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# ERROR ESTIMATES OF ENERGY REGULARIZATION FOR THE LOGARITHMIC SCHRÖDINGER EQUATION

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Abstract. The logarithmic nonlinearity has been used in many partial differential equations (PDEs) for modeling problems in different applications. Due to the singularity of the logarithmic function, it introduces tremendous difficulties in establishing mathematical theories and in designing and analyzing numerical methods for PDEs with logarithmic nonlinearity. Here we take the logarithmic Schrödinger equation (LogSE) as a prototype model. Instead of regularizing $f(\rho) = \ln \rho$ in the LogSE directly as being done in the literature, we propose an energy regularization for the LogSE by first regularizing $F(\rho) = \rho \ln \rho - \rho$ near $\rho = 0^+$ with a polynomial approximation in the energy functional of the LogSE and then obtaining an energy regularized logarithmic Schrödinger equation (ERLogSE) via energy variation. Linear convergence is established between the solutions of ERLogSE and LogSE in terms of a small regularization parameter $0 < \varepsilon \ll 1$. Moreover, the conserved energy of the ERLogSE converges to that of LogSE quadratically. Error estimates are also established for solving the ERLogSE by using Lie-Trotter splitting integrators. Numerical results are reported to confirm our error estimates of the energy regularization and of the time-splitting integrators for the ERLogSE. Finally our results suggest that energy regularization performs better than regularizing the logarithmic nonlinearity in the LogSE directly.

## 1. Introduction

Nowadays, the logarithmic nonlinearity has been adopted frequently in many fields. For example, it admits applications to quantum mechanics and quantum optics (logarithmic Schrödinger equation (LogSE) [13–15,42]), oceanography and fluid dynamics (logarithmic Korteweg-de Vries (KdV) equation or logarithmic Kadomtsev-Petviashvili (KP) equation [37, 48, 49]), quantum field theory and inflation cosmology (logarithmic Klein-Gordon equation [11, 33, 47]), material sciences (Cahn-Hilliard (CH) equation with logarithmic potentials [22, 26, 31]). Recently, the heat equation with a logarithmic nonlinearity has been investigated mathematically [1, 20].

The logarithmic nonlinearity was selected by assuming the separability of non-interacting subsystems property (cf. [13]), which means that a solution of the nonlinear equation for the whole system can be constructed, as in the linear theory, by taking the product of two arbitrary solutions of the nonlinear equations for the

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subsystems. In other words, no correlations are introduced for noninteracting subsystems. As for the physical reality, robust physical grounds have been found for the application of the equations with logarithmic nonlinearity. For instance, it was found in the stochastic formulation of quantum mechanics [43, 46] that the logarithmic nonlinear term originates naturally from an internal stochastic force due to quantum fluctuations. Such kind of nonlinearity also appears naturally in inflation cosmology and in supersymmetric field theories [10, 28].

There are some special analytic solutions for the logarithmic mechanics (cf. [41]). For example, the logarithmic KdV equation, the logarithmic KP equation, the logarithmic Klein-Gordon equation give Gaussons: solitary wave solutions with Gaussian shapes [48, 49]. Particularly, the solution of the LogSE remains Gaussian for all time if the initial data is Gaussian and solving the LogSE is equivalent to solving ordinary differential equations [16]; this property remains in the case of the logarithmic heat equation ( [1]), but not in the case of, e.g., the logarithmic KdV equation, the logarithmic KP equation, or the logarithmic Klein-Gordon equation (as can be seen directly by trying to plug time dependent Gaussian function into these equations).

The well-posedness of the Cauchy problem for the logarithmic equations is not trivial since the logarithmic nonlinearity is not locally Lipschitz continuous due to the singularity of the logarithm. The regularized free energy has been widely used for proving the existence of the CH equation with a logarithmic potential [27] and the LogSE [17] by a compactness argument.

It is also very challenging to design and analyze numerical schemes for the logarithmic equations due to the singularity of the logarithmic nonlinearity. There have been extensive numerical works for the CH equation with a logarithmic Flory Huggins energy potential [21, 23, 32, 38, 39, 50] and recently several works for the LogSE [4, 5, 44] where some kinds of regularization were proposed for either the PDEs or the energy functional. Specifically, the regularized energy functional was adopted for the CH equation with a logarithmic free energy [23, 50], and a regularized logarithmic Schrödinger equation was introduced and analyzed for solving for the LogSE [4, 5].

In this paper, we aim to design and analyze numerical methods for the logarithmic equations via an energy regularization. We take the LogSE as an example and the regularization can be extended to other logarithmic equations. Consider the LogSE which arises in a model of nonlinear wave mechanics (cf. [13]),

$$\begin{cases} i\partial_t u(\mathbf{x},t) = -\Delta u(\mathbf{x},t) + \lambda\, u(\mathbf{x},t)\, f(|u(\mathbf{x},t)|^2), \quad \mathbf{x}\in\Omega, \quad t>0, \\ u(\mathbf{x},0) = u_0(\mathbf{x}), \quad \mathbf{x}\in\overline{\Omega}, \end{cases} \tag{1.1}$$

where $t$ and $\mathbf{x}\in\mathbb{R}^d$ ($d=1,2,3$) represent the temporal and spatial coordinates, respectively, $\lambda\in\mathbb{R}\backslash\{0\}$ measures the force of the nonlinear interaction, $u := u(\mathbf{x},t)\in\mathbb{C}$ is the dimensionless wave function, $\Omega=\mathbb{R}^d$ or $\Omega\subset\mathbb{R}^d$ is a bounded domain with homogeneous Dirichlet or periodic boundary condition fixed on a Lipschitz continuous boundary, and

$$f(\rho) = \ln\rho, \qquad \rho>0, \qquad \text{with} \quad \rho=|u|^2. \tag{1.2}$$

This model has been widely applied in quantum mechanics, nuclear physics, geophysics, open quantum systems and Bose-Einstein condensation [3, 24, 35, 36, 51]. There have been extensive literatures concerning the LogSE. It is easy to see that the LogSE (1.1) conserves the *mass* defined as

$$N(t) := N(u(\cdot,t)) = \|u\|^2 = \int_\Omega |u(\mathbf{x},t)|^2 d\mathbf{x} \equiv N(u_0), \qquad t \geq 0, \tag{1.3}$$

and the *energy* defined as

$$\begin{aligned} E(t) :&= E(u(\cdot,t)) = \int_\Omega \left[|\nabla u(\mathbf{x},t)|^2 d\mathbf{x} + \lambda F(|u(\mathbf{x},t)|^2)\right] d\mathbf{x} \\ &\equiv \int_\Omega \left[|\nabla u_0(\mathbf{x})|^2 + \lambda F(|u_0(\mathbf{x})|^2)\right] d\mathbf{x} = E(u_0), \qquad t \geq 0, \end{aligned} \tag{1.4}$$

where

$$F(\rho) = \int_0^\rho f(s) ds = \int_0^\rho \ln s \, ds = \rho \ln \rho - \rho, \qquad \rho \geq 0. \tag{1.5}$$

For the Cauchy problem (1.1) in a suitable functional framework, we refer to [16, 18, 34]. For the stability properties of standing waves for (1.1), we refer to [2, 17, 19]. For the analysis of breathers and the existence of multisolitons, see [29, 30].

In order to avoid numerical blow-up of the logarithmic nonlinearity at the origin by directly regularizing the logarithmic function $f(\rho)$ in (1.2) (and thus in (1.1)) with a small regularized parameter $0 < \varepsilon \ll 1$, two models of regularized logarithmic Schrödinger equation (RLogSE) were proposed in [5]:

$$\begin{cases} i\partial_t u^\varepsilon(\mathbf{x},t) = -\Delta u^\varepsilon(\mathbf{x},t) + \lambda \, u^\varepsilon(\mathbf{x},t) \, \widetilde{f}^\varepsilon(|u^\varepsilon(\mathbf{x},t)|)^2), \quad \mathbf{x} \in \Omega, \quad t > 0, \\ u^\varepsilon(\mathbf{x},0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \overline{\Omega}, \end{cases} \tag{1.6}$$

and

$$\begin{cases} i\partial_t u^\varepsilon(\mathbf{x},t) = -\Delta u^\varepsilon(\mathbf{x},t) + \lambda \, u^\varepsilon(\mathbf{x},t) \, \widehat{f}^\varepsilon(|u^\varepsilon(\mathbf{x},t)|^2)), \quad \mathbf{x} \in \Omega, \quad t > 0, \\ u^\varepsilon(\mathbf{x},0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \overline{\Omega}, \end{cases} \tag{1.7}$$

where $\widetilde{f}^\varepsilon(\rho)$ and $\widehat{f}^\varepsilon(\rho)$ are two different types of regularization for $f(\rho)$ and they are given as

$$\widetilde{f}^\varepsilon(\rho) = \ln(\varepsilon + \sqrt{\rho})^2, \quad \widehat{f}^\varepsilon(\rho) = \ln(\varepsilon^2 + \rho), \quad \rho \geq 0, \qquad \text{with} \quad \rho = |u^\varepsilon|^2. \tag{1.8}$$

Again, the RLogSEs (1.6) and (1.7) conserve the mass (1.3) with $u = u^\varepsilon$, and conserve the *energies*

$$\widetilde{E}^\varepsilon(t) := \widetilde{E}^\varepsilon(u^\varepsilon(\cdot,t)) = \int_\Omega \left[|\nabla u^\varepsilon(\mathbf{x},t)|^2 d\mathbf{x} + \lambda \widetilde{F}^\varepsilon(|u^\varepsilon(\mathbf{x},t)|^2)\right] d\mathbf{x} \equiv \widetilde{E}^\varepsilon(u_0), \tag{1.9}$$

and

$$\widehat{E}^\varepsilon(t) := \widehat{E}^\varepsilon(u^\varepsilon(\cdot,t)) = \int_\Omega \left[|\nabla u^\varepsilon(\mathbf{x},t)|^2 d\mathbf{x} + \lambda \widehat{F}^\varepsilon(|u^\varepsilon(\mathbf{x},t)|^2)\right] d\mathbf{x} \equiv \widehat{E}^\varepsilon(u_0), \tag{1.10}$$

respectively, with

$$\begin{aligned} &\widetilde{F}^\varepsilon(\rho) = \int_0^\rho \widetilde{f}^\varepsilon(s) ds = \rho \ln(\varepsilon + \sqrt{\rho})^2 + 2\varepsilon\sqrt{\rho} - \rho - \varepsilon^2 \ln(1 + \sqrt{\rho}/\varepsilon)^2, \\ &\widehat{F}^\varepsilon(\rho) = \int_0^\rho \widehat{f}^\varepsilon(s) ds = (\varepsilon^2 + \rho) \ln(\varepsilon^2 + \rho) - \rho - 2\varepsilon^2 \ln \varepsilon, \end{aligned} \qquad \rho \geq 0. \tag{1.11}$$

The idea of this regularization is that the function $\rho \mapsto \ln \rho$ causes no (analytical or numerical) problem for large values of $\rho$, but is singular at $\rho = 0$. A linear convergence was established between the solutions of the LogSE (1.1) and the regularized model (1.6) or (1.7) for bounded $\Omega$ in terms of the small regularization parameter $0<\varepsilon\ll 1$, i.e.,

$$\sup_{t\in[0,T]}\|u^\varepsilon(t)-u(t)\|_{L^2(\Omega)}=O(\varepsilon),\quad \forall\, T>0.$$

Applying this regularized model, a semi-implicit FDM or a time-splitting method was proposed and analyzed for the LogSE (1.6) [4, 5]. The above regularization saturates the nonlinearity in the region $\{\rho<\varepsilon^2\}$ (where $\rho=|u^\varepsilon|^2$), but of course also has some (smaller) effect in the other region $\{\rho>\varepsilon^2\}$.

Energy regularization has been widely adapted in different areas for dealing with singularity and/or roughness, such as in materials science for establishing the well-posedness of the Cauchy problem for the CH equation with a logarithmic potential [27] and for treating strongly anisotropic surface energy [7, 40], in mathematical physics for the well-posedness of the LogSE [17], and in scientific computing on designing regularized numerical methods for treating singularities [9, 23, 50]. The main aim of this paper is to present an energy regularization for the LogSE (1.1) with three key features as: (i) to first regularize $F(\rho)\in C^0([0,+\infty))$ in (1.5) (and thus in the energy functional (1.4)), (ii) to regularize the nonlinearity $F(\rho)$ only in the region $\{\rho<\varepsilon^2\}$ by a sequence of polynomials and keep it unchanged in $\{\rho>\varepsilon^2\}$ such that the regularized function has given regularity; and (iii) to obtain a sequence of energy regularized logarithmic Schrödinger equations (ERLogSEs) from the regularized energy functional via energy variation. Unlike in [23,50] where the energy is approximated by a second order polynomial near the origin, here we present a systematic way to regularize the energy density near the origin by a sequence of polynomials such that the overall regularized energy has up to arbitrary order of smoothness. We will establish convergence rates between the solutions of ERLogSEs and LogSE in terms of the small regularized parameter $0<\varepsilon\ll 1$. In addition, we will also prove error estimates of numerical approximations of ERLogSEs by using time-splitting integrators.

