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Abstract: In this paper, we analyze the longtime behavior of the wave equation with local Kelvin-Voigt Damping. Through introducing proper class symbol and pseudo-diff-calculus, we obtain a Carleman estimate, and then establish an estimate on the corresponding resolvent operator. As a result, we show the logarithmic decay rate for energy of the system without any geometric assumption on the subdomain on which the damping is effective.

Keywords: Carleman estimate; wave equation; Kelvin-Voigt damping; logarithmic stability

MSC: Primary 93B05; Secondary 93B07 and 35B37

1. Introduction

In this paper, a wave equation with local Kelvin-Voigt damping is considered. More precisely, we assume that the wave propagates through two segments consisting of an elastic and a Kelvin-Voigt medium. The latter material is a viscoelastic material having the properties both of elasticity and viscosity. We analyze long time behaviour of energy of solution of the system. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary $\Gamma = \partial\Omega$. Denote by ∂_n the unit outward normal vector on boundary Γ. The PDE model is as follows.

$$\begin{cases} y_{tt}(t,x) - \operatorname{div}\left[\nabla y(t,x) + a(x)\nabla y_t(t,x)\right] = 0 & \text{in} \quad (0,\infty)\times\Omega, \\ y(t,x) = 0 & \text{on} \quad (0,\infty)\times\Gamma, \\ y(0,x) = y^0, \quad y_t(0,x) = y^1 & \text{in} \quad \Omega, \end{cases} \tag{1}$$

where the coefficient function $a(\cdot) \in L^1(\Omega)$ is non-negative and not identically null.

The natural energy of system (1) is

$$E(t) = \frac{1}{2}\Big[\int_\Omega \big(|\nabla y(t)|^2 + |y_t(t)|^2\big)dx\Big]. \tag{2}$$

A direct computation gives that

$$\frac{d}{dt}E(t) = -\int_{\operatorname{supp} a} a(x)\,|\nabla y_t(t)|^2 dx. \tag{3}$$

Formula (3) shows that the only dissipative mechanism acting on the system is the viscoelastic damping $\operatorname{div}[a\nabla y_t]$, which is only effective on $\operatorname{supp} a$.

To rewrite the system as an evolution equation, we set the energy space as

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega), \tag{4}$$

with norm

$$\|Y\|_{\mathcal{H}} = \sqrt{\|\nabla y_1\|^2_{L^2(\Omega)} + \|y_2\|^2_{L^2(\Omega)}}, \quad \forall\, Y = (y_1, y_2) \in \mathcal{H}, \tag{5}$$

Define an unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ by

$$\mathcal{A}Y = \big(y_2, \operatorname{div}(\nabla y_1 + a\nabla y_2)\big), \quad \forall\, Y = (y_1, y_2) \in D(\mathcal{A}),$$

and

$$\mathcal{D}(\mathcal{A}) = \Big\{(y_1, y_2) \in \mathcal{H} \,:\, y_2 \in H_0^1(\Omega), \operatorname{div}(\nabla y_1 + a\nabla y_2)) \in L^2(\Omega)\Big\}.$$

Let $Y(t) = (y(t), y_t(t))$. Then system (1) can be written as

$$\frac{d}{dt}Y(t) = \mathcal{A}Y(t), \quad \forall\, t \geq 0, \quad Y(0) = (y_0, y_1). \tag{6}$$

It is known from Reference [1–3] that if supp a is non-empty, the operator $\mathcal{A}$ generates a contractive C_0 semigroup $e^{t\mathcal{A}}$ on $\mathcal{H}$ and $i\mathbb{R} \subset \rho(\mathcal{A})$, the resolvent of $\mathcal{A}$. Consequently, the semigroup $e^{t\mathcal{A}}$ is strongly stable. Moreover, if the entire medium is of the viscoelastic type (i.e., supp $a = \overline{\Omega}$), the damping for the wave equation not only induces exponential energy decay, but also restricts the spectrum of the associated semigroup generator to a sector in the left half plane, and the associated semigroup is analytic [4]. However, when the Kelvin-Voigt damping is local and the material coefficient $a(\cdot)$ is a positive constant on supp a, the energy of system (1) does not decay exponentially for any geometry of Ω and supp a [1,5]. The reason is that the strong damping leads to the reflection of waves at the interface $\gamma \doteq \partial(\text{supp}\, a) \setminus \Gamma$, which then fails to be effectively damped because they do not enter the region of damping [6–8]. It turns out that the viscoelastic damping does not follow the assumption that the "geometric optics" condition implies exponential stability [9].

On the other hand, it has been proved that the properties of regularity and the stability of the 1-d system (1) depend on the continuousness of coefficient function $a(\cdot)$. More precisely, assume that $\Omega = (-1, 1)$ and $a(x)$ behaviours like x^α with $\alpha > 0$ in supp $a = [0, 1]$. Then the solution of (1) is eventually differentiable for $\alpha > 1$, exponentially stable for $\alpha \geq 1$, polynomially stable of order $\frac{1}{1-\alpha}$ for $0 < \alpha < 1$, and polynomially stable of optimal decay rate 2 for $\alpha = 0$ (see [10–13]).

For the higher dimensional system, the corresponding semigroup is exponentially stable when $a(\cdot) \in \mathscr{C}^2(\Omega)$ and supp $a \supset \Gamma$, polynomially stable of order $\frac{1}{2}$ when $a(\cdot) \equiv a_0 > 0$ on supp a and supp a satisfies certain geometry conditions [14,15]. Then, a natural problem is—how about the decay rate when Ω and supp $a \neq \emptyset$ is arbitrary? In this paper, we analyze the logarithmic decay properties of the solution to (1) and obtain that the decay rate of system energy stays at a rate $1/[\log(t+1)]^{8/5}$. This result was recently improved by Burq [16].

The main result reads as follows.

Theorem 1. *Suppose that the coefficient function $a(\cdot) \in \mathscr{C}_0^\infty(\Omega)$ is non-negative and* supp $a \subset \Omega$ *is non-empty. Then the energy of the solution of (1) decays at logarithmic speed. More precisely, one has that there exists a positive constant C such that*

$$\|e^{t\mathcal{A}}Y_0\|_{\mathcal{H}} \leq \frac{C}{[\log(t+1)]^{4k/5}} \Big\|Y_0\Big\|_{D(\mathcal{A}^k)}, \quad \forall\, t > 0, \;\; Y_0 = (y_0, y_1) \in D(\mathcal{A}^k). \tag{7}$$

We shall prove Theorem 1 through the resolvent estimate [17]. The main idea is to introduce proper operators, class of symbol and pseudo-diff-calculus. Then, a Carleman estimate on the subdomain far away from the boundary will be proven. Combining these with a classical Carleman estimation up to

the boundary, one can obtain the desired Carleman estimate and resolvent estimate. This method was developed in References [18–23] and the references cited therein.

The rest of the paper is organized as follows—we present some preliminaries in Section 2. Sections 3 and 4 are devoted to the proof of the Carleman estimate and the logarithmic stability of system (1). Finally, some proofs of the classic result are given in Appendix A to complete the paper.

Throughout this paper, we use $\|\cdot\|$, $(\cdot\,|\,\cdot)$ to denote the norm and inner product in $L^2(\Omega)$ if there are no comments. When writing $f \lesssim g$ (or $f \gtrsim g$), we mean that there exists a positive constant C such that $f \leq Cg$ (or $f \geq Cg$).

2. Preliminaries

We shall prove Theorem 1 by Weyl-Hörmander calculus, which was introduced Hörmander [24,25]. First, let's define some definitions and results on the class of symbol and pseudo-diff-calculus.

2.1. Symbol and Symbolic Calculus

Let V be a bounded open set in $\mathbb{R}^d$. For any $(x,\xi) \in \overline{V} \times \mathbb{R}^d$, $\lambda \in \mathbb{R}$ and $\tau > 0$, we introduce the metric

$$g = g_{x,\xi} = \lambda dx^2 + \mu^{-2} d\xi^2, \qquad \text{where } \mu^2 = \mu(\tau,\xi)^2 = \tau^2 + |\,\xi\,|^2, \tag{8}$$

and the weight

$$\nu = \nu(x,\lambda) = \sqrt{1+\lambda^2 a(x)^2}. \tag{9}$$

Note that $g_{x,\xi}(X,\Xi) = \lambda\,|\,X\,|^2 + \mu^{-2}(\tau,\xi)\,|\,\Xi\,|^2$ for all $X,\Xi \in \mathbb{R}^d$. Then we have the following results. It's proof is given in Appendix A for the sake of completeness.

Lemma 1. *Assume that there exist positive constants $\mathcal{C}$ and λ_0 such that $\lambda \geq \lambda_0$ and $\tau \geq \max\{\,\mathcal{C}\lambda,\ 1\,\}$. It holds*

(i) *The metric $g = g_{x,\xi}$ defined by (8) is admissible, i.e., it is slowly varying and temperate.*
(ii) *The weight $\nu = \nu(x,\lambda)$ defined by (9) is admissible, that is, it is g-continuous and g-temperate.*

Definition 1 (Section 18.4.2 in [24]). *Let the weight $m(x,\xi)$ be admissible and the metric g be defined by (8). Assume $q(x,\xi,\lambda,\tau) \in \mathscr{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, where parameters λ, τ satisfy conditions in Lemma 1. $q(x,\xi,\lambda,\tau)$ is a symbol in class $S(m,g)$ if for all $\alpha,\beta \in \mathbb{N}^d$ there exist $C_{\alpha,\beta}$ independent of τ and λ such that*

$$|\,\partial_x^\alpha \partial_\xi^\beta q(x,\xi,\lambda,\tau)\,| \leq C_{\alpha,\beta} m(x,\xi) \lambda^{|\,\alpha\,|/2} \mu(\tau,\xi)^{-|\,\beta\,|}.$$

Remark 1. (i) *It is clear that $\mu = \sqrt{\tau^2 + |\,\xi\,|^2} \in S(\mu,g)$ since $|\,\partial_\xi^\beta \mu(\tau,\xi)\,| \lesssim \mu^{1-|\,\beta\,|}$ for all $\beta \in \mathbb{R}^d$.*

(ii) *Let ν be the weight defined by (9). It is easy to get that $\lambda a \in S(\nu,g)$. In fact, if $|\,\alpha\,| \geq 2$, it holds that $|\,\partial_x^\alpha(\lambda a(x))\,| \leq C_\alpha \lambda \leq C_\alpha \lambda^{|\,\alpha\,|/2} \nu(x)$, where $C_\alpha > 0$. For the case $|\,\alpha\,| = 1$, it follows from (A1) that*

$$|\,\partial_x^\alpha(\lambda a(x))\,| \leq \sqrt{2}\,\|a''\|_{L^\infty(\Omega)}^{\frac{1}{2}} \lambda^{\frac{1}{2}} \big(\lambda a(x)\big)^{\frac{1}{2}}$$

Note that $|\,\lambda a(x)\,| < C\nu^2(x)$ for some $C > 0$. This together with the above inequality, we have that there exists a positive constant C such that

$$|\,\partial_x^\alpha(\lambda a(x))\,| < \sqrt{2C}\,\|a''\|_{L^\infty(\Omega)}^{\frac{1}{2}} \lambda^{\frac{1}{2}} \nu(x).$$

(iii) *It is known from Lemma 18.4.3 of [24] that if the metric g and weights m_1, m_2 are admissible, symbols $a \in S(m_1,g)$ and $b \in S(m_2,g)$, then $ab \in S(m_1 m_2,g)$. In particular, $(\lambda a)^j \mu^k \in S(\nu^j \mu^k, g)$ for all $j,k \in \mathbb{N} \cup \{0\}$.*

Definition 2. *Let $b \in S(m,g)$ be a symbol and $u \in \mathscr{S}(\mathbb{R}^d)$, we set*

$$b(x,D,\tau)u(x) = \mathrm{Op}(b)u(x) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi}\, b(x,\xi,\tau)\, \widehat{u}(\xi)\, d\xi.$$

It is known that $\mathrm{Op}(b) : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ is continuous and $\mathrm{Op}(b)$ can be uniquely extended to $\mathscr{S}'(\mathbb{R}^d)$ continuously. The following two lemmas are consequences of Theorem 18.5.4 and 18.5.10 in Reference [24].

