



**HAL**  
open science

# Strong approximation of particular one-dimensional diffusions

Madalina Deaconu, Samuel Herrmann

► **To cite this version:**

Madalina Deaconu, Samuel Herrmann. Strong approximation of particular one-dimensional diffusions. 2020. hal-02799638v1

**HAL Id: hal-02799638**

**<https://hal.science/hal-02799638v1>**

Preprint submitted on 5 Jun 2020 (v1), last revised 19 Sep 2023 (v4)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Strong approximation of particular one-dimensional diffusions

Madalina Deaconu<sup>1</sup> and Samuel Herrmann<sup>2</sup>

<sup>1</sup>Université de Lorraine, CNRS, Inria, IECL, F-54000 Nancy, France,  
Madalina.Deaconu@inria.fr

<sup>2</sup>Institut de Mathématiques de Bourgogne (IMB) - UMR 5584, CNRS,  
Université de Bourgogne Franche-Comté, F-21000 Dijon, France  
Samuel.Herrmann@u-bourgogne.fr

June 8, 2020

## Abstract

This paper develops a new technique for the path approximation of one-dimensional stochastic processes, more precisely the Brownian motion and families of stochastic differential equations sharply linked to the Brownian motion (usually known as  $L$  and  $G$ -classes). We are interested here in the  $\varepsilon$ -strong approximation. We propose an explicit and easy to implement procedure that constructs jointly, the sequences of exit times and corresponding exit positions of some well chosen domains. The main results control the number of steps to cover a fixed time interval and the convergence theorems for our scheme. We combine results on Brownian exit times from time-dependent domains (one-dimensional heat balls) and classical renewal theory. Numerical examples and issues are also described in order to complete the theoretical results.

**Key words:** Strong approximation, path simulation, Brownian motion, linear diffusion.

**2010 AMS subject classifications:** primary 65C05; secondary 60J60, 60J65, 60G17.

## An introduction to strong approximation

The aim of this study is to describe a path approximation of  $(X_t, 0 \leq t \leq T)$  where  $X_t$  stands for the one-dimensional Brownian motion starting in  $x$  or for a class of one-dimensional diffusions with non-homogeneous coefficients, and  $T$  is a fixed positive time. The usual and classical approximation procedure of any diffusion process consists in constructing numerical schemes like the Euler scheme: the time interval is split into subintervals  $0 < \frac{T}{n} < \dots < \frac{n-1}{n}T < T$ . For each of these time slots, the value of the process is given

or approximated. The convergence result of the proposed approximation is then based on stochastic convergence theorems: we obtain usually some  $L^p$ -convergence between the path built by the scheme and the real path of the process. The approximation error is not a.s. bounded by a constant.

In this study, we focus our attention on a different approach: for any  $\varepsilon > 0$ , we construct a suitable sequence of increasing random times  $(s_n^\varepsilon)_{n \geq 0}$  with  $s_0^\varepsilon = 0$ ,  $\lim_{n \rightarrow \infty} s_n^\varepsilon = +\infty$  and random points  $x_0^\varepsilon, x_1^\varepsilon, \dots, x_n^\varepsilon, \dots$  in such a way that

$$\sup_{t \in [0, T]} |X_t - x_t^\varepsilon| \leq \varepsilon \quad \text{a.s.} \quad (0.1)$$

where  $x_t^\varepsilon = \sum_{n \geq 0} x_n^\varepsilon 1_{\{s_n^\varepsilon \leq t < s_{n+1}^\varepsilon\}}$ . The procedure is quite simple to describe, the sequence  $(s_n^\varepsilon, x_n^\varepsilon)$  is associated to exit times and exit locations of well-chosen time-space domains for the process  $(t, X_t)$ .

Let us sketch the main steps of the method here. For this, let us consider a continuous function  $\phi_\varepsilon(t)$  which satisfies: there exists  $r_\varepsilon > 0$  s.t.

$$\text{Supp}(\phi_\varepsilon) = [0, r_\varepsilon] \text{ and } 0 < \phi_\varepsilon(t) \leq \varepsilon \text{ for any } t \in \text{Supp}(\phi_\varepsilon).$$

We start with  $(s_0^\varepsilon, x_0^\varepsilon) = (0, x)$  where  $x$  is the initial point of the path  $(X_t)$ . Then we define, for any  $n \geq 0$

$$s_{n+1}^\varepsilon := \inf\{t \geq s_n^\varepsilon : |X_t - X_{s_n^\varepsilon}| \geq \phi_\varepsilon(t - s_n^\varepsilon)\}$$

and  $x_{n+1}^\varepsilon := X_{s_{n+1}^\varepsilon}$ . In other words,  $s_{n+1}^\varepsilon$  is related to the first exit time of the stochastic process  $(t, X_{s_n^\varepsilon+t} - x_n^\varepsilon)_{t \geq 0}$  from the time-space domain  $\{(t, x) : |x| \leq \phi_\varepsilon(t)\}$ , called  $\phi_\varepsilon$ -domain in the sequel.

We observe that:

- the bound (0.1) is satisfied due to the boundedness of the function  $\phi_\varepsilon$
- the sequence  $(s_n^\varepsilon)$  satisfies  $s_{n+1}^\varepsilon - s_n^\varepsilon \leq r_\varepsilon$ , for any  $n \geq 0$ , due to the boundedness of the support of  $\phi_\varepsilon$ .

For such an approximation of the paths, the challenge consists in the choice of an appropriate function  $\phi_\varepsilon$  defining the  $\phi_\varepsilon$ -domain in such a way that the simulation of both the exit time and the exit location is easy to construct and implement. Moreover the analysis of the random scheme is based on a precise description of the number of random intervals  $[s_n^\varepsilon, s_{n+1}^\varepsilon[$  required in order to cover  $[0, T]$ . Such an analysis is developed in the next section.

Our main motivation is to develop a new approach that gives the  $\varepsilon$ -strong approximation for a large class of multidimensional SDEs. In this paper the main tools and results of this topic are developed for some particular SDEs in one dimension. We intend to pursue this research for more general situations starting with the multidimensional Brownian motion and Bessel processes.

The study of the strong behaviour of an approximation scheme, and in particular the characterisation by some bounds depending on  $\varepsilon$  of  $\sup_{t \in [0, T]} \|X_t - x_t^\varepsilon\|$  where  $x_t^\varepsilon$  stands for an approximation scheme, was considered recently by some other authors. In Blanchet, Chen

and Dong [3] the authors study the approximation of multidimensional SDEs by considering transformations of the underlying Brownian motion (the so-called Itô-Lyons map) and follow a rough path theory approach. In this paper the authors refer to the class of procedures which achieve the construction of such an approximation as Tolerance-Enforced Simulation (TES) or  $\varepsilon$ -strong simulation methods. In Chen and Huang [4] a similar question is considered but the result is obtained only for SDEs in one dimension and the effective construction of an approximation scheme is not obvious. This last procedure was extended by Beskos, Peluchetti and Roberts [2] where an iterative sampling method, which delivers upper and lower bounding processes for the Brownian path, is given.

In a more general context, Hefter, Herzwurm and Müller-Gronbach [10] give lower error bounds for the pathwise approximation of scalar SDEs, the results are based on the observations of the driving Brownian motion. Previously the notion of strong convergence was studied also intensively for particular processes like the CIR process. Strong convergence without rate was obtained by Alfonsi [1] or Hutzenthaler and Jentzen [15]. Optimal lower and upper bounds were also given. For stochastic differential equations with Lipschitz coefficients Müller-Gronbach [16] and Hofmann, Müller-Gronbach and Ritter [14] obtained lower error bounds.

All these results give a new and interesting highlight in this topic of pathwise and  $\varepsilon$ -strong approximation, but are not quite effective for a numerical purpose due to the limitations of current techniques. We believe that the construction we give and the results we prove in this paper provide an important step in the development of the  $\varepsilon$ -strong convergence for general SDEs. The main advantage of our approach is that we give an explicit and constructive procedure of the scheme and this conducts to an easy to implement algorithm. Furthermore the construction is based on explicit distributions of the exit time for time-space domains, closely related to the behaviour of the underlying process. This construction is thus deeply guided with the dynamic of the process.

For practical purposes the approximation scheme is the object of interest and in order to characterize and control its behaviour we are looking for sequences which have the same distribution. We need thus to introduce the following definition:

**Definition 0.1.** *The random process  $(y_t^\varepsilon)$  is an  $\varepsilon$ -strong approximation of the diffusion process  $(X_t)$  if there exists  $(x_t^\varepsilon)$  satisfying (0.1) such that  $(y_t^\varepsilon)$  and  $(x_t^\varepsilon)$  are identically distributed.*

The material is organized as follows. In Section 1, we focus our attention on the number of space-time domains used for building the approximated path on a given fixed time interval  $[0, T]$ . This number is denoted by  $N_T^\varepsilon$ . The main specific feature related to our approach is the randomness associated to the time splitting. A sharp description of the random number of time steps  $N_T^\varepsilon$  permits to emphasize the efficiency of the  $\varepsilon$ -strong simulation. The first section points some information in a quite general framework, that is  $\varepsilon^2 \mathbb{E}[N_T^\varepsilon]$  is upper bounded in the  $\varepsilon$  small limit, while the forthcoming sections permit to go into details for specific diffusion processes. Section 2 introduces the particular Brownian case and families of one-dimensional diffusions ( $L$ -class and  $G$ -class of diffusion in particular) are further explored in Section 3. In each case, an algorithm based on a specific  $\phi_\varepsilon$ -domain (heat ball) is presented (Theorem 2.1 and Theorem 3.8) and the efficiency of the approximation is investigated (Proposition

2.2 and Theorem 3.13). We obtain the convergence towards an explicit limit for the average expression  $\varepsilon^2 \mathbb{E}[N_T^\varepsilon]$  as  $\varepsilon$  tends to 0. In the particular diffusion case, there exists a constant  $\mu > 0$  such that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \mathbb{E}[N_T^\varepsilon] = \mu \mathbb{E} \left[ \int_0^{\rho(T)} \frac{1}{\eta^2(x + B_s)} ds \right], \quad \forall (T, x) \in \mathbb{R}_+ \times \mathbb{R} \quad (0.2)$$

where  $\eta$  and  $\rho$  are both functions related to the approximation procedure and  $(B_t)_{t \geq 0}$  stands for a standard one-dimensional Brownian motion.

Finally numerical examples permit to illustrate the convergence result of the algorithm in the last section.

## 1 Number of random intervals needed for covering the time interval $[0, T]$

The sharpness of the approximation is deeply related to the number of random intervals  $[s_n^\varepsilon, s_{n+1}^\varepsilon[$  used to cover  $[0, T]$ . If  $(X_t)$  is a homogeneous Markovian process, then we observe that  $U_{n+1}^\varepsilon = s_{n+1}^\varepsilon - s_n^\varepsilon$ , for  $n \geq 0$ , is a sequence of i.i.d. a.s. bounded variables. Obviously we have:

$$s_n^\varepsilon = \sum_{i=1}^n U_i^\varepsilon.$$

The number of variates corresponds to

$$N_T^\varepsilon := \inf\{n \geq 1 : s_n^\varepsilon \geq T\}.$$

We can control, for any  $j \in \mathbb{N}$  and  $\lambda > 0$ :

$$\mathbb{P}(N_T^\varepsilon > j) = \mathbb{P}(s_j^\varepsilon < T) = \mathbb{P}(e^{-\lambda s_j^\varepsilon} > e^{-\lambda T}) \leq e^{\lambda T} \mathbb{E}[e^{-\lambda s_j^\varepsilon}] = e^{\lambda T} \mathbb{E}[e^{-\lambda U_1^\varepsilon}]^j. \quad (1.1)$$

This calculus proves that the upper-bound essentially depends on the Laplace transform of  $U_1^\varepsilon$ .

**Nota 1.1.** *Before stating a first result let us give an important convention. All along the text we need to control (upper or lower bounds) several quantities. In order to do this we use  $C$  and  $\kappa$  to design positive constants, whose value may change from one line to the other. When the constants depend on parameters of prime interest, we use, for example, the notation  $C_{T,\alpha}$  to suggest that the constant  $C$  depends in some way on  $T$  and  $\alpha$ , where  $T$  and  $\alpha$  denote here some parameters.*

**Proposition 1.2.** *For  $\varepsilon > 0$ , let us assume that  $U_1^\varepsilon \stackrel{(d)}{=} \varepsilon^2 U$  where  $U$  is a positive random variable independent of the parameter  $\varepsilon$ , (here and all along the paper  $\stackrel{(d)}{=}$  stands for equality*

in distribution).

1. If there exist two constants  $C > 0$  and  $\kappa > 0$  such that  $\mathbb{E}[e^{-\lambda U}] \leq \frac{C}{\lambda^\kappa}$  for all  $\lambda > 0$ , then

$$\mathbb{P}(N_T^\varepsilon > j) \leq \left( \frac{eTC^{1/\kappa}}{j\kappa\varepsilon^2} \right)^{j\kappa}, \quad \forall j \in \mathbb{N}^*. \quad (1.2)$$

2. If  $\mathbb{E}[U^2] < \infty$ , then for any  $\delta > 1$ , there exists  $\varepsilon_0 > 0$  such that

$$\mathbb{E}[N_T^\varepsilon] \leq \frac{\delta eT}{\varepsilon^2 \mathbb{E}[U]}, \quad \forall \varepsilon \leq \varepsilon_0.$$

*Proof.* For the result in 1., by using both the Markov property (1.1) and the condition concerning the Laplace transform of  $U$ , we obtain:

$$\mathbb{P}(N_T^\varepsilon > j) \leq e^{\lambda T} \left( \frac{C}{\varepsilon^{2\kappa} \lambda^\kappa} \right)^j, \quad \forall \lambda > 0.$$

By choosing the optimal value of  $\lambda$  given by  $\lambda = \frac{j\kappa}{T}$  we obtain (1.2). For the result in 2., we can also remark that (1.1) leads to

$$\mathbb{E}[N_T^\varepsilon] = \sum_{j \geq 0} \mathbb{P}(N_T^\varepsilon > j) \leq \frac{e^{\lambda T}}{1 - \mathbb{E}[e^{-\lambda U_1^\varepsilon}]}. \quad (1.3)$$

If  $\mathbb{E}[U^2] < \infty$ , we get

$$\mathbb{E}[e^{-\lambda U_1^\varepsilon}] = 1 - \lambda \varepsilon^2 \mathbb{E}[U] + o(\lambda \varepsilon^2), \quad \lambda > 0.$$

The particular choice  $\lambda = 1/T$  implies the announced result.  $\square$

**Remark 1.3.** If the condition  $\mathbb{E}[U^2] < \infty$ , is not satisfied, we can construct another approach which leads to less sharp bounds. Indeed if  $N_T^\varepsilon$  denotes the number of r.v.  $(U_n^\varepsilon)$  such that  $s_{N_T^\varepsilon} \geq T$ , then the strong Markov property implies

$$\mathbb{E}[N_T^\varepsilon] \leq k \mathbb{E}[N_{T/k}^\varepsilon], \quad \forall k \in \mathbb{N}^*.$$

Taking  $\lambda = k/T$  in (1.3) and afterwards  $k = \lfloor T/\varepsilon^2 \rfloor$  we obtain

$$\mathbb{E}[N_T^\varepsilon] \leq \frac{ke}{1 - \mathbb{E}[e^{-k\varepsilon^2 U/T}]} = \frac{\lfloor T/\varepsilon^2 \rfloor e}{1 - \mathbb{E}[e^{-\lfloor T/\varepsilon^2 \rfloor \varepsilon^2 U/T}]} \approx \frac{eT}{\varepsilon^2 (1 - \mathbb{E}[e^{-U}])} \text{ as } \varepsilon \rightarrow 0.$$

This result is less sharp than the statement of Proposition 1.2 since  $1 - \mathbb{E}[e^{-U}] \leq \mathbb{E}[U]$  but it works even if the second moment of  $U$  is not finite.