The rest of this paper is organized as follows. In Section 2, we introduce a sequence of regularization for the logarithmic potential. A regularized model is derived and analyzed in Section 3 via the energy regularization of the LogSE. Some numerical methods are proposed and analyzed in Section 4. Section 5 is devoted to displaying some numerical experiments. Throughout the paper, we adopt the standard $L^2$-based Sobolev spaces as well as the corresponding norms and denote $C$ to represent a generic constant independent of $\varepsilon$, time step $\tau$ and the function $u$, and $C(c)$ is a generic constant depending on $c$.

## 2. Regularization for $F(\rho)=\rho\ln\rho-\rho$

We consider a regularization starting from an approximation to the energy density $F(\rho)$ in (1.5) (and thus in (1.4)).

2.1. **A sequence of regularization.** In order to make a comparison with the former regularization (1.6) sensible, we again distinguish the regions $\{\rho>\varepsilon^2\}$ and $\{\rho<\varepsilon^2\}$, but instead of saturating the nonlinearity in the second region, we regularize it as follows. Let $n\ge 2$ be an integer, we approximate $F(\rho)$ by a piecewise smooth function with a polynomial near the origin, i.e.,

$$F_n^\varepsilon(\rho)=F(\rho)\chi_{\{\rho\ge\varepsilon^2\}}+P_{n+1}^\varepsilon(\rho)\chi_{\{\rho<\varepsilon^2\}},\quad n\ge 2, \tag{2.1}$$

where $\varepsilon$ is a small parameter, $\chi_A$ is the characteristic function of the set $A$ and $P_{n+1}^{\varepsilon}$ is a polynomial of degree $n+1$, which satisfies $F_n^{\varepsilon} \in C^n([0,+\infty))$ and $F_n^{\varepsilon}(0) = F(0) = 0$ (this allows the regularized energy to be well-defined on the whole space). Next we derive the explicit expressions of $P_{n+1}^{\varepsilon}(\rho)$. In view of $P_{n+1}^{\varepsilon}(0) = 0$, write

$$P_{n+1}^{\varepsilon}(\rho) = \rho\, Q_n^{\varepsilon}(\rho), \tag{2.2}$$

with $Q_n^{\varepsilon}$ being a polynomial of degree of $n$. Correspondingly, denote $F(\rho) = \rho\, Q(\rho)$ with $Q(\rho) = \ln \rho - 1$. By the continuity conditions, we get

$$P_{n+1}^{\varepsilon}(\varepsilon^2) = F(\varepsilon^2), \quad (P_{n+1}^{\varepsilon})'(\varepsilon^2) = F'(\varepsilon^2), \quad \ldots, \quad (P_{n+1}^{\varepsilon})^{(n)}(\varepsilon^2) = F^{(n)}(\varepsilon^2),$$

which implies that

$$Q_n^{\varepsilon}(\varepsilon^2) = Q(\varepsilon^2), \quad (Q_n^{\varepsilon})'(\varepsilon^2) = Q'(\varepsilon^2), \quad \ldots, \quad (Q_n^{\varepsilon})^{(n)}(\varepsilon^2) = Q^{(n)}(\varepsilon^2).$$

Thus $Q_n^{\varepsilon}$ is nothing else but the $n$-th degree Taylor polynomial of $Q$ at $\rho = \varepsilon^2$, i.e.,

$$Q_n^{\varepsilon}(\rho) = Q(\varepsilon^2) + \sum_{k=1}^{n} \frac{Q^{(k)}(\varepsilon^2)}{k!}(\rho - \varepsilon^2)^k = \ln \varepsilon^2 - 1 - \sum_{k=1}^{n} \frac{1}{k}\left(1 - \frac{\rho}{\varepsilon^2}\right)^k. \tag{2.3}$$

In particular, Taylor's formula yields

$$Q(\rho) - Q_n^{\varepsilon}(\rho) = \int_{\varepsilon}^{\rho} Q^{(n+1)}(s)\frac{(\rho - s)^n}{n!}ds = \int_{\varepsilon}^{\rho} \frac{(s-\rho)^n}{s^{n+1}}ds. \tag{2.4}$$

Plugging (2.3) into (2.2), one gets the explicit formula of $P_{n+1}^{\varepsilon}(\rho)$. We emphasize a formula which will be convenient for convergence results:

$$\left(Q_n^{\varepsilon}\right)'(\rho) = \frac{1}{\varepsilon^2}\sum_{k=1}^{n}\left(1 - \frac{\rho}{\varepsilon^2}\right)^{k-1} = \frac{1}{\rho}\left(1 - \left(1 - \frac{\rho}{\varepsilon^2}\right)^n\right), \qquad 0 \le \rho \le \varepsilon^2. \tag{2.5}$$

2.2. **Properties of the regularization functions.** Differentiating (2.1) with respect to $\rho$ and noting (2.2), (2.3) and (2.5), we get

$$f_n^{\varepsilon}(\rho) = (F_n^{\varepsilon})'(\rho) = \ln \rho\, \chi_{\{\rho \ge \varepsilon^2\}} + q_n^{\varepsilon}(\rho)\chi_{\{\rho < \varepsilon^2\}}, \qquad \rho \ge 0, \tag{2.6}$$

where

$$\begin{aligned} q_n^{\varepsilon}(\rho) &= (P_{n+1}^{\varepsilon})'(\rho) = Q_n^{\varepsilon}(\rho) + \rho\,(Q_n^{\varepsilon})'(\rho) \\ &= \ln(\varepsilon^2) - \frac{n+1}{n}\left(1 - \frac{\rho}{\varepsilon^2}\right)^n - \sum_{k=1}^{n-1}\frac{1}{k}\left(1 - \frac{\rho}{\varepsilon^2}\right)^k. \end{aligned}$$

Noticing that $q_n^{\varepsilon}$ is increasing in $[0, \varepsilon^2]$, $\widetilde{f}^{\varepsilon}$ and $\widehat{f}^{\varepsilon}$ are obviously increasing in $[0,\infty)$, thus all the three types of regularization (2.1) and (1.11) preserve the convexity of $F$. Moreover, as a sequence of regularization (or approximation) for the semi-smooth function $F(\rho) \in C^0([0,\infty)) \cap C^{\infty}((0,\infty))$, we have $F_n^{\varepsilon} \in C^n([0,+\infty))$ for $n \ge 2$, while $\widetilde{F}^{\varepsilon} \in C^1([0,\infty)) \cap C^{\infty}((0,\infty))$ and $\widehat{F}^{\varepsilon} \in C^{\infty}([0,\infty))$. Similarly, as a sequence of regularization (or approximation) for the logarithmic function $f(\rho) = \ln \rho \in C^{\infty}((0,\infty))$, we observe that $f_n^{\varepsilon} \in C^{n-1}([0,\infty))$ for $n \ge 2$, while $\widehat{f}^{\varepsilon} \in C^{\infty}([0,\infty))$ and $\widetilde{f}^{\varepsilon} \in C^0([0,\infty)) \cap C^{\infty}((0,\infty))$.

Next we highlight some properties of $f_n^{\varepsilon}$. Before this, we present the following lemma established initially in [18, Lemma 1.1.1].

**Lemma 2.1.** *For $z_1, z_2 \in \mathbb{C}$, we have*

$$\left|\mathrm{Im}\left(\left(z_1\ln(|z_1|^2) - z_2\ln(|z_2|^2)\right)\left(\overline{z_1}-\overline{z_2}\right)\right)\right| \le 2|z_1-z_2|^2,$$

*where* $\mathrm{Im}(z)$ *and* $\bar(z)$ *denote the imaginary part and complex conjugate of $z$, respectively.*

**Lemma 2.2.** *Let $n \ge 2$ and $\varepsilon > 0$. For $z_1,\ z_2 \in \mathbb{C}$, we have*

$$|f_n^\varepsilon(|z_1|^2) - f_n^\varepsilon(|z_2|^2)| \le \frac{C(n)|z_1-z_2|}{\max\{\varepsilon, \min\{|z_1|,|z_2|\}\}}, \tag{2.7}$$

$$\left|\mathrm{Im}\left[\left(z_1 f_n^\varepsilon(|z_1|^2) - z_2 f_n^\varepsilon(|z_2|^2)\right)\left(\overline{z_1}-\overline{z_2}\right)\right]\right| \le C(n)|z_1-z_2|^2, \tag{2.8}$$

$$|\rho(f_n^\varepsilon)'(\rho)| \le n+1, \quad |\sqrt{\rho}(f_n^\varepsilon)'(\rho)| \le \frac{2n}{\varepsilon}, \quad |\rho^{3/2}(f_n^\varepsilon)''(\rho)| \le \frac{C(n)}{\varepsilon}, \quad \rho \ge 0, \tag{2.9}$$

where $C(n) > 0$ is a general constant depending on $n$, but independent of $\varepsilon \in (0,1]$.

*Proof.* When $|z_1|, |z_2| \ge \varepsilon$, we have

$$\left|f_n^\varepsilon(|z_1|^2) - f_n^\varepsilon(|z_2|^2)\right| = 2\ln\left(1 + \frac{\max\{|z_1|,|z_2|\} - \min\{|z_1|,|z_2|\}}{\min\{|z_1|,|z_2|\}}\right) \le \frac{2|z_1-z_2|}{\min\{|z_1|,|z_2|\}}.$$

When $|z_1|, |z_2| \le \varepsilon$, direct calculation gives

$$\begin{aligned}
&\left|f_n^\varepsilon(|z_1|^2) - f_n^\varepsilon(|z_2|^2)\right| \\
&\le \left|\left(1-\frac{|z_1|^2}{\varepsilon^2}\right)^n - \left(1-\frac{|z_2|^2}{\varepsilon^2}\right)^n\right| + \sum_{k=1}^n \frac{1}{k}\left|\left(1-\frac{|z_1|^2}{\varepsilon^2}\right)^k - \left(1-\frac{|z_2|^2}{\varepsilon^2}\right)^k\right| \\
&\le \sum_{j=1}^n \binom{n}{j}\left(1-\frac{|z_2|^2}{\varepsilon^2}\right)^{n-j}\left|\frac{|z_1|^2-|z_2|^2}{\varepsilon^2}\right|^j \\
&\quad + \sum_{k=1}^n \frac{1}{k}\sum_{j=1}^k \binom{k}{j}\left(1-\frac{|z_2|^2}{\varepsilon^2}\right)^{k-j}\left|\frac{|z_1|^2-|z_2|^2}{\varepsilon^2}\right|^j \\
&\le \frac{|z_1-z_2|}{2\varepsilon}\sum_{j=1}^n \binom{n}{j}4^j + \frac{|z_1-z_2|}{2\varepsilon}\sum_{k=1}^n \frac{1}{k}\sum_{j=1}^k \binom{k}{j}4^j \\
&\le \frac{C(n)}{\varepsilon}|z_1-z_2|.
\end{aligned}$$