Lemma 2. *Let $b \in S(m,g)$ where m is an admissible weight and g is defined by (8). Then there exists $c \in S(m,g)$ such that $\mathrm{Op}(b)^* = \mathrm{Op}(c)$ and $c(x,\xi) = \overline{b(x,\xi)} + r(x,\xi)$ where the remainder $r \in S(\lambda^{\frac{1}{2}}\mu^{-1}m, g)$.*

Lemma 3. *Let $b \in S(m_1,g)$ and $c \in S(m_2,g)$ where m_j are admissible weights for $j = 1,2$ and g is defined by (8). Then,*

(i) *there exists $d \in S(m_1m_2,g)$ such that $\mathrm{Op}(b)\,\mathrm{Op}(c) = \mathrm{Op}(d)$ and $d(x,\xi) = b(x,\xi)c(x,\xi) + r(x,\xi)$ where $r \in S(\lambda^{\frac{1}{2}}\mu^{-1}m_1m_2, g)$.*

(ii) *for commutator $i[\mathrm{Op}(b), \mathrm{Op}(c)] = \mathrm{Op}(f)$, it holds that $f \in S(\lambda^{\frac{1}{2}}\mu^{-1}m_1m_2, g)$ and $f(x,\xi) = \{b,c\}(x,\xi) + r(x,\xi)$ where $r \in S(\lambda\mu^{-2}m_1m_2, g)$.*

The operators in $S(\nu^j\mu^k, g)$ act on Sobolev spaces adapted to the class of symbol. Let $b \in S(\nu^j\mu^k, g)$, where μ and g are defined by (8). Then there exists $C > 0$ such that

$$\|\,\mathrm{Op}(b)u\|_{L^2(\mathbb{R}^d)} \le C\|\nu^j\,\mathrm{Op}(\mu^k)u\|_{L^2(\mathbb{R}^d)}, \quad \forall\, u \in \mathscr{S}(\mathbb{R}^d).$$

By symbolic calculus, the above estimate is equivalent to $\mathrm{Op}(\mu^{-k}\nu^{-j})\,\mathrm{Op}(b)$ acts on $L^2(\mathbb{R}^d)$ since the operators associated with symbol in $S(1,g)$ act on $L^2(\mathbb{R}^d)$. In particular, if $b \in S(\nu^j\mu, g)$, then for any $\lambda \ge \lambda_0$, $\tau \ge \max\{\,\mathcal{C}\lambda,\ 1\,\}$ and $u \in \mathscr{S}(\mathbb{R}^d)$, it holds that

$$\|\,\mathrm{Op}(b)u\|_{L^2(\mathbb{R}^d)} \le C\tau\|\nu^j u\|_{L^2(\mathbb{R}^d)} + C\|\nu^j Du\|_{L^2(\mathbb{R}^d)},$$

where $C > 0$ depends on λ_0 and $\mathcal{C}$.

2.2. Commutator Estimate

In this subsection, we suppose that $\lambda = 1$ since the symbol does not depend on λ. The the metric in (8) becomes

$$\tilde{g} = dx^2 + \mu^{-2}d\xi^2, \qquad \text{where } \ \mu \ \text{ is defined by (8).} \tag{10}$$

To get the commutator estimate, we shall use the following Gårding inequality [24].

Lemma 4. *Let $b \in S(\mu^{2k}, \tilde{g})$ be real valued. μ and $\tilde{g}$ are defined by (10). We assume there exists $C > 0$ such that $b(x,\xi,\tau) \ge C\mu^{2k}$. Then there exists $\widetilde{C} > 0$ such that*

$$\mathrm{Re}(\mathrm{Op}(b)w \,|\, w) \ge \widetilde{C}\|\,\mathrm{Op}(\mu^k)w\|^2, \qquad \forall\, w \in \mathscr{S}(\mathbb{R}^d). \tag{11}$$

Define the operator $P(x,D,\lambda) = D^2 + i\lambda Da(x)D - \lambda^2$, where $D^2 = \sum_{j=1}^d D_j^2$, $D_j = -i\partial_{x_j}$ and $Da(x)D = \sum_{j=1}^d D_j a(x) D_j$. Introduce the weight function $\varphi \in \mathscr{C}^\infty(\mathbb{R}^d;\mathbb{R})$. The associated conjugate operator of $P(x,D,\lambda)$ is $P_\varphi(x,D,\lambda) = e^{\tau\varphi}P(x,D,\lambda)e^{-\tau\varphi}$. Then,

$$P_\varphi = (D + i\tau\nabla\varphi(x))^2 + i\lambda(D + i\tau\nabla\varphi(x))a(x)(D + i\tau\nabla\varphi(x)) - \lambda^2.$$

By setting $Q_2 = \frac{1}{2}(P_\varphi + P_\varphi^*)$ and $Q_1 = \frac{1}{2i}(P_\varphi - P_\varphi^*)$, we have $P_\varphi = Q_2 + iQ_1$, where

$$\begin{aligned} Q_2 &= D^2 - \tau^2|\nabla\varphi|^2 - \lambda\tau Da(x)\nabla\varphi(x) - \lambda\tau\nabla\varphi(x)a(x)D - \lambda^2, \\ Q_1 &= \tau D\nabla\varphi(x) + \tau\nabla\varphi(x)D + \lambda Da(x)D - \lambda\tau^2 a(x)|\nabla\varphi|^2. \end{aligned} \tag{12}$$

Definition 3. *Let V be a bounded open set in $\mathbb{R}^d$. We say that the weight function $\varphi \in \mathscr{C}^\infty(\mathbb{R}^d;\mathbb{R})$ satisfies the* **sub-ellipticity** *condition in $\overline{V}$ if $|\, d\varphi\,| > 0$ in $\overline{V}$ and there exists constant $C > 0$,*

$$\mathfrak{p}_\varphi(x,\xi,\tau) = 0,\ \ \forall\,(x,\xi) \in \overline{V}\times\mathbb{R}^d,\ \ \tau > 0 \quad\Rightarrow\quad \{\mathfrak{q}_{12},\mathfrak{q}_{11}\}(x,\xi,\tau) \geq C(\,|\,\xi\,|^2 + \tau^2)^{3/2}, \tag{13}$$

where $\mathfrak{p}_\varphi(x,\xi,\tau) = |\xi + i\tau\nabla\varphi(x)|^2 = \mathfrak{q}_{12}(x,\xi,\tau) + i\mathfrak{q}_{11}(x,\xi,\tau)$ and $\mathfrak{q}_{11},\ \mathfrak{q}_{12}$ are real valued.

Lemma 5 ([24])**.** *Let V be a bounded open set in $\mathbb{R}^d$ and $\psi \in \mathscr{C}^\infty(\mathbb{R}^d;\mathbb{R})$ be such that $|\,\nabla\psi\,| > 0$ in $\overline{V}$. Then, for $\gamma > 0$ sufficiently large, $\varphi = e^{\gamma\psi}$ fulfills the sub-ellipticity property in V.*

Lemma 6. *Assume that φ satisfies the sub-ellipticity in Definition 3. For all $w \in \mathscr{C}_0^\infty(\Omega)$, there exist $C_1, C_2 > 0$ and $\tau_0 > 0$ such that the following inequality holds for all $\tau \geq \tau_0$,*

$$\begin{aligned} & C_1\tau^3\|w\|^2 + C_1\tau\|Dw\|^2 \\ \leq\ & \big(\operatorname{Op}(\{\xi^2 - \tau^2\nabla\varphi(x)^2,\, 2\tau\xi\cdot\nabla\varphi(x)\})w \mid w\big) + C_2\tau^{-1}\|\operatorname{Op}(\xi^2 - \tau^2\nabla\varphi(x)^2)w\|^2 \\ & + C_2\tau^{-1}\|\operatorname{Op}(2\tau\xi\cdot\nabla\varphi(x))w\|^2. \end{aligned} \tag{14}$$

The proof of Lemma 6 can be found in Appendix A.

3. Carleman Estimate

In this section, we shall prove the Carleman estimate by the tool introduced by Hörmander [24,25] and called Weyl-Hörmander calculus. It allows us to define proper operators, class of symbol and pseudo-diff-calculus. Throughout this section, we denote by $\|\cdot\|_V$, $(\cdot\,|\,\cdot)_V$ the norm and inner product in $L^2(V)$ for $V \subset \Omega$, respectively.

Let the metric g and weight ν be defined by (8) and (9). Furthermore, due to the results in Section 2.1, we know that $D + i\tau\nabla\varphi(x)$ is an operator with a symbol in $S(\mu,g)$ class, λa is in $S(\nu,g)$, and $(1 + i\lambda a(x))(\xi + i\tau\nabla\varphi(x))^2$, the principal symbol of P_φ belongs to $S(\nu\mu^2,g)$. It follows from Lemmas 2 and 3 that

$$\begin{aligned} P_\varphi &= \operatorname{Op}\left((1 + i\lambda a(x))(\xi + i\tau\nabla\varphi(x))^2\right) - \lambda^2 + R_3 \\ Q_2 &= \operatorname{Op}(q_2) - \lambda^2 + R_2 \\ Q_1 &= \operatorname{Op}(q_1) + R_1, \end{aligned} \tag{15}$$

where $q_2 = |\xi|^2 - \tau^2|\nabla\varphi(x)|^2 - 2\lambda\tau a(x)\xi\cdot\nabla\varphi(x)$, $q_1 = 2\tau\xi\cdot\nabla\varphi(x) + \lambda a(x)(|\xi|^2 - \tau^2|\nabla\varphi(x)|^2)$ and the symbols of R_j are in $S(\lambda^{\frac{1}{2}}\nu\mu, g)$ for $j = 1,2,3$. It is clear that

$$\|P_\varphi v\|_V^2 = \|Q_2 v\|_V^2 + \|Q_1 v\|_V^2 + 2\operatorname{Re}(Q_2 v\,|\,iQ_1 v)_V. \tag{16}$$

In what follows, several Carleman estimates are introduced. First, we give an estimation on the subdomain which is far away from the boundary Γ.