Let us just note that the large deviations theory cannot lead to interesting bounds in our case. Indeed the rate function  $I$  used in Cramer's theorem satisfies:

$$\limsup_{n \rightarrow \infty} n \ln \mathbb{P}(N_T^\varepsilon > n) \leq \limsup_{n \rightarrow \infty} n \ln \mathbb{P}(s_n^\varepsilon \leq T) = - \inf_{x \in [0, T]} I(x) = -\infty.$$

## 2 Approximation of one-dimensional Brownian paths

We recall that for our approach it is essential to find a function  $\phi_\varepsilon$  with compact support  $[0, r_\varepsilon]$  which satisfies  $\sup_{t \in [0, r_\varepsilon]} \phi_\varepsilon(t) = \varepsilon$  and such that the exit time  $s_1^\varepsilon$  of the  $\phi_\varepsilon$ -domain is simple to generate.

The choice of  $\phi_\varepsilon$  is directly related to the method of images described by Lerche [11] and to the heat equation on some particular domain, called heat-balls and defined in Evans, Section 2.3.2, [9]. More recent results on this subject can be found in [8], [7] and [6].

### BROWNIAN SKELETON $(BS)_\eta$

1. Let  $\varepsilon > 0$ . We define  $\phi_\varepsilon(t) := \sqrt{t \ln(\varepsilon^2 e/t)}$ , for  $t \in I_\varepsilon := [0, r_\varepsilon]$  with  $r_\varepsilon = e\varepsilon^2$ .
2. Let  $(A_n)_{n \geq 1}$  be a sequence of independent random variables with gamma distribution  $\text{Gamma}(3/2, 2)$
3. Let  $(Z_n)_{n \geq 1}$  be a sequence of i.i.d. Rademacher random variables (taking values  $+1$  or  $-1$  with probability  $1/2$ ). The sequences  $(A_n)_{n \geq 1}$  and  $(Z_n)_{n \geq 1}$  are independent.

**Definition:** For  $\varepsilon > 0$  and for any function  $\eta : \mathbb{R} \rightarrow \mathbb{R}_+$ , the *Brownian skeleton*  $(BS)_\eta$  corresponds to

$$\left( (U_n^\varepsilon)_{n \geq 1}, (s_n^\varepsilon)_{n \geq 1}, (x_n^\varepsilon)_{n \geq 0} \right) \quad \text{with} \quad \begin{cases} U_n^\varepsilon = \varepsilon^2 \eta^2(x_{n-1}^\varepsilon) e^{1-A_n}, & s_n^\varepsilon = \sum_{k=1}^n U_k^\varepsilon, \\ x_n^\varepsilon = x_{n-1}^\varepsilon + Z_n \eta(x_{n-1}^\varepsilon) \phi_\varepsilon(\eta^{-2}(x_{n-1}^\varepsilon) U_n^\varepsilon), & \forall n \geq 1 \end{cases}$$

and  $x_0^\varepsilon = x$ .

**Theorem 2.1.** *Let  $\varepsilon > 0$  and let us consider a Brownian skeleton  $(BS)_\eta$  with  $\eta \equiv 1$ . Then  $x_t^\varepsilon = \sum_{n \geq 0} x_n^\varepsilon 1_{\{s_n^\varepsilon \leq t < s_{n+1}^\varepsilon\}}$  is an  $\varepsilon$ -strong approximation of the Brownian paths starting in  $x$ . Moreover the number of approximation points on the fixed interval  $[0, T]$  satisfies:*

$$\mathbb{P}(N_T^\varepsilon > j) \leq \left( \frac{\beta' T \omega(3/2, 2, \beta')^{\beta'}}{j \varepsilon^2} \right)^{j/\beta'}, \quad \forall j \in \mathbb{N}, \quad \forall \beta' > 2, \quad (2.1)$$

with  $\omega$  a constant defined in the appendix, (4.5). Moreover, for every  $\delta > 1$  there exists  $\varepsilon_0 > 0$ , such that the following upper-bound holds,

$$\mathbb{E}[N_T^\varepsilon] \leq \frac{3\sqrt{3}\delta T}{\varepsilon^2}, \quad \forall \varepsilon \leq \varepsilon_0.$$

*Proof.* First we remark easily that  $\sup_{t \in I_\varepsilon} \phi_\varepsilon(t) = \varepsilon$  as required. So we start the skeleton of the Brownian paths  $(BS)_1$  with the starting time-space value  $(0, x_0^\varepsilon = x)$ . Then  $(0 + U_1^\varepsilon, x_0^\varepsilon +$

$Z_1\phi_\varepsilon(U_1^\varepsilon)$  stands for the first exit time and exit location of the time-space domain originated in  $(0, x_0^\varepsilon)$  whose boundary is defined by  $\phi_\varepsilon$ .

The second step is like the first one, it suffices to consider the new starting point  $(s_1^\varepsilon, x_1^\varepsilon) := (U_1^\varepsilon, x_0^\varepsilon + Z_1\phi_\varepsilon(U_1^\varepsilon))$  and so on... Using the results obtained in Lerche [11] and Deaconu - Herrmann [7], we know that these exit times are distributed like exponentials of gamma random variables (see, for instance, [7]). In particular, the probability distribution function of  $U_1^\varepsilon$  satisfies:

$$f_{U_1^\varepsilon}(t) = \frac{\phi_\varepsilon(t)}{\varepsilon\sqrt{2e\pi t}} = \frac{\sqrt{\ln(\varepsilon^2 e/t)}}{\varepsilon\sqrt{2e\pi t}} 1_{I_\varepsilon}(t), \quad \forall t \in \mathbb{R}.$$

We deduce that  $U_1^\varepsilon$  and  $e\varepsilon^2 W$  defined in Lemma 4.1 are identically distributed (with the parameters  $\alpha = \frac{3}{2}$  and  $\beta = 2$ ). By Lemma 4.1, we get for any  $\beta' > 2$

$$\mathbb{E}[e^{-\lambda U_1^\varepsilon}] \leq \omega\left(\frac{3}{2}, 2, \beta'\right) \left(\frac{1}{e\varepsilon^2 \lambda}\right)^{1/\beta'}.$$

Proposition 1.2 permits to obtain the bounds of the number of points needed to approximate the Brownian paths on the interval  $[0, T]$ , as  $\mathbb{E}(W) = 3\sqrt{3}$ .  $\square$

Let us just notice that for  $U$  a standard uniformly distributed r.v. and  $G$  a standard Gaussian r.v. independent of  $U$ ,  $W = U^2 e^{-G^2}$  is random variable with the PDF presented in Lemma 4.1 associated to the parameters  $\alpha = \frac{3}{2}$  and  $\beta = 2$ .

We can easily improve the description of the number of approximation points. Since  $(U_n^\varepsilon)_{n \geq 0}$  is a sequence of independent random variables,  $(N_t^\varepsilon)_{t \geq 0}$  is a renewal process and the classical asymptotic description holds:

**Proposition 2.2.** *The number of approximation points satisfies:*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \mathbb{E}[N_T^\varepsilon] = \frac{T}{e} 3^{3/2}.$$

Moreover the following CLT is observed:

$$\lim_{\varepsilon \rightarrow 0} \sqrt{\frac{\mu^3}{\varepsilon^2 \sigma^2 T}} \left( \varepsilon^2 N_T^\varepsilon - \frac{T}{e} 3^{3/2} \right) = G \quad \text{in distribution}$$

with  $G$  a  $\mathcal{N}(0, 1)$  standard Gaussian random variable,  $\mu = e 3^{-3/2} \approx 0.5231336$  and  $\sigma^2 = (5^{-3/2} - 3^{-3}) e^2 \approx 0.3872285$ .

*Proof.* Let us consider  $(\bar{N}_t)_{t \geq 0}$  a renewal process with arrivals  $(e^{1-A_n})_{n \geq 1}$  independent random variables defined in Theorem 2.1. Since  $A_1$  is gamma distributed, the arrival variable satisfies  $\mathbb{E}[e^{1-A_1}] = e \mathcal{L}_A(1)$  where  $\mathcal{L}_A$  stands for the Laplace transform of  $A_1$ . It is well known that  $\mathcal{L}_A(s) = (2s + 1)^{-3/2}$ .

We use here classical results for the renewal theory, see for example [5]. The elementary renewal theorem leads to

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}[\bar{N}_t]}{t} = \frac{1}{\mathbb{E}[e^{1-A_1}]} = \frac{3^{3/2}}{e}.$$



In order to obtain the first part of the statement, it suffices to observe that:

$$N_T^\varepsilon = \inf \left\{ n \geq 0 : \sum_{k=1}^n e^{1-A_k} \geq \frac{T}{\varepsilon^2} \right\} = \overline{N}_{T/\varepsilon^2}.$$

We deduce that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \mathbb{E}[N_T^\varepsilon] = \lim_{\varepsilon \rightarrow 0} \varepsilon^2 \mathbb{E}[\overline{N}_{T/\varepsilon^2}] = \lim_{t \rightarrow \infty} T \frac{\mathbb{E}[\overline{N}_t]}{t} = \frac{T}{e} 3^{3/2}.$$

The same argument holds for the CLT: if we denote by  $\mu = \mathbb{E}[e^{1-A_1}]$  and  $\sigma^2 = \text{Var}(e^{1-A_1})$  then

$$\lim_{\varepsilon \rightarrow 0} \sqrt{\frac{t\mu^3}{\sigma^2}} \left( \frac{\overline{N}_t}{t} - \frac{1}{\mu} \right) = G \quad \text{in distribution,}$$

where  $G$  is a  $\mathcal{N}(0, 1)$  standard Gaussian random variable. The statement is therefore a consequence of the link between  $N_T^\varepsilon$  and  $\overline{N}_{T/\varepsilon^2}$ .  $\square$

### 3 The particular L and G classes of diffusion

Let us now consider some generalizations of the Brownian paths study. We introduce solutions of the following one-dimensional stochastic differential equation:

$$dX_t = \sigma(t, X_t)dB_t + \mu(t, X_t) dt, \quad X_0 = x_0, \quad (3.1)$$

where  $(B_t, t \geq 0)$  stands for a standard one-dimensional Brownian motion and  $\sigma, \mu : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$ . Let us consider two families of diffusions introduced in Wang - Pötzelberger [17]:

1. ( $L$ -class) for  $\sigma(t, x) = \overline{\sigma}(t)$  and  $\mu(t, x) = a(t)x + b(t)$ ,  $x \in \mathbb{R}$
2. ( $G$ -class) for  $\sigma(t, x) = \underline{\sigma}x$  and  $\mu(t, x) = a(t)x + b(t)x \ln(x)$ ,  $x \in \mathbb{R}_+$ ,

where  $\overline{\sigma}, a, b : \mathbb{R}_+ \rightarrow \mathbb{R}$  are continuous functions and  $\underline{\sigma} \in \mathbb{R}$ .

Let us note that, in such particular cases, the solution of the SDE (3.1) has the same distribution as a function of the time-changed Brownian motion:

$$X_t = f(t, x_0 + B_{\rho(t)}), \quad t \geq 0, \quad (3.2)$$

(where  $f$  and  $\rho$  denote functions that we specify for each class afterwards).

For  $L$ -class diffusions for instance one particular choice of the function  $f$  (this choice is not unique) is given by (see, for instance, Karatzas and Shreve [12], p. 354, Section 5.6 for classical formulas and Herrmann and Massin [13] for new developments in this topic):

$$f(t, x) = \frac{\overline{\sigma}(t)}{\sqrt{\rho'(t)}} x + c(t), \quad (3.3)$$

with

$$c(t) = e^{\int_0^t a(s) ds} \int_0^t b(s) e^{-\int_0^s a(u) du} ds, \quad \text{and} \quad \rho(t) = \int_0^t \overline{\sigma}^2(s) e^{-2 \int_0^s a(u) du} ds.$$

**Remark 3.1.** *If we have a diffusion in the L-class characterized by some fixed function  $f(t, x)$  given in (3.2) then we can obtain a diffusion of G-class by using the function  $e^{f(t, x)}$  instead of  $f(t, x)$ . Obviously the corresponding coefficients  $a, b$ , and  $\underline{\sigma}$  need to be specified with respect to those connected to  $f(t, x)$ .*

**Proposition 3.2.** *Let us define the following diffusion process*

$$X_t = f(t, x_0 + B_{\rho(t)}), \quad t \geq 0, \quad (3.4)$$

where  $f$  is given by (3.3). Then  $X_t$  is a weak solution of the stochastic differential equation (3.1).