Supposing, for example, $|z_2| < \varepsilon < |z_1|$, denote $z_3$ by the intersection point of the circle $\{z \in \mathbb{C} : |z| = \varepsilon\}$ and the line segment connecting $z_1$ and $z_2$. Combining the inequalities above, we have

$$\begin{aligned}
\left|f_n^\varepsilon(|z_1|^2) - f_n^\varepsilon(|z_2|^2)\right| &\le |f_n^\varepsilon(|z_2|^2) - f_n^\varepsilon(|z_3|^2)| + |\ln(|z_1|^2) - \ln(|z_3|^2)| \\
&\le \frac{C(n)}{\varepsilon}|z_2-z_3| + \frac{2}{\varepsilon}|z_1-z_3| \\
&\le \frac{2+C(n)}{\varepsilon}|z_1-z_2|,
\end{aligned}$$

which completes the proof for (2.7). Noticing that

$$\mathrm{Im}\left[\left(z_1 f_n^\varepsilon(|z_1|^2) - z_2 f_n^\varepsilon(|z_2|^2)\right)\left(\overline{z_1}-\overline{z_2}\right)\right] = \frac{1}{2}\left[f_n^\varepsilon(|z_1|^2) - f_n^\varepsilon(|z_2|^2)\right]\mathrm{Im}(\overline{z_1}z_2 - z_1\overline{z_2}),$$

and

$$|\mathrm{Im}(\overline{z_1}z_2 - z_1\overline{z_2})| = |z_2(\overline{z_1} - \overline{z_2}) + \overline{z_2}(z_2 - z_1)| \le 2|z_2|\,|z_1 - z_2|,$$
$$|\mathrm{Im}(\overline{z_1}z_2 - z_1\overline{z_2})| = |\overline{z_1}(z_2 - z_1) + z_1(\overline{z_1} - \overline{z_2})| \le 2|z_1|\,|z_1 - z_2|,$$

which, together with (2.7), yields (2.8). Direct computation gives

$$\begin{aligned}(f_n^\varepsilon)'(\rho) &= \frac{1}{\rho}\chi_{\{\rho\ge\varepsilon^2\}} + \left(\frac{n}{\varepsilon^2}\big(1-\frac{\rho}{\varepsilon^2}\big)^{n-1} + \frac{1}{\varepsilon^2}\sum_{k=0}^{n-1}\big(1-\frac{\rho}{\varepsilon^2}\big)^k\right)\chi_{\{\rho<\varepsilon^2\}} \\ &= \frac{1}{\rho}\chi_{\{\rho\ge\varepsilon^2\}} + \left(\frac{n}{\varepsilon^2}\big(1-\frac{\rho}{\varepsilon^2}\big)^{n-1} + \frac{1}{\rho}\left[1-\big(1-\frac{\rho}{\varepsilon^2}\big)^n\right]\right)\chi_{\{\rho<\varepsilon^2\}}, \\ (f_n^\varepsilon)''(\rho) &= -\frac{1}{\rho^2}\chi_{\{\rho\ge\varepsilon^2\}} - \left(\frac{n^2-1}{\varepsilon^4}\big(1-\frac{\rho}{\varepsilon^2}\big)^{n-2} + \frac{1}{\varepsilon^4}\sum_{k=0}^{n-3}(k+1)\big(1-\frac{\rho}{\varepsilon^2}\big)^k\right)\chi_{\{\rho<\varepsilon^2\}},\end{aligned}$$

which leads to (2.9) immediately. □

2.3. **Comparison between different regularization.** To compare different regularization for $F(\rho)$ (and thus for $f(\rho)$), Fig. 1 shows different regularization $F_n^\varepsilon$ ($n = 2, 4, 100, 500$), $\widetilde{F}^\varepsilon$ and $\widehat{F}^\varepsilon$ for different $\varepsilon$, from which we can see that the newly proposed regularization $F_n^\varepsilon$ approximates $F$ more accurately.

Fig. 2 shows different regularization $f_n^\varepsilon$ ($n = 2, 4, 100, 500$), $\widetilde{f}^\varepsilon$ and $\widehat{f}^\varepsilon$ for different $\varepsilon$, while Figs. 3 & 4 show their first- and second-order derivatives. From these figures, we can see that the newly proposed regularization $f_n^\varepsilon$ (and its derivatives with larger $n$) approximates the nonlinearity $f$ (and its derivatives) more accurately.

## 3. Energy regularization for the logarithmic Schrödinger equation

In this section, we consider the regularized energy

$$E_n^\varepsilon(u) := \int_\Omega \left[|\nabla u|^2 + \lambda F_n^\varepsilon(|u|^2)\right] d\mathbf{x}, \tag{3.1}$$

where $F_n^\varepsilon$ is defined as (2.1). Then the Hamiltonian flow of the regularized energy $i\partial_t u = \frac{\delta E_n^\varepsilon(u)}{\delta \overline{u}}$ yields the following energy regularized logarithmic Schrödinger equation (ERLogSE) with a small regularized parameter $0 < \varepsilon \ll 1$ as

$$\begin{cases} i\partial_t u^\varepsilon(\mathbf{x},t) = -\Delta u^\varepsilon(\mathbf{x},t) + \lambda\, u^\varepsilon(\mathbf{x},t)\, f_n^\varepsilon(|u^\varepsilon(\mathbf{x},t)|^2), & \mathbf{x}\in\Omega, \quad t>0, \\ u^\varepsilon(\mathbf{x},0) = u_0(\mathbf{x}), & \mathbf{x}\in\overline{\Omega}, \end{cases} \tag{3.2}$$

where $f_n^\varepsilon$ is defined by (2.6).

3.1. **The Cauchy problem.** To investigate the well-posedness of the problem (3.2), we first introduce some appropriate space. For $\alpha > 0$ and $\Omega = \mathbb{R}^d$, denote by $L^2_\alpha$ the weighted $L^2$ space

$$L^2_\alpha := \{v \in L^2(\mathbb{R}^d), \quad \mathbf{x} \longmapsto \langle\mathbf{x}\rangle^\alpha v(\mathbf{x}) \in L^2(\mathbb{R}^d)\},$$

where $\langle\mathbf{x}\rangle := \sqrt{1+|\mathbf{x}|^2}$, with norm $\|v\|_{L^2_\alpha} := \|\langle\mathbf{x}\rangle^\alpha v(\mathbf{x})\|_{L^2(\mathbb{R}^d)}$. Regarding the Cauchy problem (3.2), we have similar results as the regularization (1.6) in [5]. For the convenience of the readers, we present the result as follows.

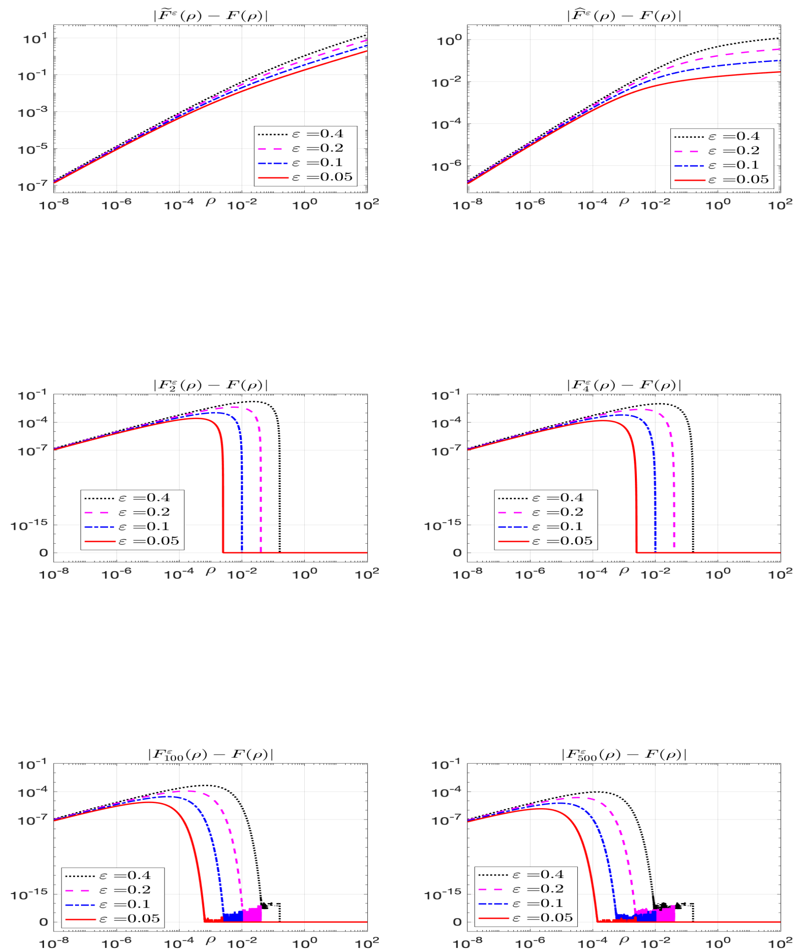


FIGURE 1. Comparison of different regularization for $F(\rho) = \rho \ln \rho - \rho$.

**Theorem 3.1.** *Let $\lambda \in \mathbb{R}$ and $0 < \varepsilon \leq 1$.*
(1). *For* (3.2) *posed on a bounded domain $\Omega$ with homogeneous Dirichlet or periodic boundary condition and initial data $u_0 \in H^1(\Omega)$,*

- *There exists a unique, global weak solution $u^\varepsilon \in L^\infty_{\text{loc}}(\mathbb{R}; H^1(\Omega))$ to* (3.2).
- *If in addition $u_0 \in H^2(\Omega)$, then $u^\varepsilon \in L^\infty_{\text{loc}}(\mathbb{R}; H^2(\Omega))$.*

(2). *For* (3.2) *on $\Omega = H^1(\mathbb{R}^d)$ with an initial data $u_0 \in H^1(\mathbb{R}^d) \cap L^2_\alpha$ $(0 < \alpha \leq 1)$,*

- *There exists a unique, global weak solution $u^\varepsilon \in L^\infty_{\text{loc}}(\mathbb{R}; H^1(\mathbb{R}^d) \cap L^2_\alpha)$ to* (3.2).
- *If in addition $u_0 \in H^2(\mathbb{R}^d)$, then $u^\varepsilon \in L^\infty_{\text{loc}}(\mathbb{R}; H^2(\mathbb{R}^d))$.*
- *If $u_0 \in H^2(\mathbb{R}^d) \cap L^2_2$, then $u^\varepsilon \in L^\infty_{\text{loc}}(\mathbb{R}; H^2(\mathbb{R}^d) \cap L^2_2)$.*

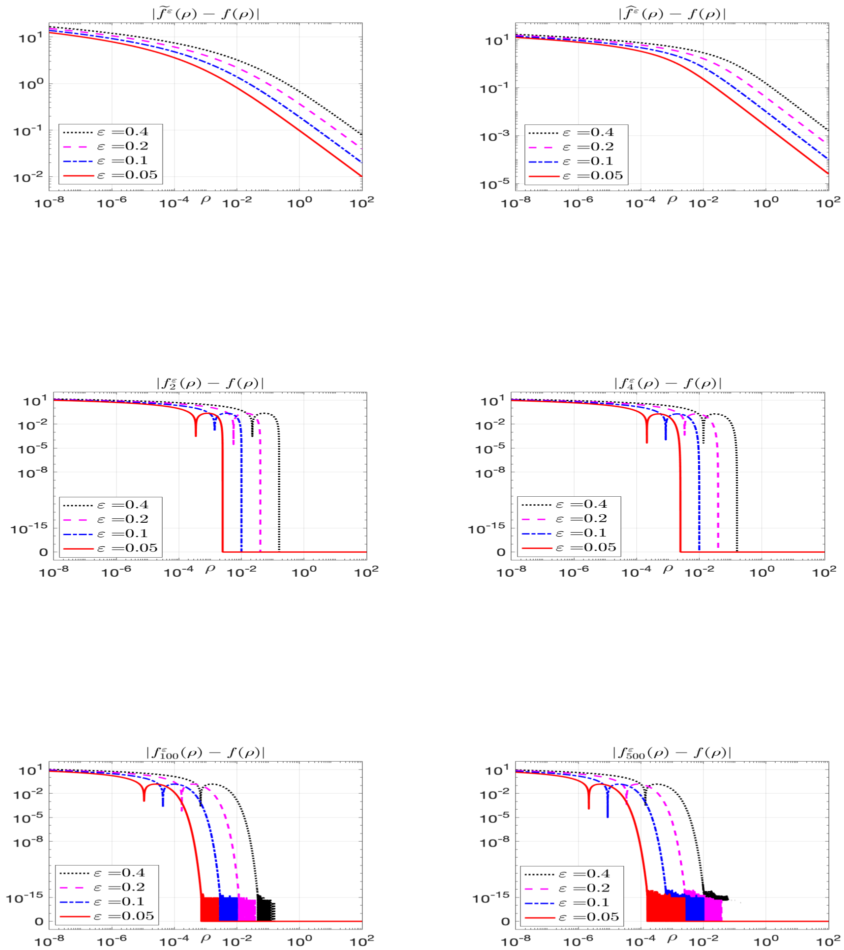


FIGURE 2. Comparison of different regularization for the nonlinearity $f(\rho) = \ln \rho$.

*Proof.* The theorem can be established by using similar arguments invoked in [5]. We refer to [5] for details. $\square$

3.2. **Convergence of the regularized model.** In this subsection, we show the approximation property of the regularized model (3.2) to (1.1).

