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# Optimal Green Technology Adoption and Policy Implementation

Jean-Marc Bourgeon\*

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## Abstract

The importance of coordination problems in the greening of the economy is analyzed using a global game approach in a simple macro-dynamic model. Two policy options to motivate firms to adopt green technology are examined: one coupling an emissions tax with subsidies, the other coupling an emissions tax with technology standards. Compared to the First-Best, these policies face coordination problems and their effectiveness depends on shocks affecting network externalities. The resulting path of the economy is stochastic and can deviate significantly from its benchmark. The underlying mechanisms are derived in a simple framework and illustrated by numerical simulations.

**Keywords:** Growth, sustainability, Technology adoption.

**JEL:** O33, O44, E37, Q55

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# 1 Introduction

Decoupling resource use and environmental discharges from economic growth is one of the main challenges facing modern economies. Among these, decarbonization, i.e. reducing carbon dioxide (CO<sub>2</sub>) emissions per unit of GDP, is a top priority. The historical average rate of decarbonization (1.5%) falls short of the progress required to meet existing climate targets: according to the IPCC (2018) Special Report, under emission pathways consistent with 1.5°C global warming, CO<sub>2</sub> emissions decline by about 45% from 2010 levels by 2030, reaching a net zero level around 2050. To limit global warming to below 2°C, they would need to decline by about 25% by 2030 in most trajectories and reach net zero by about 2070. Non-CO<sub>2</sub> emissions in scenarios that limit global warming to 1.5°C show significant reductions, similar to those in scenarios that limit warming to 2°C. To operate these radical changes, firms must reconsider their production processes. Investments needed are very large and involve all sectors.

As the International Energy Agency (IEA, 2021) points out, most of the global reductions in CO<sub>2</sub> emissions through 2030 in its model come from technologies readily available today, but by 2050, nearly half of the reductions come from technologies that are currently in the demonstration or prototype stage. In heavy industry and long-distance transportation, the share of emission reductions from technologies still in development today is even higher. This is a significant challenge for decarbonization because, as noted by Battisti (2008), the diffusion of existing cost-effective clean technologies has been notoriously slow and below potential. The diffusion of green technologies is hindered by multiple obstacles. At first, there is a conversion cost: Green technologies are often more costly and less productive than the technologies that firms are using, and may require retraining of their workforce and radical changes in their supply chains. Because of network externalities and technological spillovers, optimal investment choices do not necessarily correspond to the latest or most innovative machines.<sup>1</sup> Indeed, the profitability of a production line also depends on elements such as the know-how of the workforce, the availability of inputs and the maintenance services required by the technology. Since a machine is a durable good, usually used for several years, it is important that the parts needed for its operation and maintenance services are readily available in the future, which is all the more likely as the technology becomes more widespread. There is also the issue of compatibility of the technology with

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<sup>1</sup>Technology spillovers refer to the unintended benefits resulting from the research and development efforts of others. Network externalities refers to a situation in which the value of a technology increases with the number of its users.

existing and future industry standards, which determines what other machines can be employed. Moreover, the more a technology is used, the higher the research effort of the machine industry sector to improve the technology. Therefore, the optimal investment choice for a firm depends on the decisions made by all firms, and the more firms use a given technology, the more attractive it is. All of these phenomena, hereafter referred to as “network effects”, are difficult to anticipate, due to the many external factors that affect the supply chain and the supporting infrastructure of firms.<sup>2</sup>

In this paper, I analyze the issue of designing policies to green the economy when firms face such a coordination problem using an original micro-founded dynamic model. A large number of firms must adopt cleaner and often newer technologies whose development and diffusion are subject to network effects. Firms’ investment decisions are based on their expectations about the evolution of these effects. Their heterogeneous assessments are reflected in an industrial sector made up of firms with diverse technological processes. This diversity is captured by the distribution of two variables, one corresponding to the productive capacity of production lines (similar to the traditional economic definition of capital), the other to their pollution intensity. The firms’ coordination problem in their investment choices is framed as a sequence of global games (Carlsson & Van Damme, 1993) played each period. Unlike other approaches that often lead to multiple equilibria, global games yield a single equilibrium by taking into account the intrinsic uncertainty affecting the economy.<sup>3</sup> The dynamic unfold as follows: Given the prevailing interest rate and wage (and the public policy), the knowledge of the current distribution of technology in the industry and private signals about the shock affecting network effects allow firms to determine their optimal investment for the period. The equilibrium investment strategy of this global game is a unique

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<sup>2</sup>Hoevenagel et al. (2007) discusses the problems of environmental technology adoption in Europe. 99.8% of firms are small and medium-sized enterprises, run by entrepreneurs with significant time and task pressure. Their basic perception of environmental technologies is one of cost and risk. Their staff is not adequately trained and familiar with the installation and operation of new technologies. They are generally uninformed about potential environmental technologies and rely heavily on the advice of their professional entourage, such as suppliers, who often have the same lack of information. A firm’s network relationships influence the adoption of environmental innovations at each stage as it gathers information about potential solutions for implementing new technologies, and conversely, a firm’s strategic orientation influences the types of external expertise and network relationships that are developed.

<sup>3</sup>Multiplicity of equilibria in coordination games is due to the strong assumptions that the economic fundamentals are common knowledge and that agents are certain about each other’s behavior in equilibrium thanks to an appropriate set of beliefs. By introducing a noisy signal about the actual state of the world and some uncertainty about what other agent knows, global games lead to a unique equilibrium for each underlying state of the world without resorting to arbitrary assumptions about self-fulfilling beliefs (see Morris & Shin, 2000, 2003).

function, but idiosyncratic signals, although unbiased, lead to different expectations. Firms' choices are thus diverse, leading to a new distribution of technological processes in the next period, when another global game takes place, and so on. The resulting path of the economy is affected only by the uncertainty of network effects. In other words, the path followed by the economy and its fluctuations in this framework do not depend on arbitrary assumptions about self-fulfilling beliefs of firms. Rather, they depend on public policy and the volatility of intrinsic shocks. This framework allows us to compare the impacts of these fluctuations on the pathway of the economy with different policy instruments.

The First-Best path of the economy is derived by solving the problem of a social planner who can decide on the levels of investment in production capacity and pollution intensity of firms, taking into account the intertemporal trade-offs between consumption, environmental quality and (total) investment. Because production processes are imposed to firms by assumption, coordination problems and network effects are not relevant. The economy follows a First-Best path to environmental neutrality which is deterministic.

I then consider actual environmental policies where firms make their own investment decisions, which leads to network effects. The regulator must use two instruments, one for each characteristic of firms' production processes. I consider two policy options that both involve an emissions tax, which is a widely used incentive instrument that makes firms responsible for the pollution they emit. It influences their choice of technology both on their production capacity and on their pollution intensity characteristics. To better guide firms in their green investment choices, these policy options also incorporate one of two alternative instruments: another incentive policy, namely a schedule of subsidies for green investments, or a schedule of technological standards, i.e. restrictions on the pollution intensity of new machines that are tightened over time. The subsidy policy allows firms to better adapt to network shocks. The standard policy reduces the firms' choices and somehow the coordination problem. To compare the two policy options, I suppose that they are designed to allow the economy to follow, in expectation, the First-Best benchmark path. They thus are possible implementations of this path.

To assess the effects of the fluctuations generated by network effects, I then simplify the framework by assuming a constant marginal rate of substitution between environmental and industrial goods, exponential consumer preferences (CARA), and a Cobb-Douglas production function. This setup allows for explicit solutions of the First-Best policy and its two implementations.

The results show that the First-Best dynamic of the economy entails three sequences. The first one corresponds to a decrease in the stock of productive capital, and thus in GDP, if the footprint of the technologies originally employed is too heavy on the environment. This sequence corresponds to a swift catching up of investment in green technologies at the expense of the production capacity of firms. The second sequence corresponds to a stagnation of GDP, with the production capacity of the industrial sector just maintained, while investment in clean technologies is still growing fast to allow environmental quality to increase. Finally, the third sequence corresponds to a growing GDP along an environmentally neutral trajectory (supposed to be the best that can be achieved with green technologies).

To analyze the impact of the economic fluctuations that occur under the two policy options considered, I assume that consumers have rational expectations. While the shocks affecting network effects are independent, they induce a path dependence driven by the consumers desire to smooth consumption over time. The resulting distribution of the interest rate is shown to follow a first-order autoregressive process during the transition period to environmental neutrality under the two policy options. Given the scale of investment needed to green the economy, long lasting deviations of the economy from the First-Best path are thus likely. Simulations show that this path dependence can be significant under the two policies examined, and that technology standards permit a transition to green technologies more likely in line with the regulator's intended trajectory than subsidies do. This is because producers have less flexibility to adapt their green investments to economic conditions when standards rather than subsidies are implemented.

There is a abundant literature on growth and sustainability. The literature on endogenous green growth focuses on productivity improvements and frontier innovation. This is the case in the AK paradigm where capital-knowledge accumulates with learning-by-doing (Stockey, 1998), and Lucas-like extensions (Bovenberg & Smulders, 1995), within a framework of product variety (Gerlagh & Kuik, 2007) or within the Schumpeterian growth paradigm of destructive creation and directed technical changes (Acemoglu et al., 2012), where the most productive innovations are adopted by firms as soon as they are discovered. This article focuses on the adoption of existing technologies that have knock-on effects leading to the gradual replacement of old and polluting machines with greener ones. The approach is thus close to the literature on endogenous growth viewed as a process of adoption of existing ideas and mutual imitation between firms, as exposed by Eaton & Kortum (1999); Lucas Jr & Moll (2014); Lucas (2009);

Perla & Tonetti (2014).<sup>4</sup> The approach here is similar for describing the adoption of technologies: although the R&D sector is not spelled out, there is a set of existing technologies whose potential is more or less exploited depending on the proportion of firms that use them. The distribution of technology used among firms changes over time as firms’ incentives to adopt new technologies evolve.<sup>5</sup> Comparisons of policy instruments to implement an environmental objective have been quite numerous since Weitzman (1974). The main focus is on the information available to the regulator on the firms’ pollution abatement costs. Few papers consider the importance of firms’ expectations in achieving an environmental goal. In a recent paper, Aldy & Armitage (2020) compare an emissions tax with a cap-and-trade instrument in which firms are subject to forecasting errors in the price of pollution allowances on the secondary market, the effectiveness of the abatement technology being the same in both cases. In what follows, the firms’ expectations about their economic environment are important because of network externalities that determine the effectiveness of the technology.

The remainder of the paper is organized as follows: Section 2 describes the dynamics of technology adoption under Laissez-Faire and the supply of capital. Each period, total demand for capital must meet supply provided by consumers, leading to the equilibrium interest rate. Since the demand for capital depends on network shocks, the interest rate is also stochastic. Its distribution depends on the expectations of the (representative) consumer, whose preferences are affected by the quality of the environment, and by her desire to smooth her consumption over time. Section 3 presents the environmental dynamic, the social planner’s program and the First-Best policy. Section 4 considers its implementations using an environmental tax supplemented by a subsidy scheme or a technological standard scheme. Both policies are affected by network effects and are defined such that the expected trajectory of the economy matches the deterministic First-Best path. In section 5, consumer preferences and the production function are specified to better characterize how these network effects affect the path of the economy. Assuming a constant marginal rate of substitution between consumption and environmental quality, the intertemporal rate of substitution in consumption is constant during the transition to environmental neutrality. The First-Best path is derived by assuming a Cobb-Douglas production function and an exponential utility function (CARA). During the transition path to environmental neutrality under the

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<sup>4</sup>These papers assume that each agent in the economy is endowed with a certain amount of knowledge (“ideas”) and this knowledge evolves through contact with the rest of the population.

<sup>5</sup>There is also a microeconomics literature that investigates the problem of network externalities, initiated by Katz & Shapiro (1985). See Guimaraes & Pereira (2016) for a recent contribution, and Shy (2011) for a survey.

two policy options, the distribution of the interest rate follows a first-order stochastic process and the economy a Gaussian random walk. These paths are illustrated with numerical simulations presented at the end of the section. The last section concludes.