*Proof.* We can write the previous expression for  $f$  on the form

$$f(t, x) = x \cdot e^{\int_0^t a(s) ds} + e^{-\int_0^t a(s) ds} \int_0^t b(s) e^{-\int_0^s a(u) du} ds. \quad (3.5)$$

We denote also by

$$\mathbb{I}_t = \int_0^t \sqrt{\rho'(s)} dB_s. \quad (3.6)$$

In order to prove the result we need to prove that  $X_t = f(t, x_0 + \mathbb{I}_t)$  satisfies the equation (3.1). For the initial condition we can see that:

$$X_0 = f(0, x_0) = x_0. \quad (3.7)$$

Let us now evaluate

$$\begin{aligned} dX_t &= \left[ a(t) e^{\int_0^t a(s) ds} (x_0 + \mathbb{I}_t) + a(t) e^{\int_0^t a(s) ds} \int_0^t b(s) e^{-\int_0^s a(u) du} ds + e^{\int_0^t a(s) ds} b(t) e^{-\int_0^t a(s) ds} \right] dt \\ &\quad + \bar{\sigma}(t) dB_t \\ &= [a(t) f(t, x_0 + \mathbb{I}_t) + b(t)] dt + \bar{\sigma}(t) dB_t = [a(t) X_t + b(t)] dt + \bar{\sigma} dB_t. \end{aligned} \quad (3.8)$$

This ends the proof of the proposition.  $\square$

In this section, we consider particular diffusion processes which are strongly related to the Brownian paths. It is therefore intuitive to replace in (3.2) the Brownian trajectory by its approximation. If the function  $f$  is Lipschitz continuous, then the error stemmed from the approximation is easily controlled (the proof is left to the reader).

**Assumption 3.3.** *The diffusion process  $(X_t)_{t \geq 0}$  satisfies*

$$X_t = f(t, x_0 + B_{\rho(t)})$$

with  $f$  a Lipschitz continuous function:

$$|f(t, x) - f(s, y)| \leq K_{Lip}(T)(|x - y| + |t - s|), \quad \forall (x, y) \in \mathbb{R}^2, \quad \forall (s, t) \in [0, T]^2 \quad (3.9)$$

where  $K_{Lip}(T)$  stands for the Lipschitz constant. The function  $\rho$  is an increasing continuous function with initial value  $\rho(0) = 0$ .

**Proposition 3.4.** Consider  $T > 0$  and  $\varepsilon > 0$  fixed. Let the diffusion process  $(X_t)$  satisfy Assumption 3.3 and let

$$x_t^\theta := \sum_{n \geq 0} x_n^\theta 1_{\{s_n^\theta \leq t < s_{n+1}^\theta\}}$$

be a  $\theta$ -strong approximation of the Brownian motion (see Theorem 2.1) with  $\theta = \varepsilon K_{Lip}^{-1}(\rho^{-1}(T))$  on the time interval  $[0, \rho^{-1}(T)]$ , where  $K_{Lip}(T)$  is defined in (3.9), then

$$y_t^\varepsilon := \sum_{n \geq 0} f(\rho^{-1}(s_n^\theta), x_n^\theta) 1_{\{s_n^\theta \leq \rho(t) < s_{n+1}^\theta\}}$$

is an  $\varepsilon$ -strong approximation of  $(X_t)$  on  $[0, T]$ .

Unfortunately the Lipschitz continuity of the function  $f$  is a restrictive condition which is not relevant for most of the diffusion processes. In particular, a typical diffusion belonging to the  $L$  or  $G$ -class does not satisfy the Lipschitz condition. Consequently we introduce a weaker framework.

**Assumption 3.5.** The diffusion process  $(X_t)_{t \geq 0}$  is a function of the time-changed Brownian motion:

$$X_t = f(t, x_0 + B_{\rho(t)})$$

where  $\rho$  is an increasing continuous function with initial value  $\rho(0) = 0$ ,  $f$  is a  $\mathcal{C}^{1,1}(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ -function satisfying

$$\sup_{t \in [0, T]} \max \left\{ \left| \frac{\partial f}{\partial t}(t, x) \right|, \left| \frac{\partial f}{\partial x}(t, x) \right| \right\} \leq F(x^2), \quad \forall x \in \mathbb{R}, \quad (3.10)$$

with a strictly increasing  $\mathcal{C}^2$ -continuous function  $F$ . Moreover there exist two constants  $\kappa_1$  and  $\kappa_2$  such that  $\max(F, F', F'')(x^2) \leq \kappa_1 e^{\kappa_2 x}$  for all  $x \geq 0$ .

**Assumption 3.6.**  $\exists \kappa_{\min} > 0$  such that  $\rho'(\rho^{-1}(x)) \geq \kappa_{\min}$  for all  $x \in \mathbb{R}$ .

**Remark 3.7.** One can check easily that the  $L$  and  $G$ -class diffusions verify these hypothesis.

Let us define the function  $\eta$  to be

$$\eta(x) = \frac{1}{(e\kappa_{\min}^{-1} + 1)F(2x^2 + 1)}. \quad (3.11)$$

This function is strictly decreasing and Lipschitz continuous.

**Theorem 3.8.** Let  $\varepsilon > 0$ . Let  $(X_t)_{t \geq 0}$  be the solution of (3.1) satisfying Assumptions 3.5 and 3.6 and let us consider the Brownian skeleton  $(BS)_\eta$  associated to the function  $\eta$  defined in (3.11), then

$$y_t^\varepsilon := \sum_{n \geq 0} f(\rho^{-1}(s_n^\varepsilon), x_n^\varepsilon) 1_{\{s_n^\varepsilon \leq \rho(t) < s_{n+1}^\varepsilon\}} \quad (3.12)$$

is an  $\varepsilon$ -strong approximation of  $(X_t)$  on  $[0, T]$ .

*Proof.* Let us assume that  $t$  satisfies  $s_n^\varepsilon \leq \rho(t) \leq s_{n+1}^\varepsilon$  for some  $n \in \mathbb{N}$ . We denote  $t_n^\varepsilon := \rho^{-1}(s_n^\varepsilon)$  and  $A_t^n := f(t, x_0^\varepsilon + B_{\rho(t)}) - f(t_n^\varepsilon, x_n^\varepsilon)$ . We obtain, there exists  $\tau \in (0, 1)$  such that:

$$\begin{aligned} A_t^n &= (t - t_n^\varepsilon) \frac{\partial f}{\partial t}(t_n^\varepsilon + \tau(t - t_n^\varepsilon), x_n^\varepsilon + \tau(x_0^\varepsilon + B_{\rho(t)} - x_n^\varepsilon)) \\ &\quad + (x_0^\varepsilon + B_{\rho(t)} - x_n^\varepsilon) \frac{\partial f}{\partial x}(t_n^\varepsilon + \tau(t - t_n^\varepsilon), x_n^\varepsilon + \tau(x_0^\varepsilon + B_{\rho(t)} - x_n^\varepsilon)). \end{aligned}$$

Under the assumption (3.10) we have

$$|A_t^n| \leq \left( |t - t_n^\varepsilon| + |x_0^\varepsilon + B_{\rho(t)} - x_n^\varepsilon| \right) \cdot F((x_n^\varepsilon + \tau(x_0^\varepsilon + B_{\rho(t)} - x_n^\varepsilon))^2).$$

Since

$$|t - t_n^\varepsilon| \leq |t_{n+1}^\varepsilon - t_n^\varepsilon| = |\rho^{-1}(s_{n+1}^\varepsilon) - \rho^{-1}(s_n^\varepsilon)| = \left| \int_{s_n^\varepsilon}^{s_{n+1}^\varepsilon} \frac{du}{\rho'(\rho^{-1}(u))} \right|,$$

we obtain

$$|t - t_n^\varepsilon| \leq \kappa_{\min}^{-1} |s_{n+1}^\varepsilon - s_n^\varepsilon| = \kappa_{\min}^{-1} |U_{n+1}^\varepsilon| \leq e \kappa_{\min}^{-1} \varepsilon^2 \eta(x_n^\varepsilon).$$

Moreover, by the definition of the BM approximation,

$$|x_0^\varepsilon + B_{\rho(t)} - x_n^\varepsilon| \leq \varepsilon \eta(x_n^\varepsilon).$$

Finally due to the monotone property of  $F$ ,

$$\begin{aligned} |A_t^n| &\leq \left( e \kappa_{\min}^{-1} \varepsilon^2 \eta^2(x_n^\varepsilon) + \varepsilon \eta(x_n^\varepsilon) \right) \cdot F((x_n^\varepsilon + \tau(x_0^\varepsilon + B_{\rho(t)} - x_n^\varepsilon))^2) \\ &\leq \left( e \kappa_{\min}^{-1} \varepsilon^2 \eta^2(x_n^\varepsilon) + \varepsilon \eta(x_n^\varepsilon) \right) \cdot F(2(x_n^\varepsilon)^2 + 2\varepsilon^2 \eta^2(x_n^\varepsilon)). \end{aligned}$$

There exists  $\varepsilon_0 > 0$  such that  $\varepsilon^2 \eta^2(x) \leq 1/2$  for all  $x \in \mathbb{R}$  and  $\varepsilon \leq \varepsilon_0$ . Then, by the definition of the function  $\eta$ , for  $\varepsilon \leq \varepsilon_0$ , we have

$$|A_t^n| \leq (e \kappa_{\min}^{-1} + 1) \varepsilon \eta(x_n^\varepsilon) \cdot F(2(x_n^\varepsilon)^2 + 1) \leq \varepsilon,$$

for any  $t \in [s_n^\varepsilon, s_{n+1}^\varepsilon]$ . We deduce that the linear interpolation between  $(y_n^\varepsilon)_n$  where  $y_n^\varepsilon := f(\rho^{-1}(s_n^\varepsilon), x_n^\varepsilon)$  and  $n \leq \inf\{k \geq 0 : s_k \geq \rho(T)\}$  is a  $\varepsilon$ -strong approximation of  $(X_t, t \in [0, T])$ .  $\square$

Let us now describe the efficiency of the  $\varepsilon$ -strong approximation. We introduce

$$N_t^\varepsilon := \inf\{n \geq 0 : s_n^\varepsilon \geq t\} \quad \text{and} \quad \hat{N}_t^\varepsilon := N_{\rho(t)}^\varepsilon, \quad (3.13)$$

where  $s_n^\varepsilon$  is issued from the Brownian skeleton  $(BS)_\eta$ .  $\hat{N}_t^\varepsilon$  corresponds therefore to the number of random points needed to approximate the diffusion paths on  $[0, t]$ . Let us observe that the random variables  $U_n^\varepsilon$  are no more i.i.d. random variables in the diffusion case (different to the Brownian case), therefore we cannot use the classical renewal theorem in order to describe  $\hat{N}_t^\varepsilon$ .

**Proposition 3.9.** *Let  $(N_t^\varepsilon, t \geq 0)$  be the counting process defined by (3.13). Then, under Assumptions 3.5 and 3.6, for any  $x \in \mathbb{R}$  and  $t \geq 0$ , the average  $\psi^\varepsilon(t, x) := \mathbb{E}[N_t^\varepsilon | x_0^\varepsilon = x]$  is finite. Moreover there exists a constant  $\lambda_0 > 0$  such that the Laplace transform  $\mathcal{L}\psi^\varepsilon(\lambda, x) := \int_0^\infty e^{-\lambda t} \psi^\varepsilon(t, x) dt$  is finite for any  $\lambda \in \mathbb{C}$  satisfying  $\text{Re}(\lambda) > \lambda_0$ .*

*Proof.* The mean of the counting process is defined by

$$\psi^\varepsilon(t, x) = \sum_{n \geq 1} \mathbb{P}(N_t^\varepsilon \geq n) = \sum_{n \geq 1} \mathbb{P}(s_n^\varepsilon \leq t).$$

Let us denote by  $m_n^\varepsilon := \min_{0 \leq k \leq n-1} \eta^2(x_k^\varepsilon)$ , where  $\eta$  is defined by (3.11) and introduce the following decomposition:

$$\mathbb{P}(s_n^\varepsilon \leq t) = \mathbb{P}(s_n^\varepsilon \leq t, m_n^\varepsilon \geq n^{-2/3}) + \mathbb{P}(s_n^\varepsilon \leq t, m_n^\varepsilon < n^{-2/3}). \quad (3.14)$$

By the definition of the sequence  $(s_n^\varepsilon)$ , we get

$$s_n^\varepsilon = \varepsilon^2 \eta^2(x_0^\varepsilon) e^{1-A_1} + \dots + \varepsilon^2 \eta^2(x_{n-1}^\varepsilon) e^{1-A_n} \geq \varepsilon^2 m_n^\varepsilon (e^{1-A_1} + \dots + e^{1-A_n}).$$

Hence, for any  $\lambda > 0$ , we have

$$\begin{aligned} u_n^\varepsilon(t) &:= \mathbb{P}(s_n \leq t, m_n^\varepsilon \geq n^{-2/3}) \leq \mathbb{P}(e^{1-A_1} + \dots + e^{1-A_n} \leq t n^{2/3} \varepsilon^{-2}) \\ &= \mathbb{P}\left(\exp\left\{-\frac{\lambda}{2} n^{-2/3} \varepsilon^2 (e^{1-A_1} + \dots + e^{1-A_n})\right\} \geq e^{-\frac{\lambda t}{2}}\right) \\ &\leq e^{\frac{\lambda t}{2}} \mathbb{E}[\exp(-n^{-2/3} \varepsilon^2 e^{1-A_1})]^n. \end{aligned} \quad (3.15)$$

Since the second moment of  $e^{1-A_1}$  is finite (see Lemma 4.1 with parameters  $\alpha = 3/2$  and  $\beta = 2$ , any moment of this random variable is in fact finite), we obtain the Taylor expansion:

$$\mathbb{E}\left[\exp\left(-\frac{\lambda}{2} n^{-2/3} \varepsilon^2 e^{1-A_1}\right)\right] = 1 - \frac{\lambda n^{-2/3} \varepsilon^2 e}{2 \cdot 3^{3/2}} + \left(\frac{\lambda}{2}\right)^2 \frac{n^{-4/3} \varepsilon^4 e^2}{2 \cdot 5^{3/2}} + o(n^{-4/3}).$$