**Lemma 3.2.** *Suppose the equation is set on $\Omega$, where $\Omega = \mathbb{R}^d$, or $\Omega \subset \mathbb{R}^d$ is a bounded domain with homogeneous Dirichlet or periodic boundary condition, then we have the general estimate:*

$$\frac{d}{dt}\|u^{\varepsilon}(t) - u(t)\|_{L^2}^2 \leq |\lambda| \left(4\|u^{\varepsilon}(t) - u(t)\|_{L^2}^2 + 6\varepsilon\|u^{\varepsilon}(t) - u(t)\|_{L^1}\right). \tag{3.3}$$

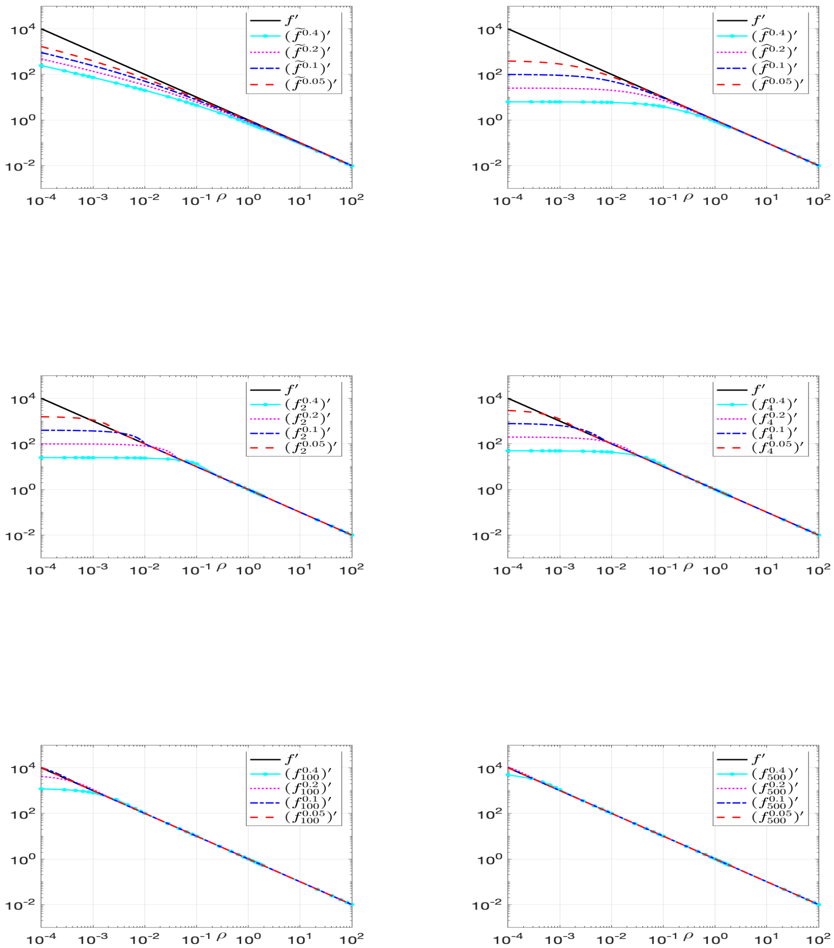


FIGURE 3. Comparison of different regularization for $f'(\rho) = 1/\rho$.

*Proof.* Subtracting (1.1) from (3.2), we see that the error function $e^\varepsilon := u^\varepsilon - u$ satisfies

$$i\partial_t e^\varepsilon + \Delta e^\varepsilon = \lambda \left[u^\varepsilon \ln(|u^\varepsilon|^2) - u\ln(|u|^2)\right] + \lambda u^\varepsilon \left[f_n^\varepsilon(|u^\varepsilon|^2) - \ln(|u^\varepsilon|^2)\right] \chi_{\{|u^\varepsilon|<\varepsilon\}}.$$

Multiplying the above error equation by $\overline{e^\varepsilon(t)}$, integrating in space and taking the imaginary parts, we can get by using Lemma 2.1, (2.4) and (2.5) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|e^\varepsilon(t)\|_{L^2}^2 &= 2\lambda \operatorname{Im}\int_\Omega \left[u^\varepsilon \ln(|u^\varepsilon|) - u\ln(|u|)\right] \overline{e^\varepsilon}(\mathbf{x},t)d\mathbf{x} \\ &\quad + \lambda \operatorname{Im}\int_{|u^\varepsilon|<\varepsilon} u^\varepsilon \left[f_n^\varepsilon(|u^\varepsilon|^2) - \ln(|u^\varepsilon|^2)\right] \overline{e^\varepsilon}(\mathbf{x},t)d\mathbf{x}\end{aligned}$$

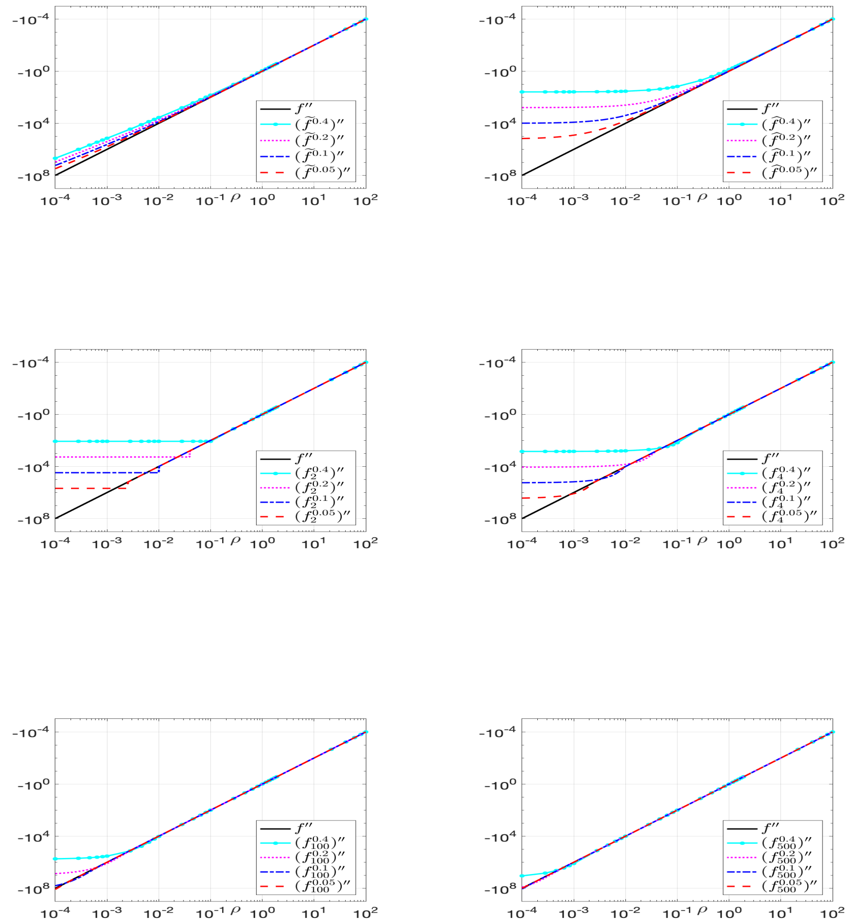


FIGURE 4. Comparison of different regularization for $f''(\rho) = -1/\rho^2$.

$$
\begin{aligned}
&\leq 2|\lambda|\|e^\varepsilon(t)\|_{L^2}^2 + |\lambda|\Big|\int_{|u^\varepsilon|<\varepsilon} u^\varepsilon\overline{e^\varepsilon}\left[Q_n^\varepsilon(|u^\varepsilon|^2) - \ln(|u^\varepsilon|^2) + |u^\varepsilon|^2(Q_n^\varepsilon)'(|u^\varepsilon|^2)\right]d\mathbf{x}\Big| \\
&\leq 2|\lambda|\|e^\varepsilon(t)\|_{L^2}^2 + |\lambda|\Big|\int_{|u^\varepsilon|<\varepsilon} \overline{e^\varepsilon}u^\varepsilon\Big[\int_{|u^\varepsilon|^2}^{\varepsilon^2}\frac{(s-|u^\varepsilon|^2)^n}{s^{n+1}}ds - 1 + |u^\varepsilon|^2(Q_n^\varepsilon)'(|u^\varepsilon|^2)\Big]d\mathbf{x}\Big| \\
&= 2|\lambda|\,\|e^\varepsilon(t)\|_{L^2}^2 + |\lambda|\Big|\int_{|u^\varepsilon|<\varepsilon} \overline{e^\varepsilon}u^\varepsilon\Big[\int_{|u^\varepsilon|^2}^{\varepsilon^2}\frac{(s-|u^\varepsilon|^2)^n}{s^{n+1}}ds - \left(1-\frac{|u^\varepsilon|^2}{\varepsilon^2}\right)^n\Big]d\mathbf{x}\Big| \\
&\leq 2|\lambda|\,\|e^\varepsilon(t)\|_{L^2}^2 + \varepsilon|\lambda|\|e^\varepsilon\|_{L^1} + |\lambda|\Big|\int_0^{\varepsilon^2} s^{-n-1}\int_{|u^\varepsilon|^2<s}\overline{e^\varepsilon}u^\varepsilon(s-|u^\varepsilon|^2)^n d\mathbf{x}ds\Big| \\
&\leq 2|\lambda|\,\|e^\varepsilon(t)\|_{L^2}^2 + 3\varepsilon|\lambda|\|e^\varepsilon\|_{L^1},
\end{aligned}
$$

which gives the result. ☐

Then applying the same arguments as in [5], based on the previous error estimate, and interpolation between $L^2$ and $H^2$, we get the convergence behavior as follows.

**Proposition 3.3.** *If $\Omega$ has finite measure and $u_0 \in H^2(\Omega)$, then for any $T > 0$, we have*

$$\|u^\varepsilon - u\|_{L^\infty(0,T;L^2(\Omega))} \le C_1\varepsilon, \quad \|u^\varepsilon - u\|_{L^\infty(0,T;H^1(\Omega))} \le C_2\varepsilon^{1/2},$$

*where $C_1$ depends on $|\lambda|$, $T$, $|\Omega|$, and $C_2$ depends on additional $\|u_0\|_{H^2(\Omega)}$; If $\Omega = \mathbb{R}^d$, $1 \le d \le 3$ and $u_0 \in H^2(\mathbb{R}^d) \cap L^2_2$, then for any $T > 0$, we have*

$$\|u^\varepsilon - u\|_{L^\infty(0,T;L^2(\mathbb{R}^d))} \le D_1\varepsilon^{\frac{4}{4+d}}, \quad \|u^\varepsilon - u\|_{L^\infty(0,T;H^1(\mathbb{R}^d)))} \le D_2\varepsilon^{\frac{2}{4+d}},$$

*where $D_1$ and $D_2$ depend on $d$, $|\lambda|$, $T$, $\|u_0\|_{L^2_2}$ and $\|u_0\|_{H^2(\mathbb{R}^d)}$.*

3.3. **Convergence of the energy.** By construction, the energy is conserved, i.e.,

$$E_n^\varepsilon(u^\varepsilon) = \int_\Omega \left[|\nabla u^\varepsilon(\mathbf{x},t)|^2 + \lambda F_n^\varepsilon(|u^\varepsilon(\mathbf{x},t)|^2)\right] d\mathbf{x} = E_n^\varepsilon(u_0).$$

For the convergence of the energy, we have the following estimate.