Theorem 2. *Suppose φ satisfies sub-ellipticity condition on $V \subset \Omega$. Then, there exist positive constants C, $\tilde{K}$ and λ_0, such that for every $u \in \mathscr{C}_0^\infty(V)$, it holds*

$$\tau^3\|(1 + \lambda^2 a(x)^2)^{\frac{1}{2}}e^{\tau\varphi}u\|_V^2 + \tau\|(1 + \lambda^2 a(x)^2)^{\frac{1}{2}}e^{\tau\varphi}Du\|_V^2 \leq C\|e^{\tau\varphi}Pu\|_V^2, \tag{17}$$

where $\lambda \geq \lambda_0$ *and* $\tau \geq \max\{\tilde{K}|\lambda|^{5/4}, 1\}$.

Proof. We shall prove that (17) is equivalent to

$$\|P_\varphi v\|_V^2 \gtrsim \tau\|\nu(x)Dv\|_V^2 + \tau^3\|\nu(x)v\|_V^2. \tag{18}$$

First, assume (18) holds. Set $v = e^{\tau\varphi}u$. Then, $Dv = e^{\tau\varphi}(Du - i\tau\nabla\varphi\, u)$ and $e^{\tau\varphi}Du = Dv + i\tau\nabla\varphi\, v$. Then there exist positive constants $c_1,\ c_2$ such that

$$\begin{aligned} c_1\left(\|\nu(x)Dv\|_V + \tau\|\nu(x)v\|_V\right) &\leq \|\nu(x)e^{\tau\varphi}Du\|_V + \tau\|\nu(x)e^{\tau\varphi}u\|_V \\ &\leq c_2\left(\|\nu(x)Dv\|_V + \tau\|\nu(x)v\|_V\right). \end{aligned} \tag{19}$$

Combining this with (18), we conclude that

$$\tau^3\|\nu(x)e^{\tau\varphi}u\|_V^2 + \tau\|\nu(x)e^{\tau\varphi}Du\|_V^2 \lesssim (\tau\|\nu(x)Dv\|_V^2 + \tau^3\|\nu(x)v\|_V^2) \lesssim \|e^{\tau\varphi}Pu\|_V^2.$$

On the other hand, (17) implies that

$$\tau^3\|\nu(x)e^{\tau\varphi}u\|_V^2 + \tau\|\nu(x)e^{\tau\varphi}Du\|_V^2 \lesssim \|P_\varphi v\|_V^2.$$

Then, we proved (18) from the above estimate and (19).
Now we are going to prove (18). Note that

$$2\,\mathrm{Re}(Q_2v \,|\, iQ_1v)_V = (Q_2v \,|\, iQ_1v)_V + (iQ_1v \,|\, Q_2v)_V = (i[Q_2, Q_1]v \,|\, v)_V, \tag{20}$$

where we denote by $[Op(a), Op(b)] = Op(a) \circ Op(b) - Op(b) \circ Op(a)$ and Poisson bracket $\{a, b\}(x, \xi, \tau) = \sum_{1\leq j\leq d}(\partial_{\xi_j}a\partial_{x_j}b - \partial_{x_j}a\partial_{\xi_j}b)(x, \xi, \tau)$. From symbolic calculus, the principal symbol of $i[Q_2, Q_1]$ is $\{q_2, q_1\}$. Due to the results in Section 2.1, we obtain that

$$i[Q_2, Q_1] = \mathrm{Op}(\{q_2, q_1\}) + R_4,$$

where $\{q_2, q_1\} \in S(\nu^2\mu^3\lambda^{\frac{1}{2}}, g)$ and the symbol of R_4 is in $S(\nu^2\mu^2\lambda, g)$.

A direct computation gives that

$$\begin{aligned} \{q_2, q_1\} &= (1 + a^2(x)\lambda^2)\{\xi^2 - \tau^2\nabla\varphi(x)^2, 2\tau\xi.\nabla\varphi(x)\} \\ &\quad + \Big(\{\xi^2 - \tau^2\nabla\varphi(x)^2, \lambda a(x)\} - \lambda a(x)\{2\tau\xi.\nabla\varphi(x), \lambda a(x)\}\Big)(\xi^2 - \tau^2\nabla\varphi(x)^2) \\ &\quad - 2\tau\xi.\nabla\varphi(x)\Big(\lambda a(x)\{\lambda a(x), \xi^2 - \tau^2\nabla\varphi(x)^2\} + \{\lambda a(x), 2\tau\xi.\nabla\varphi(x)\}\Big). \end{aligned}$$

Since

$$\begin{aligned} q_2(x, \xi) + \lambda a(x)q_1(x, \xi) &= (1 + \lambda^2 a(x)^2)(\xi^2 - \tau^2\nabla\varphi(x)^2), \\ q_1(x, \xi) - \lambda a(x)q_2(x, \xi) &= 2\tau\xi.\nabla\varphi(x)(1 + \lambda^2 a^2(x)), \end{aligned} \tag{21}$$

we have

$$\begin{aligned} &\{q_2, q_1\} \\ = \;& \nu^2(x)\{\xi^2 - \tau^2\nabla\varphi(x)^2, 2\tau\xi.\nabla\varphi(x)\} \\ &+\nu^{-2}(x)(q_2(x, \xi) + \lambda a(x)q_1(x, \xi))\Big(\{\xi^2 - \tau^2\nabla\varphi(x)^2, \lambda a(x)\} - \lambda a(x)\{2\tau\xi.\nabla\varphi(x), \lambda a(x)\}\Big) \\ &-\nu^{-2}(x)(q_1(x, \xi) - \lambda a(x)q_2(x, \xi))\Big(\lambda a(x)\{\lambda a(x), \xi^2 - \tau^2\nabla\varphi(x)^2\} + \{\lambda a(x), 2\tau\xi.\nabla\varphi(x)\}\Big). \end{aligned}$$

Then, it follows from symbolic calculus that

$$\begin{aligned} i[Q_2, Q_1] &= \nu(x)\,\mathrm{Op}\left(\{\xi^2-\tau^2\nabla\varphi(x)^2, 2\tau\xi.\nabla\varphi(x)\}\right)\nu(x) \\ &\quad +B_1\nu^{-1}(x)\,\mathrm{Op}\left(q_2(x,\xi)+\lambda a(x)q_1(x,\xi)+\lambda^2\right) \\ &\quad -B_2\nu^{-1}(x)\,\mathrm{Op}\left(q_1(x,\xi)-\lambda a(x)q_2(x,\xi)-\lambda^3 a(x)\right)+R_5, \end{aligned} \tag{22}$$

where the symbol of B_1 is $\nu^{-1}(x)\left[\{\xi^2-\tau^2\nabla\varphi(x)^2, \lambda a(x)\}-\lambda a(x)\{2\tau\xi.\nabla\varphi(x), \lambda a(x)\}\right] \in S(\lambda^{\frac{1}{2}}\nu\mu, g)$, the symbol of B_2 is $\nu^{-1}(x)\left[\lambda a(x)\{\lambda a(x), \xi^2-\tau^2\nabla\varphi(x)^2\}+\{\lambda a(x), 2\tau\xi.\nabla\varphi(x)\}\right] \in S(\lambda^{\frac{1}{2}}\nu\mu, g)$ and the symbol of R_5 is in $S(\lambda\nu^2\mu^2, g)$. We refer to Section 2.1 where the rules on symbolic calculus are given and precise. Therefore, by the continuity of pseudo-differential operator, we have that for $j=1,2$, $k=0,1$ and $\ell=1,2$,

$$\begin{aligned} & |\,(B_j\nu(x)^{-1}(\lambda a(x))^k Q_\ell v\,|\,v)_V\,| \\ = \;& |\,(\nu(x)^{-1}(\lambda a(x))^k Q_\ell v\,|\,B_j^* v)\,| \;\le C\|Q_\ell v\|_V\|B_j^* v\|_V \\ \le \;& (1/10)\|Q_\ell v\|_V^2 + C'\lambda\tau^2\|\nu(x)v\|_V^2 + C'\lambda\|\nu(x)Dv\|_V^2, \end{aligned} \tag{23}$$

$$\begin{aligned} |\,(B_j\nu^{-1}(x)\lambda^2(\lambda a(x))^k v\,|\,v)_V\,| &\le C\lambda^{5/2}\big(\tau\|\nu(x)v\|_V+\|\nu(x)Dv\|_V\big)\|\nu(x)v\|_V \\ &\le C\lambda^{5/2}\tau\|\nu(x)v\|_V^2 + C\lambda^{5/2}\tau^{-1}\|\nu(x)Dv\|_V^2, \end{aligned} \tag{24}$$

and

$$|\,(R_5 v\,|\,v)_V\,| \lesssim \lambda\tau^2\|\nu(x)v\|_V^2+\lambda\|\nu(x)Dv\|_V^2. \tag{25}$$

Let $w=(1+\lambda^2 a(x)^2)^{\frac{1}{2}}v=\nu v$ in (14). We obtain

$$\begin{aligned} C_1\tau^3\|\nu v\|_V^2 + C_1\tau\|D(\nu(x)v)\|_V^2 &\le (\mathrm{Op}(\{\xi^2-\tau^2\varphi(x)^2, 2\tau\xi.\nabla\varphi(x)\})\nu v\,|\,\nu v)_V \\ &\quad +C_2\tau^{-1}\|\,\mathrm{Op}(\xi^2-\tau^2\nabla\varphi(x)^2)\nu v\|_V^2 \\ &\quad +C_2\tau^{-1}\|\,\mathrm{Op}(2\tau\xi.\nabla\varphi(x))\nu v\|_V^2,\;\; C_1,\, C_2>0. \end{aligned} \tag{26}$$

Since the symbol of $[D,\nu(x)]$ is in $S(\lambda^{\frac{1}{2}}\nu, g)$, we have

$$\|\nu(x)Dv\|_V^2 \le \|D(\nu(x)v)\|_V^2+\|[D,\nu(x)]v\|_V^2 \le \|D(\nu(x)v)\|_V^2+C\lambda\|\nu(x)v\|_V^2.$$

Consequently, for $\tau\ge C\lambda$ with $C>0$ sufficiently large, it holds

$$\tau^3\|\nu(x)v\|_V^2+\tau\|\nu(x)Dv\|_V^2 \lesssim \tau^3\|\nu(x)v\|_V^2+\tau\|D(\nu(x)v)\|_V^2. \tag{27}$$