By using the classical relation  $\ln(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \dots\right)$ , for  $x \in (0, 1)$ , we can deduce that

$$\mathbb{E}\left[\exp\left(-\frac{\lambda}{2} n^{-2/3} \varepsilon^2 e^{1-A_1}\right)\right]^n = \exp\left[-\frac{\lambda e n^{1/3} \varepsilon^2}{2 \cdot 3^{3/2}}\right] + o(1) \text{ as } n \rightarrow +\infty \quad (3.16)$$

which implies that

$$\sum_{n \geq 1} \mathbb{P}(s_n^\varepsilon \leq t, m_n^\varepsilon \geq n^{-2/3}) \leq \sum_{n \geq 1} \exp\left[-\frac{\lambda e n^{1/3} \varepsilon^2}{2 \cdot 3^{3/2}}\right] < \infty. \quad (3.17)$$

Let us just note that this result is still true if we consider the terms  $\bar{u}_n^\varepsilon(\lambda) := \int_0^\infty e^{-\lambda t} u_n^\varepsilon(t) dt$ . Indeed (3.15) leads to

$$\bar{u}_n^\varepsilon(\lambda) \leq \int_0^\infty e^{-\lambda t} e^{\frac{\lambda t}{2}} \mathbb{E}\left[\exp\left(-\frac{\lambda}{2} n^{-2/3} \varepsilon^2 e^{1-A_1}\right)\right]^n dt = \frac{2}{\lambda} \mathbb{E}\left[\exp\left(-\frac{\lambda}{2} n^{-2/3} \varepsilon^2 e^{1-A_1}\right)\right]^n.$$

Since the upper bound is the term of a convergent series, we deduce by comparison that

$$\sum_{n \geq 1} \bar{u}_n^\varepsilon(\lambda) < \infty, \quad \forall \lambda > 0. \quad (3.18)$$

Let us now focus our attention to the second term of the r.h.s in (3.14). Since we consider the Brownian skeleton  $(BS)_\eta$ , the sequence  $(s_n^\varepsilon, x_n^\varepsilon)$  belongs to the graph of a Brownian trajectory. Consequently the condition  $m_n^\varepsilon < n^{-2/3}$  can be related to a condition on the Brownian paths:

$$v_n^\varepsilon(t) := \mathbb{P}(s_n^\varepsilon \leq t, m_n^\varepsilon < n^{-2/3}) \leq \mathbb{P}(\exists s \leq t \text{ s.t. } \eta^2(x_0^\varepsilon + B_s) < n^{-2/3}),$$

where  $B$  is a one-dimensional standard Brownian motion. Using the upper-bound of the function  $F$  in Assumption 3.5 and the definition of the function  $\eta$ , we obtain the bound: there exists  $C > 0$  and  $\kappa > 0$  such that  $\eta(x) \geq Ce^{-\kappa|x|}$ . Let us note that  $x_0^\varepsilon = x$ . The Brownian reflection principle leads to

$$\begin{aligned} v_n^\varepsilon(t) &\leq \mathbb{P}\left(\eta^2\left(\sup_{s \in [0, t]} |x + B_s|\right) < n^{-2/3}\right) \\ &\leq \mathbb{P}\left(\sup_{s \in [0, t]} |x + B_s| > \frac{1}{3\kappa} \ln(nC^3)\right) \\ &\leq \mathbb{P}\left(\sup_{s \in [0, t]} |B_s| > \frac{1}{3\kappa} \ln(nC^3) - |x|\right) \\ &\leq 2\mathbb{P}\left(|B_t| > \frac{1}{3\kappa} \ln(nC^3) - |x|\right) \\ &\leq \frac{12\kappa\sqrt{t}}{\ln(nC^3/e^{3\kappa|x|})\sqrt{2\pi}} \exp\left(-\frac{\ln^2(nC^3/e^{3\kappa|x|})}{18\kappa^2 t}\right). \end{aligned}$$

Since  $\frac{\ln^2(nC^3/e^{3\kappa|x|})}{18\kappa^2 t} \geq 2 \ln(n)$  for large values of  $n$ , we deduce that the r.h.s of the previous equality corresponds to the term of a convergent series. Therefore

$$\sum_{n \geq 1} \mathbb{P}(s_n^\varepsilon \leq t, m_n^\varepsilon < n^{-2/3}) < \infty. \quad (3.19)$$

Let us now define  $\bar{v}_n^\varepsilon(\lambda) := \int_0^\infty e^{-\lambda t} v_n^\varepsilon(t) dt$ . The previous inequalities permit to obtain:

$$\bar{v}_n^\varepsilon(\lambda) \leq \int_0^\infty e^{-\lambda t} 2\mathbb{P}\left(|B_t| > \frac{1}{3\kappa} \ln(nC^3/e^{3\kappa|x|})\right) dt = \sqrt{\frac{2}{\pi}} \iint_{\mathbb{R}_+^2} e^{-\lambda t - \frac{u^2}{2}} 1_{\{u \geq \alpha_n/\sqrt{t}\}} dt du$$

where  $\alpha_n = \frac{1}{3\kappa} \ln(nC^3/e^{3\kappa|x|})$ . Hence

$$\bar{v}_n^\varepsilon(\lambda) \leq \sqrt{\frac{2}{\pi}} \frac{1}{\lambda} \int_0^\infty e^{-\frac{\lambda \alpha_n^2}{u^2} - \frac{u^2}{2}} du.$$

By the change of variable  $u = \sqrt{r}(2\lambda\alpha_n^2)^{1/4}$ , we have

$$\begin{aligned}\bar{v}_n^\varepsilon(\lambda) &\leq \frac{(2\lambda\alpha_n^2)^{1/4}}{\lambda\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{r}} e^{-\alpha_n\sqrt{\frac{\lambda}{2}}(\frac{1}{r}+r)} dr \\ &= \frac{(2\lambda\alpha_n^2)^{1/4}}{\lambda\sqrt{2\pi}} \left\{ \int_0^1 \frac{1}{\sqrt{r}} e^{-\alpha_n\sqrt{\frac{\lambda}{2}}(\frac{1}{r}+r)} dr + \int_1^\infty \frac{1}{\sqrt{r}} e^{-\alpha_n\sqrt{\frac{\lambda}{2}}(\frac{1}{r}+r)} dr \right\}.\end{aligned}$$

Using the change of variable  $r \mapsto \frac{1}{r}$  in the first integral leads to

$$\begin{aligned}\bar{v}_n^\varepsilon(\lambda) &\leq \frac{(2\lambda\alpha_n^2)^{1/4}}{\lambda\sqrt{2\pi}} \int_1^\infty \left( \frac{1}{\sqrt{r}} + \frac{1}{r^{3/2}} \right) e^{-\alpha_n\sqrt{\frac{\lambda}{2}}(\frac{1}{r}+r)} dr \leq \frac{2(2\lambda\alpha_n^2)^{1/4}}{\lambda\sqrt{2\pi}} \int_1^\infty e^{-\alpha_n\sqrt{\frac{\lambda}{2}}r} dr \\ &\leq \frac{1}{\sqrt{\pi\alpha_n}} \left( \frac{2}{\lambda} \right)^{5/4} e^{-\alpha_n\sqrt{\frac{\lambda}{2}}}.\end{aligned}$$

Since  $\alpha_n \sim \frac{1}{3\kappa} \ln n$ , the upper bound is a term of a convergent series as soon as  $\lambda > \lambda_0 := 18\kappa^2$ . Therefore, by comparison,

$$\sum_{n \geq 1} \bar{v}_n^\varepsilon(\lambda) < \infty \quad \text{for } \lambda > \lambda_0. \quad (3.20)$$

Combining (3.14), (3.17) and (3.19) leads to the announced statement  $\psi^\varepsilon(t, x) < \infty$ . Since

$$\int_0^\infty e^{-\lambda t} \psi^\varepsilon(t, x) dt = \sum_{n \geq 1} \bar{u}_n^\varepsilon(\lambda) + \sum_{n \geq 1} \bar{v}_n^\varepsilon(\lambda),$$

the convergence (3.18) and (3.20) of both series for  $\lambda > \lambda_0$  implies that the Laplace transform is well defined for  $\lambda > \lambda_0$ . Of course, this result can be extended for complex values  $\lambda \in \mathbb{C}$  satisfying  $\text{Re}(\lambda) > \lambda_0$ .  $\square$

**Proposition 3.10.** *Under Assumption 3.6, the function  $(t, x) \mapsto \psi^\varepsilon(t, x) = \mathbb{E}[N_t^\varepsilon | x_0^\varepsilon = x]$  is continuous.*

*Proof.* Let us consider the Brownian skeleton  $(BS)_\eta$  which corresponds to the sequences  $(U_n^\varepsilon)_{n \geq 1}$ ,  $(s_n^\varepsilon)_{n \geq 1}$  and  $(x_n^\varepsilon)_{n \geq 0}$  with  $x_0^\varepsilon = x$ . We consider also a second Brownian approximation  $(\hat{U}_n^\varepsilon)_{n \geq 1}$ ,  $(\hat{s}_n^\varepsilon)_{n \geq 1}$  and  $(\hat{x}_n^\varepsilon)_{n \geq 0}$  with  $\hat{x}_0^\varepsilon = \hat{x}$ , both approximations being constructed with respect to the same r.v.  $(A_n)_{n \geq 1}$  and  $(Z_n)_{n \geq 1}$ . The corresponding counting processes are denoted by  $N_t^\varepsilon$  and  $\hat{N}_t^\varepsilon$ .

**Step 1.** Let us describe the distance between these two schemes. The function  $\eta$  is bounded so we denote by  $M = \sup_{x \in \mathbb{R}} \eta(x)$  and  $L_{\text{Lip}}$  the Lipschitz constant of  $\eta$ . Hence

$$|\eta^2(x_n^\varepsilon) - \eta^2(\hat{x}_n^\varepsilon)| \leq 2ML_{\text{Lip}}|x_n^\varepsilon - \hat{x}_n^\varepsilon|, \quad \forall n \geq 0.$$

Using the definition of the approximations  $(BS)_\eta$ , we have

$$\begin{aligned}|x_n^\varepsilon - \hat{x}_n^\varepsilon| &= |x_{n-1}^\varepsilon - \hat{x}_{n-1}^\varepsilon + \varepsilon Z_n \phi_1(e^{1-A_n})(\eta(x_{n-1}^\varepsilon) - \eta(\hat{x}_{n-1}^\varepsilon))| \\ &\leq |x_{n-1}^\varepsilon - \hat{x}_{n-1}^\varepsilon| + \varepsilon |\eta(x_{n-1}^\varepsilon) - \eta(\hat{x}_{n-1}^\varepsilon)| \\ &\leq (1 + \varepsilon L_{\text{Lip}})|x_{n-1}^\varepsilon - \hat{x}_{n-1}^\varepsilon| \leq (1 + \varepsilon L_{\text{Lip}})^n |x_0^\varepsilon - \hat{x}_0^\varepsilon|.\end{aligned}$$

We deduce

$$\max_{0 \leq k \leq n} |\eta^2(x_k^\varepsilon) - \eta^2(\hat{x}_k^\varepsilon)| \leq 2ML_{\text{Lip}}(1 + \varepsilon L_{\text{Lip}})^n |x - \hat{x}|. \quad (3.21)$$

**Step 2.** Since  $N_t^\varepsilon$  is a  $\mathbb{N}$ -valued random variable, we get

$$\begin{aligned} \psi^\varepsilon(t, x) &= \sum_{n \geq 1} \mathbb{P}(N_t^\varepsilon \geq n) = \sum_{n \geq 1} \mathbb{P}(s_n^\varepsilon \leq t) = \sum_{n \geq 1} \mathbb{P}(s_n^\varepsilon \leq t, \hat{s}_n^\varepsilon \leq t) + \sum_{n \geq 1} \mathbb{P}(s_n^\varepsilon \leq t, \hat{s}_n^\varepsilon > t) \\ &= \sum_{n \geq 1} \mathbb{P}(\hat{s}_n^\varepsilon \leq t) - \sum_{n \geq 1} \mathbb{P}(\hat{s}_n^\varepsilon \leq t, s_n^\varepsilon > t) + \sum_{n \geq 1} \mathbb{P}(s_n^\varepsilon \leq t, \hat{s}_n^\varepsilon > t). \end{aligned}$$

We deduce that

$$|\psi^\varepsilon(t, x) - \psi^\varepsilon(t, \hat{x})| \leq \sum_{n \geq 1} u_n^\varepsilon(t) + \sum_{n \geq 1} \bar{u}_n^\varepsilon(t), \quad (3.22)$$

where  $u_n^\varepsilon(t) = \mathbb{P}(\hat{s}_n^\varepsilon \leq t, s_n^\varepsilon > t)$  and  $\bar{u}_n^\varepsilon(t) = \mathbb{P}(s_n^\varepsilon \leq t, \hat{s}_n^\varepsilon > t)$ . Since  $\bar{u}_n^\varepsilon(t) \leq \mathbb{P}(s_n^\varepsilon \leq t)$  which is the term of a convergent series (see Proposition 3.9), then for any  $\rho > 0$  there exists  $n_0^\varepsilon \in \mathbb{N}$  such that

$$\sum_{n > n_0^\varepsilon} \bar{u}_n^\varepsilon(t) < \rho. \quad (3.23)$$

Moreover

$$\bar{u}_n^\varepsilon(t) = \mathbb{P}(s_n^\varepsilon \leq t, s_n^\varepsilon + (\hat{s}_n^\varepsilon - s_n^\varepsilon) > t) \leq \mathbb{P}(t - \delta < s_n^\varepsilon \leq t) + \mathbb{P}(\hat{s}_n^\varepsilon - s_n^\varepsilon > \delta).$$

The random variables  $(s_n^\varepsilon)_{1 \leq n \leq n_0^\varepsilon}$  are absolutely continuous with respect to the Lebesgue measure. Consequently

$$\forall \rho > 0, \quad \exists \delta_\varepsilon > 0 \quad \text{such that} \quad \sum_{n=1}^{n_0^\varepsilon} \mathbb{P}(t - \delta_\varepsilon < s_n^\varepsilon \leq t) \leq \rho. \quad (3.24)$$