**Proposition 3.4.** *For $u_0 \in H^1(\Omega) \cap L^\alpha(\Omega)$ with $\alpha \in (0,2)$, the energy $E_n^\varepsilon(u_0)$ converges to $E(u_0)$ with*

$$|E_n^\varepsilon(u_0) - E(u_0)| \le |\lambda| \|u_0\|_{L^\alpha}^\alpha \frac{\varepsilon^{2-\alpha}}{1-\alpha/2}.$$

*In addition, for bounded $\Omega$, we have*

$$|E_n^\varepsilon(u_0) - E(u_0)| \le |\lambda|\,|\Omega|\,\varepsilon^2.$$

*Proof.* It can be deduced from the definition and (2.4) that

$$\begin{aligned}
|E_n^\varepsilon(u_0) - E(u_0)| &= |\lambda| \left| \int_\Omega [F(|u_0(\mathbf{x})|^2) - F_n^\varepsilon(|u_0(\mathbf{x})|^2)] d\mathbf{x} \right| \\
&= |\lambda| \left| \int_{|u_0(\mathbf{x})|<\varepsilon} |u_0(\mathbf{x})|^2 [Q(|u_0(\mathbf{x})|^2) - Q_n^\varepsilon(|u_0(\mathbf{x})|^2)] d\mathbf{x} \right| \\
&= |\lambda| \int_{|u_0(\mathbf{x})|<\varepsilon} |u_0(\mathbf{x})|^2 \int_{|u_0(\mathbf{x})|^2}^{\varepsilon^2} s^{-n-1}(s - |u_0(\mathbf{x})|^2)^n ds d\mathbf{x} \\
&= |\lambda| \int_0^{\varepsilon^2} s^{-n-1} \int_{|u_0(\mathbf{x})|^2<s} |u_0(\mathbf{x})|^2 (s - |u_0(\mathbf{x})|^2)^n d\mathbf{x} ds.
\end{aligned}$$

If $\Omega$ is bounded, we immediately get

$$|E_n^\varepsilon(u_0) - E(u_0)| \le |\lambda|\,|\Omega|\,\varepsilon^2.$$

For unbounded $\Omega$, one gets

$$|E_n^\varepsilon(u_0) - E(u_0)| \le |\lambda| \int_0^{\varepsilon^2} s^{-n-1} s^{n+1-\alpha/2} \|u_0\|_{L^\alpha}^\alpha ds = |\lambda|\,\|u_0\|_{L^\alpha}^\alpha \frac{\varepsilon^{2-\alpha}}{1-\alpha/2},$$

which completes the proof. ☐

*Remark* 3.5. Recall that it was shown in [5] that for the regularized model (1.6) with energy density (1.11), the energy

$$\widetilde{E}^\varepsilon(u_0) = \|\nabla u_0\|_{L^2}^2 + \lambda \int_\Omega \widetilde{F}^\varepsilon(|u_0|^2) d\mathbf{x} \tag{3.4}$$

converges to $E(u_0)$ at first order. While for the regularization (1.7) with energy density (1.11) and regularized energy as

$$\widehat{E}^\varepsilon(u_0) = \|\nabla u_0\|_{L^2}^2 + \lambda \int_\Omega \widehat{F}^\varepsilon(|u_0|^2) d\mathbf{x}, \tag{3.5}$$

we have

$$\begin{aligned}
\left|\widehat{E}^\varepsilon(u_0) - E(u_0)\right| &= |\lambda| \left| \int_\Omega [F(|u_0(\mathbf{x})|^2) - \widehat{F}^\varepsilon(|u_0(\mathbf{x})|^2)] d\mathbf{x} \right| \\
&= |\lambda| \left| \int_\Omega \left[ (\varepsilon^2 + |u_0|^2) \ln(\varepsilon^2 + |u_0|^2) - \varepsilon^2 \ln(\varepsilon^2) - |u_0|^2 \ln(|u_0|^2) \right] d\mathbf{x} \right| \\
&\le |\lambda| \, \varepsilon^2 \int_\Omega \ln\left(1 + \frac{|u_0|^2}{\varepsilon^2}\right) d\mathbf{x} + |\lambda| \int_\Omega |u_0|^2 \ln\left(1 + \frac{\varepsilon^2}{|u_0|^2}\right) d\mathbf{x} \\
&\le |\lambda| \, \varepsilon^{2-\alpha} C(\alpha) \int_\Omega |u_0|^\alpha d\mathbf{x} \\
&= |\lambda| \, \varepsilon^{2-\alpha} C(\alpha) \|u_0\|_{L^\alpha}^\alpha,
\end{aligned}$$

where we have used the inequality $\ln(1+x) \le C(\beta) x^\beta$ for $\beta \in (0,1]$ and $x \ge 0$. Hence for $u_0 \in H^1(\Omega) \cap L^\alpha(\Omega)$ with $\alpha \in (0,2)$, we derive

$$\left|\widehat{E}^\varepsilon(u_0) - E(u_0)\right| \le |\lambda| \, \varepsilon^{2-\alpha} C(\alpha) \|u_0\|_{L^\alpha}^\alpha,$$

which shows the same convergence rate as $E_n^\varepsilon$. Thus the newly proposed regularization $F_n^\varepsilon$ is more accurate than $\widetilde{F}^\varepsilon$, and than $\widehat{F}^\varepsilon$ in the case of bounded domains, from the viewpoint of energy.

## 4. Regularized Lie-Trotter splitting methods

In this section, we investigate the approximation property of Lie-Trotter splitting methods [12, 25, 45] for solving the regularized model (3.2) in 1D. Extensions to higher dimensions are straightforward. For simplicity of notation, we set $\lambda = 1$.

4.1. **A time-splitting for** (3.2)**.** The operator splitting methods are based on a decomposition of the flow of (3.2):

$$\partial_t u^\varepsilon = A(u^\varepsilon) + B(u^\varepsilon),$$

where

$$A(v) = i\Delta v, \quad B(v) = -iv f_n^\varepsilon(|v|^2),$$

and the solution of the sub-equations

$$\begin{cases} \partial_t v(x,t) = A(v(x,t)), & x \in \Omega, \quad t > 0, \\ v(x,0) = v_0(x), \end{cases} \tag{4.1}$$

$$\begin{cases} \partial_t \omega(x,t) = B(\omega(x,t)), & x \in \Omega, \quad t > 0, \\ \omega(x,0) = \omega_0(x), \end{cases} \tag{4.2}$$

where $\Omega = \mathbb{R}$ or $\Omega \subset \mathbb{R}$ is a bounded domain with homogeneous Dirichlet or periodic boundary condition on the boundary. Denote the flow of (4.1) and (4.2) as

$$v(\cdot,t) = \Phi_A^t(v_0) = e^{it\Delta}v_0, \quad \omega(\cdot,t) = \Phi_B^t(\omega_0) = \omega_0 e^{-itf_n^\varepsilon(|\omega_0|^2)}, \quad t \geq 0. \tag{4.3}$$

As is well-known, the flow $\Phi_A^t$ satisfies the isometry relation, i.e.,

$$\|\Phi_A^t(v_0)\|_{H^s} = \|v_0\|_{H^s}, \quad s \in \mathbb{N}, \quad t \geq 0. \tag{4.4}$$

Regarding the exact flow $\Phi_B^t$, we have the following properties.

**Lemma 4.1.** *Assume $\tau > 0$ and $\omega_0 \in H^1(\Omega)$, then*

$$\|\Phi_B^\tau(\omega_0)\|_{L^2} = \|\omega_0\|_{L^2}, \quad \|\Phi_B^\tau(\omega_0)\|_{H^1} \leq [1 + 2(n+1)\tau]\, \|\omega_0\|_{H^1}. \tag{4.5}$$

*For $v$, $w \in L^2(\Omega)$,*

$$\|\Phi_B^\tau(v) - \Phi_B^\tau(w)\|_{L^2} \leq [1 + C(n)\tau]\, \|v - w\|_{L^2}. \tag{4.6}$$

*Proof.* By direct calculation, we get

$$\partial_x \Phi_B^\tau(\omega_0) = e^{-i\tau f_n^\varepsilon(|\omega_0|)^2}\left[\partial_x\omega_0 - i\tau (f_n^\varepsilon)'(|\omega_0|^2)(\omega_0^2 \partial_x \overline{\omega_0} + |\omega_0|^2 \partial_x \omega_0)\right],$$

which gives (4.5) immediately by recalling (2.9). We claim that for any $x \in \Omega$,

$$|\Phi_B^\tau(v)(x) - \Phi_B^\tau(w)(x)| \leq [1 + C(n)\tau]\, |v(x) - w(x)|.$$

Assuming, for example, $|v(x)| \leq |w(x)|$, then by inserting a term $v(x)e^{-i\tau f_n^\varepsilon(|w(x)|)^2}$, we can get that

$$\begin{aligned}
|\Phi_B^\tau(v)(x) - \Phi_B^\tau(w)(x)| &= \left| v(x)e^{-i\tau f_n^\varepsilon(|v(x)|)^2} - w(x)e^{-i\tau f_n^\varepsilon(|w(x)|)^2} \right| \\
&= \left| v(x) - w(x) + v(x)\left(e^{i\tau[f_n^\varepsilon(|w(x)|^2) - f_n^\varepsilon(|v(x)|^2)]} - 1\right)\right| \\
&\leq |v(x) - w(x)| + 2|v(x)| \left| \sin\left(\frac{\tau}{2}\left[f_n^\varepsilon(|w(x)|^2) - f_n^\varepsilon(|v(x)|^2)\right]\right)\right| \\
&\leq |v(x) - w(x)| + \tau|v(x)|\, |f_n^\varepsilon(|w(x)|^2) - f_n^\varepsilon(|v(x)|^2)| \\
&\leq [1 + C(n)\tau]\, |v(x) - w(x)|,
\end{aligned}$$

where we used the estimate (2.7). When $|v(x)| \geq |w(x)|$, the same inequality can be obtained by exchanging $v$ and $w$ in the above computation. Thus the proof for (4.6) is completed. □

4.2. **Error estimates for $\Phi^\tau = \Phi_A^\tau \Phi_B^\tau$.** We consider the Lie-Trotter splitting

$$u^{\varepsilon,k+1} = \Phi^\tau(u^{\varepsilon,k}) = \Phi_A^\tau(\Phi_B^\tau(u^{\varepsilon,k})), \quad k \geq 0; \quad u^{\varepsilon,0} = u_0, \quad \tau > 0. \tag{4.7}$$

For $u_0 \in H^1(\Omega)$, it follows from (4.4) and (4.5) that

$$\begin{aligned}
&\|u^{\varepsilon,k}\|_{L^2} = \|u^{\varepsilon,k-1}\|_{L^2} \equiv \|u^{\varepsilon,0}\|_{L^2} = \|u_0\|_{L^2}, \\
&\|u^{\varepsilon,k}\|_{H^1} \leq [1 + 2(n+1)\tau]\, \|u^{\varepsilon,k-1}\|_{H^1} \leq e^{2(n+1)k\tau}\|u_0\|_{H^1}, \quad k \geq 0.
\end{aligned} \tag{4.8}$$

**Theorem 4.2.** *Let $T > 0$. Assume that the solution of (3.2) satisfies $u^\varepsilon \in L^\infty(0,T;H^1(\Omega))$. Then there exists $0 < \varepsilon_0 < 1$ depending on $n$ and $\|u^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))}$ such that when $\varepsilon \leq \varepsilon_0$ and $k\tau \leq T$, we have*

$$\|u^{\varepsilon,k} - u^\varepsilon(t_k)\|_{L^2} \leq C\left(n, T, \|u^\varepsilon\|_{L^\infty([0,T];H^1(\Omega))}\right) \ln(\varepsilon^{-1})\tau^{1/2}, \tag{4.9}$$

*where $C(\cdot,\cdot,\cdot)$ is independent of $0 < \varepsilon \leq \varepsilon_0$.*

*Proof.* Denote the exact flow of (3.2) by $u^\varepsilon(t) = \Psi^t(u_0)$. Firstly we give the local error for $v \in H^1(\Omega)$:

$$\|\Psi^\tau(v) - \Phi^\tau(v)\|_{L^2} \le C\|v\|_{H^1} \ln(\varepsilon^{-1})\tau^{3/2}, \quad \tau \le 1, \quad \varepsilon \le \varepsilon_0, \tag{4.10}$$

where $\varepsilon_0$ depends on $n$ and $\|v\|_{H^1}$. It can be obtained from the definition that

$$\begin{aligned} i\partial_t \Psi^t(v) + \Delta \Psi^t(v) &= \Psi^t(v) f_n^\varepsilon(|\Psi^t(v)|^2), \\ i\partial_t \Phi^t(v) + \Delta \Phi^t(v) &= \Phi_A^t\left(\Phi_B^t(v) f_n^\varepsilon(|\Phi_B^t(v)|^2)\right). \end{aligned}$$

Denote $\mathcal{E}^t(v) = \Psi^t(v) - \Phi^t(v)$, we have

$$i\partial_t \mathcal{E}^t(v) + \Delta \mathcal{E}^t(v) = \Psi^t(v) f_n^\varepsilon(|\Psi^t(v)|^2) - \Phi_A^t\left(\Phi_B^t(v) f_n^\varepsilon(|\Phi_B^t(v)|^2)\right). \tag{4.11}$$

Multiplying (4.11) by $\overline{\mathcal{E}^t(v)}$, integrating in space and taking the imaginary part, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\mathcal{E}^t(v)\|_{L^2}^2 &= \operatorname{Im}\left(\Psi^t(v) f_n^\varepsilon(|\Psi^t(v)|^2) - \Phi_A^t\left(\Phi_B^t(v) f_n^\varepsilon(|\Phi_B^t(v)|^2)\right), \mathcal{E}^t(v)\right) \\ &= \operatorname{Im}\left(\Psi^t(v) f_n^\varepsilon(|\Psi^t(v)|^2) - \Phi^t(v) f_n^\varepsilon(|\Phi^t(v)|^2), \mathcal{E}^t(v)\right) \\ &\quad + \operatorname{Im}\left(\Phi^t(v) f_n^\varepsilon(|\Phi^t(v)|^2) - \Phi_A^t\left(\Phi_B^t(v) f_n^\varepsilon(|\Phi_B^t(v)|^2)\right), \mathcal{E}^t(v)\right) \\ &\le C(n)\|\mathcal{E}^t(v)\|_{L^2}^2 \\ &\quad + \left\|\Phi^t(v) f_n^\varepsilon(|\Phi^t(v)|^2) - \Phi_A^t\left(\Phi_B^t(v) f_n^\varepsilon(|\Phi_B^t(v)|^2)\right)\right\|_{L^2} \|\mathcal{E}^t(v)\|_{L^2}, \end{aligned}$$

where we have used (2.8). This implies

$$\frac{d}{dt}\|\mathcal{E}^t(v)\|_{L^2} \le C(n)\|\mathcal{E}^t(v)\|_{L^2} + J_1 + J_2, \tag{4.12}$$

where

$$\begin{aligned} J_1 &= \|\Phi^t(v) f_n^\varepsilon(|\Phi^t(v)|^2) - \Phi_B^t(v) f_n^\varepsilon(|\Phi_B^t(v)|^2)\|_{L^2}, \\ J_2 &= \|\Phi_B^t(v) f_n^\varepsilon(|\Phi_B^t(v)|^2) - \Phi_A^t\left(\Phi_B^t(v) f_n^\varepsilon(|\Phi_B^t(v)|^2)\right)\|_{L^2}. \end{aligned}$$