It follows from (26) and (27) that

$$\begin{aligned} C_1\big(\tau^3\|\nu v\|_V^2+\tau\|\nu(x)Dv\|_V^2\big) &\le (\mathrm{Op}(\{\xi^2-\tau^2\varphi(x)^2, 2\tau\xi.\nabla\varphi(x)\})\nu v\,|\,\nu v)_V \\ &\quad +C_2\tau^{-1}\|\,\mathrm{Op}(\xi^2-\tau^2\nabla\varphi(x)^2)\nu v\|_V^2 \\ &\quad +C_2\tau^{-1}\|\,\mathrm{Op}(2\tau\xi.\nabla\varphi(x))\nu v\|_V^2. \end{aligned} \tag{28}$$

Now we are going to estimate the terms $\| \mathrm{Op}(\xi^2 - \tau^2 \nabla\varphi(x)^2) \nu v\|_V^2$ and $\| \mathrm{Op}(2\tau\xi.\nabla\varphi(x)) \nu v\|_V^2$. It follows from (15) and (21) that

$$\begin{array}{rl} & \| \mathrm{Op}(\xi^2 - \tau^2 \nabla\varphi(x)^2) \nu v\|_V^2 + \| \mathrm{Op}(2\tau\xi.\nabla\varphi(x)) \nu v\|_V^2 \\ = & \| \mathrm{Op}(\nu^{-2}(q_2 + \lambda a(x) q_1) \nu v\|_V^2 + \| \mathrm{Op}(\nu^{-2}(q_1 - \lambda a(x) q_2) \nu v\|_V^2 \\ = & \|\nu^{-1}\big[Q_2 + \lambda^2 - R_2 + \lambda a(x)(Q_1 - R_1)\big] v\|_V^2 \\ & + \|\nu^{-1}\big[Q_1 - R_1 - \lambda a(x)(Q_2 + \lambda^2 - R_2)\big] v\|_V^2. \end{array} \tag{29}$$

Combining this with the fact that the symbols of R_j are in $S(\lambda^{\frac{1}{2}} \nu\mu, g)$ for $j = 1, 2, 3$, we have that there exists a positive constant C such that

$$\begin{array}{rl} & \| \mathrm{Op}(\xi^2 - \tau^2 \nabla\varphi(x)^2) \nu v\|_V^2 + \| \mathrm{Op}(2\tau\xi.\nabla\varphi(x)) \nu v\|_V^2 \\ \lesssim & \sum_{j=1,2} \|Q_j v\|_V^2 + \|R_1 v\|_V^2 + \|R_2 v\|_V^2 + \lambda^4 \|v\|_V^2 \\ \lesssim & \sum_{j=1,2} \|Q_j v\|_V^2 + \lambda\tau^2 \|\nu v\|_V^2 + \lambda \|\nu D v\|_V^2 + \lambda^4 \|v\|_V^2. \end{array} \tag{30}$$

From (28) and (30), we have that for $\tau \geq C\lambda$, there exist positive constants $C_1,\ C_2$ such that

$$\begin{array}{rl} & C_1\big(\tau^3 \|\nu(x) v\|_V^2 + \tau \|\nu(x) D v\|_V^2\big) \\ \leq & (\mathrm{Op}(\{\xi^2 - \tau^2 \varphi(x)^2, 2\tau\xi.\nabla\varphi(x)\}) \nu v \,|\, \nu v)_V \\ & + C_2\big((1/\tau) \sum_{j=1,2} \|Q_j v\|_V^2 + \lambda\tau \|\nu v\|_V^2 + \|\nu D v\|_V^2 + \lambda^3 \|v\|_V^2\big). \end{array} \tag{31}$$

On the other hand, due to (22)–(24), we have that there exists a positive constant C' such that

$$\begin{array}{rl} & \big(\mathrm{Op}(\{\xi^2 - \tau^2 \varphi(x)^2, 2\tau\xi.\nabla\varphi(x)\}) \nu v \,|\, \nu v\big)_V \\ \leq & (i[Q_2, Q_1] v \,|\, v)_V + (1/4) \sum_{j=1,2} \|Q_j v\|_V^2 + C'\big(\lambda\tau^2 \|\nu(x) v\|_V^2 \\ & + \lambda \|\nu(x) D v\|_V^2 + \lambda^{5/2}\tau \|\nu(x) v\|_V^2 + \lambda^{5/2}\tau^{-1} \|\nu(x) D v\|_V^2\big). \end{array} \tag{32}$$

Finally, by (31) and (32), one can choose $\tau \geq \tilde{K}|\lambda|^{5/4}$ with $\tilde{K}$ sufficiently large and $\tau \geq \tau_0$ with τ_0 sufficiently large, such that for some $C > 0$,

$$\begin{array}{rl} C\big(\tau^3 \|\nu(x) v\|_V^2 + \tau \|\nu(x) D v\|_V^2\big) \leq & (i[Q_2, Q_1] v \,|\, v)_V + (1/2) \sum_{j=1,2} \|Q_j v\|_V^2. \\ & + \varepsilon\tau^3 \|\nu(x) v\|_V^2 + \varepsilon\tau \|\nu(x) D v\|_V^2, \end{array} \tag{33}$$

where $\varepsilon > 0$ is arbitrary. Choosing ε small with respect C_1, using (16), (20) and (33), we obtain

$$\begin{array}{ll} C_1\tau^3 \|\nu(x) v\|_V^2 + C_1\tau \|\nu(x) D v\|_V^2 & + (1/2)\|Q_1 v\|_V^2 + (1/2)\|Q_2 v\|_V^2 \\ & \leq (i[Q_2, Q_1] v, v) + \|Q_1 v\|_V^2 + \|Q_2 v\|_V^2 = \|P_\varphi v\|_V^2, \end{array} \tag{34}$$

this implies (18). □

Remark 2. *The estimates on the previous terms impose the assumption $\tau \geq \tilde{K}|\lambda|^{5/4}$. The other remainder terms only impose the condition $\tau \geq C\lambda$. This condition is related with the principal normal condition. Indeed for a complex operator, with symbol $p_1 + ip_2$ where p_1, p_2 are both real valued, the Carleman estimate is only true if $\{p_1, p_2\} = 0$ on $p_1 = p_2 = 0$. Here the symbol of operator before conjugaison by weight is $\xi^2 - \lambda^2 + i\lambda a(x)\xi^2$, and the Poisson bracket is $\{\xi^2 - \lambda^2, \lambda a(x)\xi^2\} = 2\lambda(\xi.a'(x))\xi^2$. We can estimate this term, uniformly in a neighborhood of $a(x) = 0$, by $C\lambda a^{\frac{1}{2}}(x) \,|\, \xi \,|^3$. This explanation does not justify the power $|\lambda|^{5/4}$ found at the end of computations but shows the difficulties.*

Since there is higher order term $div(a(x)\nabla y_t)$ in system (1), it is necessary to deal with the term $div(a(x)\nabla f)$ for $f \in H^1(\Omega)$ when proving the resolvent estimate. The following result is analogue to the work by [26] .

Theorem 3. *Suppose φ satisfies sub-ellipticity condition on $V \subset \Omega$. Then, there exist $C, \tilde{K} > 0$, $\lambda_0 > 0$, such that for all $u \in \mathscr{C}_0^\infty(V)$ satisfying*

$$Pu = g_0 + \sum_{j=0}^{d} \partial_{x_j} g_j, \quad \text{where} \quad g_j \in L^2(V), \ j = 0, 1, \cdots, d, \tag{35}$$

it holds

$$\tau \|(1+\lambda^2 a(x)^2)^{\frac{1}{2}} e^{\tau\varphi} u\|^2_{L^2(V)} + \tau^{-1} \|(1+\lambda^2 a(x)^2)^{\frac{1}{2}} e^{\tau\varphi} Du\|^2_{L^2(V)} \leq C \sum_{j=0}^{d} \|e^{\tau\varphi} g_j\|^2_{L^2(V)}, \tag{36}$$

where $\lambda \geq \lambda_0$ and $\tau \geq \max\{\tilde{K}|\lambda|^{5/4}, 1\}$.

Proof. First, from (15), we have $D^2 = Q_2 + S_2$ and $\lambda a D^2 = Q_1 + S_1$ where S_1 and S_2 have symbols in $S(\tau\nu\mu, g)$ if $\tau \gtrsim \lambda$. It follows that

$$\begin{aligned} \|D^2 v\|^2 &\lesssim \|Q_2 v\|^2 + \tau^2(\tau^2 \|\nu(x) v\|^2 + \|\nu(x) Dv\|^2), \\ \|\lambda a D^2 v\|^2 &\lesssim \|Q_1 v\|^2 + \tau^2(\tau^2 \|\nu(x) v\|^2 + \|\nu(x) Dv\|^2). \end{aligned} \tag{37}$$

Using the fact that $\|\nu(x) D^2 v\|^2 \leq 2(\|D^2 v\|^2 + \|\lambda a D^2 v\|^2)$, (34), we obtain

$$\tau^3 \|\nu(x) v\|^2 + \tau \|\nu(x) Dv\|^2 + \tau^{-1} \|\nu(x) D^2 v\|^2 \lesssim \|P_\varphi v\|^2. \tag{38}$$

Let w and χ be in $\mathscr{C}_0^\infty(\Omega)$ such that $\chi = 1$ on a neighborhood of supp w. From (27), we obtain

$$\begin{aligned} \tau \|\nu(x) w\|^2 + \tau^{-1} \|\nu(x) Dw\|^2 &\lesssim \tau \|\nu(x) w\|^2 + \tau^{-1} \|D(\nu(x) w)\|^2 \\ &\lesssim \tau^3 \|\operatorname{Op}(\mu^{-1}) \nu(x) \chi w\|^2 + \tau^{-1} \|D^2 \operatorname{Op}(\mu^{-1}) \nu \chi w\|^2, \end{aligned} \tag{39}$$

where the last estimate is obtained by Fourier transform and by the inequality

$$\tau + \frac{|\xi|^2}{\tau} \lesssim \frac{\tau^3}{\tau^2 + |\xi|^2} + \frac{|\xi|^4}{\tau(\tau^2 + |\xi|^2)}.$$

From the results in Section 2.1, we have $\operatorname{Op}(\mu^{-1}) \nu\chi = \nu\chi \operatorname{Op}(\mu^{-1}) + R_1$, where R_1 has a symbol in $S(\mu^{-2}\nu\lambda^{\frac{1}{2}}, g)$, and $D^2 \operatorname{Op}(\mu^{-1}) \nu\chi = \nu D^2 \chi \operatorname{Op}(\mu^{-1}) + R_2$, where R_2 has a symbol in $S(\nu\lambda^{\frac{1}{2}}, g)$. Then, it follows from (39) that

$$\tau \|\nu(x) w\|^2 + \tau^{-1} \|\nu(x) Dw\|^2 \lesssim \tau^3 \|\nu(x) \chi \operatorname{Op}(\mu^{-1}) w\|^2 + \tau^{-1} \|\nu D^2 \chi \operatorname{Op}(\mu^{-1}) w\|^2 + \lambda \|\nu(x) w\|^2.$$