It suffices therefore to deal with the remaining expression:  $\sum_{n=1}^{n_0^\varepsilon} \mathbb{P}(\hat{s}_n^\varepsilon - s_n^\varepsilon > \delta_\varepsilon)$ . By Step 1 of the proof and by the definition of  $(s_n^\varepsilon)$  and  $(\hat{s}_n^\varepsilon)$ , we have for  $n \leq n_0^\varepsilon$

$$\begin{aligned} |s_n^\varepsilon - \hat{s}_n^\varepsilon| &= \varepsilon^2 |(\eta(x_0^\varepsilon) - \eta(\hat{x}_0^\varepsilon))e^{1-A_1} + \dots + (\eta^2(x_{n-1}^\varepsilon) - \eta^2(\hat{x}_{n-1}^\varepsilon))e^{1-A_n}| \\ &\leq \varepsilon^2 \max_{0 \leq k \leq n-1} |\eta^2(x_k^\varepsilon) - \eta^2(\hat{x}_k^\varepsilon)| (e^{1-A_1} + \dots + e^{1-A_n}) \\ &\leq C \varepsilon^2 |x - \hat{x}| (e^{1-A_1} + \dots + e^{1-A_n}), \end{aligned}$$

with  $C = 2ML_{\text{Lip}}(1 + \varepsilon L_{\text{Lip}})^{n_0^\varepsilon}$ . Hence

$$\forall \rho > 0, \quad \exists \kappa_\varepsilon > 0 \quad \text{such that} \quad |x - \hat{x}| < \kappa_\varepsilon \Rightarrow \sum_{n=1}^{n_0^\varepsilon} \mathbb{P}(\hat{s}_n^\varepsilon - s_n^\varepsilon > \delta_\varepsilon) \leq \rho. \quad (3.25)$$

Combining (3.23), (3.24) and (3.25), we obtain

$$\forall \rho > 0, \quad \exists \kappa_\varepsilon > 0 \quad \text{such that} \quad |x - \hat{x}| < \kappa_\varepsilon \Rightarrow \sum_{n \geq 1} \bar{u}_n^\varepsilon(t) \leq 3\rho.$$



Let us use now similar arguments in order to bound the series associated to  $u_n^\varepsilon(t)$ . Using the following upper-bound,

$$u_n^\varepsilon(t) = \mathbb{P}(s_n^\varepsilon > t, s_n^\varepsilon + (\hat{s}_n^\varepsilon - s_n^\varepsilon) \leq t) \leq \mathbb{P}(t < s_n^\varepsilon \leq t + \delta) + \mathbb{P}(\hat{s}_n^\varepsilon - s_n^\varepsilon > \delta),$$

we deduce that all arguments presented so far and concerning  $\bar{u}_n^\varepsilon(t)$  can be used for  $u_n^\varepsilon(t)$ . Finally (3.22) leads to the continuity of  $x \mapsto \psi^\varepsilon(t, x)$ :

$$\forall \rho > 0, \quad \exists \kappa_\varepsilon > 0 \quad \text{such that} \quad |x - \hat{x}| < \kappa \Rightarrow |\psi^\varepsilon(t, x) - \psi^\varepsilon(t, \hat{x})| \leq 6\rho.$$

Let us end the proof by focusing our attention on the continuity with respect to the time variable. Since

$$\psi^\varepsilon(t, x) = \sum_{n \geq 1} \mathbb{P}(s_n^\varepsilon \leq t),$$

where  $s_n^\varepsilon$  is an absolutely continuous random variable, the Lebesgue monotone convergence theorem implies the continuity of  $t \mapsto \psi^\varepsilon(t, x)$ .  $\square$

We give now an important result concerning the function  $\psi^\varepsilon(t, x) = \mathbb{E}[N_t^\varepsilon | x_0^\varepsilon = x]$ .

**Proposition 3.11.** *Under Assumption 3.5 and Assumption 3.6, there exist  $C_T > 0$ ,  $\kappa > 0$  and  $\varepsilon_0 > 0$  such that*

$$\varepsilon^2 \psi^\varepsilon(t, x) \leq C_T e^{\kappa|x|}, \quad \forall t \in [0, T], \quad \forall x \in \mathbb{R}, \quad \forall \varepsilon \leq \varepsilon_0. \quad (3.26)$$

Let us note that the constant  $\kappa$  is explicit: it suffices to choose  $\kappa = 3\sqrt{2}\kappa_2$  where  $\kappa_2$  corresponds to the constant introduced in Assumption 3.5.

*Proof.* The proof follows similar ideas as those developed in Proposition 3.9. We introduce here  $m_n^\varepsilon := \min_{0 \leq k \leq n-1} \eta^2(x_k^\varepsilon)$  and recall that

$$\psi^\varepsilon(t, x) = \sum_{n \geq 1} \mathbb{P}(s_n^\varepsilon \leq t). \quad (3.27)$$

We aim to control

$$\varepsilon^2 \psi^\varepsilon(t, x) = \sum_{n \geq 1} (\varepsilon^2 u_n^\varepsilon(t, x) + \varepsilon^2 v_n^\varepsilon(t, x)) \quad (3.28)$$

by using similar notations as those presented in Proposition 3.9, that is

$$u_n^\varepsilon(t, x) = \mathbb{P}(s_n^\varepsilon \leq t; m_n^\varepsilon \geq \varepsilon^\gamma n^{-2/3} | x_0^\varepsilon = x), \quad (3.29)$$

$$v_n^\varepsilon(t, x) = \mathbb{P}(s_n^\varepsilon \leq t; m_n^\varepsilon < \varepsilon^\gamma n^{-2/3} | x_0^\varepsilon = x). \quad (3.30)$$

Here a suitable choice of the exponent  $\gamma$  should ensure the required boundedness. We shall discuss about this choice in the following.

**Step 1.** Consider first the sequence  $(u_n^\varepsilon(t, x))_{n \geq 1}$ . The definition (3.29) leads to

$$\begin{aligned} u_n^\varepsilon(t, x) &\leq \mathbb{P}(e^{1-A_1} + \dots + e^{1-A_n} \leq tn^{2/3} \varepsilon^{-(\gamma+2)}) \\ &\leq \mathbb{P}\left(\exp\left[-\frac{\lambda}{2} n^{-2/3} \varepsilon^{\gamma+2} (e^{1-A_1} + \dots + e^{1-A_n})\right] \geq e^{-\frac{\lambda t}{2}}\right) \\ &\leq e^{\frac{\lambda t}{2}} \left[ \mathbb{E}\left(\exp\left(-\frac{\lambda}{2} \varepsilon^{\gamma+2} n^{-2/3} e^{1-A_1}\right)\right) \right]^n, \quad \forall (t, x) \in [0, T] \times \mathbb{R}, \quad \forall n \geq 1. \end{aligned} \quad (3.31)$$

Using the explicit distribution of the random variable  $A_1$ , we obtain that

$$\varepsilon^2 u_n^\varepsilon(t, x) \leq \varepsilon^2 e^{\frac{\lambda T}{2}} \exp\left(-\frac{\lambda \varepsilon^{\gamma+2} n^{1/3} e}{2 \cdot 3^{3/2}}\right), \quad \forall (t, x) \in [0, T] \times \mathbb{R}, \forall n \geq 1. \quad (3.32)$$

Since the function  $z \mapsto \exp(-z)$  is non increasing,

$$\varepsilon^2 \sum_{n \geq 1} \exp\left(-\frac{\lambda \varepsilon^{\gamma+2} n^{1/3} e}{2 \cdot 3^{3/2}}\right) \leq \varepsilon^2 \int_0^{+\infty} \exp\left(-\frac{\lambda \varepsilon^{\gamma+2} z^{1/3} e}{2 \cdot 3^{3/2}}\right) dz. \quad (3.33)$$

By performing the change of variable  $y = \varepsilon^2 z$  we have

$$\varepsilon^2 \int_0^{+\infty} \exp\left(-\frac{\lambda \varepsilon^{\gamma+2} z^{1/3} e}{2 \cdot 3^{3/2}}\right) dz = \int_0^{+\infty} \exp\left(-\frac{\lambda \varepsilon^{\gamma+2-\frac{2}{3}} y^{1/3} e}{2 \cdot 3^{3/2}}\right) dy. \quad (3.34)$$

Therefore choosing  $\gamma = -\frac{4}{3}$  gives

$$\sum_{n \geq 1} \varepsilon^2 u_n^\varepsilon(t, x) \leq \varepsilon^2 e^{\frac{\lambda T}{2}} \sum_{n \geq 1} \exp\left(-\frac{\lambda \varepsilon^{\gamma+2} n^{1/3} e}{2 \cdot 3^{3/2}}\right) \quad (3.35)$$

$$\leq e^{\frac{\lambda T}{2}} \int_0^{+\infty} \exp\left(-\frac{\lambda y^{1/3} e}{2 \cdot 3^{3/2}}\right) dy =: C_T^0 < +\infty. \quad (3.36)$$

Let us just note that the upper-bound does not depend on the space variable  $x$ .

**Step 2.** Let us focus now on the second part, that is, the terms  $\varepsilon^2 v_n^\varepsilon(t, x)$ . Using the properties of  $F$  (see Assumption 3.5), there exist  $C > 0$  and  $\kappa > 0$  such that  $\eta(z) \geq C e^{-\kappa|z|}$  (the value of  $\kappa$  here corresponds to  $\sqrt{2}\kappa_2$  where  $\kappa_2$  is the constant appearing in Assumption 3.5)

$$\begin{aligned} \varepsilon^2 v_n^\varepsilon(t, x) &\leq \varepsilon^2 \mathbb{P}\left(\exists s \leq t \text{ s.t. } \eta^2(B_s + x) < \varepsilon^\gamma n^{-2/3}\right) \\ &\leq \varepsilon^2 \mathbb{P}\left(C^2 \exp\left(-2\kappa \sup_{s \in [0, t]} |B_s + x|\right) < \varepsilon^\gamma n^{-2/3}\right) \\ &\leq \varepsilon^2 \mathbb{P}\left(\exp\left(-\sup_{s \in [0, t]} |B_s + x|\right) < \left(C^{-1} \varepsilon^{\frac{\gamma}{2}} n^{-1/3}\right)^{1/\kappa}\right), \end{aligned} \quad (3.37)$$

for all  $(t, x) \in [0, T] \times \mathbb{R}$ . We need here an auxiliary result.

**Lemma 3.12.** *Let us define the function*

$$\mathcal{R}(x, z) := \mathbb{P}\left(\sup_{s \in [0, t]} |x + B_s| > \ln(z)\right). \quad (3.38)$$

For any  $(x, z) \in \mathbb{R} \times \mathbb{R}_+^*$  we have  $\mathcal{R}(x, z) \geq 0$  and  $z \mapsto \mathcal{R}(x, z)$  is non increasing. Furthermore for any  $\delta > 0$ , there exists  $C_T > 0$  such that

$$\int_0^{+\infty} \mathcal{R}(x, z^{1/\delta}) dz \leq C_T e^{\delta|x|}, \quad \forall (t, x) \in [0, T] \times \mathbb{R}. \quad (3.39)$$

We postpone the proof of this lemma. We observe therefore

$$v_n^\varepsilon(t, x) \leq \mathcal{R} \left( x, \left( \frac{Cn^{1/3}}{\varepsilon^{\gamma/2}} \right)^{1/\kappa} \right) = \mathbb{P} \left( \sup_{s \in [0, t]} |x + B_s| > \frac{1}{\kappa} \ln \left( \frac{Cn^{1/3}}{\varepsilon^{\gamma/2}} \right) \right). \quad (3.40)$$

We define  $n(\varepsilon) = \inf\{n \geq 0 \text{ s.t. } Cn^{1/3} \geq \varepsilon^{\gamma/2}\}$ . Then

$$\begin{aligned} \varepsilon^2 \sum_{n \geq 1} v_n^\varepsilon(t, x) &= \varepsilon^2 \sum_{n=1}^{n(\varepsilon)} v_n^\varepsilon(t, x) + \varepsilon^2 \sum_{n \geq n(\varepsilon)+1} v_n^\varepsilon(t, x) \\ &\leq \varepsilon^2 n(\varepsilon) + \varepsilon^2 \sum_{n \geq n(\varepsilon)+1} \mathcal{R} \left( x, \left( \frac{Cn^{1/3}}{\varepsilon^{\gamma/2}} \right)^{1/\kappa} \right) \\ &\leq \varepsilon^2 n(\varepsilon) + \varepsilon^2 \int_{n(\varepsilon)}^{+\infty} \mathcal{R} \left( x, \left( \frac{Cz^{1/3}}{\varepsilon^{\gamma/2}} \right)^{1/\kappa} \right) dz. \end{aligned} \quad (3.41)$$

In the last expression only the term under the integral depends on  $x$ . We perform the change of variable in this term of the form  $y = \varepsilon^2 z$  and obtain:

$$\begin{aligned} \varepsilon^2 \sum_{n \geq 1} v_n^\varepsilon(t, x) &\leq \varepsilon^2 n(\varepsilon) + \int_{\varepsilon^2 n(\varepsilon)}^{+\infty} \mathcal{R} \left( x, \left( \frac{Cy^{1/3}}{\varepsilon^{2/3 + \gamma/2}} \right)^{1/\kappa} \right) dy \\ &\leq \varepsilon^2 n(\varepsilon) + \int_0^{+\infty} \mathcal{R}(x, C^{1/\kappa} y^{1/3\kappa}) dy \leq \varepsilon^2 n(\varepsilon) + C_T e^{3\kappa|x|}, \end{aligned} \quad (3.42)$$

by using the particular value  $\gamma = -4/3$  and Lemma 3.12. In order to conclude we need to control  $\varepsilon^2 n(\varepsilon)$ . By the definition of  $n(\varepsilon)$  we have

$$n(\varepsilon) = \inf \left\{ n \geq 0 \text{ s.t. } n \geq \frac{1}{\varepsilon^2 C^3} \right\} \leq \frac{1}{\varepsilon^2 C^3} + 1. \quad (3.43)$$

This allows us to conclude that  $\varepsilon^2 n(\varepsilon) \leq \frac{1}{C^3} + 1$ . Combining the two steps of the proof leads to the announced upper-bound (3.26).  $\square$