## 2 Technology adoption and capital markets under Laissez-Faire

Consider a discrete time economy composed of a continuum of firms, of total mass equal to one, that collectively produce at date  $t$  an amount  $q_t$  of output, taken as the numeraire and corresponding to the GDP. The firms' production may come from various technological processes with different environmental impacts. More precisely, I suppose that for each particular production task, firms have the choice among a large set of machines characterized by their productivity and polluting emissions level. These sets of machines evolve over time, but each period, comparing machines with similar productivity levels, the cheaper they are, the more they pollute. A firm's production line is characterized by machines acquired over time and renewed or replaced when deemed necessary. Each period, firms' technological processes may thus be very different, and to capture this heterogeneity in a simple way, the process of firm  $i$  at date  $t$  is described by two parameters,  $k_{it}$  and  $x_{it}$ , dubbed 'productive capacity' and 'green technology index' (or 'technology mix') respectively. The productive capacity of the technology is similar to the standard economic notion of capital: it corresponds to the input that, once combined with labor, allows the firm to produce items or services. It yields a gross revenue  $q_{it} = Q(k_{it}, \ell_{it})$  where  $\ell_{it}$  is the firm's employment level, and  $Q$  a production function, homogeneous of degree one and satisfying the Inada conditions. The green technology index captures the firm's effort to diminish the environmental impact of its production, given by  $\iota_{it}q_{it}$  where

$$\iota_{it} = \max\{0, \varphi - \xi x_{it}/q_{it}\} \quad (1)$$

corresponds to the emission intensity of its production process. It is positive because I suppose that green technologies can only reduce the firm's emissions. The emission intensity diminishes with  $x_{it}$ , with a maximum equal to  $\varphi$  if  $x_{it} = 0$ . Hence, the higher  $x_{it}$ , the lower the environmental impact of the firm's production. This impact depends on  $\xi$ , a positive parameter that measures the environmental efficiency of green investments.



Each period, part of the machines used by firms is depreciated, due to wear and tear, or because some of them are obsolete or so deteriorated that maintenance costs are too high. I suppose a constant depreciation rate  $1 - g$  that affects both the productive capacity and the green technology index. To renew and adapt their production process, firms buy new machines that correspond to an investment in productive capacity and/or in environmental quality. Firm  $i$ 's investment in productive capacity,  $I_{it}$ , leads to a dynamic of  $k_{it}$  given by  $k_{it+1} = gk_{it} + I_{it}$ . Similarly, firm  $i$ 's spending in environmental quality,  $o_{it}$ , modifies its technology mix according to  $x_{it+1} = gx_{it} + o_{it}$ .<sup>6</sup>

Depending on their technology choices, firms benefit from (or suffer from the lack of) network effects related to their input suppliers, maintenance services (how easy it is to find specific inputs and parts to service the machines), the know-how of workers, and more generally, the peculiarities of the machines they use. The more widespread is the technology used, the less a firm encounters problems and the easier it is to achieve a production target.

As an illustration, consider the problem of a firm having to renew (part of) its fleet of vehicles. Electric vehicles (EVs) are an option, but their main drawback (in addition to being pricey) is a limited operating range due to the battery capacity. To reduce the dependency on charging stations, different types of EVs have emerged with different impacts on the environment: in addition to electric vehicles that are entirely battery-powered, there are several so-called hybrid vehicles, such as plug-in hybrids that operate on a battery with the help of an internal combustion engine (ICE) or hybrids that have their battery charged by an ICE. Similarly, fuel cell electric vehicles that produce electricity by combining hydrogen from their tanks with oxygen from the air have recently emerged, but the hydrogen distribution network is very limited. This multiplicity of options makes it difficult for users to reach a decision, even though some of these technologies require a large number of users to make the charging station networks and all other supporting services profitable, which hinders their development in initial stages.

Hence, depending on the environmental characteristics of the machines chosen, network effects are more or less significant. These effects are not investment costs (due to the price of the machines), but losses or savings related to the ease or difficulty of their use given the existing supporting infrastructure of the economy (e.g., delays due to maintenance services, charging times and location of charging stations in the case

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<sup>6</sup>These investments may be negative, i.e. firms may consider resealing some of their machines. Observe that we can have  $I_{it} = 0$  and  $o_{it} > 0$ , in which case firm  $i$  buys pollution abatement equipment, i.e. devices that do not produce any items but reduce the polluting emissions of the firm.

of EVs). To benefit from these effects, firms must somehow anticipate the extent of the total green investment when making their own, which leads to an intertemporal coordination problem that is formalized as a succession of global games taking place each period.<sup>7</sup> More precisely, I suppose that under the Laissez-Faire situation, these effects correspond to monetary amounts given by<sup>8</sup>

$$G_{it} = G(x_{it}; \mu_{t-1}, \mathcal{O}_{t-1}, \omega_t) = (\mu_{t-1} + \lambda \mathcal{O}_{t-1} + \omega_t) x_{it} - x_{it}^2/2 - \bar{G}_t, \quad (2)$$

for firm  $i$  at date  $t$ , where  $\mu_{t-1} \equiv \int_0^1 x_{it-1} di$  is the green technology index (GTI) of the economy at the previous period,  $\mathcal{O}_{t-1} \equiv \int_0^1 o_{jt-1} dj$  the sum of green investments made simultaneously by firms at  $t - 1$  for period  $t$ , and  $\omega_t$  the realization of  $\tilde{\omega}_t$ , a time-independent normally distributed noise, with variance  $\sigma_\omega^2$ , that summarizes the many unmodeled shocks affecting the development and the diffusion of technology.<sup>9</sup>  $G_{it}$  may be positive or negative, depending on whether firm  $i$  benefits or suffers from network effects a date  $t$ . The last term,  $\bar{G}_t$ , ensures that on average over all firms, the network returns are zero.<sup>10</sup>

GTI  $\mu_{t-1}$  reflects the set of the most commonly used machines at  $t - 1$ . Through knowledge diffusion and supply chain adaptation over time, they are well served by the existing service network. Date- $t$  network externalities depend on  $\mu_{t-1}$  because only part of the capital stock is renewed in each period, and thus the remaining part still requires a supporting infrastructure close to the existing one, which leads firms to make conservative investments (e.g. ICE or hybrid vehicles may reveal the most convenient choices because they rely on the existing network of gas and service stations).<sup>11</sup> Changes in the relative attractiveness of green technologies from one period to the next (e.g., due

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<sup>7</sup>These coordination problems with strategic complementarity are known as “beauty contests” (see, e.g., Angeletos & Pavan, 2004, 2007 and Morris & Shin, 2002).

<sup>8</sup>This expression can be considered as a second-order approximation (around zero) of a more general expression. It leads to close-form solutions of the firms’ coordination game, and more importantly, it preserves the structure of the game from one period to the next (the distribution of  $x_{it}$  at equilibrium remains Gaussian with mean  $\mu_t$  and a constant standard deviation).

<sup>9</sup>These shocks may come from the service network, input suppliers or machines producers, reflecting the expectations of these operators on technology development. To streamline the analysis, I assume that these shocks do not directly affect the firms’ production level  $q_{it}$ .

<sup>10</sup>Since green investments are not productive per se, firms’ monetary gains and losses from network effects must cancel out. For the sake of simplicity, I suppose that they offset each other in each period. Of course, that does not mean that firms must neglect their signal about  $\omega_t$ . On the contrary, by following the optimal strategy that depends on their private information, as described below, they maximize their expected profit from network effects, a profit which is null on average.

<sup>11</sup>The situation is different when firms are constrained by technological standards that prevent them from replacing old machines with new ones that have the same negative impact on the environment. See Section 4 for details.

to technological improvements or exogenous changes in the supporting infrastructure) are summarized by  $\omega_t$  which is unknown at  $t - 1$ , when firms make their investments. The adaptation of the service network depends on this shock and on choices made by firms at  $t - 1$ , a sum equal to  $\mathcal{O}_{t-1}$  of investments that are made simultaneously. This adaptation may be more or less rapid, depending on the extent of immediate network effects which is captured by parameter  $\lambda \in [0, 1]$ . If  $\lambda = 0$ , the first term of (2) reduces to  $\mu_{t-1} + \omega_t$ , i.e. the former average mix plus a noise which may be positive or negative. This noise is not amplified or reduced by the firms' investments because firms do not expect that workers or the supporting infrastructure will adapt rapidly. When immediate network effects are at work ( $\lambda > 0$ ), the first term of (2) also depends on the most recent investments made by all firms,  $\mathcal{O}_{t-1}$ . The economic infrastructure evolves more rapidly the larger  $\lambda$  is, the extreme case where  $\lambda$  is close to 1 corresponding to an immediate adaptation of the workers and the supporting infrastructure to new investment choices.<sup>12</sup>

Firms must anticipate at  $t - 1$  their optimal date- $t$  technology mix that will result from the choices made simultaneously by other firms according to their expectations of  $\omega_t$ . It is easily shown that the ideal technology mix at date  $t$  is<sup>13</sup>

$$x_t^* \equiv \mu_{t-1} + \lambda \mathcal{O}_{t-1} + \omega_t. \quad (3)$$

Indeed, using the fact that the average network effect is zero, it comes

$$\int_0^1 G(\mu_{t-1}, \mathcal{O}_{t-1}, x_{it}, \omega_t) di = \int_0^1 (x_t^* x_{it} - x_{it}^2/2) di - \bar{G}_t = 0$$

which leads to

$$\bar{G}_t = x_t^* \mu_t - (\sigma_{x_t}^2 + \mu_t^2)/2$$

where  $\sigma_{x_t}^2 \equiv \int_0^1 (x_{jt} - \mu_t)^2 dj$  is the variance of the technology indexes at date  $t$ . Substituting in (2) and collecting terms gives

$$G_{it} = [(x_t^* - \mu_t)^2 + \sigma_{x_t}^2 - (x_{it} - x_t^*)^2]/2, \quad (4)$$

where  $(x_{it} - x_t^*)^2/2$  corresponds to firm  $i$ 's loss due a mix  $x_{it}$  different from the ideal

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<sup>12</sup>It is likely that the value of  $\lambda$  is low. As stressed by Battisti (2008), a consistent literature has shown that, even when a clean or a cost-reducing technology is readily available in the market, its spreading takes several years.

<sup>13</sup>As  $\omega_t$  is normally distributed,  $x_t^*$  can take negative values, and all the more probably if  $\mu_t$  is small. I will neglect this possibility in the following by assuming that  $\mu_t$  remains far enough from 0.

one  $x_t^*$ . Observe that if all firms were equipped with the ideal mix, i.e.  $x_{it} = x_t^*$  for all  $i$ , we would have  $G_{it} = 0$  for all  $i$ , meaning that no firm would gain or lose from network effects due to perfect coordination. However, firms cannot perfectly assess the ideal mix  $x_t^*$  because they don't know at date  $t - 1$  the realization of  $\tilde{\omega}_t$  and the total of green investments  $\mathcal{O}_{t-1}$  made simultaneously by all firms.

This dynamic setup is solved sequentially, focusing on Markov perfect equilibria where  $x_t^*$  is a state variable. In period  $t - 1$ , firms learn the distribution of the green indexes  $x_{it-1}$  and must anticipate the realization of the next period ideal technology mix,  $\tilde{x}_t^*$ , a random variable which distribution depends on the behavior of all firms. Firm  $i$  infers the other firms' decisions according to what it knows about them and its beliefs on the realization of the next period shock  $\tilde{\omega}_t$ . I suppose that these beliefs are unbiased idiosyncratic private signals of the shock: they are formed according to  $\tilde{\eta}_{it-1} = \omega_t + \tilde{v}_{it-1}$ , where  $\omega_t$  is the realization of  $\tilde{\omega}_t$ ,  $\tilde{v}_{it-1}$  a time-independent noise, normally distributed with variance  $\sigma_v^2$ , verifying  $\mathbb{E}[\tilde{v}_{it-1}] = \mathbb{E}[\tilde{v}_{it-1}\tilde{v}_{jt-1}] = 0$  for all  $i, j$ , and  $\int_0^1 v_{it-1} di = 0$ .