For $\tau \geq \max\{C\lambda, 1\}$ with C large enough, one has the following result from the above inequality.

$$\tau \|\nu(x) w\|^2 + \tau^{-1} \|\nu(x) Dw\|^2 \lesssim \tau^3 \|\nu(x) \chi \operatorname{Op}(\mu^{-1}) w\|^2 + \tau^{-1} \|\nu D^2 \chi \operatorname{Op}(\mu^{-1}) w\|^2. \tag{40}$$

Now, we apply (38) to $v = \chi \operatorname{Op}(\mu^{-1}) w$ to have

$$\begin{aligned} & \tau^3 \|\nu(x) \chi \operatorname{Op}(\mu^{-1}) w\|^2 + \tau \|\nu(x) D\chi \operatorname{Op}(\mu^{-1}) w\|^2 + \tau^{-1} \|\nu(x) D^2 \chi \operatorname{Op}(\mu^{-1}) w\|^2 \\ \lesssim \ & \|P_\varphi \chi \operatorname{Op}(\mu^{-1}) w\|^2. \end{aligned}$$

Thus, combining this with (40) yields

$$\tau\|\nu(x)w\|^2+\tau^{-1}\|\nu(x)Dw\|^2\lesssim\|P_\varphi\chi\,\mathrm{Op}(\mu^{-1})w\|^2. \tag{41}$$

Finally, note that P_φ has a symbol in $S(\nu\mu^2,g)$. Consequently, $P_\varphi\chi\,\mathrm{Op}(\mu^{-1})=\mathrm{Op}(\mu^{-1})P_\varphi\chi+R$, where R has a symbol in $S(\nu\lambda^{\frac{1}{2}},g)$. Then, we can deduce from (41) that

$$\tau\|\nu(x)w\|^2+\tau^{-1}\|\nu(x)Dw\|^2\lesssim\|\,\mathrm{Op}(\mu^{-1})P_\varphi w\|^2+\lambda\|\nu w\|^2.$$

When $\tau\geq C\lambda$, with C large enough, the error term $\|\nu\lambda^{\frac{1}{2}}w\|^2$ can be absorbed by the left hand side. Hence,

$$\tau\|\nu(x)w\|^2+\tau^{-1}\|\nu(x)Dw\|^2\lesssim\|\,\mathrm{Op}(\mu^{-1})P_\varphi w\|^2. \tag{42}$$

For $w=e^{\tau\varphi}u$, we have

$$P_\varphi w=e^{\tau\varphi}Pu=e^{\tau\varphi}g_0+\sum_{j=1}^{d}e^{\tau\varphi}\partial_{x_j}g_j=e^{\tau\varphi}g_0+\sum_{j=1}^{d}\big(\partial_{x_j}(e^{\tau\varphi}g_j)-\tau g_j\partial_{x_j}\varphi\big).$$

Obviously, one has that $\|\,\mathrm{Op}(\mu^{-1})P_\varphi w\|^2\lesssim\sum_{j=0}^{d}\|e^{\tau\varphi}g_j\|^2$. Combining this with (42), we obtain Theorem 3. □

Remark 3. *Since $a(\cdot)$ is non-negative and not identically null, there exists $\delta>0$ such that $\{x\in\Omega:a(x)>\delta\}\neq\emptyset$. We introduce several sets as follows.*

$$\begin{aligned}
W_1&=\Omega\setminus\operatorname{supp}a,\\
W_2&=\Omega\setminus\big(\{x\in\Omega:a(x)>\delta\}\cup\mathcal{O}(\Gamma)\big),\\
W_3&=\overline{\Omega}\setminus\mathcal{O}\big(\{x\in\Omega:a(x)\geq0\}\big),\\
W_4&=\overline{\Omega}\setminus\{x\in\Omega:a(x)>\delta/2\},
\end{aligned}$$

where $\mathcal{O}(\Gamma)$ means the neighborhood of Γ.

It is known that there exists a function $\psi\in\mathscr{C}^\infty(\Omega)$ such that [25]

(1) $\psi(x)=0$ *for* $x\in\partial\Omega$.
(2) $\partial_n\psi(x)<0$ *for* $x\in\partial\Omega$.
(3) $\nabla\psi(x)\neq0$ *for* $x\in W_1$.

Let $\varphi=e^{\gamma\psi}$. It follows from Lemma 5 that φ satisfies the sub-ellipticity condition on $x\in W_2$ if $\gamma>0$ is sufficiently large. Then, in Theorem 2 and 3, one can choose V as W_2.

The following result can be obtained from the classical Carleman estimate up to the boundary [27] (Proposition 2). Note that this estimate corresponds to the Laplacian with Dirichlet boundary condition.

Lemma 7. *Suppose φ satisfies the sub-ellipticity condition on W_4. Then, there exist $C,K>0$, $\lambda_0>0$, such that for all $u\in\mathscr{C}^\infty(\overline{\Omega})$ satisfying* $\operatorname{supp}u\subset W_3$ *and $u=0$ on Γ, it holds that*

$$\tau^3\|e^{\tau\varphi}u\|^2+\tau\|e^{\tau\varphi}Du\|^2\leq C\|e^{\tau\varphi}(D^2-\lambda^2)u\|^2, \tag{43}$$

where $\lambda\geq\lambda_0$ and $\tau\geq\max\{K\lambda,1\}$.

Theorem 4. *Suppose φ satisfies sub-ellipticity condition on W_4. Let $u \in \mathscr{C}^\infty(\overline{\Omega})$ satisfy*

$$\begin{aligned} Pu &= f_0 + \sum_{j=1}^{d} \partial_{x_j} f_j \quad \text{in} \quad \Omega, \\ u &= 0 \qquad\qquad\qquad\quad \text{on} \quad \Gamma, \end{aligned}$$

where $f_j \in L^2(\Omega)$, $\operatorname{supp} f_0 \subset \Omega$ and $\operatorname{supp} f_j \subset \{x \in \Omega : a(x) \geq 0\}$ for $j = 1, \cdots, d$. Then, there exist $C, \tilde{K} > 0$, $\lambda_0 > 0$, such that for all $\lambda \geq \lambda_0$ and $\tau \geq \max\{\tilde{K}|\lambda|^{5/4}, 1\}$, it holds

$$\tau \|e^{\tau\varphi} u\|^2 + \tau^{-1} \|e^{\tau\varphi} Du\|^2 \leq C\Big(\sum_{j=0}^{d} \|e^{\tau\varphi} f_j\|^2 + \lambda \|e^{\tau\varphi} u\|^2_{L^2(\{x\in\Omega,\, a(x)\geq\delta/2\})} \Big),$$

where the positive constant δ is defined as in Remark 3.

Proof. Let $\chi_1,\ \chi_2 \in \mathscr{C}^\infty(\Omega)$ satisfy χ_1 and χ_2 are supported on W_3 and W_2, respectively. We assume that $\chi_1 + \chi_2 = 1$ on W_3. In particular $\chi_1 = 1$ on a neighborhood of $\partial\Omega$, and $\chi_2 = 1$ in a neighborhood of the frontier of $\operatorname{supp} a$.

Since $\chi_1 \partial_{x_j} f_j = 0$ for $j = 1, \cdots, d$, we have $P\chi_1 u = \chi_1 f_0 + [P,\, \chi_1]u$, where $[P,\, \chi_1]$ is a first order operator and supported on $\operatorname{supp} \chi_1$. By using $\chi_1 u$ instead of u in Lemma 7, we obtain

$$\tau^3 \|e^{\tau\varphi} \chi_1 u\|^2 + \tau \|e^{\tau\varphi} D\chi_1 u\|^2 \lesssim \|e^{\tau\varphi} \chi_1 f_0\|^2 + \|e^{\tau\varphi} [P, \chi_1] u\|^2. \tag{44}$$

On the other hand, it is clear that $P\chi_2 u = \chi_2 f_0 + \sum_{j=1}^{d} \left(\partial_{x_j}(\chi_2 f_j) - f_j \partial_{x_j} \chi_2\right) + [P, \chi_2]u$. Since $[P, \chi_2]$ is a first order operator, we have that there exist a_0, a_1, $\cdots$, a_d and b_0, b_1, $\cdots$, b_d such that $[P, \chi_2]u = a_0 u + \sum_{j=1}^{d} \partial_{x_j}(a_j u) + \lambda \sum_{j=1}^{d} \partial_{x_j}(b_j u)$, where a_j and b_j $(j = 0, 1, \cdots, d)$ are supported on the set $\{x \in \Omega : \delta \geq a(x) \geq \delta/2\} \cup (\Omega \setminus \operatorname{supp} a)$. Then, applying Theorem 3 with $\chi_2 u$ instead of u, we obtain

$$\begin{aligned} \tau \|(1 + \lambda^2 a(x)^2)^{\frac{1}{2}} e^{\tau\varphi} \chi_2 u\|^2 &+ \tau^{-1} \|(1 + \lambda^2 a(x)^2)^{\frac{1}{2}} e^{\tau\varphi} D\chi_2 u\|^2 \\ &\lesssim \sum_{j=0}^{d} \|e^{\tau\varphi} f_j\|^2 + \|e^{\tau\varphi} u\|^2_{L^2(\Omega\setminus\operatorname{supp} a)} + \lambda \|e^{\tau\varphi} u\|^2_{L^2(\{x\in\Omega : a(x)\geq\delta/2\})}. \end{aligned}$$

Summing this estimate with (44) multiply by τ^{-2}, using $(1 + \lambda^2 a(x)^2) \geq 1$, we obtain

$$\begin{aligned} \tau \|e^{\tau\varphi}(\chi_1 + \chi_2) u\|^2 + \tau^{-1} \|e^{\tau\varphi} D(\chi_1 + \chi_2) u\|^2 &\lesssim \sum_{j=0}^{d} \|e^{\tau\varphi} f_j\|^2 + \|e^{\tau\varphi} u\|^2_{L^2(\Omega\setminus\operatorname{supp} a)} \\ &\quad + \lambda \|e^{\tau\varphi} u\|^2_{L^2(\{x\in\Omega,\, a(x)\geq\delta/2\})} + \tau^{-2} \|e^{\tau\varphi} [P, \chi_1] u\|^2. \end{aligned}$$

As $D(\chi_1 + \chi_2)$ is supported on $\{x \in \Omega : a(x) \geq \delta/2\}$, $\chi_1 + \chi_2 = 1$ on $\Omega \setminus \operatorname{supp} a$, and $[P, \chi_1]$ supported where $\chi_1 + \chi_2 = 1$, we have, for τ sufficiently large

$$\tau \|e^{\tau\varphi}(\chi_1 + \chi_2) u\|^2 + \tau^{-1} \|e^{\tau\varphi}(\chi_1 + \chi_2) Du\|^2 \lesssim \sum_{j=0}^{d} \|e^{\tau\varphi} f_j\|^2 + \lambda \|e^{\tau\varphi} u\|^2_{L^2(\{x\in\Omega : a(x)\geq\delta/2\})}.$$

Then as $\chi_1 + \chi_2 = 1$ on $\Omega \setminus \{x \in \Omega : a(x) > \delta/2\}$, we obtain the statement of Theorem 4. □

4. Resolvent Estimate

In this section, we prove the main result. From Batty-Duyckaerts [17], the estimate of the energy decay in Theorem 1 can be obtained through the following result.