*Proof of Lemma 3.12.* The proof of the first two properties is obvious by using the definition of  $\mathcal{R}$ . Let us show that (3.39) is true. By using the reflection principle of the Brownian motion we can evaluate

$$\begin{aligned} \int_0^{+\infty} \mathcal{R}(x, z^{1/\delta}) dz &= \int_0^{e^{\delta(1+|x|)}} \mathcal{R}(x, z^{1/\delta}) dz + \int_{e^{\delta(1+|x|)}}^{+\infty} \mathcal{R}(x, z^{1/\delta}) dz \\ &\leq e^{\delta(1+|x|)} + 4 \int_{e^{\delta(1+|x|)}}^{+\infty} \mathbb{P} \left( G > \frac{1}{\sqrt{t}} \left( \frac{1}{\delta} \ln(z) - |x| \right) \right) dz \\ &\leq e^{\delta(1+|x|)} + 4 \int_{e^{\delta(1+|x|)}}^{+\infty} \mathbb{P} \left( G > \frac{1}{\sqrt{T}} \left( \frac{1}{\delta} \ln(z) - |x| \right) \right) dz, \end{aligned} \quad (3.44)$$

where  $G$  denotes a standard normal random variable  $\mathcal{N}(0, 1)$ . We used the fact that for any  $u > 0$  we have  $\mathbb{P}(G \geq u) \leq \frac{1}{u\sqrt{2\pi}}e^{-\frac{u^2}{2}}$ . Hence, for  $z \geq e^{\delta(1+|x|)}$ ,

$$\begin{aligned} \mathbb{P}\left(G > \frac{1}{\sqrt{T}}\left(\frac{1}{\delta}\ln(z) - |x|\right)\right) &\leq \frac{\delta\sqrt{T}}{(\ln(z) - \delta|x|)\sqrt{2\pi}} \exp\left\{-\frac{1}{2T}\left(\frac{1}{\delta}\ln(z) - |x|\right)^2\right\} \\ &\leq \frac{\sqrt{T}}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2T}\left(\frac{1}{\delta}\ln(z) - |x|\right)^2\right\}. \end{aligned}$$

We deduce

$$\int_0^{+\infty} \mathcal{R}(x, z^{1/\delta})dz \leq e^{\delta(1+|x|)} + \frac{4\sqrt{T}}{\sqrt{2\pi}} \int_{e^{\delta(1+|x|)}}^{+\infty} \exp\left\{-\frac{1}{2T}\left(\frac{1}{\delta}\ln(z) - |x|\right)^2\right\}dz.$$

By doing the change of variable  $z = e^{\delta(1+|x|)}y$ , we have

$$\begin{aligned} \int_0^{+\infty} \mathcal{R}(x, z^{1/\delta})dz &\leq e^{\delta(1+|x|)} \left[1 + \frac{4\sqrt{T}}{\sqrt{2\pi}} \int_1^{+\infty} e^{-\frac{1}{2T}\left(\frac{1}{\delta}(\delta(1+|x|) + \ln y) - |x|\right)^2} dy\right] \\ &\leq e^{\delta(1+|x|)} \left[1 + \frac{4\sqrt{T}}{\sqrt{2\pi}} \int_1^{+\infty} e^{-\frac{1}{2T}\left(1 + \frac{1}{\delta}\ln(y)\right)^2} dy\right] =: C_T e^{\delta|x|}. \end{aligned}$$

The upper-bound holds for any  $(t, x) \in [0, T] \times \mathbb{R}$  as announced.  $\square$

Since Proposition 3.9 and Proposition 3.10 point out different preliminary properties of the average number of steps needed by the Brownian skeleton to cover the time interval  $[0, T]$ , the study of the  $\varepsilon$ -strong approximation of both the linear and growth diffusions can be achieved.

**Theorem 3.13.** *Let  $(X_t)_{0 \leq t \leq T}$  be a solution of the stochastic differential equation (3.1) satisfying both Assumptions 3.5 and 3.6. Let  $(y_t^\varepsilon)_{0 \leq t \leq T}$  be the  $\varepsilon$ -strong approximation of  $(X_t)_{0 \leq t \leq T}$  given by (3.12) and  $\hat{N}_T^\varepsilon$  the random number of points needed to build this approximation. Then, there exist  $\mu > 0$  such that*

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^2 \mathbb{E}[\hat{N}_T^\varepsilon] = \mu \mathbb{E} \left[ \int_0^{\rho(T)} \frac{1}{\eta^2(x + B_s)} ds \right], \quad \forall (T, x) \in \mathbb{R}_+ \times \mathbb{R} \quad (3.45)$$

where  $(B_t)_{t \geq 0}$  stands for a standard one-dimensional Brownian motion.

**Remark 3.14.** 1. *The constant appearing in the statement is explicitly known. Let us introduce  $M$  the cumulative distribution function associated to the random variable  $e^{1-A}$  with  $A \sim \text{Gamma}(3/2, 2)$ . We denote by  $M(f) = \int_0^\infty f(s) dM(s)$ , for any nonnegative function  $f$ . Then*

$$\mu = \frac{1}{M(\phi_1^2)} \quad \text{and} \quad M(\phi_1^2) = M(\text{Id}) = e 3^{-3/2} \approx 0.5231336. \quad (3.46)$$

2. Let us note that the link between the function  $\eta$ , defining the approximation scheme, and the function  $F$  introduced in Assumption 3.5, permits to write

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \varepsilon^2 \mathbb{E}[\hat{N}_T^\varepsilon] &= (e\kappa_{\min}^{-1} + 1)^2 \mu \mathbb{E} \left[ \int_0^{\rho(T)} F^2(2(x + B_s)^2 + 1) ds \right] \\ &= (e\kappa_{\min}^{-1} + 1)^2 \mu \mathbb{E} \left[ \int_{\mathbb{R}} F^2(2(x + y)^2 + 1) L_{\rho(T)}(y) dy \right]. \end{aligned}$$

The last equality is just an immediate application of the occupation time formula (see, for instance, Corollary 1.6 page 209 in [18]),  $L_t(y)$  standing for the local time of the standard Brownian motion. A proof of Theorem 3.13 based on the local times of the Brownian motion and therefore on a precise description of the Brownian paths could be investigated, we prefer here to propose a proof involving a renewal property of the average number of points in the numerical scheme.

*Proof of Theorem 3.13.* We start to mention that the notation of the constants is generic through this proof:  $C := C_\theta$  or  $\kappa := \kappa_\theta$  if the constants depend on a parameter  $\theta$ .

The proof of the theorem is based on the study of a particular renewal inequality. The material is organized as follows: on one hand we shall prove that  $u^\varepsilon(t, x) := \varepsilon^2 \mathbb{E}[N_t^\varepsilon | x_0^\varepsilon = x]$ ,  $N_t^\varepsilon$  being defined in (3.13), satisfies a renewal equation. On the other hand, we describe  $U(t, x)$  defined by

$$U(t, x) := \mu \mathbb{E} \left[ \int_0^t \frac{1}{\eta^2(x + B_s)} ds \right] = \frac{(e\kappa_{\min}^{-1} + 1)^2}{M(\phi_1^2)} \mathbb{E} \left[ \int_0^t F^2(2(x + B_s)^2 + 1) ds \right], \quad (3.47)$$

where  $\mu$  corresponds to the constant introduced in the statement of the theorem and described in Remark 3.14. Then we observe that the difference:

$$D^\varepsilon(t, x) := u^\varepsilon(t, x) - U(t, x) \quad (3.48)$$

satisfies a renewal inequality which leads to  $\lim_{\varepsilon \rightarrow 0} D^\varepsilon(t, x) = 0$ .

**Step 1. Renewal equation satisfied by  $\psi^\varepsilon(t, x) = \mathbb{E}[N_t^\varepsilon | \mathbf{x}_0^\varepsilon = \mathbf{x}]$ .** Let us note that  $\psi^\varepsilon(t, x)$  satisfies the following renewal equation:

$$\psi^\varepsilon(t, x) = M(t/(\varepsilon^2 \eta^2(x))) + \sum_{i=\pm 1} \frac{1}{2} \int_0^{t/(\varepsilon^2 \eta^2(x))} \psi^\varepsilon\left(t - s\varepsilon^2 \eta^2(x), x + i\varepsilon \eta(x) \phi_1(s)\right) dM(s) \quad (3.49)$$

where  $M$  corresponds to the cumulative distribution function associated to the random variable  $e^{1-A}$  with  $A \sim \text{Gamma}(3/2, 2)$ . Indeed, we focus our attention to  $U_1^\varepsilon$  the first positive abscissa of the Brownian paths skeleton  $(BS)_\eta$ . Either  $U_1^\varepsilon > t$  and consequently  $N_t^\varepsilon = 0$  either  $U_1^\varepsilon = s\varepsilon^2 \eta^2(x) \leq t$  which implies that  $N_t^\varepsilon$  given  $x_0^\varepsilon = x$  and  $1 + N_{t-s\varepsilon^2 \eta^2(x)}^\varepsilon$

with initial position  $x_1^\varepsilon$  are identically distributed (Markov property). Hence

$$\begin{aligned}\psi^\varepsilon(t, x) &= \int_0^{t/(\varepsilon^2\eta^2(x))} \mathbb{E} \left[ \mathbb{E} \left[ 1 + N_{t-s\varepsilon^2\eta^2(x)}^\varepsilon \Big| x_1^\varepsilon \right] \Big| x_0^\varepsilon = x \right] dM(s) \\ &= M(t/(\varepsilon^2\eta^2(x))) + \int_0^{t/(\varepsilon^2\eta^2(x))} \mathbb{E} \left[ \psi^\varepsilon(t - s\varepsilon^2\eta^2(x), x_1^\varepsilon) \Big| x_0^\varepsilon = x \right] dM(s).\end{aligned}$$

Let us now introduce  $u^\varepsilon(t, x) := \varepsilon^2\psi^\varepsilon(t, x)$  and define for any nonnegative function  $h$ :

$$\mathcal{M}_\varepsilon h(t, x) = \sum_{i=\pm 1} \frac{1}{2} \int_0^{t/(\varepsilon^2\eta^2(x))} h\left(t - s\varepsilon^2\eta^2(x), x + i\varepsilon\eta(x)\phi_1(s)\right) dM(s). \quad (3.50)$$

Then the following renewal equation holds

$$u^\varepsilon(t, x) = \varepsilon^2 M(t/(\varepsilon^2\eta^2(x))) + \mathcal{M}_\varepsilon u^\varepsilon(t, x), \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}. \quad (3.51)$$

**Step 2. Description of the function  $U(t, x)$  introduced in (3.47).** Due to Assumption 3.5,  $F$  is assumed to be a  $\mathcal{C}^2$ -continuous function with at most exponential growth. Lebesgue's theorem permits therefore to obtain that  $x \mapsto U(t, x)$  is also a  $\mathcal{C}^2$ -continuous function. Moreover, combining Itô's formula and Lebesgue's theorem leads to the  $\mathcal{C}^2$ -regularity with respect to both variables  $t$  and  $x$ . Since  $U$  is regular and has at most exponential growth, it corresponds to the probabilistic representation (see for example Karatzas and Shreve [12], p. 270, Corollary 4.5) of the unique solution:

$$M(\text{Id}) \frac{\partial U}{\partial t}(t, x) = \frac{M(\phi_1^2)}{2} \frac{\partial^2 U}{\partial x^2}(t, x) + (e\kappa_{\min}^{-1} + 1)^2 F^2(2x^2 + 1), \quad U(0, x) = 0. \quad (3.52)$$

We just recall that  $M(\text{Id}) = M(\phi_1^2)$  (see Remark 3.14).

Let  $R > 0$ . Using the Taylor expansion in order to compute the operator defined in (3.50), we obtain

$$\begin{aligned}\mathcal{M}_\varepsilon U(t, x) &= U(t, x)M(t/(\varepsilon^2\eta^2(x))) - \varepsilon^2\eta^2(x)M(\text{Id})\frac{\partial U}{\partial t}(t, x) \\ &\quad + \frac{\varepsilon^2}{2}\eta^2(x)M(\phi_1^2)\frac{\partial^2 U}{\partial x^2}(t, x) + \varepsilon^2 o_R(1),\end{aligned}$$

where  $o_R(1)$  tends uniformly towards 0 on  $[0, T] \times [-R, R]$  as  $\varepsilon \rightarrow 0$  (let us just note that  $o_R(1)$  is a generic notation in the sequel). The equation (3.52) and the particular link between both functions  $F$  and  $\eta$  imply

$$\mathcal{M}_\varepsilon U(t, x) = U(t, x)M(t/(\varepsilon^2\eta^2(x))) - \varepsilon^2 + \varepsilon^2 o_R(1), \quad \forall (t, x) \in [0, T] \times [-R, R]. \quad (3.53)$$

**Step 3. Study of the difference  $D^\varepsilon(t, x)$  introduced in (3.48).** Since both  $u^\varepsilon$  and  $U$  are continuous functions satisfying an exponential bound (immediate consequence of the

regularity and growth property of  $F$  for  $U$  and statement of Proposition 3.11 for  $u^\varepsilon$ ), so is  $D^\varepsilon$ . Hence, there exists  $C > 0$  and  $\kappa > 0$  such that

$$|D^\varepsilon(t, x)| \leq C e^{\kappa|x|}, \quad \forall (t, x) \in [0, T] \times \mathbb{R}. \quad (3.54)$$

Moreover combining (3.53) and (3.51) implies

$$D^\varepsilon(t, x) = (U(t, x) - \varepsilon^2)(1 - M(t/(\varepsilon^2\eta^2(x)))) + \mathcal{M}_\varepsilon D^\varepsilon(t, x) + \varepsilon^2 o_R(1). \quad (3.55)$$

The support of the distribution associated to  $M$  is compact. Moreover  $\eta$  defined in (3.11) is upper-bounded. Consequently there exists  $\rho > 0$  (independent of  $x$  and  $\varepsilon$ ) such that  $M(t/(\varepsilon^2\eta^2(x))) = 1$  for all  $t \geq \rho\varepsilon^2$  and  $x \in \mathbb{R}$ . For small values of  $t$ , that is  $t \leq \rho\varepsilon^2$ , it suffices to use the regularity of  $U$  with respect to that variable in order to get a constant  $C_R > 0$  such that  $|U(t, x)| \leq C_R \varepsilon^2$  for all  $x \in [-R, R]$ . To sum up the observations for any value of  $t$ : there exists  $C_R > 0$  such that

$$|D^\varepsilon(t, x)| \leq \mathcal{M}_\varepsilon |D^\varepsilon|(t, x) + C_R \varepsilon^2 1_{\{t \leq \rho\varepsilon^2\}} + \varepsilon^2 o_R(1), \quad \forall (t, x) \in [0, T] \times [-R, R]. \quad (3.56)$$