Consider the Laissez-Faire situation from date  $t = 0$  (the present period) assuming that all firm  $i$  is endowed with technology  $(k_{i0}, x_{i0}) > 0$ . Each following period  $t$ , given  $w_t$  and  $r_t$  the date- $t$  wage and interest rate, firm  $i$  plans its technological investments and employment level  $\{(o_{it}, I_{it}, \ell_{it}), t = 0, 1, 2, \dots\}$  to maximize the expected discounted sum of its profits  $\mathbb{E}_t[\sum_{h \geq 0} (\prod_{\tau=1}^h \delta_{t+\tau})(q_{it+h} + G_{it+h} - w_t \ell_{it+h} - I_{it+h} - o_{it+h})]$  where  $\delta_t = 1/(1 + r_t)$  is the date- $t$  discount factor.<sup>14</sup> With a total supply of labor equal to one each period, i.e.  $\int_0^1 \ell_{it} di = \ell_t = 1$  for all  $t$ , it is shown in the appendix that

**Proposition 1** *Under Laissez-Faire, firms' equilibrium investment in productive capacity at time  $t$  is given by*

$$I_{it} = q'^{-1}(1 + r_t - g) - gk_{it} \quad (5)$$

where  $q(k) \equiv Q(k, 1)$  is an increasing and concave function, leading to  $k_{it+1} = k_{t+1} = q'^{-1}(1 + r_t - g)$  for all  $i$ . The equilibrium wage is given by  $w_t = q(k_t) - r_{t-1}k_t$ . Assuming  $\mu_t$  large enough, firm  $i$ 's investment strategy in green technology satisfies

$$o_{it} = \mathbb{E}[\tilde{x}_{t+1}^* | \eta_{it}, \mu_t, r_t] - gx_{it} - 1 - r_t + g, \quad (6)$$

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<sup>14</sup> $\mathbb{E}_t[\tilde{X}]$  is a shorthand for  $\mathbb{E}[\tilde{X} | \mathcal{I}_t]$ , i.e. the expectation of the random variable  $\tilde{X}$  given the information  $\mathcal{I}_t$  available at date  $t$ .

leading to

$$o_{it} = g(\mu_t - x_{it}) + \frac{(1-g)(\mu_t - 1) + \eta_{it} - r_t}{1-\lambda} \quad (7)$$

at equilibrium. The resulting firms' technology mixes at  $t+1$  are normally distributed with mean

$$\mu_{t+1} = g\mu_t + \frac{(1-g)(\mu_t - 1) + \omega_{t+1} - r_t}{1-\lambda}, \quad (8)$$

corresponding to the date- $t+1$  GTI, and standard deviation  $\sigma_{x_t} = \sigma_x \equiv \sigma_v/(1-\lambda)$  for all  $t$ .

Because firms have the same production function  $Q$ , the productive capacity of all firms is the same, determined by choices that are guided unambiguously by the interest rate. This is not the case for their spending in green technology (6) that depend on individual estimates of the most efficient mix  $\mathbb{E}[\tilde{x}_{t+1}^*|\eta_{it}, \mu_t, r_t]$ . The trade-offs that the firm is facing when investing are the following. The date- $t$  borrowing cost of one monetary unit is  $1 + r_t$ . Invested in productive capital, its return is equal to the sum of the marginal productivity of capital,  $q'(k_{t+1})$ , and  $g$ , the remaining value of capital (the rest being depreciated). With investment  $I_t$ , the date- $t+1$  capital stock is equal to  $k_{t+1} = gk_t + I_t$ , hence an optimal productive investment that solves  $q'(gk_t + I_t) + g = 1 + r_t$ , a deterministic equation. (6) is derived similarly: a monetary unit invested in green technology has a marginal return equal to the sum of the marginal network gain  $dG_{it+1}/dx_{it+1} = \tilde{x}_{t+1}^* - x_{it+1}$  and  $g$ . With investment  $o_{it}$ , the firm date- $t+1$  green index is given by  $x_{it+1} = gx_{it} + o_{it}$ . Firm  $i$ 's investment rule is thus to equalize  $1 + r_t$  to the expected return  $\mathbb{E}[\tilde{x}_{t+1}^*|\eta_{it}, \mu_t, r_t] - (gx_{it} + o_{it}) + g$ , since  $\tilde{x}_{t+1}^*$  is unknown. Firm  $i$ 's expectation takes into account the fact all other firms are behaving the same way and form similar expectations given their signals. At equilibrium, firm  $i$  adopts the green investment strategy (7) that is a linear function of the publicly observable variables  $\mu_t$  and  $r_t$ , its own technology index  $x_{it}$ , and its private signal  $\eta_{it}$ .<sup>15</sup> The first term in (7),  $g(\mu_t - x_{it})$ , indicates that firm  $i$  makes-up for the difference between its green index and that of the economy on its undepreciated capital stock. The second term shows that to replace old capital, the firm considers the marginal net return of investing at the GTI level given its appraisal of the network shock,  $(1-g)\mu_t + g - (1+r_t) + \eta_{it}$ , multiplied by the magnification effect of the network externality,  $1/(1-\lambda)$ , since all firms operate the same way. For firms with a low mix and a high signal, (7) corresponds to buying machines with a low environmental impact. For those with a high mix and

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<sup>15</sup>The proof follows Angeletos & Pavan (2004). Morris & Shin (2002) show that this linear, symmetric, rational-expectations strategy leads to the unique (per period) equilibrium.

a low signal, their investment is directed in the opposite direction: they save on new equipment spending by buying less expensive brown technologies. Thanks to unbiased idiosyncratic signals, while firms' green indexes are distributed normally around the GTI, the network shock  $\omega_t$  is correctly taken into account by firms on average as shown by (8).<sup>16</sup> This equation also shows that, as the other relevant variables, this shock is magnified by  $1/(1 - \lambda)$ , a factor that is larger the more responsive the supporting infrastructure. Hence, the more reactive is the supporting infrastructure, the larger are the effects of the network shock and of the cost of capital  $r_t$  on the next period GTI. However, the interest rate is determined on financial markets and its equilibrium value depends on the total demand of capital, and thus on  $\omega_t$  as discussed below.

The expected dynamic of the GTI under Laissez-Faire (8) can be negative, in which case firms acquire increasingly dirty technologies, or positive if  $\mu_t$  is large enough, greater than  $[1 + r_t/(1 - g)]/\lambda$ , i.e. if the technology spillovers are sufficiently large, the depreciation rate large and/or the interest rate low. Indeed, as firms have to renew their machines, they may invest in less polluting ones if they expect that the supporting infrastructure and the workforce know-how adapt rapidly. This is however very unlikely if the initial GTI  $\mu_0$  is low. Indeed, firms will only purchase green technologies in the absence of governmental incentives if these technologies are already the most widely used in the economy, and thus the most easily exploitable.

Due to the idiosyncratic shocks on beliefs,  $v_{it}$ , firms have different expectations on  $x_{t+1}^*$ , hence choose machines with similar productive capacities but different environmental impacts. These discrepancies lead to a Gaussian distribution of firms' green indexes around the GTI, given by  $x_{it} = \mu_t + v_{it}/(1 - \lambda)$ . Hence, the industry sector can be viewed as a "cloud" of firms whose green technology levels are drawn in each period from a normal distribution centered on  $\mu_t$  with standard deviation  $\sigma_v/(1 - \lambda)$  which is all the greater as  $\lambda$  is large.

## 2.1 Consumers and capital markets

Capital supply comes from consumers who maximize their intertemporal utility by arbitraging between consumption and savings each period. Consumers derive well-being from manufactured and environmental goods and services. Environmental goods are

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<sup>16</sup>Observe that (8) is not biased by the lack of information. This is because the signals are private and affected by independent idiosyncratic noises. If firms also shared a public signal, their choices would be distorted in the same direction, as would the resulting dynamics of  $\mu_t$ , to an extent that depends on the relative reliability of the public signal: the more precise the signal, the larger the distortion.

freely available and their consumption is subsumed by the environmental quality (EQ) index  $e_t$ . The consumption of manufactured goods is denoted by  $c_t$ . I suppose that consumers do not try to modify the environment through their consumption and saving plans.<sup>17</sup> Consumers' per-period preferences are represented by a concave utility function  $u(c_t, e_t)$ , and their behavior is modeled by considering a representative consumer whose saving and consumption plans solve

$$\max \mathbb{E}_t \left\{ \sum_{h=0}^{+\infty} \beta^h u(\tilde{c}_{t+h}, \tilde{e}_{t+h}) : \tilde{c}_{t+h} = \tilde{R}_{t+h} + \tilde{r}_{t+h-1} \tilde{S}_{t+h-1} - \tilde{s}_{t+h}, \tilde{s}_{t+h} = \tilde{S}_{t+h} - \tilde{S}_{t+h-1} \right\} \quad (9)$$

each period, where  $R_t$  is her date- $t$  revenue,  $S_{t-1}$  her savings from the previous period,  $r_{t-1}S_{t-1}$  the corresponding date- $t$  capital earnings,  $s_t$  the savings adjustment of period  $t$ , and  $\beta = \exp(-\psi)$  the intrinsic discount factor corresponding to a psychological discount rate  $\psi > 0$ . Solving (9), we obtain

**Lemma 1** *The consumption rule that solves (9) satisfies*

$$h_t^a \equiv \frac{\partial u(c_t, e_t) / \partial c}{\beta \mathbb{E}_t [\partial u(\tilde{c}_{t+1}, \tilde{e}_{t+1}) / \partial c]} = 1 + r_t \quad (10)$$

at each date  $t$ .

Equation (10) corresponds to the Ramsey-Euler rule which states that  $h_t^a$ , the expected intertemporal rate of substitution in consumption (IRS), is equal to the return of capital each period. It also defines the supply function of capital, while (5) and (8) are the demand side coming from firms.

At equilibrium, aggregate production net of investment must be equal to total consumption of manufactured goods, i.e.

$$c_t = q_t - I_t - \mathcal{O}_t = q(k_t) - (k_{t+1} - gk_t) - (\mu_{t+1} - g\mu_t). \quad (11)$$

Hence, at equilibrium on financial markets, i.e. when the total demand for capital by firms meets the supply provided by consumers, the interest rate is affected by the network shocks. Its distribution over time depends on accumulated capital, on the consumer' expectations whose preferences are affected by the quality of the environment, and on her desire to smooth consumption over time. Since productive investment de-

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<sup>17</sup>This could be because they consider that they are too numerous for their individual behavior to have a significant impact on it.

depends on the interest rate, GDP is also affected by these fluctuations and thus follows a random path.

### 3 First-Best path of the economy

Because market prices do not account for the environmental footprint of the economy, public policies must be designed to guide firms in their investment choices. In this section, I present the way the environment is affected by the industrial production and consider the problem of a benevolent social planner whose task is to determine the socially optimal path of the economy. I suppose in this section that he can force firms to adopt the socially optimal set of machines in each period. Since production processes are imposed on firms, coordination problems and network effects are irrelevant. In the absence of frictions and shocks, the economy follows a deterministic path to environmental neutrality that corresponds to the First-Best.<sup>18</sup>

#### Environmental dynamic

Production generates pollution that deteriorates the quality and the availability of environmental goods and services provided by Nature. These effects are summarized in the dynamic of the environmental quality index  $e_t$ , which is given by

$$e_{t+1} = \theta e_t + \hat{e} - \iota_t q_t \quad (12)$$

where  $\theta \in (0, 1)$  is the environmental inertia rate,  $\hat{e}$  the per-period maximum regeneration capacity of the environment, and

$$\iota_t = \int_0^1 q_{it} l_{it} di / q_t \quad (13)$$

the emission intensity of the economy at date  $t$ , which measures the total environmental damage per unit of GDP. Without human interference ( $\iota_t = 0$ ), the EQ index is at its pristine level  $e_N = \hat{e}/(1 - \theta)$ . More generally, it comes using  $\iota_t \approx \varphi - \xi \mu_t / q_t$  that the dynamic of EQ follows approximatively the linear first-order recursive equation

$$e_{t+1} = \theta e_t + \xi \mu_t - \varphi q_t + \hat{e}. \quad (14)$$

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<sup>18</sup>While the supporting infrastructure can adapt well in advance to the policy, shocks may still exist, requiring the social planner to revise the policy plan each period. I neglect these shocks in the determination of the optimal policy because they do not cause a coordination problem.



Since green technologies can only reduce emissions and do not allow for direct improvement of EQ, environmental neutrality is the best society can achieve.