Theorem 5. *There exists $C > 0$, such that for every $\nu \in \mathbb{R}$ with λ large, we have*

$$\left\|(\mathcal{A} - i\lambda)^{-1}\right\|_{\mathcal{L}(\mathcal{H})} \leq Ce^{C|\lambda|^{5/4}}. \tag{45}$$

Let λ be a real number such that $|\lambda|$ is large. Consider the resolvent equation:

$$F = (\mathcal{A} - i\lambda)X, \quad \text{where } X = (y_1, y_2) \in D(\mathcal{A}), \quad F = (f_1, f_2) \in \mathcal{H}, \tag{46}$$

which yields

$$\begin{cases} \operatorname{div}(\nabla y_1 + a\nabla y_2) + \lambda^2 y_1 = i\lambda f_1 + f_2, & \text{in } \Omega, \\ y_2 = i\lambda y_1 + f_1 & \text{in } \Omega, \\ y_1|_\Gamma = 0. \end{cases} \tag{47}$$

In what follows, we shall prove the solution to (46) satisfies Theorem 5. As a result, the logarithmic decay of (7) is arrived at. The proof is divided into two lemmas.

Lemma 8. *Let $\eta > 0$ and $\chi \in \mathscr{C}_0^\infty(\{x \in \Omega : a(x) > \eta\})$ be real valued functions, there exists $C > 0$ such that*

$$\|\chi y_1\| \leq C(\|\nabla y_1\|_{L^2(\{x\in\Omega : a(x)>\eta\})} + \|f_1\|_{H^1(\Omega)} + \|f_2\|_{L^2(\Omega)}),$$

where y_1, satisfies (47).

Proof. By (47), one has that y_1 satisfies

$$\operatorname{div}(\nabla y_1 + i\lambda a\nabla y_1) + \lambda^2 y_1 = i\lambda f_1 + f_2 - \operatorname{div} a\nabla f_1. \tag{48}$$

Multiplying this equation by $\chi^2\overline{y}_1$, and integrating on Ω, we obtain,

$$\begin{aligned} -\int_\Omega \nabla y_1 \nabla(\chi^2\overline{y}_1)dx - i\lambda\int_\Omega a(x)\nabla y_1\nabla(\chi^2\overline{y}_1)dx + \lambda^2\|\chi y_1\|^2 \\ = i\lambda\int_\Omega f_1\chi^2\overline{y}_1 dx + \int_\Omega f_2\chi^2\overline{y}_1 dx + \int_\Omega a(x)\nabla f_1\nabla(\chi^2\overline{y}_1)dx. \end{aligned} \tag{49}$$

Since $\nabla(\chi^2\overline{y}_1) = \chi^2\nabla\overline{y}_1 + 2\overline{y}_1\chi\nabla\chi$, we have

$$\begin{aligned} \left|\int_\Omega a(x)\nabla y_1\nabla(\chi^2\overline{y}_1)dx\right| &\lesssim \|Dy_1\|^2_{L^2(\{x\in\Omega : a(x)>\eta\})} + \|\chi y_1\|\|Dy_1\|_{L^2(\{x\in\Omega : a(x)>\eta\})} \\ &\leq 2\|Dy_1\|^2_{L^2(\{x\in\Omega : a(x)>\eta\})} + 1/8\|\chi y_1\|^2. \end{aligned} \tag{50}$$

Substituting (50) into (49), we obtain

$$\begin{aligned} \lambda^2\|\chi y_1\|^2 \lesssim \quad & (2|\lambda|+1)\|Dy_1\|^2_{L^2(\{x\in\Omega : a(x)>\eta\})} + \lambda^2/8\|\chi y_1\|^2 \\ & + \lambda^2/4\|\chi y_1\|^2 + \|f_1\|^2 + \lambda^{-2}\|f_2\|^2 + 2(1+\lambda^{-2})\|f_1\|^2_{H^1(\Omega)} \\ & + \lambda^2/8\|\chi y_1\|^2 + \|Dy_1\|^2_{L^2(\{x\in\Omega :\ a(x)>\eta\})}. \end{aligned} \tag{51}$$

Then, the proof is finished. □

Lemma 9. *For every y_1, f_1 and f_2 satisfying (47), there exists $C > 0$ such that*

$$\int_\Omega a\,|\nabla y_1|^2 dx \leq C\big(\|f_1\|^2_{H^1(\Omega)} + (\|f_1\| + \|f_2\|)\|y_1\|\big).$$

Proof. From (47), multiplying by $\overline{y}_1$, we have

$$\int_\Omega (\operatorname{div}(\nabla y_1 + a\nabla y_2) + \lambda^2 y_1)\overline{y}_1 dx = \int_\Omega (i\lambda f_1 + f_2)\overline{y}_1 dx.$$

Replacing y_2 by $f_1 + i\lambda y_1$ and integrating by parts, we have

$$\int_\Omega (-\nabla y_1 \cdot \nabla \overline{y}_1 + \lambda^2 |\, y_1 \,|^2 - a\nabla(i\lambda y_1 + f_1)\cdot\nabla\overline{y}_1)dx = \int_\Omega (i\lambda f_1 + f_2)\overline{y}_1 dx.$$

Consequently,

$$\int_\Omega [-(i\lambda a + 1)\,|\,\nabla y_1\,|^2 + \lambda^2\,|\,y_1\,|^2]dx = \int_\Omega [a\nabla f_1\cdot\nabla\overline{y}_1 + (i\lambda f_1 + f_2)\overline{y}_1]dx.$$

Taking the imaginary part, we obtain

$$\int_\Omega a\,|\,\nabla y_1\,|^2 dx = -\lambda^{-1}\operatorname{Im}\int_\Omega [a\nabla f_1\cdot\nabla\overline{y_1} + (i\lambda f_1 + f_2)\overline{y_1}]dx.$$

Using the Cauchy-Schwarz inequality in the above equality, one has

$$\begin{aligned}\int_\Omega a\,|\,\nabla y_1\,|^2 dx &\le |\,\lambda\,|^{-1}\Big(\int_\Omega a\,|\,\nabla y_1\,|^2 dx\Big)^{\frac{1}{2}}\|f_1\|_{H^1(\Omega)} + |\,\lambda\,|^{-1}(|\,\lambda\,|\,\|f_1\| + \|f_2\|)\|y_1\| \\ &\le 1/4\int_\Omega a\,|\,\nabla y_1\,|^2 dx + |\,\lambda\,|^{-2}\|f_1\|^2_{H^1(\Omega)} + |\,\lambda\,|^{-1}(|\,\lambda\,|\,\|f_1\| + \|f_2\|)\|y_1\|.\end{aligned}$$

Thus, we obtain the result. □

Proof of Theorem 5. In Lemma 8, taking $\chi(x) = 1$ is $a(x) \ge 2\eta$, we have

$$\|y_1\|^2_{L^2(\{x\in\Omega\,:\,a(x)>2\eta\})} \le C(\|\nabla y_1\|^2_{L^2(\{x\in\Omega\,:\,a(x)>\eta\})} + \|f_1\|^2_{H^1(\Omega)} + \|f_2\|^2_{L^2(\Omega)}).$$

It follows from Lemma 9 that

$$\|\nabla y_1\|^2_{L^2(\{x\in\Omega\,:\,a(x)>\eta\})} \le C\big(\|f_1\|^2_{H^1(\Omega)} + (\|f_1\| + \|f_2\|)\|y_1\|\big).$$

Both inequalities imply

$$\|y_1\|^2_{H^1(\{x\in\Omega\,:\,a(x)>2\eta\})} \le C\big(\|f_1\|^2_{H^1(\Omega)} + \|f_2\|^2_{L^2(\Omega)} + (\|f_1\| + \|f_2\|)\|y_1\|\big). \tag{52}$$

Applying Theorem 4 to y_1 satisfying (48), we obtain

$$\begin{aligned}\tau\|e^{\tau\varphi}y_1\|^2 + \tau^{-1}\|e^{\tau\varphi}Dy_1\|^2 \lesssim \lambda^2\|e^{\tau\varphi}f_1\|^2 + \|e^{\tau\varphi}f_2\|^2 + \sum_{j=1}^d \|e^{\tau\varphi}a\,\partial_{x_j}f_1\|^2 \\ + \lambda\|e^{\tau\varphi}y_1\|^2_{L^2(\{x\in\Omega\,:\,a(x)\ge\delta/2\})}.\end{aligned}$$

Let $c_1 = \min_{x\in\Omega}\varphi(x)$ and $c_2 = \max_{x\in\Omega}\varphi(x)$, we conclude from the above inequality that

$$\begin{aligned}& e^{2c_1\tau}\tau\|y_1\|^2 + \tau^{-1}e^{2c_1\tau}\|Dy_1\|^2 \\ \lesssim\ & \lambda^2 e^{2c_2\tau}\|f_1\|^2_{H^1(\Omega)} + e^{2c_2\tau}\|f_2\|^2 + \lambda e^{2c_2\tau}\|y_1\|^2_{H^1(\{x\in\Omega\,:\,a(x)\ge\delta/2\})}.\end{aligned} \tag{53}$$

Substituting (52) with $\eta = \delta/2$ into (53), we obtain

$$e^{2c_1\tau}\tau\|y_1\|^2 + \tau^{-1}e^{2c_1\tau}\|Dy_1\|^2 \lesssim \lambda^2 e^{2c_2\tau}\|f_1\|^2_{H^1(\Omega)} + e^{2c_2\tau}\|f_2\|^2 + e^{2c_2\tau}(\|f_1\| + \|f_2\|)\|y_1\|.$$

Let $c_3 = 2(c_2 - c_1) + 1$. For $\tau \geq \max\{\tilde{K}|\lambda|^{5/4}, 1\}$, One has

$$\|y_1\|^2 + \|Dy_1\|^2 \lesssim e^{c_3\tau}\|f_1\|^2_{H^1(\Omega)} + e^{c_3\tau}\|f_2\|^2 + e^{c_3\tau}(\|f_1\| + \|f_2\|)\|y_1\|.$$

Using $e^{c_3\tau}(\|f_1\| + \|f_2\|)\|y_1\| \leq \varepsilon\|y_1\|^2 + \varepsilon^{-1}e^{2c_3\tau}(\|f_1\| + \|f_2\|)^2$ in the above estimate, we conclude that

$$\|y_1\|^2 + \|Dy_1\|^2 \lesssim e^{2c_3\tau}\|f_1\|^2_{H^1(\Omega)} + e^{2c_3\tau}\|f_2\|^2,$$

which gives the desired result taking $\tau = K\,|\,\lambda\,|^{5/4}$ with K large enough. □

5. Conclusions

It is known that the stability property of the wave equation system with local viscoelastic damping depends on both continuousness and geometry of the support set of the damping function. In this paper, we obtain the logarithmic decay of a wave equation system with local Kelvin-Voigt damping, where the damping function is smooth and its support is an arbitrary non-empty subset. The approach is based on Batty-Duyckaerts' result that the resolvent estimate (45) implies the logarithmic decay of the semigroup. Through introducing proper operators, class of symbol and pseudo-diff-calculus, we obtain a Carleman estimate on the subdomain far away from the boundary. Combining these with a classical Carleman estimate up to the boundary, we arrive at the desired Carleman estimate and resolvent estimate.