**Step 4. Asymptotic behaviour of  $D^\varepsilon(t, x)$ .** It is possible to link the operator  $\mathcal{M}_\varepsilon$  to the approximation scheme of the Brownian motion: the Brownian skeleton  $(BS)_\eta$ . We recall that  $s_n^\varepsilon = \sum_{k=1}^n U_k^\varepsilon$  and that  $(s_n^\varepsilon, x_n^\varepsilon)_{n \geq 0}$  is a skeleton of the Brownian paths: the sequence (Markov chain) has the same distribution than points belonging to a Brownian trajectory. It represents the successive exit times and positions of small  $\phi_\varepsilon$ -domains also called heat-balls, the radius of any heat-ball being upper-bounded by  $\varepsilon\eta(0)$ . We observe that

$$\mathcal{M}_\varepsilon(h)(t, x) = \mathbb{E}[h(t - U_1^\varepsilon, x_1^\varepsilon) 1_{\{U_1^\varepsilon \leq t\}} | x_0^\varepsilon = x], \quad \text{for any nonnegative function } h.$$

Consequently, for any  $(t, x) \in [0, T] \times [-R, R]$ , (3.56) becomes

$$|D^\varepsilon(t, x)| \leq \mathbb{E}[|D^\varepsilon(t - U_1^\varepsilon, x_1^\varepsilon)| 1_{\{U_1^\varepsilon \leq t\}} | x_0^\varepsilon = x] + C_R \varepsilon^2 1_{\{t \leq \rho\varepsilon^2\}} + \varepsilon^2 o_R(1). \quad (3.57)$$

Since the sequence  $(s_n^\varepsilon, x_n^\varepsilon)_{n \geq 0}$  is a Markov chain, the aim is to iterate the upper-bound a large number of times. In order to achieve such a procedure, we need to ensure that  $x_1^\varepsilon, \dots, x_n^\varepsilon$  belong to the interval  $[-R, R]$ . We introduce

$$\tau_{R, \varepsilon} = \inf\{n \geq 0 : x_n^\varepsilon \notin [-R, R]\}.$$

The  $\phi_\varepsilon$ -domains associated to the Brownian approximation are bounded (their radius is less than  $\varepsilon\eta(0)$ ), we therefore obtain that  $|x_{\tau_{R, \varepsilon}}^\varepsilon| \leq R + \varepsilon\eta(0)$  and (3.54) implies the existence of  $C > 0$  and  $\kappa > 0$  such that  $|D^\varepsilon(t, x_{\tau_{R, \varepsilon}}^\varepsilon)| \leq C e^{\kappa R}$ , for any  $t \leq T$  and  $\varepsilon \leq 1$ . Let us note that for notational convenience we use  $\mathbb{P}_x$  (resp.  $\mathbb{E}_x$ ) for the conditional probability (resp. expectation) with respect to the event  $x_0^\varepsilon = x$ . Hence (3.56) gives

$$\begin{aligned} |D^\varepsilon(t, x)| &\leq \mathbb{E}_x[|D^\varepsilon(t - s_1^\varepsilon, x_1^\varepsilon)| 1_{\{s_1^\varepsilon \leq t; |x_1^\varepsilon| \leq R\}}] + \mathbb{E}_x[|D^\varepsilon(t - s_1^\varepsilon, x_1^\varepsilon)| 1_{\{s_{\tau_{R, \varepsilon}}^\varepsilon \leq t; \tau_{R, \varepsilon} = 1\}}] \\ &\quad + C_R \varepsilon^2 1_{\{t \leq \rho\varepsilon^2\}} + \varepsilon^2 o_R(1) \\ &\leq \mathbb{E}_x[|D^\varepsilon(t - s_1^\varepsilon, x_1^\varepsilon)| 1_{\{s_1^\varepsilon \leq t; |x_1^\varepsilon| \leq R\}}] + C e^{\kappa R} \mathbb{P}_x(s_{\tau_{R, \varepsilon}}^\varepsilon \leq t; \tau_{R, \varepsilon} = 1) \\ &\quad + C_R \varepsilon^2 1_{\{t \leq \rho\varepsilon^2\}} + \varepsilon^2 o_R(1). \end{aligned}$$

In order to simplify the notations when iterating the procedure, we introduce the following events:

$$\mathcal{J}_{R,\varepsilon}^n := \{s_n^\varepsilon \leq t\} \cap \{\tau_{R,\varepsilon} > n\}.$$

By iterating the upper-bound, we obtain

$$\begin{aligned} |D^\varepsilon(t, x)| &\leq \mathbb{E}_x[|D^\varepsilon(t - s_2^\varepsilon, x_2^\varepsilon)|1_{\mathcal{J}_{R,\varepsilon}^2}] + C_R \varepsilon^2 \mathbb{P}_x(t - \rho\varepsilon^2 \leq s_1^\varepsilon \leq t; \mathcal{J}_{R,\varepsilon}^2) \\ &\quad + C e^{\kappa R} \mathbb{P}_x(s_{\tau_{R,\varepsilon}}^\varepsilon \leq t; \tau_{R,\varepsilon} \leq 2) + C_R \varepsilon^2 1_{\{t \leq \rho\varepsilon^2\}} + 2\varepsilon^2 o_R(1) \\ &\leq \mathbb{E}_x[|D^\varepsilon(t - s_n^\varepsilon, x_n^\varepsilon)|1_{\mathcal{J}_{R,\varepsilon}^n}] + C_R \varepsilon^2 \sum_{k \geq 1} \mathbb{P}_x(t - \rho\varepsilon^2 \leq s_k^\varepsilon \leq t; \mathcal{J}_{R,\varepsilon}^k) \\ &\quad + C e^{\kappa R} \mathbb{P}_x(s_{\tau_{R,\varepsilon}}^\varepsilon \leq t; \tau_{R,\varepsilon} \leq n) + C_R \varepsilon^2 1_{\{t \leq \rho\varepsilon^2\}} + n\varepsilon^2 o_R(1) \\ &\leq C_R \left( \mathcal{A}_1(R, \varepsilon, n) + \mathcal{A}_2(R, \varepsilon) + \mathcal{A}_3(R) + \mathcal{A}_4(R, \varepsilon, n) \right), \end{aligned} \quad (3.58)$$

with

$$\begin{aligned} \mathcal{A}_1(R, \varepsilon, n) &:= \mathbb{P}_x(\mathcal{J}_{R,\varepsilon}^n), & \mathcal{A}_2(R, \varepsilon) &= \varepsilon^2 \sum_{k \geq 1} \mathbb{P}_x(t - \rho\varepsilon^2 \leq s_k^\varepsilon \leq t; \mathcal{J}_{R,\varepsilon}^k), \\ \mathcal{A}_3(R) &= e^{\kappa R} \mathbb{P}(\exists s \leq t : x + B_s \notin [-R, R]), & \mathcal{A}_4(R, \varepsilon, n) &= \varepsilon^2 1_{\{t \leq \rho\varepsilon^2\}} + n\varepsilon^2 o_R(1). \end{aligned}$$

We shall now describe precisely the bound of each of these terms. The crucial idea is to first fix  $R$  sufficiently large and then to choose  $n = \xi \lfloor 1/\varepsilon^2 \rfloor$  for  $\xi$  large enough and depending on  $R$ . Let  $\delta > 0$ . We shall prove that there exists  $\varepsilon_0$  such that  $|D^\varepsilon(t, x)| \leq \delta$  for  $\varepsilon \leq \varepsilon_0$ .

1. Due to the reflection principle of the Brownian motion, there exists  $R$  large enough such that

$$\mathcal{A}_3(R) \leq 4e^{\kappa R} \mathbb{P}(B_t \geq R - |x|) = 4e^{\kappa R} \mathbb{P}\left(G \geq \frac{R - |x|}{\sqrt{t}}\right) \leq \delta/4, \quad (3.59)$$

where  $G$  is a standard Gaussian variate. From now on,  $R$  is fixed s.t. (3.59) is satisfied.

2. Let us consider the term  $\mathcal{A}_2$ . We introduce the particular choice  $n = \xi \lfloor 1/\varepsilon^2 \rfloor$  with  $\xi \in \mathbb{N}$ . By the definition of the Brownian skeleton,  $s_n^\varepsilon \leq t$  corresponds to

$$\sum_{k=1}^{\xi \lfloor 1/\varepsilon^2 \rfloor} \varepsilon^2 \eta^2(x_{k-1}^\varepsilon) e^{1-A_k} \leq t,$$

where  $(A_k)_{k \geq 1}$  is a sequence of i.i.d random variables. Since  $\eta$  is an even function and decreases on  $\mathbb{R}_+$ , we observe :

$$\mathcal{J}_{R,\varepsilon}^n \subset \left\{ \frac{\varepsilon^2}{\xi} \sum_{k=1}^{\xi \lfloor 1/\varepsilon^2 \rfloor} e^{1-A_k} \leq \frac{t}{\xi \eta^2(R)} \right\}, \quad \forall \xi \in \mathbb{N}.$$

By the law of large numbers, the left hand side of the inequality converges towards  $\mathbb{E}[e^{1-A_1}]$  as  $\varepsilon \rightarrow 0$ . Hence, as soon as  $\xi > t/(\eta^2(R)\mathbb{E}[e^{1-A_1}])$ , there exists  $\varepsilon_1 > 0$  such that  $\mathcal{A}_1(R, \varepsilon, n) \leq \delta/4$  for  $\varepsilon \leq \varepsilon_1$  and  $n = \xi \lfloor 1/\varepsilon^2 \rfloor$ .



3. Let us now deal with  $\mathcal{A}_4$ . The parameters  $R$  and  $\xi$  have already been fixed and  $n = \xi \lfloor 1/\varepsilon^2 \rfloor$ . It is therefore obvious that there exists a constant  $\varepsilon_2 > 0$  such that  $\mathcal{A}_4(R, \varepsilon, n) \leq \delta/4$  for  $\varepsilon \leq \varepsilon_2$ .
4. Finally we focus our attention on the last term  $\mathcal{A}_2$  ( $R$  being fixed). We introduce the notation  $\chi(A, B) = 1_{A \cap B}$  and the stopping time

$$\mathfrak{z} = \inf\{n \geq 0 : s_n^\varepsilon \geq t - \rho\varepsilon^2\}.$$

Then

$$\begin{aligned} \varepsilon^{-2} \mathcal{A}_2(R, \varepsilon) &= \mathbb{E} \left[ \sum_{k \geq 1} \chi(t - \rho\varepsilon^2 \leq s_k^\varepsilon \leq t, k < \tau_{R, \varepsilon}) \right] = \mathbb{E} \left[ \sum_{k \geq \mathfrak{z}} \chi(s_k^\varepsilon \leq t, k < \tau_{R, \varepsilon}) \right] \\ &\leq 1 + \mathbb{E} \left[ \sum_{k \geq 1} \chi(U_{\mathfrak{z}+1}^\varepsilon + \dots + U_{\mathfrak{z}+k}^\varepsilon \leq t - s_{\mathfrak{z}}^\varepsilon, \mathfrak{z} + k < \tau_{R, \varepsilon}) \right]. \end{aligned}$$

By definition  $U_n^\varepsilon = \varepsilon^2 \eta^2(x_{n-1}^\varepsilon) e^{1-A_n} \geq \varepsilon^2 \eta^2(R) e^{1-A_n}$  for any  $n < \tau_{R, \varepsilon}$ , since  $\eta$  is decreasing on  $\mathbb{R}_+$  and corresponds to an even function. Moreover the definition of  $\mathfrak{z}$  implies  $t - s_{\mathfrak{z}}^\varepsilon \leq \rho\varepsilon^2$ . We deduce that

$$\varepsilon^{-2} \mathcal{A}_2(R, \varepsilon) \leq 1 + \mathbb{E} \left[ \sum_{k \geq 1} \chi \left( e^{1-A_{\mathfrak{z}+1}} + \dots + e^{1-A_{\mathfrak{z}+k}} \leq \frac{\rho}{\eta^2(R)}, \mathfrak{z} + k < \tau_{R, \varepsilon} \right) \right].$$

Since  $(e^{1-A_n})_{n \geq 1}$  is a sequence of i.i.d random variables, we can define the associate renewal process  $(\overline{N}_t)_{t \geq 0}$  already introduced in the proof of Proposition 2.2. We obtain

$$\varepsilon^{-2} \mathcal{A}_2(R, \varepsilon) \leq 1 + \mathbb{E} \left[ \sum_{k \geq 1} \chi \left( e^{1-A_1} + \dots + e^{1-A_k} \leq \frac{\rho}{\eta^2(R)} \right) \right] \leq 1 + \mathbb{E}[\overline{N}_{\frac{\rho}{\eta^2(R)}}] < \infty.$$

In other words, there exists  $\varepsilon_3 > 0$  s.t.  $\mathcal{A}_2(R, \varepsilon) \leq \delta/4$  for any  $\varepsilon \leq \varepsilon_3$ .

Let us combine the asymptotic analysis of each term in (3.58). Then, for any  $\delta > 0$ , we define  $\varepsilon_0 := \min(\varepsilon_1, \varepsilon_2, \varepsilon_3)$  which insures the announced statement:  $|D^\varepsilon(t, x)| \leq \delta$  for any  $\varepsilon \leq \varepsilon_0$ .  $\square$

## 4 Numerical application

Let us focus our attention on particular examples of  $L$ -class diffusion processes. We recall that these diffusion processes are characterized by their drift term  $a(t)x + b(t)$  and their diffusion coefficient  $\overline{\sigma}(t)$ . In many situations, both the particular function  $f(t, x)$  and the time scale  $\rho(t)$  which permit to write the diffusion process as a function of the time-changed Brownian motion  $X_t = f(t, x_0 + B_{\rho(t)})$  have an explicit formula. We propose two particular cases already introduced in exit problem studies [13].

For each one of these cases we first illustrate some of the results obtained in the theoretical part. Secondly we compare our approach with classical schemes like the Euler scheme. Even if this comparison is quite difficult as our method looks for a control on the path with an  $\varepsilon$  approximation while the classical methods do not follow this objective, we construct a rough comparison that we explain later on.