**Definition 1 (Environmentally Neutral Path)** *The economy has reached at date  $T$  an Environmentally Neutral Path (ENP) if for all  $t \geq T$ ,  $\iota_t = 0$ .*

An ENP is a sustainable situation in which the emission intensity of the economy is nil. Along an ENP, thanks to the natural regeneration capacity of the environment, EQ increases and tends toward its pristine level  $e_N$ .

## The regulator's program

Consider a social planner who determines the dynamics of the productive capacity and of the GTI of the economy to maximize consumer welfare given their impact on consumption and the environment. I suppose that total investment cannot be negative so that this maximization is constrained by

$$k_{t+1} \geq gk_t, \quad (15)$$

$$\varphi q(k_t) \geq \xi \mu_t, \quad (16)$$

where the last inequality corresponds to the ENP constraint.<sup>19</sup> In absence of shocks, the planner's problem is to solve

$$\max_{\{\mu_t, k_t, c_t, e_t\}_{t>0}} \left\{ \sum_{t=0}^{+\infty} \beta^t u(c_t, e_t) : (11), (14) - (16) \right\} \quad (17)$$

given  $e_0, k_0$  and  $\mu_0$ . I suppose  $\xi \mu_0 < \varphi q(k_0)$ , so that society is not already on a ENP. Denoting

$$h_t = \frac{\partial u(c_t, e_t) / \partial c}{\beta \partial u(c_{t+1}, e_{t+1}) / \partial c} \quad (18)$$

the date- $t$  IRS, it is shown in the appendix that:

**Proposition 2** *The First-Best path of the economy  $\{\mu_t^*, k_t^*, c_t^*, e_t^*, h_t^*\}_{t>0}$  satisfies (11), (14),*

$$q'(k_{t+1}^*) \leq \frac{h_t^* - g}{1 - (h_t^* - g)\varphi/\xi}, \quad (19)$$

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<sup>19</sup>Constraint (15) implies that the social planner cannot forbid the use of machines already bought that are too environmentally damaging. Hence, the policy solution of (17) is not strictly speaking "First-Best".

for all  $t > 0$ , with an equality when (15) is not binding, and

$$h_{t+1}^*(h_t^* - g - \theta) + \theta g = \xi \frac{\partial u(c_{t+2}^*, e_{t+2}^*) / \partial e}{\partial u(c_{t+2}^*, e_{t+2}^*) / \partial c} \quad (20)$$

when (16) is not binding.

Condition (19) indicates that when (15) is not binding, the IRS is larger than the rental rate of capital  $g + q'(k_{t+1})$  due to the impact of production on the environment (i.e. unless  $\varphi = 0$ ). The First-Best policy is also characterized by a sequence of IRS  $\{h_t^*\}_{t>0}$  that solves (20) when (16) is not binding. To interpret this condition, first observe that investment in green technology at time  $t$  has an impact on the environment after two periods, i.e. at  $t + 2$ . The IRS over these two periods is given by  $h_{t+1}h_t$ . Absent stock effects, i.e.  $g = \theta = 0$ , (20) states that the optimal date- $t$  two-period IRS must be equal to the marginal rate of substitution (MRS) of consumption for EQ at date  $t + 2$ . The regenerative capacity of the environment ( $\theta > 0$ ) and the durability of capital ( $g > 0$ ) allow for larger IRSs, i.e. larger increases in consumption over time.

## 4 Policy implementation

The First-Best path is derived assuming that it is possible to dictate their production processes to firms. This is hardly possible (let alone desirable) in practice. In this section, I examine the problem of policy implementation with actual environmental policy instruments. More precisely, I consider two policy options that both involve the use of an emission tax to limit the productive capacity of firms, but differ in their instrument promoting green technologies: subsidies or technological standards. Lemmas 2 and 3 presents the resulting firms' behavior. In both cases, I assume that the social planner can credibly commit to implement these policies. They are thus open loop policies allowing economic agents to have consistent expectations.<sup>20</sup> Proposition 3 then specifies the two policy schemes that implement the First-Best path in expectation. Compared to the First-Best path, both policies are affected by network and spillover effects and the corresponding paths are stochastic. As this uncertainty is detrimental for the consumer, optimal policies should account for the cost of this risk (leading to second-best paths that are likely to be different in expectation). By considering

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<sup>20</sup>An open-loop policy is designed before the regulatory period and is not revised thereafter: the levels of taxes and subsidies or standards that have been chosen for each period by the regulator are implemented regardless of the actual path followed by the economy.

policies that induce the same expected path, we can assess and compare the impacts of fluctuations resulting from network and spillover effects. Such a comparison is conducted in the next section.

## Subsidy programs

Suppose that the government implements its environmental policy using only incentive instruments, namely an emissions tax scheme  $\{\tau_t\}_{t>0}$  and a green technology subsidy scheme  $\{z_t\}_{t>0}$ . Given this policy, firm  $i$ 's per period profit becomes

$$\pi_{it} \equiv Q(k_{it}, \ell_{it}) + G_{it} - w_t \ell_{it} - I_{it} - o_{it}(1 - z_t) - \tau_t (\varphi Q(k_{it}, \ell_{it}) - \xi x_{it})$$

where the last term corresponds to the environmental tax payment, and the term  $o_{it}(1 - z_t)$  to the net payment for green technology investment. It is shown in the appendix that

**Lemma 2** *Under a tax-subsidy scheme  $\{\tau_t, z_t\}_{t>0}$ , firms' investment in productive capacity at date  $t$  satisfies*

$$q'(k_{t+1}) = \frac{1 + r_t - g}{1 - \tau_{t+1}\varphi}, \quad (21)$$

and the dynamic of GTI is given by

$$\mu_{t+1} = g\mu_t + \frac{(1 - g)(\mu_t - 1) + \omega_{t+1} - r_t + \xi\tau_{t+1} + (1 + r_t)z_t - gz_{t+1}}{1 - \lambda}. \quad (22)$$

The impact of the environmental tax on productive capital appears in the denominator of (21): the higher the tax level, the lower the denominator and thus the higher the marginal productivity of  $k_{t+1}$ , i.e. the lower its level. As expected, the environmental tax also has a positive impact on the dynamic of GTI (22). The difference with the Laissez-Faire dynamic (8) appears in the numerator, given by the additional terms  $\xi\tau_{t+1} + (1 + r_t)z_t - gz_{t+1}$ . Interestingly, the subsidy policy has countervailing effects: on the one hand, the current subsidy level  $z_t$  impacts positively GTI, but that of the next period decreases it: indeed, the anticipation of a large subsidy in the next period encourages firms to postpone their green investments.

## Environmental standards

As an alternative to the subsidy policy, consider that the government decides to complement the environmental tax program with environmental standards, i.e. policy in-

struments that constrain investment choices. These standards limit the set of machines that can be used or that are allowed to be offered by machine suppliers on the market. They correspond to restrictions on the pollution intensity of the machines that are tightened over time.<sup>21</sup> Denote by  $\underline{x}_t$  the date- $t$  green index targeted for new machines by the government with a technological standard policy, a strict policy corresponding to large  $\underline{x}_t$ . Absent network effects, firms would renew their old machines (a proportion  $1 - g$  of their stocks) with new ones that just meet the standard. The corresponding dynamic of firm  $i$ 's green index would thus be given by  $x_{it} = gx_{it-1} + (1 - g)\underline{x}_t$ , and summing over all firms, GTI would evolve according to  $g\mu_{t-1} + (1 - g)\underline{x}_t$ . Taking into account networks effects that induce firms to buy the same machines, they have to anticipate an ideal technology mix given by

$$x_t^* = g\mu_{t-1} + (1 - g)\underline{x}_t + \lambda\mathcal{O}_{t-1} + \omega_t. \quad (23)$$

Compared to (3), the index of the most used machines in the last period,  $\mu_{t-1}$ , is replaced by  $g\mu_{t-1} + (1 - g)\underline{x}_t$  in (23) due to the governmental restrictions on new machines that prohibit the replacement of old machines with ones that have the same environmental impact. The remaining terms in (23) are the same as in (3): the ideal mix depends on the shock that will affect the service network,  $\omega_t$ , and on its sensitivity  $\lambda$  to total green investment  $\mathcal{O}_{t-1}$ .

**Lemma 3** *With a tax-standard policy  $\{\tau_t, \underline{x}_t\}_{t>0}$ , firms' investments satisfy (21) and*

$$o_{it} = g(\mu_t - x_{it}) + \frac{(1 - g)(\underline{x}_t - 1) + \eta_{it} - r_t + \xi\tau_{t+1}}{1 - \lambda}.$$

*The resulting firms' technology mixes at  $t + 1$  are normally distributed with mean*

$$\mu_{t+1} = g\mu_t + \frac{(1 - g)(\underline{x}_t - 1) + \omega_{t+1} - r_t + \xi\tau_{t+1}}{1 - \lambda}, \quad (24)$$

*and standard deviation  $\sigma_x$  for all  $t$ .*

Hence, the effect of the environmental tax is the same under the two policy options, given by (21). It produces the same positive effect on the GTI dynamics (24) and (22). The main difference between these dynamics is that the term  $(1 - g)(\mu_t - 1)/(1 - \lambda)$  in (22) is replaced by  $(1 - g)(\underline{x}_t - 1)/(1 - \lambda)$  in (24), which anchors the GTI level to its

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<sup>21</sup>This is the case in the European Union which imposes emission standards for vehicles that have evolved in stages of 4 to 5 years (from the Euro 1 standard in 1992 to the Euro 6 standard currently in force for light-duty vehicles).

First-Best path under the standard policy (assuming, of course, that  $\underline{x}_t$  is adequately chosen). Therefore, although they have the same first term  $g\mu_t$ , the path dependence of GTI can be expected to be reduced under the standard policy relative to the subsidy policy, i.e., the standard policy should be a more effective instrument than the subsidy policy at driving the greening of the economy. Both (22) and (24) show that whatever the implementation chosen by the government, the shock  $\omega_{t+1}$  affects the diffusion of technology, rendering the path of GTI stochastic. Under both policy regimes, this uncertainty affects EQ but also financial markets through the total demand of capital, hence the interest rate.

The next proposition specifies the policy schemes that implement the First-Best path in expectation.

**Proposition 3** *The tax-subsidy and the tax-standard schemes that implement the First-Best path in expectation satisfy*

(i)  $\tau_{t+1} \leq (h_t^* - g)/\xi$  given by

$$\tau_{t+1} = \begin{cases} (1 - (h_t^* - g)/q'(gk_t^*)) / \varphi & t < t_0 \\ (h_t^* - g)/\xi & t \geq t_0 \end{cases}, \quad (25)$$

(ii)

$$z_t = \frac{(1 - \lambda)\mu_{t+1}^* - (1 - \lambda g)\mu_t^* + gz_{t+1}}{h_t^*} + \frac{h_t^* - g - \xi\tau_{t+1}}{h_t^*} \quad (26)$$

for all  $t > 0$ , with  $\lim_{t \rightarrow \infty} z_t < 0$ , and

(iii)

$$\underline{x}_t = \frac{(1 - \lambda)(\mu_{t+1}^* - g\mu_t^*)}{1 - g} + \frac{h_t^* - g - \xi\tau_{t+1}}{1 - g} \quad (27)$$

for all  $t > 0$ .