To estimate $J_1$ in (4.12), firstly we prove that $\|\Phi^t(v)\|_{L^\infty}, \|\Phi_B^t(v)\|_{L^\infty} \le C(n)$. It follows from (4.4) and (4.5) that

$$\|\Phi^t(v)\|_{H^1} = \|\Phi_B^t(v)\|_{H^1} \le (1 + 2(n+1)t)\|v\|_{H^1}.$$

Hence by Sobolev imbedding, when $t \le 1$, we have $\|\Phi^t(v)\|_{L^\infty}$, $\|\Phi_B^t(v)\|_{L^\infty} \le C(n)\|v\|_{H^1}$. Next we claim that for $y$, $z$ satisfying $|y|, |z| \le D$, it can be established that

$$|y f_n^\varepsilon(|y|^2) - z f_n^\varepsilon(|z|^2)| \le C(C(n) + \ln(\varepsilon^{-1}))|y - z|,$$

when $\varepsilon$ is sufficiently small. Assuming, for example, $0 \le |z| \le |y|$, then by applying (2.7) and $|f_n^\varepsilon(|y|^2)| \le 2\ln(\varepsilon^{-1}) + n + 1$, when $\varepsilon \le \varepsilon_1 := \min\{1, 1/D\}$, we get

$$\begin{aligned} |y f_n^\varepsilon(|y|^2) - z f_n^\varepsilon(|z|^2)| &= |(y - z) f_n^\varepsilon(|y|^2)| + |z||f_n^\varepsilon(|y|^2) - f_n^\varepsilon(|z|^2)| \\ &\le C(C(n) + \ln(\varepsilon^{-1}))|y - z| + |z|\frac{C(n)|y - z|}{|z|} \\ &\le C(C(n) + \ln(\varepsilon^{-1}))|y - z|. \end{aligned} \tag{4.13}$$

Thus, we obtain when $\varepsilon \le \varepsilon_2 := \min\{1, 1/(C(n)\|v\|_{H^1})\}$,

$$\begin{aligned} J_1 &\le C(C(n) + \ln(\varepsilon^{-1}))\|\Phi^t(v) - \Phi_B^t(v)\|_{L^2} \\ &\le C(C(n) + \ln(\varepsilon^{-1}))\sqrt{2t}\|\Phi_B^t(v)\|_{H^1} \end{aligned}$$

$$\leq C(C(n)+\ln(\varepsilon^{-1}))\sqrt{2t}(1+2(n+1)t)\|v\|_{H^1}, \tag{4.14}$$

where we used the estimate

$$\left\|\omega-\Phi_A^t(\omega)\right\|_{L^2} \leq \sqrt{2t}\,\|\omega\|_{H^1}, \tag{4.15}$$

which was adopted in [4], instead of the estimate in [12]

$$\left\|\omega-\Phi_A^t(\omega)\right\|_{L^2} \leq 2t\|\omega\|_{H^2},$$

which yields an extra $1/\varepsilon$ factor in the error estimate.

To estimate $J_2$, we first claim that

$$\|\Phi_B^t(v)f_n^\varepsilon(|\Phi_B^t(v)|^2)\|_{H^1} \leq C(C(n)+\ln(\varepsilon^{-1}))(1+t)\|v\|_{H^1}, \tag{4.16}$$

when $\varepsilon \leq \varepsilon_3 := \min\{1, 1/\|v\|_{L^\infty}\}$. Recalling that

$$\Phi_B^t(v)f_n^\varepsilon(|\Phi_B^t(v)|^2) = vf_n^\varepsilon(|v|^2)e^{-itf_n^\varepsilon(|v|^2)},$$

and $|f_n^\varepsilon(|v|^2)| \leq 2\ln(\varepsilon^{-1})+n+1$, when $\varepsilon \leq \varepsilon_3$, this implies

$$\|(\Phi_B^t(v))f_n^\varepsilon(|\Phi_B^t(v)|^2)\|_{L^2} \leq 2(C(n)+\ln(\varepsilon^{-1}))\|v||_{L^2}.$$

Noticing that

$$\begin{aligned}\partial_x[\Phi_B^t(v)f_n^\varepsilon(|\Phi_B^t(v)|^2)] = e^{-itf_n^\varepsilon(|v|^2)}\big[&v_xf_n^\varepsilon(|v|^2)\\ &+(1-itf_n^\varepsilon(|v|^2))(f_n^\varepsilon)'(|v|^2)(v^2\overline{v_x}+|v|^2v_x)\big],\end{aligned}$$

which together with (2.9) yields

$$|\partial_x[\Phi_B^t(v)f_n^\varepsilon(|\Phi_B^t(v)|^2)]| \leq C(C(n)+\ln(\varepsilon^{-1}))(1+t)|v_x|,$$

which immediately gives (4.16). Applying (4.15) again yields that

$$\begin{aligned}J_2 &\leq \sqrt{2t}\,\|(\Phi_B^t(v))f_n^\varepsilon(|\Phi_B^t(v)|^2)\|_{H^1}\\ &\leq C(C(n)+\ln(\varepsilon^{-1}))\sqrt{t}(1+t)\|v\|_{H^1},\end{aligned} \tag{4.17}$$

for $\varepsilon \leq \varepsilon_3$. Combining (4.12), (4.14) and (4.17), we get

$$\frac{d}{dt}\|\mathcal{E}^t(v)\|_{L^2} \leq C(n)\|\mathcal{E}^t(v)\|_{L^2} + C(C(n)+\ln(\varepsilon^{-1}))\sqrt{t}(1+t)\|v\|_{H^1}.$$

Applying the Gronwall's inequality, when $\tau \leq 1$, we have

$$\begin{aligned}\|\mathcal{E}^\tau(v)\|_{L^2} &\leq C(C(n)+\ln(\varepsilon^{-1}))\sqrt{\tau}(1+\tau)(e^{\tau C(n)}-1)\|v\|_{H^1}\\ &\leq C(n)(C(n)+\ln(\varepsilon^{-1}))\tau^{3/2}\|v\|_{H^1} \leq C(n)\ln(\varepsilon^{-1})\tau^{3/2}\|v\|_{H^1},\end{aligned}$$

when $\varepsilon \leq \varepsilon_0 = \min\{\varepsilon_2, \varepsilon_3, e^{-C(n)}\}$ depending on $n$ and $\|v\|_{H^1}$, which completes the proof for (4.10).

Next we give the stability analysis for the operator $\Phi^t$:

$$\|\Phi^\tau(v)-\Phi^\tau(w)\|_{L^2} \leq (1+C(n)\tau)\|v-w\|_{L^2}, \quad \text{for} \quad v,w \in L^2(\Omega). \tag{4.18}$$

Noticing that $\Phi_A^\tau$ is a linear isometry on $H^s(\Omega)$, (4.6) gives (4.18) directly. Thus the error (4.9) can be established by combining the local error (4.10), the stability property (4.18) and a standard argument (cf. [4,12]). $\square$

*Remark* 4.3. As established in Theorem 3.1, for an arbitrarily large fixed $T > 0$, the above assumptions are satisfied as soon as $u_0 \in H^1(\Omega)$ when $\Omega$ is bounded. More specifically,

$$\sup_{t\in[0,T]} \|u^\varepsilon(t)\|_{H^j} \le C\left(\|u_0\|_{H^j}\right), \quad j = 1, 2,$$

for a constant $C$ depending only on $\|u_0\|_{H^j}$. When $\Omega = \mathbb{R}^d$, we require in addition $u_0 \in L^2_\alpha$ for some $0 < \alpha \le 1$.

*Remark* 4.4. For $d = 2, 3$, the error estimate (4.9) can be established under the more restrictive condition $u^\varepsilon \in L^\infty(0, T; H^2(\Omega))$, and then $\varepsilon_0$ depends on $n$ and $\|u^\varepsilon\|_{L^\infty(0,T;H^2(\Omega))}$ by applying similar arguments as in [4], where $\|\Phi^t_B(v)\|_{H^2}$ has to be further investigated due to the Sobolev inequality $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ for $d = 2, 3$. For the details, we refer to [4].

4.3. **Error estimates for $\Phi^\tau = \Phi^\tau_B\Phi^\tau_A$.** We consider another Lie-Trotter splitting

$$u^{\varepsilon,k+1} = \Phi^\tau(u^{\varepsilon,k}) = \Phi^\tau_B(\Phi^\tau_A(u^{\varepsilon,k})), \quad k \ge 0; \quad u^{\varepsilon,0} = u_0, \quad \tau > 0. \tag{4.19}$$

Similar as before, we have

$$\|u^{\varepsilon,k}\|_{L^2} = \|u_0\|_{L^2}, \quad \|u^{\varepsilon,k}\|_{H^1} \le e^{2(n+1)k\tau}\|u_0\|_{H^1}, \quad k \ge 0. \tag{4.20}$$

**Theorem 4.5.** *Let $T > 0$. Assume that the solution of* (3.2) *satisfies $u^\varepsilon \in L^\infty(0, T; H^2(\Omega))$. Then there exists $\varepsilon_0 > 0$ depending on $n$ and $\|u^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))}$ such that when $\varepsilon \le \varepsilon_0$ and $k\tau \le T$, we have*

$$\|u^{\varepsilon,k} - u^\varepsilon(t_k)\|_{L^2} \le C\left(n, T, \|u^\varepsilon\|_{L^\infty([0,T];H^2(\Omega))}\right)\frac{\tau}{\varepsilon}, \tag{4.21}$$

*where $C(\cdot,\cdot,\cdot)$ is independent of $\varepsilon > 0$.*

*Proof.* Firstly we give the local error for $v_0 \in H^1(\Omega)$:

$$\|\Psi^\tau(v_0) - \Phi^\tau(v_0)\|_{L^2} \le C(n, \|v_0\|_{H^2})\frac{\tau^2}{\varepsilon}, \quad \tau \le 1, \quad \varepsilon \le \varepsilon_0, \tag{4.22}$$

where $\varepsilon_0$ depends on $n$ and $\|v_0\|_{H^1}$. We start from the Duhamel's formula for $v(t) = \Psi^t(v_0)$:

$$\Psi^t(v_0) = e^{it\Delta}v_0 + \int_0^t e^{i(t-s)\Delta}B(v(s))ds. \tag{4.23}$$

Recalling

$$B(v(s)) = B(e^{is\Delta}v_0) + \int_0^s dB(e^{i(s-y)\Delta}v(y))[e^{i(s-y)\Delta}B(v(y))]dy, \tag{4.24}$$

which is the variation-of-constants formula $B(g(s)) - B(g(0)) = \int_0^s dB(g(y))[g'(y)]dy$ for $g(y) = e^{i(s-y)\Delta}v(y)$. Here $dB$ is defined by

$$dB(w_1)[w_2] = -iw_2 f^\varepsilon_n(|w_1|^2) - 2iw_1(f^\varepsilon_n)'(|w_1|^2)[w_1\overline{w_2} + \overline{w_1}w_2].$$

Plugging (4.24) into (4.23) with $t = \tau$, we get

$$\Psi^\tau(v_0) = e^{i\tau\Delta}v_0 + \int_0^\tau e^{i(\tau-s)\Delta}B(e^{is\Delta}v_0)ds + e_1,$$

where

$$e_1 = \int_0^\tau \int_0^s e^{i(\tau-s)\Delta}dB(e^{i(s-y)\Delta}v(y))[e^{i(s-y)\Delta}B(v(y))]dyds.$$