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Appendix A

In this Appendix, we shall prove Lemmas 1 and 6. First, we claim that for a compactly supported and nonnegative function $a \in \mathscr{C}^2(\Omega)$, the following inequality holds:

$$|\,a'(x)\,|^2 \leq 2a(x)\|a''\|_\infty, \quad \forall\, x \in \Omega. \tag{A1}$$

In fact, from the following identity

$$a(x+h) = a(x) + a'(x)h + \int_0^1 (1-t)a''(x+th)h^2dt, \quad \forall\, h \in \mathbb{R},$$

one can get

$$a(x) + a'(x)h + \frac{1}{2}\|a''\|_\infty\,|\,h\,|^2 \geq 0.$$

Let $h = y\,a'(x)$, where $y \in \mathbb{R}$ and $x \in \Omega$ are arbitrary. It follows from the above inequality that

$$a(x) + |\,a'(x)\,|^2 y + \frac{1}{2}\|a''\|_\infty\,|\,a'(x)\,|^2 y^2 \geq 0, \quad \forall\, x \in \Omega,\ y \in \mathbb{R}.$$

Then,

$$|\,a'(x)\,|^4 - 2a(x)\|a''\|_\infty\,|\,a'(x)\,|^2 \leq 0,$$

and (A1) is proved.

Proof of Lemma 1. (i) From Definition 18.4.1 in [24], the metric $g_{x,\xi}$ defined by (8) is slowly varying if there exist $\delta > 0$ and $C > 0$ such that

$$g_{x,\xi}(y-x,\eta-\xi) \le \delta \text{ implies } g_{y,\eta}(X,\Xi) \le C g_{x,\xi}(X,\Xi), \ \ \forall\, x,y,\xi,\eta,X,\Xi \in \mathbb{R}^d,$$

where the constants δ and C are independent on the parameters λ and τ.

Suppose $0 < \delta \le 1/4$ and

$$g_{x,\xi}(y-x,\eta-\xi) = \lambda\,|\,y-x\,|^2 + (\tau^2 + |\,\xi\,|^2)^{-1}\,|\,\eta-\xi\,|^2 \le \delta.$$

Then, we have

$$\begin{aligned} \tau^2 + |\,\xi\,|^2 &\le \tau^2 + 2\,|\,\xi-\eta\,|^2 + 2\,|\,\eta\,|^2 \\ &\le \tau^2 + 2\delta(\tau^2 + |\,\xi\,|^2) + 2\,|\,\eta\,|^2. \end{aligned}$$

This implies that $\tau^2 + |\,\xi\,|^2 \le 4(\tau^2 + |\,\eta\,|^2)$. Consequently,

$$g_{y,\eta}(X,\Xi) = \lambda\,|\,X\,|^2 + (\tau^2 + |\,\eta\,|^2)^{-1}\,|\,\Xi\,|^2 \le \lambda\,|\,X\,|^2 + \frac{1}{4}(\tau^2 + |\,\xi\,|^2)^{-1}\,|\,\Xi\,|^2 \le g_{x,\xi}(X,\Xi).$$

Therefore, g is slowly varying.

For a given metric $g_{x,\xi}$, the associated metric $g^{\sigma}_{x,\xi}$ is defined by $g^{\sigma}_{x,\xi} = (\tau^2 + |\,\xi\,|^2)dx^2 + \lambda^{-1}d\xi^2$. The metric $g_{x,\xi}$ is temperate if there exist $C > 0$ and $N > 0$, such that

$$g_{x,\xi}(X,\Xi) \le C g_{y,\eta}(X,\Xi)\big(1 + g^{\sigma}_{x,\xi}(x-y,\xi-\eta)\big)^N, \ \ \forall\, x,y,\xi,\eta,X,\Xi \in \mathbb{R}^d, \tag{A2}$$

where the constants C and N are independent on the parameters λ and τ (Definition 18.5.1 in [24]).

For the metric $g = g_{x,\xi}$ defined by (8), (A2) is equivalent to

$$\begin{aligned} &\lambda\,|\,X\,|^2 + (\tau^2 + |\,\xi\,|^2)^{-1}\,|\,\Xi\,|^2 \\ \le\ & C\big(\lambda\,|\,X\,|^2 + (\tau^2 + |\,\eta\,|^2)^{-1}\,|\,\Xi\,|^2\big)\big(1 + (\tau^2 + |\,\xi\,|^2)\,|\,x-y\,|^2 + \lambda^{-1}\,|\,\xi-\eta\,|^2\big)^N. \end{aligned} \tag{A3}$$

First, assume that $\tau^2 + |\,\eta\,|^2 \le 4(\tau^2 + |\,\xi\,|^2)$. It follows that

$$(\tau^2 + |\,\eta\,|^2) \le C(\tau^2 + |\,\xi\,|^2)\big(1 + \lambda^{-1}\,|\,\xi-\eta\,|^2\big)^N, \quad C > 0,\ N > 0. \tag{A4}$$

Then it is easy to obtain (A3) from (A4).

Secondly, consider the case $\tau^2 + |\,\eta\,|^2 > 4(\tau^2 + |\,\xi\,|^2)$. Then

$$|\,\eta\,| > 2\,|\,\xi\,|, \quad |\,\eta\,| > \sqrt{3}\tau, \tag{A5}$$

and

$$|\,\xi-\eta\,| > \frac{1}{2}\,|\,\eta\,| > \frac{\sqrt{3}}{2}\tau > \frac{\sqrt{3}}{2}\mathcal{C}\lambda. \tag{A6}$$

It follows from (A5) and (A6) that

$$\lambda^{-1}\,|\,\xi-\eta\,|^2 > \frac{\sqrt{3}}{2}\mathcal{C}\,|\,\xi-\eta\,| > \frac{\sqrt{3}}{4}\mathcal{C}\,|\,\eta\,|.$$

Consequently,

$$\big(1 + \lambda^{-1}\,|\,\xi-\eta\,|^2\big)^2 > \frac{3}{16}\mathcal{C}^2\,|\,\eta\,|^2 > \frac{3}{32}\mathcal{C}^2(\,|\,\eta\,|^2 + 3\tau^2).$$

This together with $\tau^2 + |\xi|^2 \geq 1$ yields that there exists a positive constant C such that (A4) holds with $N = 2$.

(ii) It is known from Definition 18.4.2 in [24] that a weight $\nu(x)$ is g-continuous if there exist $\delta > 0$ and $C > 0$ such that

$$g_{x,\xi}(y - x, \eta - \xi) \leq \delta \text{ implies } C^{-1}\nu(x) \leq \nu(y) \leq C\nu(x), \ \forall\, x, y, \xi, \eta \in \mathbb{R}^d.$$

where the constants δ and C are independent on the parameters λ and τ. Since the weight $\nu(x)$ defined by (9) does not depend on ξ, the above condition is reduced to

$$\lambda |x - y|^2 \leq \delta \text{ implies } C^{-1}\nu(x) \leq \nu(y) \leq C\nu(x), \ \forall\, x, y \in \mathbb{R}^d.$$

The weight $\nu(x)$ is g-temperate if there exist $C > 0$ and $N > 0$ such that

$$\nu(y) \leq C\nu(x)\big(1 + g^{\sigma}_{y,\eta}(x - y, \xi - \eta)\big)^N, \ \forall\, x, y, \xi, \eta \in \mathbb{R}^d, \tag{A7}$$

where the constants C and N do not depend on the parameters λ and τ (18.5.1 in [24]). The weight $\nu(x)$ is admissible if it is g-continuous and g-temperate. When a weight is admissible, all the powers of this weight are g-continuous and g-temperate. Therefore, it suffice to prove that $1 + \lambda a(x)$ is admissible.

Let $s \in [0, t]$ and $t \in [0, 1]$. Define $f(s) = \lambda a(x + s(y - x))$ and $F(t) = \sup_{s\in[0,t]} f(s)$ where $x,\ y \in \Omega$ satisfying $\lambda |x - y|^2 \leq \delta$. It is clear that $f'(s) = \lambda\, a'(x + s(y - x))\,(y - x)$. Combining this with (A1) yields

$$|f'(s)| \leq \lambda |a'(x + s(y - x))|\,|y - x| \leq 2\lambda \|a''\|_\infty^{\frac{1}{2}} [a(x + s(y - x))]^{\frac{1}{2}} |y - x|.$$

Consequently,

$$\sup_{s\in[0,t]} |f'(s)| \leq 2\lambda^{\frac{1}{2}} \|a''\|_\infty^{\frac{1}{2}} F(t)^{\frac{1}{2}} |y - x|.$$

Since $f(t) \leq f(0) + t \sup_{s\in[0,t]} |f'(s)|$, F is non-decreasing and $\lambda |x - y|^2 \leq \delta$, we obtain that for all $t \in [0, 1]$,

$$f(t) \leq f(0) + C\lambda^{\frac{1}{2}} F(t)^{\frac{1}{2}} |y - x| \leq f(0) + C\sqrt{\delta} F(t)^{\frac{1}{2}} \leq f(0) + C\sqrt{\delta} F(\alpha)^{\frac{1}{2}},$$

where $C = 2\|a''\|_\infty^{\frac{1}{2}}$ and $\alpha \in [t, 1]$. Note that $f(0) = F(0)$. It follows that

$$F(\alpha) = \sup_{t\in[0,\alpha]} f(t) \leq F(0) + C\sqrt{\delta} F(\alpha)^{\frac{1}{2}}.$$

This yields

$$1 + F(\alpha) \leq 1 + F(0) + C\sqrt{\delta}\big(1 + F(\alpha)\big)^{\frac{1}{2}} \leq 1 + F(0) + C\sqrt{\delta}\big(1 + F(\alpha)\big). \tag{A8}$$

By choosing δ sufficiently small such that $C\sqrt{\delta} \leq 1/2$, one can deduce from (A8) that

$$1 + F(\alpha) \leq 2(1 + F(0)), \quad \forall\, \alpha \in [t, 1].$$

In particular, we have

$$1 + \lambda a(y) \leq 2(1 + \lambda a(x)).$$

The above inequality remains true if we exchange x and y. Therefore, the weight $1 + \lambda a(x)$ is g-continuous.