Not surprisingly, the emissions tax (25) is the same for both policy implementations. In the case the constraint  $k_{t+1} \geq gk_t$  is not binding, it equates the rental price of capital,  $h_t^* - g = 1 + r_t - g$ , with the marginal savings from investing in green technology,  $\xi\tau_{t+1}$ . When the stock of productive capital is large, i.e. when the constraint  $k_{t+1} \geq gk_t$  is binding, that tax level is reduced so that firms use all the productive capital that is not depreciated. This reduced tax level is compensated by a large subsidy level as shown by the last term of (26) that is positive when  $t < t_0$  and null when  $t \geq t_0$ . Similarly, in the standard policy case, the last term of (27) is positive when  $t < t_0$  and null when  $t \geq t_0$ ,

implying standards that are stringent (i.e.  $\bar{x}_t$  large) when the tax level is reduced. Proposition 3 also shows that the subsidy schedule entails negative values passed a certain date, i.e. it becomes a tax scheme. This is due to the herd behavior of firms generated by the network effects: firms over-invest in green technology compared to the First-Best path once GTI is high enough. As noted earlier, the expected dynamic of GTI under Laissez-Faire is positive if  $\mu_t > [1 + r_t/(1 - g)]/\lambda$ . With environmental taxes as given by (25) and absent a complementary subsidy or standard policy, this condition becomes  $\mu_t > [h_t^* - (1 + r_t)]/[(1 - g)\lambda]$ , i.e.  $\mu_t > 0$  since  $\mathbb{E}[r_t] = h_t^* - 1$ . Hence, firms' investment in green technology could be greater than  $g\mu_t$ , the level necessary to renew the green capital that is depreciated, without further governmental intervention than the environmental tax. However, the resulting investment could still be lower than  $\mu_{t+1}^*$ , the First-Best next period level. More generally, it is different from it, and a policy complementing the environmental tax is necessary to guide firms along the First-Best path. As GTI increases, firms are very likely to over-invest after a while, and it is then optimal to tax green investment.<sup>22</sup>

## 5 Illustrative example

To assess the extent of the fluctuations affecting the economy, I consider in this section a framework that allows for explicit solutions of the First-Best policy and its implementations. First, Lemma 4 shows that when MRS is constant, the First-Best dynamic of the economy entails at most 3 sequences. The first corresponds to a decrease in the stock of productive capital, and thus in GDP. This sequence is likely to occur if the footprint of the technologies originally employed is too heavy on the environment. It corresponds to a swift catching up of investments in green technologies at the expense of the productive capacity of firms. The second sequence corresponds to a stagnation of GDP, with the productive capacity of the industrial sector just maintained, while investment in clean technologies is still growing fast to allow EQ to increase. Finally, the third sequence corresponds to a growing GDP along an ENP.

The First-Best path is then fully specified Proposition 4 assuming a CARA utility function and a Cobb-Douglas production function. The two policy implementations are

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<sup>22</sup>Because the First-Best policy imposes restrictions in the long term, the public intervention is long lasting and results in taxing both polluting emissions and green investments. The government may consider alternatively a policy without public intervention in the long run, perhaps as soon as an ENP is reached. The resulting over-investment in green technology allows for larger productive investments than optimal levels, and thus an higher GDP level in the long run. However, because consumption of manufactured goods is reduced in the meantime, it is detrimental for consumers.

also examined under these assumptions and assuming rational expectations on the part of consumers. Proposition 5 shows that during the transition period to environmental neutrality, the distribution of the interest rate can be approximated by a first-order autoregressive process and the path of the economy by Gaussian random walks under these policies. They are specified and compared with simulations.

## Constant MRS and the First-Best policy

To characterize the First-Best policy, assume that the consumption of manufactured and environmental goods and services can be subsumed in a ‘global wealth index’  $y_t \equiv c_t + pe_t$  where  $p$  is the constant value of the environment, so that the consumer’s MRS is the same whatever the GDP, equal to  $p$ .<sup>23</sup> With a constant MRS, (20) simplifies to

$$h_{t+1}^* h_t^* - h_{t+1}^* (g + \theta) + \theta g = p\xi, \quad (28)$$

which must hold for all  $t$  as long as (16) is not binding. Without stock effects, i.e.  $g = \theta = 0$ , this equation reduces to  $h_{t+1}^* h_t^* = p\xi$  for all  $t$ , and thus  $h_t^* = \sqrt{p\xi} \equiv \bar{h}$ , a constant (as  $h_t$  must be positive, and ruling out cyclical solutions that are suboptimal since  $u$  is concave). If  $\beta > 1/\sqrt{p\xi}$ , i.e. if consumers are not too impatient, the First-Best policy corresponds to an increasing path of the global wealth index  $y_t$ . More generally, a constant IRS that solves (28) is a root of  $P(\bar{h}) = 0$  where

$$P(h) \equiv p\xi - (h - g)(h - \theta). \quad (29)$$

To avoid that the solution of (28) corresponds to a degenerate situation, we make the following assumptions:

**H1:**  $\xi/\varphi > 1/\beta - (3g - \theta)/2$ .

**H2:**  $(1/\beta - \theta)(1/\beta - g) < p\xi < (\xi/\varphi)^2 + (g - \theta)\xi/\varphi$ .

Assumption H1 defines a minimum efficiency threshold for green technologies. It is a necessary condition for H2 to be possible.<sup>24</sup> The first condition in H2 allows for an increasing First-Best path of the global wealth index. Compared to the the case  $g = \theta = 0$  discussed above, stock effects ( $g$  and  $\theta$  positive) relax the condition on the consumer’s patience. The second condition in H2 allows for a positive consumption

<sup>23</sup>The following results easily generalize to the case where  $p$  takes several discrete values depending on the relative levels of consumption and EQ.

<sup>24</sup>See the proof of Lemma 4 for more details.

(and production) of industrial goods along the First-Best path. This last condition is deduced from (15) that is indefinitely binding otherwise: if  $p$  is too large, EQ would completely supplant consumption: the stock of capital would always decrease over time, i.e.  $k_{t+1} = gk_t$  for all  $t$ .

The following lemma gives the principal properties of the First-Best path when the MRS is constant.

**Lemma 4** *As long as the ENP constraint (16) is not binding, the First-Best path is characterized by:*

- (i) *A constant IRS :  $h_t^* = \bar{h} = (g + \theta + \sqrt{(g - \theta)^2 + 4p\xi})/2 > 1/\beta$ , corresponding to a level of productive capital given by*

$$\bar{k} \equiv q'^{-1} \left( \frac{\bar{h} - g}{1 - (\bar{h} - g)\varphi/\xi} \right) > 0. \quad (30)$$

- (ii) *If  $gk_0 > \bar{k}$ ,  $k_{t+1}^* = gk_t^*$  for all  $t < t_0 \equiv \ln(\bar{k}/k_0)/\ln g$ .*

- (iii)  *$k_t^* = \bar{k}$  for all  $t_0 \leq t \leq T + 1$  where  $T$  is given by  $\mu_T^* = q(\bar{k})\varphi/\xi$ .*

Point (i) states that, as long as environmental neutrality is not reached, the First-Best path of the economy is characterized by a constant IRS  $\bar{h}$  that corresponds to the largest root of (29). Under assumptions H1 and H2,  $\bar{h} > 1/\beta$ , so that the global wealth index  $y_t$  follows an increasing path, while the productive capital stays at level  $\bar{k}$  after an eventual adjustment period. More precisely, Lemma 4 indicates that the economy may go through up to 3 sequences depending on the parameter values (and at least two if  $\xi\mu_0 \ll \varphi q_0$ ). The first sequence is spelled out in Point (ii): if the initial stock of capital  $k_0$  is large, the depreciated productive capital is not replaced during several periods (from  $t = 1$  to  $t = t_0$ ), until it reaches  $\bar{k}$ . This is the case when  $p$ , the social value of the environment, is large (since  $\bar{h}$  increases with  $p$  and  $\bar{k}$  given by (30) is a decreasing function of  $\bar{h}$ ). Hence, while the IRS is constant, the stock of capital may have to adapt over several periods before reaching  $\bar{k}$ . The GDP thus progressively decreases during this sequence. Point (iii) states that the second sequence,  $t_0 \leq t \leq T + 1$ , corresponds to a stagnant GDP, equal to  $q(\bar{k})$ , where the investment in productive capital allows firms to maintain the stock to  $\bar{k}$ , while the investment in clean technology allows the economy to increase GTI and EQ. Finally, the third sequence,  $t > T + 1$ , corresponds to an increasing GDP along an ENP: the increase in GDP is proportional to the increase



in clean technology (the factor of proportionality being equal to  $\xi/\varphi$ ).<sup>25</sup>

In the following proposition, the First-Best dynamic of the economy is derived assuming a Cobb-Douglas production function  $q_t = Ak_t^\alpha$  and CARA (exponential) consumer's utility, i.e.  $u(c_t, e_t) = -e^{-\gamma(c_t + pe_t)}$ .

**Proposition 4** *With constant MRS, exponential utility function and a Cobb-Douglas production function,*

(i) *the First-Best sequence of the EQ index  $\{e_t^*\}_{t>1}$  is given by*

$$e_t^* = e_1 + \left( \frac{g^\alpha - g^{\alpha t}}{1 - g^\alpha} - \left( \frac{g^\alpha}{\bar{h}} \right)^{t_0-1} \frac{\bar{h}^t - \bar{h}}{\bar{h} - 1} \right) \nu_0 + (t-1)\nu_1 - \frac{(1-\theta)(1-g)\nu_1}{\xi p \bar{h}^T} \frac{\bar{h}^t - \bar{h}}{\bar{h} - 1}$$

for all  $1 < t < t_0$ ,

$$e_t^* = e_{t_0-1}^* + (t - t_0 + 1)\nu_1 - \frac{(1-\theta)(1-g)\nu_1}{\xi p \bar{h}^T} \frac{\bar{h}^t - \bar{h}^{t_0-1}}{\bar{h} - 1} \quad (31)$$

for all  $t_0 \leq t \leq T$ , and

$$e_t^* = e_N - \theta^{t-T}(e_N - e_T^*)$$

for all  $t > T$ , where  $\nu_1 > 0$ ,  $\nu_0 > 0$  if  $\xi/\varphi \geq 1 - g$ , and where  $T$  is deduced from  $e_T^* = e_N - \ln(\beta \bar{h})^{1/(1-\theta)p\gamma}$ .

(ii) *EQ increases and consumption decreases at decreasing rates for all  $t \in \{1, \dots, T\}$ .*

As expected, the path followed by EQ is composed of three sequences, the first of which being affected by the constraint on productive capital ( $k_{t+1} \geq k_t$ ) relative to the second, while the third corresponds to the ENP. Because the IRS is constant as long as environmental neutrality is not reached, the increase in global wealth is constant: we have  $\bar{h} = e^{\gamma(y_{t+1} - y_t)}/\beta$  and thus  $c_{t+1} - c_t + p(e_{t+1} - e_t)$  constant. However, while EQ is increasing, consumption decreases. This is due to the investment in green technology since the investment in productive capital is either null or constant during this period. Hence, during the first two sequences, when the GDP decreases ( $1 \leq t < t_0$ ) and when it is stabilized at level  $q(\bar{k})$  ( $t_0 \leq t \leq T$ ), consumption of industrial goods decreases.

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<sup>25</sup>From this result, we can infer that if  $p$  decreases with  $e_t/c_t$  by discrete steps as mentioned footnote 23, the optimal solution entails decreasing optimal IRS  $\bar{h}_i$ , and thus increasing levels of productive capital  $\bar{k}_i$  and GDP levels when the economy as not yet reached an ENP. In term of industrial production, such a policy is thus very restrictive at first, and then progressively more permissive as GTI and EQ improve.

Table 1: Calibration parameters and equilibrium values

$q_0$	Pop	$r_0$	$g$	$\alpha$	$E_0$	$\iota_0$	$\xi$	$\varphi$	$\mu_0$	$A$
85.91	7.7	0.06	$4/5$	$1/3$	37.1	0.432	0.84	0.84	41.74	17.92
CO <sub>20</sub>	CO <sub>2N</sub>	TP	$\theta$	$\hat{e}$	$e_0$	$e_N$	$p$	$\gamma$	$\psi$	
407.4	280	450	0.985	19.81	331	1,320.9	$15/1000$	0.2	0.015	
$k_0$	$\bar{h}$	$\bar{k}$	$\bar{q}$	$\bar{\tau}$	$t_0$	$T$	$k_\infty$	$q_\infty$	$\lambda$	$\sigma_\omega$
110.14	1.038	83.69	78.39	0.283	2	60	101.77	83.68	0.005	$1/2$

Date  $t = 0$  is 2018. GDP and capital stocks ( $q_t$ ,  $k_t$ ,  $\mu_t$ ) are expressed in US\$ trillions, world population in billions, CO<sub>2</sub> concentrations (Tipping Point TP, CO<sub>20</sub>, CO<sub>2N</sub>) in ppm, 2018 emissions  $E_0$  and carbon budgets ( $e_0$ ,  $e_N$ ,  $\hat{e}$ ) in Gt of CO<sub>2</sub> (see footnote 26 for unit definitions). Emission intensities  $\iota_0$  and  $\varphi$  in kg of CO<sub>2</sub> for US\$1.  $\varphi$  corresponds to the emission intensity for 1960 (as estimated by the World Bank).