**Example 1 (periodic functions).** We set:

$$a(t) = \frac{\cos(t)}{2 + \sin(t)}, \quad b(t) = \cos(t), \quad \text{and} \quad \bar{\sigma}(t) = 2 + \sin(t). \quad (4.1)$$

Then the three basic components of the  $\varepsilon$ -strong approximation (see Theorem 3.8) are given by  $\rho(t) = 4t$ ,

$$f(t, x) = (2 + \sin(x)) \left( \frac{x}{2} + \ln \left( 1 + \frac{\sin(t)}{2} \right) \right) \quad \text{and} \quad F(x) = 3 + \frac{\sqrt{|x|}}{2}.$$

We observe that the simulation of a  $\varepsilon$ -strong approximation of the diffusion paths ( $X_t, t \in [0, 1]$ ) requires a random number of  $\phi_\varepsilon$ -domains illustrated by the histogram in Figure 1.

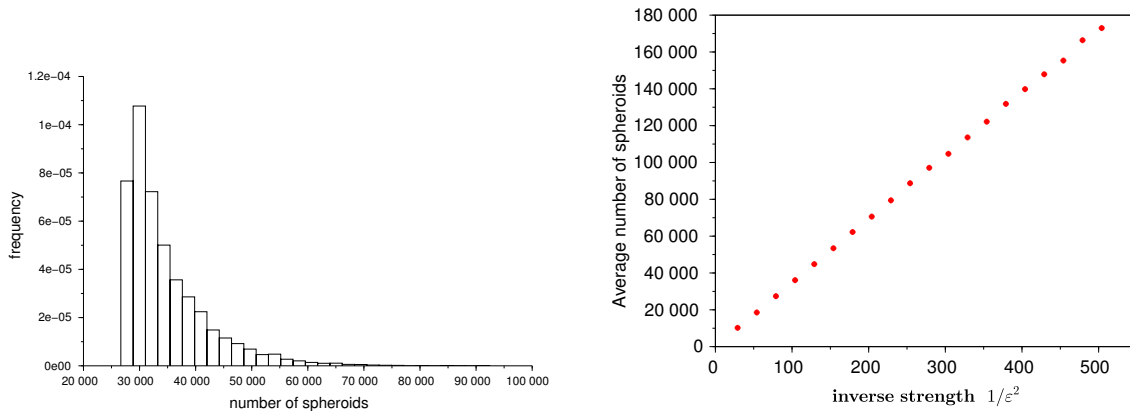


Figure 1: Histogram of the number of  $\phi_\varepsilon$ -domains used to cover the time interval  $[0, 1]$  for  $\varepsilon = 0.1$  (left, sample of size 10 000) – Average number of  $\phi_\varepsilon$ -domains versus the inverse strength  $1/\varepsilon^2$  (right, sample of size for each point: 1 000). For both pictures:  $x_0 = 0$ .

As said before, it is quite difficult to compare such a method with other numerical approximations of diffusion processes: the other methods don't lead to build paths which are a.s.  $\varepsilon$ -close to the diffusion ones. Let us nevertheless sketch a rough comparison: the simulation of 10 000 paths on  $[0, 1]$  with  $\varepsilon = 0.1$  requires about 255.7 sec and one can observe that the average time step is about  $3 \cdot 10^{-5}$ . If we consider the classical Euler-scheme with the corresponding constant step size, then a similar sample of paths requires about 41.3 sec (on the same computer). One argument which permits to explain the difference in speed is that the  $\varepsilon$ -strong approximation needs at each step to test if the number of  $\phi_\varepsilon$ -domains used

so far is sufficient to cover the time interval, such a test is quite time-consuming. Let us also note that the  $\varepsilon$ -strong approximation permits to be precise not only in the approximation of the marginal distribution but also in the approximation of the whole trajectory. In other words, it is a useful tool for Monte Carlo estimation of an integral, of a supremum, of any functional of the diffusion.

This first example illustrates also the convergence result presented in Theorem 3.13. Since the limiting value is expressed as an average integral of a Brownian motion path, the use of the Monte Carlo procedure permits to get an approximated value: 347.1 on one hand and on the other hand the estimation of the regression line in Figure 1 (right) indicates

$$\mathcal{M}(\hat{N}_1^\varepsilon) \approx 344.3 \times \frac{1}{\varepsilon^2} + 418.5$$

where  $\mathcal{M}$  corresponds to the estimated average value for the sample of size 10 000.

**Example 2 (polynomial decrease).** We consider on the time interval  $[0, 1]$  the  $\varepsilon$ -strong approximation of the mean reverting diffusion process given by

$$a(t) = \frac{1}{2} \frac{1}{1+t}, \quad b(t) = 0, \quad \text{and} \quad \bar{\sigma}(t) = 2. \quad (4.2)$$

Then we obtain the time-scale function  $\rho(t) = 4 \ln(1+t)$  and  $f(t, x) = x\sqrt{1+t}$ . We choose therefore  $F(x) = \sqrt{2} + \frac{\sqrt{|x|}}{2}$ . The number of  $\phi_\varepsilon$ -domains is illustrated in Fig 2. The simulation of a sample of trajectories on the time interval  $[0, 1]$  of size 10 000 requires also about 261 sec for the particular choice  $\varepsilon = 0.01$  while the classical Euler scheme generated with a comparable step size  $3.3 \cdot 10^{-5}$  requires 30 sec.

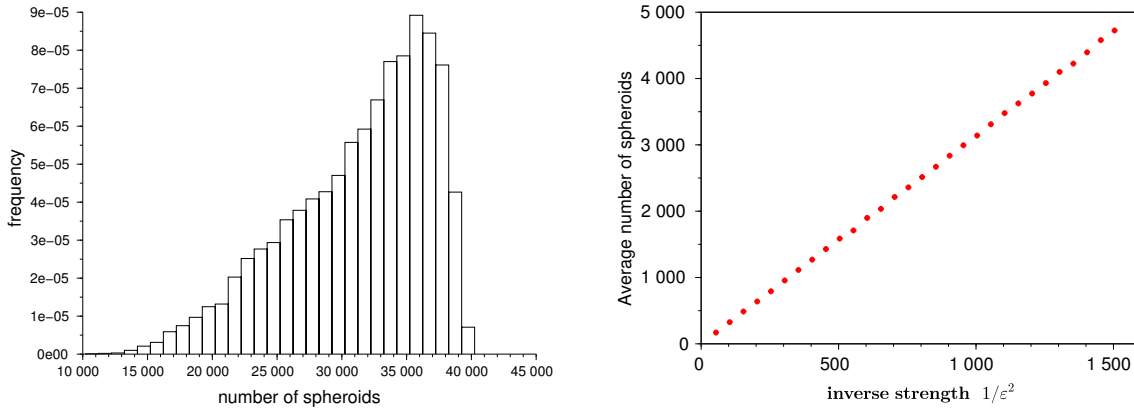


Figure 2: Histogram of the number of  $\phi_\varepsilon$ -domains used to cover the time interval  $[0, 1]$  for  $\varepsilon = 0.01$  (left, sample size: 10 000) – Average number of  $\phi_\varepsilon$ -domains versus the inverse strength  $1/\varepsilon^2$  (right, sample size 1 000). For both pictures:  $x_0 = 0$ .

## Appendix

Let us just present here useful upper-bounds related to the log-gamma distribution.

**Lemma 4.1.** *Let  $\alpha \geq 1$  and  $\beta \geq 1$  and let us assume that  $W$  is a random variable of a log-gamma distribution type. Its probability distribution function is*

$$f_W(t) := \frac{1}{\Gamma(\alpha)\beta^\alpha} (-\ln t)^{\alpha-1} t^{1/\beta-1} \mathbf{1}_{[0,1]}(t), \quad \forall t \in \mathbb{R}.$$

(1) Then

$$\mathbb{E}[e^{-\lambda W}] = \sum_{k \geq 0} \frac{(-1)^k \lambda^k}{k! (1 + k\beta)^\alpha}. \quad (4.3)$$

(2) In particular, for  $\alpha = 1$ , we get

$$\mathbb{E}[e^{-\lambda W}] \leq \frac{\Gamma(1/\beta)}{\beta \lambda^{1/\beta}}. \quad (4.4)$$

(3) In the case:  $\alpha > 1$ , for any  $\beta' > \beta$ , we obtain

$$\mathbb{E}[e^{-\lambda W}] \leq \omega(\alpha, \beta, \beta') \frac{1}{\lambda^{1/\beta'}}, \quad \text{with } \omega(\alpha, \beta, \beta') = \frac{(\alpha - 1)^{\alpha-1} \Gamma(1/\beta')}{\Gamma(\alpha) \beta^\alpha e^{\alpha-1} (\beta^{-1} - \beta'^{-1})^{\alpha-1}}. \quad (4.5)$$

*Proof.* For (1), let us first note that an easy computation leads to the following moments, for any  $k \geq 1$ :

$$\mathbb{E}[W^k] = (1 + k\beta)^{-\alpha}. \quad (4.6)$$

After summing over  $k$  (4.6) we deduce the expression of the Laplace transform (4.3).

For (2), let us first consider the particular case:  $\alpha = 1$ . Using the expression of the PDF and the change of variable  $u = \lambda x$ , we obtain

$$\mathbb{E}[e^{-\lambda W}] = \frac{1}{\beta} \int_0^1 e^{-\lambda x} x^{1/\beta-1} dx = \frac{\Gamma(1/\beta)}{\beta \lambda^{1/\beta}} I_\beta(\lambda),$$

where  $I_\beta(\lambda) := \frac{1}{\Gamma(1/\beta)} \int_0^\lambda e^{-u} u^{1/\beta-1} du$ . We observe that  $\lambda \mapsto I_\beta(\lambda)$  is increasing and  $\lim_{\lambda \rightarrow \infty} I_\beta(\lambda) = 1$  which leads to (4.4).

For (3), let us now assume that  $\alpha > 1$  and consider  $\beta' > \beta$ . Then

$$\begin{aligned} \mathbb{E}[e^{-\lambda W}] &= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^1 e^{-\lambda x} \left( -x^{(1/\beta-1/\beta')/(\alpha-1)} \ln x \right)^{\alpha-1} x^{1/\beta'-1} dx \\ &\leq \left( \frac{\alpha - 1}{e \beta^{\alpha/(\alpha-1)} (\beta^{-1} - \beta'^{-1})} \right)^{\alpha-1} \frac{1}{\Gamma(\alpha)} \int_0^1 e^{-\lambda x} x^{1/\beta'-1} dx, \end{aligned}$$

since  $-u^r \ln u \leq (re)^{-1}$ . Moreover the change of variable  $u = \lambda x$  leads to

$$\mathbb{E}[e^{-\lambda W}] \leq \frac{\omega(\alpha, \beta, \beta')}{\lambda^{1/\beta'}} I_{\beta'}(\lambda).$$

The bound  $I_{\beta'}(\lambda) \leq 1$  directly leads to (4.5). □

## References

- [1] A. Alfonsi. On the discretization schemes for the CIR (and Bessel squared) processes. *Monte Carlo Methods Applications*, 11 (4):355–384, 2005.
- [2] A. Beskos, S. Peluchetti and G. Roberts.  $\varepsilon$ -strong simulation of the Brownian path. *Bernoulli*, 18 (4):1223–1248, 2012.
- [3] J. Blanchet, X. Chen and J. Dong.  $\varepsilon$ -strong simulation for multidimensional stochastic differential equations via rough path analysis. *The Annals of Applied Probability*, 27 (1):275–336, 2017.
- [4] N. Chen and Z. Huang. Localization and exact simulation of Brownian motion-driven stochastic differential equations. *Math. Oper. Res.*, 38:591–616, 2013.
- [5] D.J. Daley and D. Vere-Jones. *An Introduction to the Theory of Point Processes*. Volume I: *Elementary Theory and Methods*. Springer, second edition, 2002.
- [6] M. Deaconu and S. Herrmann. Simulation of hitting times for Bessel processes with non integer dimension. *Bernoulli*, 23 (4B):3744–3771, 2017.
- [7] M. Deaconu, S. Maire and S. Herrmann. The walk on moving spheres: a new tool for simulating Brownian motion’s exit time from a domain. *Mathematics and Computers in Simulation*, 135:28–38, 2017.
- [8] M. Deaconu and S. Herrmann. Hitting time for Bessel processes—walk on moving spheres algorithm (WoMS). *The Annals of Applied Probability*, 23 (6):2259–2289, 2013.
- [9] L.C. Evans. Partial differential equations. *Graduate Studies in Mathematics, Vol. 19*, American Mathematical Society, Providence, RI, Second Edition, 2010.
- [10] M. Hefter, A. Herzwurm and T. Müller-Gronbach. Lower error bounds for strong approximation of scalar SDEs with non-Lipschitzian coefficients. *The Annals of Applied Probability*, 29 (1):178–216, 2019.
- [11] H. R. Lerche. Boundary crossing of Brownian motion. *Lecture Notes in Statistics*, 40, Springer-Verlag, Berlin, 1986.
- [12] I. Karatzas and Steven E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.
- [13] S. Herrmann and N. Massin. Approximation of exit times for one-dimensional linear and growth diffusion processes. Preprint, <https://hal.archives-ouvertes.fr/hal-02147874v2>, 2019.

- [14] N. Hofmann, T. Müller-Gronbach and K. Ritter. Linear vs standard information for scalar stochastic differential equations. Algorithms and complexity for continuous problems/Algorithms, computational complexity, and models of computation for nonlinear and multivariate problems (Dagstuhl/South Hadley, MA, 2000). *J. Complexity*, 18 (2):394–414, 2002.
- [15] M. Hutzenthaler and A. Jentzen. Numerical approximations of stochastic differential equations with non-globally Lipschitz continuous coefficients. *Mem. Amer. Math. Soc.*, 236, no. 1112, 2015.
- [16] T. Müller-Gronbach. The optimal uniform approximation of systems of stochastic differential equations. *Ann. Appl. Probab.* 12 (2):664–690, 2002.
- [17] K. Pötzelberger and L. Wang. Crossing probabilities for diffusions with piecewise continuous boundaries. *Methodology and Computing in Applied Probability*, 9 (1): 21-40, 2007.
- [18] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion*. Springer Verlag, Berlin, Heidelberg, 1999.