On the other hand, note that $1+\lambda a(x)$ is independent to ξ. Then, to obtain the weight $1+\lambda a(x)$ is σ-temperate, it is sufficient to prove that

$$1+\lambda a(y) \leq C(1+\lambda a(x))(1+\tau^2 \,|\, x-y \,|^2)^N. \tag{A9}$$

In fact, it is clear that $1+\lambda a(y) \leq 1+\lambda(a(x)+C \,|\, x-y \,|)$ where $C=\|a'\|_\infty$. Therefore, there exists positive constant $C'=C\mathcal{C}^{-1}$ such that

$$1+\lambda a(y) \leq (1+\lambda a(x))(1+C'\tau \,|\, x-y \,|) \leq (1+\lambda a(x))(2+2(C'\tau \,|\, x-y \,|)^2)^{\frac{1}{2}}.$$

Thus, we obtain (A9) with $N=\frac{1}{2}$, $C=2\max\{1,\ C'\}$. □

Proof of Lemma 6. In what follows, we use the symbolic calculus with $\lambda=1$ since the symbol does not depend on λ. First, by homogeneity in (ξ,τ), compactness arguments and sub-ellipticity condition, we claim that there exist constants C, $\delta>0$ such that

$$C\big[\,|\,2\xi\cdot\nabla\varphi(x)\,|^2+\mu^{-2}\,(\,|\xi|^2-\tau^2|\nabla\varphi(x)|^2\,)^2\big]+\{|\xi|^2-\tau^2|\nabla\varphi(x)|^2,\ 2\xi\cdot\nabla\varphi(x)\}\geq\delta\mu^2. \tag{A10}$$

The proof of (A10) is classical. In fact, set

$$\mathcal{K}=\{(x,\xi,\tau)\in\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}\ :\ x\in\overline{\Omega},\ |\,\xi\,|^2+\tau^2=1,\ \tau\geq 0\},$$

and for $(x,\xi,\tau)\in\mathcal{K},\ \kappa>0$,

$$G(x,\xi,\tau,\kappa)=\kappa\big[\,|\,2\xi\cdot\nabla\varphi(x)\,|^2+\mu^{-2}\,(\,|\xi|^2-\tau^2|\nabla\varphi(x)|^2\,)^2\big]+\{\xi^2-\tau^2|\nabla\varphi(x)|^2,\ 2\xi\cdot\nabla\varphi(x)\}.$$

If $p_\varphi=0$ for $(x,\xi,\tau)\in\mathcal{K}$, then $|\,2\xi\cdot\nabla\varphi(x)\,|^2+\mu^{-2}\,(\,|\xi|^2-\tau^2|\nabla\varphi(x)|^2\,)^2=0$. it is clear that there exists a positive constant δ such that (A10) holds due to the fact that ϕ is sub-elliptic. When $|\,2\xi\cdot\nabla\varphi(x)\,|^2+\mu^{-2}\,(\,|\xi|^2-\tau^2|\nabla\varphi(x)|^2\,)^2>0$, there exists a positive constant $\kappa_{x,\xi,\tau}$ such that $G(x,\xi,\tau,\kappa)>0$ for every $\kappa\geq\kappa_{x,\xi,\tau}$ since $\{\xi^2-\tau^2|\nabla\varphi(x)|^2,\ 2\xi\cdot\nabla\varphi(x)\}$ is bounded on $\mathcal{K}$. By continuity of $G(x,\xi,\tau,\kappa)$, there exists a neighborhood of (x,ξ,τ), denoted by $V_{x,\xi,\tau}$, such that $G(x,\xi,\tau,\kappa)>0$ for all $(x,\xi,\tau)\in V_{x,\xi,\tau}$ and $\kappa\geq\kappa_{x,\xi,\tau}$. Since $\mathcal{K}$ is compact, there exist finite sets $V_j=V_{x_j,\xi_j,\tau_j}$ and corresponding constants $\kappa_j=\kappa_{x_j,\xi_j,\tau_j}$ $(i=1,2,\cdots,n)$, such that $\mathcal{K}\subset\cup_{i=1}^n V_j$ and $G(x,\xi,\tau,\kappa)>0$ for all $(x,\xi,\tau)\in V_j$ and $\kappa>\kappa_j$. Let $\tilde{\kappa}=\max\{\kappa_j\ :\ j=1,2,\cdots,n\}$. It follows that $G(x,\xi,\tau,\tilde{\kappa})>0$ for all $(x,\xi,\tau)\in\mathcal{K}$ and $\kappa\geq\tilde{\kappa}$. Finally, using the compactness of $\mathcal{K}$ again, we conclude that there exists $\delta>0$ such that $G(x,\xi,\tau,\tilde{\kappa})\geq\delta$. Thus, (A10) is reached since g is a homogeneous function of degree 2 with respect to variables (ξ,τ).

By Gårding inequality (11), there exists a constant $\widetilde{C}>0$ such that, for $\tau\geq\tau_0$ where τ_0 sufficiently large,

$$\begin{aligned}\widetilde{C}\|\,\mathrm{Op}(\mu)w\|^2\leq\ \ &\mathrm{Re}\,\big(\,\mathrm{Op}\,(\,|\,2\xi\cdot\nabla\varphi(x)\,|^2+\mu^{-2}\,(\,|\xi|^2-\tau^2|\nabla\varphi(x)|^2\,)^2\\&+\{|\xi|^2-\tau^2|\nabla\varphi(x)|^2,\ 2\xi\cdot\nabla\varphi(x)\}\big)w\,|\,w\big).\end{aligned} \tag{A11}$$

Now we are going to estimate the terms $\tau\,\mathrm{Op}(\,|\,2\xi\cdot\nabla\varphi(x)\,|^2)$ and $\tau\,\mathrm{Op}(\mu^{-2}\,(\,|\xi|^2-\tau^2|\nabla\varphi(x)|^2\,)^2)$. Firstly, it follows from Lemma 3 that

$$\tau^{-1}\,\mathrm{Op}(\,|\,2\tau\xi\cdot\nabla\varphi(x)\,|^2)=\tau^{-1}\,\mathrm{Op}(2\tau\xi\cdot\nabla\varphi(x))^*\,\mathrm{Op}(2\tau\xi\cdot\nabla\varphi(x))+\tau\,\mathrm{Op}(r_1), \tag{A12}$$

where $r_1 \in S(\mu, \tilde{g})$ and $\tilde{g}$ is defined by (10). Therefore, for any $\varepsilon > 0$, there exists a positive constant C_ε such that

$$\begin{aligned} & \left|\left(\tau^{-1}\operatorname{Op}(\,|\,2\tau\xi\cdot\nabla\varphi(x)\,|\,^2)w\,|\,w\right)\right| \\ \leq\ & \tau^{-1}\|\operatorname{Op}(2\tau\xi\cdot\nabla\varphi(x))w\|^2+\tau\,|\,(\operatorname{Op}(r_1)w\,|\,w)\,| \\ \leq\ & \tau^{-1}\|\operatorname{Op}(2\tau\xi\cdot\nabla\varphi(x))w\|^2+\varepsilon\,\tau\|\operatorname{Op}(\mu)w\|^2+C_\varepsilon\tau\|w\|^2. \end{aligned} \tag{A13}$$

Substituting (A13) into (A11) and choosing ε small enough, we have

$$\begin{aligned} & C\tau\|\operatorname{Op}(\mu)w\|^2 \\ \leq\ & \operatorname{Re}\left(\operatorname{Op}\left(\tau\mu^{-2}\left(\,|\xi|^2-\tau^2|\nabla\varphi(x)|^2\,\right)^2+\tau\left\{|\xi|^2-\tau^2|\nabla\varphi(x)|^2,\ 2\xi\cdot\nabla\varphi(x)\right\}\right)w\,\middle|\,w\right) \\ & +\tau^{-1}\|\operatorname{Op}(2\tau\xi\cdot\nabla\varphi(x))w\|^2+C_\varepsilon\tau\|w\|^2. \end{aligned} \tag{A14}$$

Secondly, by symbolic calculus, we have that

$$\tau\operatorname{Op}(\mu^{-2}\left(\,|\xi|^2-\tau^2|\nabla\varphi(x)|^2\,\right)^2)=\tau\operatorname{Op}(r_0)\operatorname{Op}(|\xi|^2-\tau^2|\nabla\varphi(x)|^2)+\tau\operatorname{Op}(r_2), \tag{A15}$$

where $r_0(x,\xi)=\mu^{-2}(|\xi|^2-\tau^2|\nabla\varphi(x)|^2)\in S(1,\tilde{g})$ and $r_2\in S(\mu,\tilde{g})$. Therefore, for all $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$\begin{aligned} & |\,(\tau\operatorname{Op}(r_0)\operatorname{Op}(|\xi|^2-\tau^2|\nabla\varphi(x)|^2)w\,|\,w)\,| \\ \leq\ & C_\varepsilon\tau^{-1}\|\operatorname{Op}(r_0)\operatorname{Op}(|\xi|^2-\tau^2|\nabla\varphi(x)|^2)w\|^2+\varepsilon\tau^3\|w\|^2 \\ \leq\ & C_\varepsilon\tau^{-1}\|\operatorname{Op}(|\xi|^2-\tau^2|\nabla\varphi(x)|^2)w\|^2+\varepsilon\tau^3\|w\|^2. \end{aligned} \tag{A16}$$

We choose ε small enough and combine (A15) and (A16) with (A14) to get

$$\begin{aligned} & C\tau\|\operatorname{Op}(\mu)w\|^2 \\ \leq\ & \operatorname{Re}\left(\tau\left\{|\xi|^2-\tau^2|\nabla\varphi(x)|^2,\ 2\xi\cdot\nabla\varphi(x)\right\}w\,\middle|\,w\right)+C_\varepsilon\tau^{-1}\|\operatorname{Op}(|\xi|^2-\tau^2|\nabla\varphi(x)|^2)w\|^2 \\ & +\tau^{-1}\|\operatorname{Op}(2\tau\xi\cdot\nabla\varphi(x))w\|^2+C_\varepsilon(\tau+\varepsilon\tau^3)\|w\|^2. \end{aligned} \tag{A17}$$

Finally, it is clear that there exist positive constant C such that

$$\tau^3\|w\|^2+\tau\|Dw\|^2\leq C\tau\|\operatorname{Op}(\mu)w\|^2. \tag{A18}$$

Thus, we obtain (14) by using (A17) and (A18), choosing ε small enough and letting $\tau>\tau_0$ big enough. □

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