These results are illustrated Fig. 1 using the calibration parameters and the resulting equilibrium values presented Table 1. The reference year  $t = 0$  is 2018. The corresponding world GDP and population are  $q_0$ =US\$85.91 trillions and Pop=7.7 billions.  $p$  is set at US\$15/tCO<sub>2</sub>, the psychological discount rate  $\psi = 1.5\%$ , hence  $\beta \approx .985$ , and the intertemporal elasticity parameter  $\gamma = .2$ . Computations are made using the per-capita global wealth level ( $y_t$ /Pop). The initial capital stock is derived from the interest rate by  $k_0 = \alpha q_0 / (1 + r_0 - g)$  where  $r_0 = 6\%$ .  $e_t$  is defined as a global “carbon budget” at date  $t$  (expressed in Gt of CO<sub>2</sub> in the atmosphere), i.e. the difference between a tipping point (TP) and the level of GHG at date  $t$  expressed in CO<sub>2</sub> equivalent.<sup>26</sup> TP is set at 450 ppm (3496.5 Gt CO<sub>2</sub>), and the initial concentration level CO<sub>20</sub> is 407.4 ppm (3165.5 Gt CO<sub>2</sub>), hence an initial EQ index  $e_0 = 331$  Gt CO<sub>2</sub>.<sup>27</sup> Accordingly, given a pre-industrial level CO<sub>2N</sub>=280 ppm (2176 Gt CO<sub>2</sub>), the pre-industrial budget is  $e_N = 1321$  Gt CO<sub>2</sub>.  $\theta$  is set at .985, leading to  $\hat{e} = (1 - \theta)e_N = 19.81$  Gt CO<sub>2</sub>. The initial emission intensity  $\iota_0$  corresponds to the ratio CO<sub>20</sub>/ $q_0 = 431.5$  g CO<sub>2</sub>/US\$, leading to an initial GTI index  $\mu_0 = q_0(\phi - \iota_0)/\xi$ =US\$41.74 trillions. Finally, given  $q_0$  and  $\mu_0$ , it comes  $e_1 = \theta e_0 + \hat{e} - \varphi q_0 + \xi \mu_0 = 310$  Gt CO<sub>2</sub>. Given these parameters, the de-growth sequence takes two periods (from  $t = 1$  to  $t = t_0 = 2$ ), and economic stagnation lasts until  $t = T = 60$ . During this second sequence, IRS is  $\bar{h} = 1.038$ .

Panel 1a shows the dynamic of EQ that increases rapidly until  $t = T$ , when the ENP is reached. Then EQ increases at a lower rate toward  $e_N$ . Panel 1b shows the

<sup>26</sup>Units are either gigaton (Gt shorthand), i.e. 10<sup>9</sup> (billions) metric tons, or part per million (ppm shorthand), which refers to the atmospheric concentration. Each ppm represents approximately 2.13 Gt of carbon in the atmosphere as a whole, equivalent to 7.77 Gt of CO<sub>2</sub>.

<sup>27</sup>According to the IPCC Fifth Assessment Report, 450 ppm lead to a temperature increase of approximately 2°C.

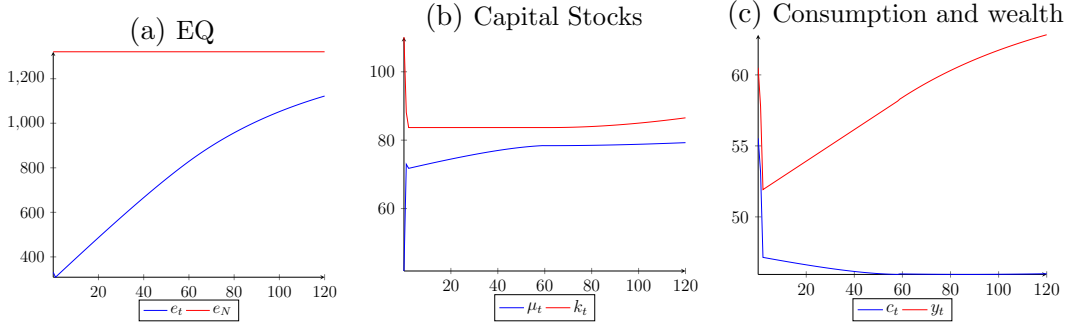


Figure 1: Optimal dynamic.

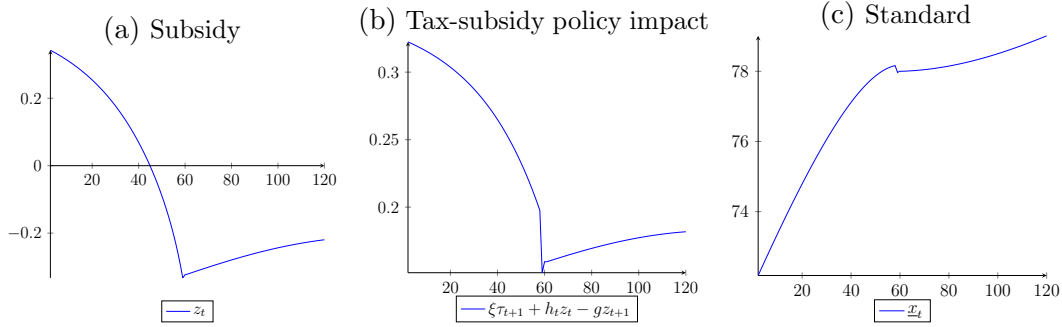


Figure 2: Tax and subsidy schemes ( $\lambda = 0.01$ ).

sharp decrease in the productive capital and the sharp increase in GTI from period 0 to period 1 (from  $k_0 = 110.14$  to  $\bar{k} = 83.69$  and  $\mu_0 = 41.74$  to  $\mu_1 = 71.6$  respectively). Then, the productive capital stays at  $\bar{k}$  until  $t = T$ , while GTI increases. Both capital stocks increase afterward, at the slow rate permitted by the ENP constraint (the long term level of capital, which determines  $\mu_\infty = q_\infty = 83.68$ , is  $k_\infty = 101.77$ ). Panel 1c shows the decrease in consumption, very sharp at first (from  $c_0 = 55.4$  to  $c_3 = 47.14$ ), then slower until period  $T$ . It increases afterward, at a very slow pace. Total wealth also decreases the first two periods, but increases sharply afterward. This simulations also show the limit of the assumption of a constant MRS: the decreases in GDP and consumption are very sharp due to the perfect substitutability between industrial and environmental goods.<sup>28</sup>

## Policy implementation

The policy schemes (25)–(27) that implement the First-Best path in expectation are illustrated Fig. 2 assuming a spillover coefficient  $\lambda = .005$ . For both policy, the emissions tax is constant at  $\bar{\tau} = .283$  (i.e. 283 US\$/t CO<sub>2</sub>) over the period  $t_0 < t < T$  (it then slowly decreases). The subsidy scheme (panel 2a) is around 34% at  $t_0 = 2$  and decreases to become negative at  $t = 45$ . As shown panel 2b, the resulting net impact of the tax-subsidy policy on GTI, i.e. the term  $\xi\tau_{t+1} + h_t z_t - g z_{t+1}$  in (22), is decreasing but positive over this period (it stays positive and increases slightly thereafter). Simulations show that the subsidy scheme is extremely sensitive to  $\lambda$  (the schedule is positive for only a decade when  $\lambda = .01$ , and entire negative when  $\lambda \geq .02$ ). The standard policy depicted panel 2c increases rapidly until environmental neutrality is reached, then slowly along the ENP.

Assuming rational expectations, it is possible to be more specific about the dynamic of the economy for  $t \in \{t_0, \dots, T\}$  under these policy implementations. Indeed, over this period the interest rate satisfies  $\tilde{r}_t = \bar{r}(1 + \tilde{\varepsilon}_t)$  where  $\bar{r} = \bar{h} - 1$  is the expected interest rate and  $\tilde{\varepsilon}_t$  is a zero-mean random shock whose distribution depends on the present and past realizations of  $\tilde{\omega}_t$ . The productive capacity and total production  $\tilde{k}_t$  and  $\tilde{q}_t$  are also randomly distributed around their stationary values  $\bar{k}$  and  $q(\bar{k})$ . Using linear approximations, the path of global wealth index  $y_t$  can be approximated by a Gaussian random walk at the rational expectations equilibrium (REE), and the supply function of capital (10) by<sup>29</sup>

$$r_t = \psi + \gamma(\mathbb{E}_t[\tilde{y}_{t+1}] - y_t) - \gamma^2 \mathbb{V}_t[\tilde{y}_{t+1}]/2 \quad (32)$$

where  $\psi$  is the intrinsic discount factor. Expression (32) exhibits the familiar effects that determine the rental price of capital: the intrinsic preference for an immediate consumption  $\psi$ , the economic trend of the global wealth index that also encourages immediate consumption if it is positive, and a precautionary effect that operates in the opposite direction and corresponds to a risk premium due to the uncertainty affecting the economy.<sup>30</sup>

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<sup>28</sup>The assumption of a constant MRS simplifies the dynamics of the transition period which can be derived explicitly. As indicated footnote 25, we may expect by relaxing this assumption that the productive capital increases during the transition period to an ENP.

<sup>29</sup>This expression is derived using  $\psi = -\ln \beta$ ,  $1 + r_t \approx e^{r_t}$ , and  $\mathbb{E}[e^{-\gamma \tilde{y}}] = e^{-\gamma(\mathbb{E}[\tilde{y}] - \gamma \mathbb{V}[\tilde{y}]/2)}$  when  $\tilde{y}$  is normally distributed.

<sup>30</sup>This simple expression is due to the CARA preferences,  $\gamma$  being the coefficient of absolute risk aversion.

**Proposition 5** *Assuming a REE with constant MRS, a CARA utility function and a Cobb-Douglas production function, the interest rate under either the tax-subsidy ( $\ell = \text{sub}$ ) or the tax-standard ( $\ell = \text{std}$ ) policy is approximated by  $\tilde{r}_t^\ell = \bar{r}(1 + \tilde{\varepsilon}_t^\ell)$  for  $t \in \{t_0, \dots, T\}$ , where*

$$\begin{aligned}\tilde{\varepsilon}_t^{\text{sub}} &= \chi^{\text{sub}}(z_t)\tilde{\omega}_{t+1} + \rho^{\text{sub}}(z_t)\varepsilon_{t-1}^{\text{sub}}, \\ \tilde{\varepsilon}_t^{\text{std}} &= \chi^{\text{std}}\tilde{\omega}_{t+1} + \rho^{\text{std}}\varepsilon_{t-1}^{\text{std}},\end{aligned}\tag{33}$$

are normally distributed. The paths of  $\tilde{e}_t^\ell$ ,  $\tilde{\mu}_t^\ell$  and  $\tilde{y}_t^\ell$  under policy  $\ell \in \{\text{sub}, \text{std}\}$ , can be approximated by Gaussian random walks deduced from (14), (11) and

$$\mu_{t+1}^{\text{sub}} = a_1^{\text{sub}}\mu_t^{\text{sub}} + a_2^{\text{sub}}e_t^{\text{sub}} + a_3^{\text{sub}}(z_t) + Z_t^{\text{sub}} + b_1^{\text{sub}}(z_t)\varepsilon_t^{\text{sub}} + b_2^{\text{sub}}\varepsilon_{t-1}^{\text{sub}}\tag{34}$$

with

$$Z_t^{\text{sub}} = a_0^{\text{sub}} \sum_{i=0}^{+\infty} (a_0^{\text{sub}}\gamma)^i (\xi\tau_{t+1+i} + h_{t+i}z_{t+i} - gz_{t+1+i}),\tag{35}$$

and

$$\mu_{t+1}^{\text{std}} = a_1^{\text{std}}\mu_t^{\text{std}} + a_2^{\text{std}}e_t^{\text{std}} + a_3^{\text{std}} + Z_t^{\text{std}} + b_1^{\text{std}}\varepsilon_t^{\text{std}} + b_2^{\text{std}}\varepsilon_{t-1}^{\text{std}}\tag{36}$$

with

$$Z_t^{\text{std}} = a_0^{\text{std}} \sum_{i=0}^{+\infty} (a_0^{\text{std}}\gamma)^i (\xi\tau_{t+1+i} + (1-g)\underline{x}_{t+i}).\tag{37}$$

The network externality shock  $\tilde{\omega}_t$  generates the stochastic distributions (33) that depend on the policy implemented. These distributions of the shocks affecting the interest rate follow first-order autoregressive processes, with parameters that are constant in the tax-standard policy case while they are functions of the subsidy level  $z_t$  in the tax-subsidy case. In both cases, the uncorrelated shocks affecting the network externalities generate path-dependent fluctuations. This business cycle is not due to changes in the environmental policy: the regulator is supposed to commit to a policy that is perfectly anticipated by the agents. Rather, it is due to the consumer's desire to smooth her consumption over time. This auto-correlation can be explained as follows: At the beginning of the transition period, assuming that the GDP and the interest rate are initially equal to their nominal values for that period, a positive shock increases the demand for green technology, all the more so if the supporting infrastructure is responsive ( $\lambda$  is large). This leads to an increase in the interest rate, which in turn decreases investment in productive capital. During the transition period, the environmental tax is such that this investment just allows firms to maintain their productivity.

