for this $\tilde{\mu}$ -walk. Denote by G_1, \dots, G_k the non compact factors of G and $p : \prod_{i=1}^s G_i \rightarrow \prod_{i=1}^k G_i, (g_i)_{i \leq s} \mapsto (g_i)_{i \leq k}$ the projection on their product (notice that $k \geq 1$ otherwise G would have a

discrete subgroup of infinite covolume). Then the projection $p(\pi^{-1}(\Lambda))$ is a discrete subgroup of infinite covolume in $\prod_{i=1}^k G_i$. It is enough to prove our result of transience for the image $p_*\tilde{\mu}$ on $\prod_{i=1}^k G_i$. But this probability measure is Zariski-dense. Hence, we have reduced Theorem 2.1 to the case of a group with no compact factor, which finishes the proof. □

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