On the other hand, for the Lie splitting $\Phi^\tau(v_0) = \Phi_B^\tau \Phi_A^\tau(v_0)$, applying the first-order Taylor expansion

$$\Phi_B^\tau(w) = w + \tau B(w) + \tau^2 \int_0^1 (1-s) dB(\Phi_B^{s\tau}(w))[B(\Phi_B^{s\tau}(w))]ds,$$

for $w = \Phi_A^\tau(v_0) = e^{i\tau\Delta}v_0$, we get

$$\Phi^\tau(v_0) = \Phi_B^\tau \Phi_A^\tau(v_0) = e^{i\tau\Delta}v_0 + \tau B(e^{i\tau\Delta}v_0) + e_2,$$

with

$$e_2 = \tau^2 \int_0^1 (1-s) dB(\Phi_B^{s\tau}(e^{i\tau\Delta}v_0))[B(\Phi_B^{s\tau}(e^{i\tau\Delta}v_0))]ds.$$

Thus

$$\Psi^\tau(v_0) - \Phi^\tau(v_0) = e_1 - e_2 + e_3,$$

where

$$e_3 = \int_0^\tau e^{i(\tau-s)\Delta} B(e^{is\Delta}v_0)ds - \tau B(e^{i\tau\Delta}v_0).$$

Noticing that $e_3$ is the quadrature error of the rectangle rule approximating the integral on $[0,\tau]$ of the function $g(s) = e^{i(\tau-s)\Delta}B(e^{is\Delta}v_0)$, this implies

$$e_3 = -\tau^2 \int_0^1 \theta g'(\theta\tau)d\theta,$$

where $g'(s) = -e^{i(\tau-s)\Delta}[A,B](e^{is\Delta}v_0)$, with

$$\begin{aligned}
[A,B](w) &= dA(w)[Bw] - dB(w)[Aw] = i\Delta(Bw) - dB(w)[Aw] \\
&= (f_n^\varepsilon)'(|w|^2)(2w_x^2\overline{w} + 4w|w_x|^2 + 3w^2\overline{w_{xx}} - |w|^2 w_{xx}) \\
&\quad + w(f_n^\varepsilon)''(|w|^2)(w_x\overline{w} + w\overline{w_x})^2.
\end{aligned}$$

Applying (2.9), we get

$$|[A,B](w)| \leq \frac{C(n)}{\varepsilon}|w_x|^2 + C(n)|w_{xx}|,$$

which implies

$$\begin{aligned}
\|[A,B](w)\|_{L^2} &\leq \frac{C(n)}{\varepsilon}\|w_x\|_{L^4}^2 + C(n)\|w_{xx}\|_{L^2} \\
&\leq \frac{C(n)}{\varepsilon}\|w_x\|_{L^\infty}\|w_x\|_{L^2} + C(n)\|w_{xx}\|_{L^2} \\
&\leq C(n)\|w\|_{H^2}(1 + \|w\|_{H^2}/\varepsilon),
\end{aligned}$$

where we have used the Sobolev imbedding $H^2(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ for $d=1$. This yields that for any $s \in [0,1]$,

$$\|g'(s)\|_{L^2} = \|[A,B](e^{is\Delta}v_0)\|_{L^2} \leq C(n)\|v_0\|_{H^2}(1 + \|v_0\|_{H^2}/\varepsilon),$$

which immediately gives

$$\|e_3\|_{L^2} \leq \tau^2 \int_0^1 \|g'(\theta\tau)\|_{L^2}d\theta \leq C(n)\tau^2\|v_0\|_{H^2}(1 + \|v_0\|_{H^2}/\varepsilon). \tag{4.25}$$

Next we estimate $e_1$ and $e_2$. In view of (2.9), we have

$$\|dB(w_1)[w_2]\|_{L^2} \leq C(\ln(\varepsilon^{-1}) + C(n))\|w_2\|_{L^2},$$

when $\varepsilon \le \varepsilon_1 := \min\{1, 1/\|w_1\|_{L^\infty}\}$. By Sobolev's imbedding,

$$\|e^{i(s-y)\Delta}v(y)\|_{L^\infty} \le C\|e^{i(s-y)\Delta}v(y)\|_{H^1} = C\|\Psi^y(v_0)\|_{H^1},$$

thus when $\varepsilon \le \varepsilon_2 := \min\{1, \frac{1/C}{\max\limits_{y\in[0,\tau]}\|\Psi^y(v_0)\|_{H^1}}\}$, we have

$$\begin{aligned}\|e_1\|_{L^2} &\le C(\ln(\varepsilon^{-1}) + C(n))\int_0^\tau\int_0^s \|e^{i(s-y)\Delta}B(v(y))\|_{L^2}dyds\\ &\le C(\ln(\varepsilon^{-1}) + C(n))\int_0^\tau\int_0^s \|B(v(y))\|_{L^2}dyds\\ &\le C(\ln(\varepsilon^{-1}) + C(n))^2\tau^2\max_{0\le y\le\tau}\|\Psi^y(v_0)\|_{L^2}\\ &\le C(\ln(\varepsilon^{-1}) + C(n))^2 C(\|v_0\|_{H^1})\tau^2. \end{aligned}\tag{4.26}$$

Similarly, by recalling

$$\|\Phi_B^{s\tau}(e^{i\tau\Delta}v_0)\|_{L^\infty} = \|e^{i\tau\Delta}v_0\|_{L^\infty} \le C\|v_0\|_{H^1},$$

when $\varepsilon \le \varepsilon_3 := \min\{1, 1/(C\|v_0\|_{H^1})\}$,

$$\begin{aligned}\|e_2\|_{L^2} &\le C(\ln(\varepsilon^{-1}) + C(n))\tau^2\int_0^1 \|B(\Phi_B^{s\tau}(e^{i\tau\Delta}v_0))\|_{L^2}ds\\ &\le C(\ln(\varepsilon^{-1}) + C(n))^2\tau^2\int_0^1 \|\Phi_B^{s\tau}(e^{i\tau\Delta}v_0)\|_{L^2}ds\\ &= C(\ln(\varepsilon^{-1}) + C(n))^2\tau^2\|v_0\|_{L^2}. \end{aligned}\tag{4.27}$$

Combining (4.25), (4.26) and (4.27), we obtain

$$\|\Psi^\tau(v_0) - \Phi^\tau(v_0)\|_{L^2} \le C(n, \|v_0\|_{H^2})\,\frac{\tau^2}{\varepsilon},$$

when $\varepsilon \le \varepsilon_0 := \min\{\varepsilon_2, \varepsilon_3\}$ which depends on $n$ and $\|v_0\|_{H^1}$ by Remark 4.3. Similarly the stability can be established by (4.6):

$$\|\Phi^\tau(v) - \Phi^\tau(w)\|_{L^2} \le (1 + C(n)\tau)\|\Phi_A^\tau(v-w)\|_{L^2} = (1 + C(n)\tau)\|v-w\|_{L^2},\tag{4.28}$$

for $v, w \in L^2(\Omega)$. Thus the error (4.21) can be similarly established by combining the local error (4.22), the stability property (4.28) and a standard argument (cf. [4,12]). $\square$

*Remark* 4.6. For $d = 2, 3$, the error estimate (4.21) can be established with $\varepsilon_0$ depending on $n$ and $\|u^\varepsilon\|_{L^\infty(0,T;H^2(\Omega))}$ by noticing that $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ and $H^2(\Omega) \hookrightarrow W^{1,4}(\Omega)$ for $d = 2, 3$.

*Remark* 4.7 (Strang splitting). When considering a Strang splitting,

$$u^{\varepsilon,k+1} = \Phi_B^{\tau/2}\left(\Phi_A^\tau\left(\Phi_B^{\tau/2}(u^{\varepsilon,k})\right)\right), \quad \text{or} \quad u^{\varepsilon,k+1} = \Phi_A^{\tau/2}\left(\Phi_B^\tau\left(\Phi_A^{\tau/2}(u^{\varepsilon,k})\right)\right),\tag{4.29}$$

by applying similar but more intricate arguments as above, one can get the error bound as

$$\|u^{\varepsilon,k} - u^\varepsilon(t_k)\|_{L^2} \le C\left(n, T, \|u^\varepsilon\|_{L^\infty([0,T];H^4(\Omega))}\right)\frac{\tau^2}{\varepsilon^3},$$

under the assumption that $u^\varepsilon \in L^\infty(0,T;H^4(\Omega))$.

*Remark* 4.8. In view of Theorem 3.1, Theorems 4.2 and 4.5 rely on a regularity that we know is available. On the other hand, the regularity assumed in the above remark on Strang splitting is unclear in general, in the sense that we don't know how to bound $u^\varepsilon$ in $L^\infty(0,T;H^4(\Omega))$.

## 5. Numerical results

In this section, we first test the convergence rate of the energy regularized model (3.2) and compare it with the other two (1.6) & (1.7). We then test the order of accuracy of the regularized Lie-Trotter splitting (LTSP) scheme (4.7) & (4.19) and Strang splitting (STSP) scheme (4.29). To simplify the presentation, we unify the regularized models (1.6), (1.7) and (3.2) as follows:

$$\begin{cases} i\partial_t u^\varepsilon(\mathbf{x},t)+\Delta u^\varepsilon(\mathbf{x},t)=\lambda u^\varepsilon(\mathbf{x},t) f^\varepsilon_{\rm reg}(|u^\varepsilon(\mathbf{x},t)|^2), \quad \mathbf{x}\in\Omega, \quad t>0,\\ u^\varepsilon(\mathbf{x},0)=u_0(\mathbf{x}), \quad \mathbf{x}\in\overline{\Omega}. \end{cases} \tag{5.1}$$

With the regularized nonlinearity $f^\varepsilon_{\rm reg}(\rho)$ being chosen as $\widetilde{f}^\varepsilon$, $\widehat{f}^\varepsilon$ and $f^\varepsilon_n$, model (5.1) recovers the regularized model (1.6), (1.7) and (3.2), respectively. In practical computation, we impose the periodic boundary condition on $\Omega$ and employ the standard Fourier pseudo-spectral method [4, 6, 8] for spatial discretization. The details are omitted here for brevity.

Hereafter, unless specified, we take $d=1$, $\Omega=[-16,16]$, $\lambda=-1$ and consider the following Gaussian initial data, i.e., $u_0(x)$ is chosen as

$$u_0(x)=b_0 e^{ivx+\frac{\lambda}{2}(x-x_0)^2}, \qquad x\in\mathbb{R}. \tag{5.2}$$

In this case, the LogSE (1.1) admits the moving Gausson solution

$$u(x,t)=b_0 e^{i(vx-(a_0+v^2)t)+\frac{\lambda}{2}(x-x_0-2vt)^2}, \qquad x\in\mathbb{R}, \quad t\geq 0, \tag{5.3}$$

with $b_0=1/\sqrt[4]{-\lambda\pi}$ and $a_0=-\lambda(1-\ln(\lambda^2/\sqrt{\pi}))$. In this paper, we fix $v=1$ and $x_0=0$. The mesh size is chosen as $h=1/64$ for all examples. To quantify the numerical errors, we define the following error functions:

$$\begin{aligned} &\check{e}^\varepsilon(t_k):=u(\cdot,t_k)-u^\varepsilon(\cdot,t_k), \quad \check{e}^\varepsilon_\rho(t_k):=\rho(\cdot,t_k)-\rho^\varepsilon(\cdot,t_k),\\ &\breve{\check{e}}^\varepsilon(t_k):=u(\cdot,t_k)-u^{\varepsilon,k}, \quad e^\varepsilon(t_k):=u^\varepsilon(\cdot,t_k)-u^{\varepsilon,k},\\ &e^\varepsilon_E:=|E(u_0)-E^\varepsilon_{\rm reg}(u_0)|. \end{aligned} \tag{5.4}$$

Here, $u$ and $u^\varepsilon$ are the exact solutions of the LogSE (1.1) and RLogSE (5.1), respectively, while $u^{\varepsilon,k}$ is the numerical solution of the RLogSE (5.1) obtained by LTSP ((4.7) or (4.19)) or STSP (4.29). The "exact" solution $u^\varepsilon$ is obtained numerically by STSP (4.29) with a very small time step, e.g., $\tau=10^{-5}$ and a very fine mesh size, e.g., $h=1/64$. The energy is obtained by the trapezoidal rule for approximating the integrals in the energy (1.4), (3.1), (3.4) and (3.5).