Table 2: REE coefficients

$a_0^\ell$	$a_1^\ell$	$a_2^\ell$	$\bar{a}_3^\ell$	$\bar{b}_1^\ell$	$b_2^\ell$	$\bar{\rho}^\ell$	$\bar{\chi}^\ell$	$\bar{\sigma}_{y_{t+1} t}^\ell$
0.866	0.998	4.696 <sub>-5</sub>	-8.045 <sub>-2</sub>	28.849	-27.878	0.965	0.035	1.536 <sub>-1</sub>
0.836	0.798	4.507 <sub>-5</sub>	-7.757 <sub>-2</sub>	23.081	-22.293	0.964	0.043	6.651 <sub>-2</sub>

Values and average values of the coefficients of (33)–(37) under the tax-subsidy ( $\ell = sub$ , first row) and the tax-standard ( $\ell = std$ , second row) policies, with  $\bar{a}_3^{sub} \equiv \sum_{t=t_0}^T a_3^{sub}(z_t)/(T-t_0+1)$ ,  $\bar{a}_3^{std} \equiv a_3^{std}$  and similarly for  $\bar{b}_1^\ell$ ,  $\bar{\rho}^\ell$ ,  $\bar{\chi}^\ell$  and  $\bar{\sigma}_{y_{t+1}|t}^\ell$ . The last column corresponds to the average value of the one-period-ahead standard deviation of the aggregate wealth. Subscripts correspond to exponents (i.e.  $2.3_{-2} = 2.3 \cdot 10^{-2}$ ).

The increase in the interest rate thus reduces the productive capital stock to a level below its nominal level. As the decline in investment reduces the next period GDP, the willingness of consumers to lend capital is reduced, which tends to increase the next period interest rate even more in the event of a new positive shock. The opposite trend occurs if the initial shock is negative.

The equilibrium dynamic of GTI given by (34) and (36) are linear first-order recursive equations with constant parameter in the latter, and some parameters ( $a_3^{sub}$  and  $b_1^{sub}$ ) function of the subsidy level  $z_t$  in the former. Both include forward looking term  $Z_t^\ell$  given by either (35) or (37) depending on the policy  $\ell \in \{sub, std\}$ . These policy indexes are exponential smoothing of future tax and subsidy levels or future tax and standard levels respectively.<sup>31</sup>

The simulations of policy implementations show that  $\rho_1^{sub}(z_t)$  and  $\chi^{sub}(z_t)$  are slightly increasing while  $a_3^{sub}(z_t)$ ,  $b_1^{sub}(z_t)$  and  $\sigma_{y_{t+1}|t}^{sub}$  are slightly decreasing. They all stay very close to their average values reported Table 2. Coefficients  $a_0^\ell$  determine the weights associated to future policy levels in (35) or (37). We have  $a_1^{sub} > a_1^{std}$  which reflects an higher path dependency under the tax-subsidy policy than under the tax-standard policy. The importance of shocks is reflected by  $\bar{b}_1^\ell$  and  $b_2^\ell$  that have large absolute values of opposite signs, allowing the consumer to adjust her anticipations of GTI to the business cycle fluctuations. As a result, the volatility of aggregate wealth  $\sigma_{y_{t+1}|t}^\ell$  from period to period is small, but larger under the tax-subsidy policy than under the tax-standard policy (more than twice as much on average), reflecting the greater latitude given to producers in their green investment choices in the former case compared to the latter. The autocorrelation coefficient of  $\tilde{\varepsilon}_t^\ell$  is slightly larger under the tax-subsidy policy than under the tax-standard policy, i.e.  $\bar{\rho}^{sub} > \rho^{std}$ , while it is the

<sup>31</sup>Under the assumption of open-loop policies, the paths (35) and (37) of  $Z_t^\ell$  and the values of the parameters that are functions of  $z_t$  in (33) and (36) are deterministic.

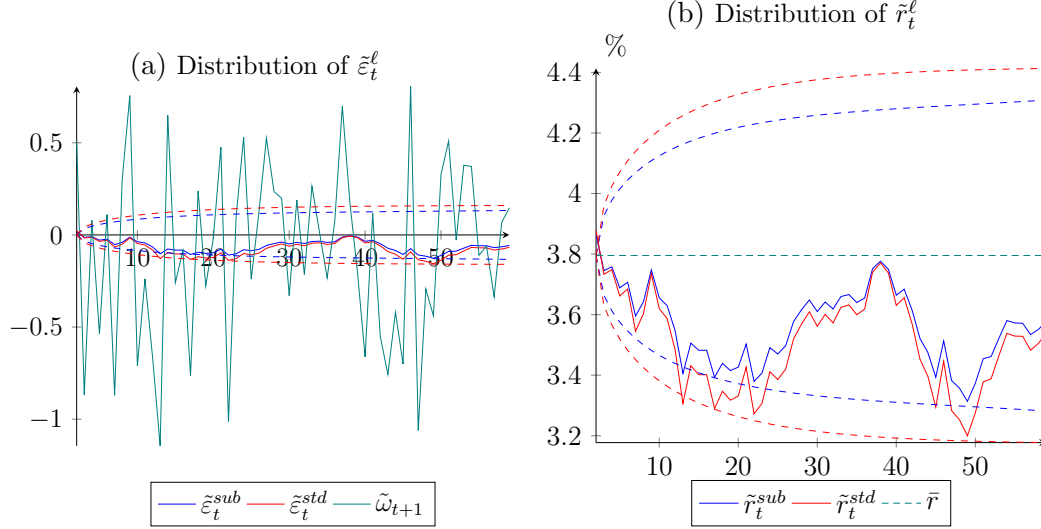


Figure 3: Shocks and the interest rate (95% CIs are delineated by the dashed black lines).

reverse for the impact of the immediate network shock: we have  $\chi^{std} > \bar{\chi}^{sub}$ .

The resulting distributions of  $\tilde{\varepsilon}_t$  and  $\tilde{r}_t$  are illustrated Fig. 3. In panel 3a, the shocks  $\tilde{\omega}_t$  are randomly distributed around 0, while  $\tilde{\varepsilon}_t^\ell$  fluctuates somehow smoothly and stay below  $\varepsilon_0^\ell = 0$  under both policies. The amplitude of its path under the tax-standard policy is often larger than under the tax-subsidy policy due to  $\chi^{std} > \bar{\chi}^{sub}$ . The large autocorrelation coefficient  $\rho^\ell$  compared to the low impact of the innovation  $\chi^\ell$  explain this paths dependency of  $\tilde{\varepsilon}_t^\ell$  under both policies. As a result, the interest rate departs from its nominal value  $\bar{r} = \bar{h} - 1 = 3.8\%$ , following the same variations as  $\varepsilon_t$  (panel 3b). On these graphs, the dashed curves above and below the horizontal lines located at  $\varepsilon_{t_0} = 0$  and  $\bar{r}$  delineate the 95% confidence interval (CI) deduced from the Gaussian distributions  $\mathcal{N}(0, \mathbb{V}[\tilde{\varepsilon}_t^\ell])$  (panel 3a) and  $\mathcal{N}(\bar{r}, \bar{r}^2 \mathbb{V}[\tilde{\varepsilon}_t^\ell])$  (panel 3b). Because  $\chi^{std} > \bar{\chi}^{sub}$ , the CI is larger at first under the tax-standard policy than under the tax-subsidy policy. This effect is progressively attenuated due to the difference in the autocorrelation of the shocks affecting the interest rate under the two policy (the one of the subsidy policy is larger on average and increasing).

The resulting impact on GTI and EQ is illustrated Fig. 4. Panel 4a shows large fluctuations of  $\mu_t^\ell$  due to the network effects. While  $\tilde{\mu}_t^{std}$  fluctuates along the First-Best path  $\mu^*$ , the fluctuations of  $\tilde{\mu}_t^{sub}$  do not show this positive trend. The consequence of these larger fluctuations on EQ is apparent Panel 4b: the stochastic path  $\tilde{e}_t^{std}$  is much closer to the First-Best path  $e_t^*$  than  $\tilde{e}_t^{sub}$ .

It is also possible to derive approximations of the CIs for the paths of  $\tilde{\mu}_t^\ell$ ,  $\tilde{e}_t^\ell$  and  $\tilde{q}_t^\ell$

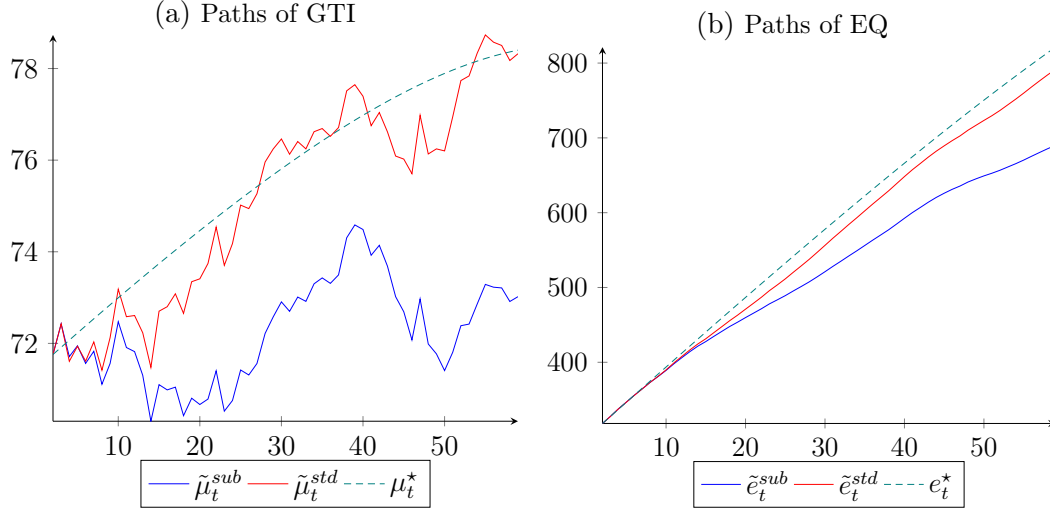


Figure 4: Paths of GTI and EQ under the First-Best policy and the two policy implementations.

over  $\{t_0, \dots, T\}$  using (14) and (34). These equations give the recursive expression

$$\tilde{Y}_t^\ell = B_t^\ell Y_{t-1}^\ell + H_t^\ell \tilde{\nu}_t \quad (38)$$

where  $\tilde{Y}_t^\ell = (\tilde{\mu}_t^\ell, \tilde{e}_t^\ell, \tilde{q}_t^\ell, 1, \tilde{\varepsilon}_t^\ell, \tilde{\varepsilon}_{t-1}^\ell)'$  is the column vector of state values (with the constant) and of the interest rate shock of date  $t$  and  $t - 1$ ,

$$B_t^\ell = \begin{bmatrix} a_1^\ell & a_2^\ell & 0 & a_{3t}^\ell + Z_t^\ell & b_{1t}^\ell & b_2^\ell \\ \xi & \theta & -\varphi & \hat{e} & 0 & 0 \\ 0 & 0 & 0 & \bar{q} & q'(\bar{k})/q''(\bar{k}) & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_t^\ell & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad H_t^\ell = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \chi_t^\ell \sigma_\omega \\ 0 \end{bmatrix},$$

and  $\tilde{\nu}_t$  is an independent standardized Gaussian variable.  $\tilde{Y}_t^\ell$  follows a Gaussian random walk with  $\mathbb{E}[\tilde{Y}_t^\ell] = (\prod_{i=0}^{t-t_0} B_i^\ell) Y_{t_0}$  and  $\mathbb{V}[\tilde{Y}_t^\ell] = \sum_{i=0}^{t-t_0} (\prod_{j=0}^i B_j^\ell) H_i^\ell H_i^{\prime} (\prod_{j=0}^i B_j^\ell)'$ . The transition matrix  $B_t^\ell$  and the column vector  $H_t^\ell$  are time-dependent because of the policy index  $Z_t^\ell$  that changes over time according to (35) or (37), and also because coefficients  $a_{3t}^{sub}$ ,  $b_{1t}^{sub}$ ,  $\rho_t^{sub}$  and  $\chi_t^{sub}$  are functions of the subsidy schedule  $z_t$ .