### 5.1. **Convergence rate of the regularized model.**

Here, we consider the error between the solutions of the RLogSE (5.1) and the LogSE (1.1). For different regularized models (i.e., different choices of regularized nonlinearity $f^\varepsilon_{\rm reg}$ in equation (5.1)), Fig. 5 shows $\|\check{e}^\varepsilon(3)\|_{H^1}$, $\|\check{e}^\varepsilon(3)\|$, $\|\check{e}^\varepsilon(3)\|_\infty$ and $\|\check{e}^\varepsilon_\rho(3)\|_1$, while Fig. 6 depicts $e^\varepsilon_E$ versus $\varepsilon$. From these figures and additional similar numerical results not shown here for brevity, we could clearly see: (i) In all norms, the solution of the RLogSE (5.1) converges linearly to that of the LogSE (1.1) in terms of $\varepsilon$. Moreover, the regularized energy $\widetilde{E}^\varepsilon$ converges linearly to the original energy $E$ in terms of $\varepsilon$, while $\widehat{E}^\varepsilon$ & $E_n^\varepsilon$ (for any $n \geq 2$) converges quadratically. These results confirm the theoretical results in Section 3.2 & 3.3. (ii) In $L^1$-norm, the density $\rho^\varepsilon$ of the solution of the RLogSE with regularized nonlinearity $\widetilde{f}^\varepsilon$ converges linearly to that of the LogSE (1.1) in terms of $\varepsilon$, while the convergence rate is not clear for those of RLogSE with other regularized nonlinearities. Generally, for fixed $\varepsilon$, the errors of the densities that measured in $L^1$-norms are smaller than those of wave functions (measured in $L^2$, $H^1$ or $L^\infty$-norm). (iii) For any fixed $\varepsilon > 0$, the proposed energy regularization (i.e., $f_{\rm reg}^\varepsilon = f_n^\varepsilon$) outperforms the other two (i.e., $f_{\rm reg}^\varepsilon = \widehat{f}^\varepsilon$ and $f_{\rm reg}^\varepsilon = \widetilde{f}^\varepsilon$) in the sense that its corresponding errors in wave function and total energy are smaller. The larger the order (i.e., $n$) of the energy-regularization is chosen, the smaller the difference between the solutions of the ERLogSE (3.2) and LogSE is obtained.

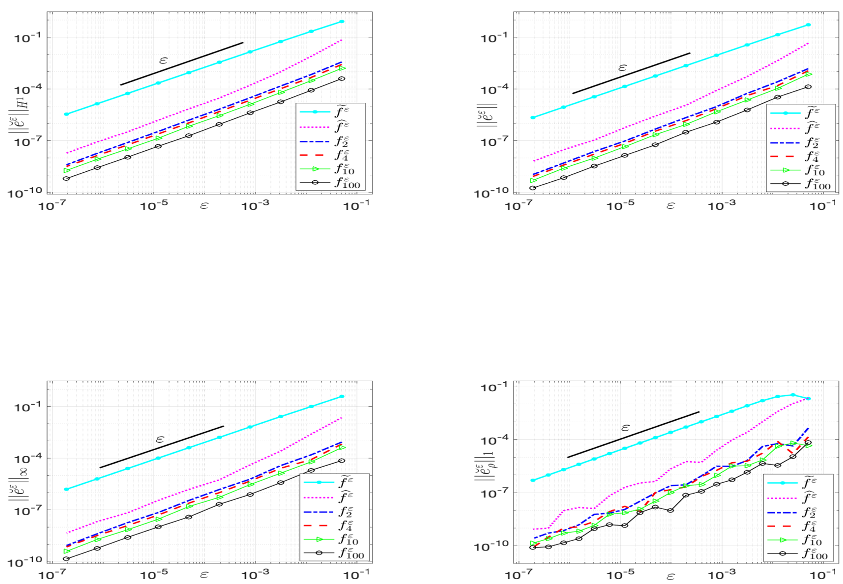

FIGURE 5. Convergence of the RLogSE (5.1) with different regularized nonlinearity $f_{\rm reg}^\varepsilon$ to the LogSE (1.1), i.e., the error $\breve{e}(3)$ in different norms as well as $\|\breve{e}_\rho(3)\|_1$ versus the regularization parameter $\varepsilon$.

5.2. **Convergence rate of the time-splitting spectral method.** Here, we investigate the model RLogSE (5.1) with $f_{\rm reg}^\varepsilon = f_n^\varepsilon$, i.e., the ERLogSE (3.2). We will test the convergence rate of type-1 LTSP (4.7) & type-2 LTSP (4.19) and the STSP (4.29) to the ERLogSE (3.2) or the LogSE (1.1) in terms of the time step $\tau$ for fixed $\varepsilon \in (0, 1)$. Fig. 7 shows the errors $\|e^\varepsilon(3)\|_{H^1}$ versus time step $\tau$ for $f_2^\varepsilon$ & $f_4^\varepsilon$. In addition, Table 1 displays $\|\breve{e}^\varepsilon(3)\|$ versus $\varepsilon$ & $\tau$ for $f_2^\varepsilon$.

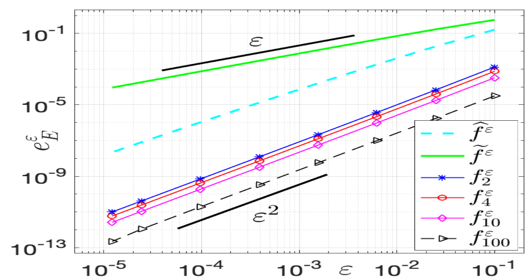


FIGURE 6. Convergence of the RLogSE (5.1) with different regularized nonlinearity $f_{\mathrm{reg}}^\varepsilon$ to the LogSE (1.1): the energy error $e_E^\varepsilon$ (5.4).

From Fig. 7, Table 1 and additional similar results not shown here for brevity, we can observe that: (i) In $H^1$ norm, for any fixed $\varepsilon \in (0,1)$ and $n \geq 2$, the LTSP scheme converges linearly while the STSP scheme converges quadratically when $\varepsilon < \varepsilon_0$ for some $\varepsilon_0 > 0$. (ii) For any $f_n^\varepsilon$ with $n \geq 2$, the STSP converges quadratically to the LogSE (1.1) only when $\varepsilon$ is sufficiently small, i.e., $\varepsilon \lesssim \tau^2$ (cf. each row in the lower triangle below the diagonal in bold letter in Table 1). (iii) When $\tau$ is sufficiently small, i.e., $\tau^2 \lesssim \varepsilon$, the ERLogSE (3.2) converge linearly at $O(\varepsilon)$ to the LogSE (1.1) (cf. each column in the upper triangle above the diagonal in bold letter in Table 1). (iv) The numerical results are similar for other $f_n^\varepsilon$ with $n \geq 4$ and when the errors are measured in $l^\infty$- and $l^2$-norm, which confirm the theoretical conclusion in Theorem 4.5 and Remark 4.7.

## 6. Conclusion

We proposed a new systematic energy regularization approach to overcome the singularity of the logarithmic nonlinearity in the logarithmic Schrödinger equation (LogSE). With a small regularized parameter $0 < \varepsilon \ll 1$, other than the existing ones that directly regularize the logarithmic nonlinearity, we regularize the corresponding nonlinear part in the energy density. The Hamiltonian flow of the new regularized energy then yields an energy regularized logarithmic Schrödinger equation (ERLogSE). Linear and quadratical convergence were established between the solutions and conserved total energy of ERLogSE and LogSE in terms of $\varepsilon$, respectively. Then we presented and analyzed time-splitting schemes to solve the ERLogSE. The classical first and second order of convergence were obtained both theoretically and numerically for Lie-Trotter and Strang splitting schemes, respectively. Numerical results suggest that the error bounds of splitting-schemes to the LogSE clearly depend on the time step $\tau$ and mesh size $h$ as well as the regularized parameter $\varepsilon$. Our numerical results confirm the error bounds and indicate that the ERLogSE model outperforms the other existing ones in accuracy.

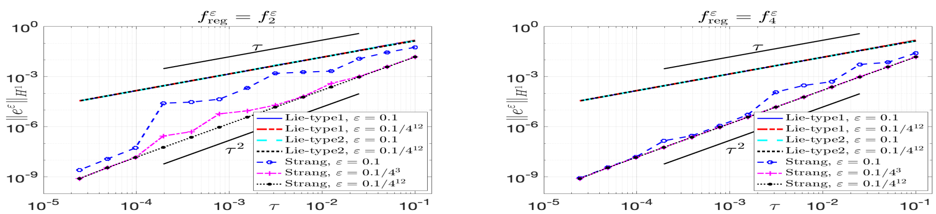


FIGURE 7. Convergence of the type-1 LTSP (4.7) & type-2 LTSP (4.19) as well as the STSP (4.29) to the ERLogSE (3.2) with regularized nonlinearity $f_2^\varepsilon$ (left) and $f_4^\varepsilon$ (right), i.e., errors $\|e^\varepsilon(3)\|_{H_1}$ versus $\tau$ for different $\varepsilon$.

TABLE 1. Convergence of the STSP (4.29) (via solving the ERLogSE (3.2) with $f_2^\varepsilon$) to the LogSE (1.1), i.e., $\|\breve{e}^\varepsilon(3)\|$ for different $\varepsilon$ and $\tau$.

| | $\tau=0.1$ | $\tau/2$ | $\tau/2^2$ | $\tau/2^3$ | $\tau/2^4$ | $\tau/2^5$ | $\tau/2^6$ | $\tau/2^7$ | $\tau/2^8$ | $\tau/2^9$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\varepsilon$=0.025 | 7.98E-3 | **2.13E-3** | 8.86E-4 | 7.28E-4 | 7.14E-4 | 7.12E-4 | 7.12E-4 | 7.12E-4 | 7.12E-4 | 7.12E-4 |
| rate | – | **1.91** | 1.27 | 0.28 | 0.03 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| $\varepsilon/4$ | 7.77E-3 | 1.96E-3 | **5.02E-4** | 1.67E-4 | 1.12E-4 | 1.08E-4 | 1.08E-4 | 1.08E-4 | 1.08E-4 | 1.08E-4 |
| rate | – | 1.99 | **1.97** | 1.59 | 0.57 | 0.06 | 0.01 | 0.00 | 0.00 | 0.00 |
| $\varepsilon/4^2$ | 7.76E-3 | 1.95E-3 | 4.88E-4 | **1.25E-4** | 3.81E-5 | 2.40E-5 | 2.28E-5 | 2.27E-5 | 2.27E-5 | 2.27E-5 |
| rate | – | 2.00 | 2.00 | **1.97** | 1.71 | 0.67 | 0.07 | 0.01 | 0.00 | 0.00 |
| $\varepsilon/4^3$ | 7.76E-3 | 1.95E-3 | 4.87E-4 | 1.22E-4 | **3.08E-5** | 8.95E-6 | 5.09E-6 | 4.74E-6 | 4.72E-6 | 4.71E-6 |
| rate | – | 2.00 | 2.00 | 2.00 | **1.98** | 1.78 | 0.82 | 0.10 | 0.01 | 0.00 |
| $\varepsilon/4^4$ | 7.76E-3 | 1.95E-3 | 4.87E-4 | 1.22E-4 | 3.04E-5 | **7.66E-6** | 2.092E-6 | 9.93E-7 | 8.80E-7 | 8.72E-7 |
| rate | – | 2.00 | 2.00 | 2.00 | 2.00 | **1.99** | 1.87 | 1.08 | 0.18 | 0.01 |
| $\varepsilon/4^5$ | 7.76E-3 | 1.95E-3 | 4.87E-4 | 1.22E-4 | 3.04E-5 | 7.61E-6 | **1.92E-6** | 5.26E-7 | 2.54E-7 | 2.27E-7 |
| rate | – | 2.00 | 2.00 | 2.00 | 2.00 | 2.00 | **1.99** | 1.87 | 1.05 | 0.16 |
| $\varepsilon/4^6$ | 7.76E-3 | 1.95E-3 | 4.87E-4 | 1.22E-4 | 3.04E-5 | 7.61E-6 | 1.90E-6 | **4.78E-7** | 1.27E-7 | 5.36E-8 |
| rate | – | 2.00 | 2.00 | 2.00 | 2.00 | 2.00 | 2.00 | **1.99** | 1.91 | 1.25 |
| $\varepsilon/4^7$ | 7.76E-3 | 1.95E-3 | 4.87E-4 | 1.22E-4 | 3.04E-5 | 7.61E-6 | 1.90E-6 | 4.76E-7 | **1.19E-7** | 3.13E-8 |
| rate | – | 2.00 | 2.00 | 2.00 | 2.00 | 2.00 | 2.00 | 2.00 | **2.00** | 1.93 |
| $\varepsilon/4^8$ | 7.76E-3 | 1.95E-3 | 4.87E-4 | 1.22E-4 | 3.04E-5 | 7.61E-6 | 1.90E-6 | 4.76E-7 | 1.19E-7 | **2.98E-8** |
| rate | – | 2.00 | 2.00 | 2.00 | 2.00 | 2.00 | 2.00 | 2.00 | 2.00 | **2.00** |

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