Fig. 5 depicts the results derived from (38) and shows a striking difference in the magnitude of the confidence intervals (panels 5a and 5b): The CIs for the tax-subsidy policy delineated by the dashed blue curves are much larger than those for the tax-



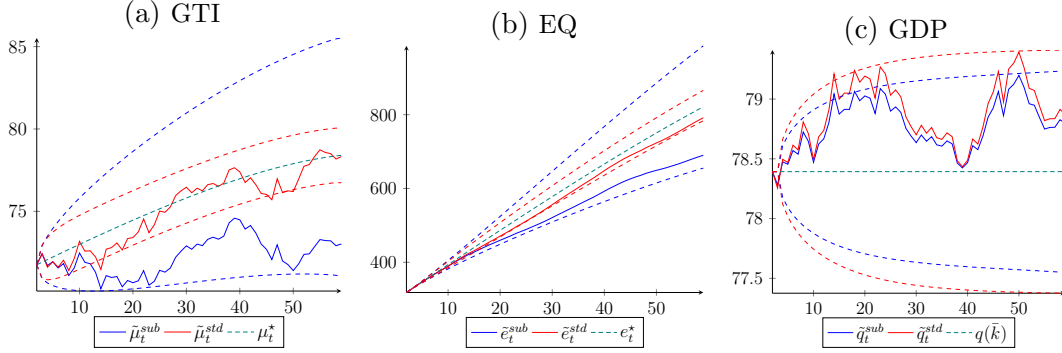


Figure 5: GTI, EQ and GDP under the two policies (95% CIs are delineated by the dashed blue (red) lines under the tax-subsidy (tax-standard) implementation).

standard policy (dashed red curves). This is not the case for the GDP as shown in panel 5c, which also present realized GDP paths under the two implementations that are (relatively) close. This is because investment in productive capacity is not directly affected by network effects, but only through the variations of the interest rate. The amplification effects of network shocks that affects EQ (which depends on  $\mu_t$ ) are absent for GDP (which depends on  $k_t$ ). The differences in GDP are the mirror image of the differences in interest rate levels shown Fig 3 (panel 3b).

## 6 Conclusion

This paper analyzes the effect of coordination problems on the adoption of green technologies. Governments design policies to drive their economies toward environmental neutrality, but investment choices are ultimately made by private agents who respond to policies according to their own expectations about the future of the economy. The effectiveness of green technologies is partly the result of their choices, through complex network and spillover effects. As a result, regulatory instruments can only imperfectly guide economies on their paths to environmental neutrality. By modeling the coordination problem of firms as a succession of global games, the trajectories of the economy that are derived in this paper are not subject to arbitrary assumptions about firms' self-fulfilling beliefs. Instead, it is assumed that agents are aware that the development and diffusion of technologies are subject to shocks that they imperfectly anticipate. While the resulting equilibrium path does not depend on firms' beliefs, it is indirectly affected by these shocks, which are subject to network and spillover effects and to the equilibrium conditions in real and financial markets. Although highly stylized, the model presented in this article shows that these shocks cause economic fluctuations that

can move the economy away from the First-Best path, or any trajectory envisioned by the authorities.

Technology standards permit a transition to green technologies more in line with the regulator's intended trajectory than subsidies do. This is because producers have less flexibility to adapt their green investments to economic conditions when standards rather than subsidies are implemented. The latter instrument allows firms to better react to changes on financial markets, for better or for worse in terms of the resulting environmental quality.

The First-Best path of the economy is derived assuming that governments know the technologies that make such a path feasible, which is unrealistic. Environmental policies are based on scenarios that depend heavily on assumptions about how technologies will evolve –both in terms of performance and cost– in the distant future. As noted by the IEA (2021), its model's forecasts based on already existing technologies have a time horizon of about a decade. Forecasts beyond 2030 (and up to 2070) rely on assumptions about the evolution of green technology. These assumptions are based on the opinion of experts who cannot have complete knowledge of all the technologies that will be deployed, but only indications of those that are under development, at the research stage or in demonstration projects. The likely evolution of these technologies, in terms of cost and performance, is subject to many uncertainties. Determining the appropriate policy is therefore a very difficult task. Technology standards, by imposing minimum environmental quality characteristics that machines must meet, seem to be the most appropriate instrument for achieving binding objectives such as those imposed by international environmental agreements. And indeed, emission standards policies such as those implemented by the European Union for vehicles, which oblige car manufacturers to produce low-emission models, seem to be more effective in guiding society towards environmental neutrality than have been the economic incentives given to car buyers to date.

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# Appendix

## A Proof of Proposition 1

As firm  $i$ 's profit is separable in revenues from productive capacity and network effects from its green index, the firm's investment problem can be split into two independent programs. Applying the principle of optimality, the investment strategy in productive capacity can be derived by using the Bellman equation

$$\mathcal{W}(k_{it}) = \max_{I_{it}, \ell_{it}} Q(k_{it}, \ell_{it}) - w\ell_{it} - I_{it} + \delta_t \mathcal{W}(gk_{it} + I_{it}). \quad (39)$$

Maximizing (39) with respect to  $\ell_{it}$  gives

$$\partial Q(k_{it}, \ell_{it}) / \partial \ell = w_t,$$

while the first-order condition with respect to  $I_{it}$  leads to

$$-1 + \delta_t \mathcal{W}'(gk_{it} + I_{it}) = 0. \quad (40)$$

The envelop condition yields

$$\mathcal{W}'(k_{it}) = \partial Q(k_{it}, \ell_{it}) / \partial k + g\delta_t \mathcal{W}'(gk_{it} + I_{it})$$

implying  $\mathcal{W}'(k_{it}) = \partial Q(k_{it}, \ell_{it}) / \partial k + g$ . Plugging this expression in (40) evaluated for period  $t + 1$  yields

$$\partial Q(gk_{it} + I_{it}, \ell_{it+1}) / \partial k = 1 + r_t - g.$$

As  $Q$  is homogeneous of degree 1, we thus get  $q'(k_{it+1}/\ell_{it+1}) = 1 + r_t - g$  where  $q(k) \equiv Q(k, 1)$  is an increasing and concave function. Inverting, it comes that  $k_{it+1}/\ell_{it+1} = q'^{-1}(1 + r_t - g)$ , and using  $\int_0^1 \ell_{it} di = \ell_t = 1$ ,  $k_{it}/\ell_{it} = k_t/\ell_t = k_t$  for all  $i$  and  $t$ , with  $k_t = q'^{-1}(1 + r_{t-1} - g)$ . Using  $w_t = \partial Q(k_{it}, \ell_{it}) / \partial \ell = d[\ell_{it} q(k_{it}/\ell_{it})] / d\ell_{it} = q(k_{it}/\ell_{it}) - q'(k_{it}/\ell_{it})k_{it}/\ell_{it}$  yields  $w_t = q_t - r_{t-1}k_t$ .

Neglecting the constants in (4), the Bellman equation corresponding to the technology mix is given by

$$\mathcal{V}(x_{it}; x_t^*) = \max_{o_{it}} -(x_{it} - x_t^*)^2 / 2 - o_{it} + \delta_t \mathbb{E}[\mathcal{V}(gx_{it} + o_{it}; \tilde{x}_{t+1}^*) | \eta_{it}, \mu_t, r_t]. \quad (41)$$

Maximizing (41) with respect to  $o_{it}$  leads to

$$-1 + \delta_t \mathbb{E} [\partial \mathcal{V}(gx_{it} + o_{it}; \tilde{x}_{t+1}^*) / \partial x | \eta_{it}, \mu_t, r_t] = 0. \quad (42)$$

The envelop condition yields

$$\partial \mathcal{V}(x_{it}; x_t^*) / \partial x = x_t^* - x_{it} + g \delta_t \mathbb{E} [\partial \mathcal{V}(gx_{it} + o_{it}; \tilde{x}_{t+1}^*) / \partial x | \eta_{it}, \mu_t, r_t] \quad (43)$$

implying  $\partial \mathcal{V}(x_{it}; x_t^*) / \partial x = x_t^* - x_{it} + g$ . Plugging this expression in (42) evaluated in expectation for period  $t + 1$  yields

$$1 + r_t = \mathbb{E} [\tilde{x}_{t+1}^* - (gx_{it} + o_{it}) + g | \eta_{it}, \mu_t, r_t] = \mathbb{E} [\tilde{x}_{t+1}^* | \eta_{it}, \mu_t, r_t] - (gx_{it} + o_{it}) + g,$$

which gives (6). Following Angeletos & Pavan (2004), the resulting investment strategy is linear in the variables observed by the firm at date  $t$ , i.e. it is given by

$$o(\mu_t, \eta_{jt}, x_{jt}, r_t) = \beta_1 \mu_t + \beta_2 \eta_{jt} + \beta_3 x_{jt} + \beta_4 r_t + \beta_5 \quad (44)$$

where the coefficients  $\beta_k, k = 1, \dots, 5$ , are derived as follows. On average, as  $\int_0^1 \eta_{jt} dj = \omega_{t+1}$  and  $\int_0^1 x_{jt} dj = \mu_t$ , we have

$$\int_0^1 o(\mu_t, \eta_t, \varepsilon_{jt}, x_{jt}, r_t) dj = (\beta_1 + \beta_3) \mu_t + \beta_2 \omega_{t+1} + \beta_4 r_t + \beta_5$$

and thus, from (3),

$$x_{t+1}^* = \mu_t + \lambda \int_0^1 o_{it} di + \omega_{t+1} = [1 + \lambda(\beta_1 + \beta_3)] \mu_t + (1 + \lambda\beta_2) \omega_{t+1} + \lambda\beta_4 r_t + \lambda\beta_5.$$

Using (6) and  $\mathbb{E}[\tilde{\omega}_{t+1} | \eta_{it}] = \eta_{it}$ , we get

$$\begin{aligned} o_{it} &= \mathbb{E}[\tilde{x}_{t+1}^* | \eta_{it}, \mu_t, r_t] - 1 + g - gx_{it} - r_t \\ &= [1 + \lambda(\beta_1 + \beta_3)] \mu_t + (1 + \lambda\beta_2) \eta_{it} - (1 - \lambda\beta_4) r_t + \lambda\beta_5 - 1 + g - gx_{it}. \end{aligned}$$

Identifying with (44) yields  $\beta_3 = -g$ ,  $\beta_1 = 1 + \lambda(\beta_1 + \beta_3) = (1 - \lambda g)/(1 - \lambda)$ ,  $\beta_2 = 1 + \lambda\beta_2 = 1/(1 - \lambda)$ ,  $\beta_4 = -1 + \lambda\beta_4 = -1/(1 - \lambda)$ ,  $\beta_5 = -(1 - g)/(1 - \lambda)$ , hence (7). Consequently,

$$x_{it+1} = gx_{it} + o_{it} = \frac{(1 - \lambda g) \mu_t + \eta_{it} + g - 1 - r_t}{1 - \lambda},$$

and, integrating,

$$\mu_{t+1} = \frac{(1 - \lambda g)\mu_t + \omega_{t+1} + g - (1 + r_t)}{1 - \lambda}$$

using  $\int_0^1 \eta_{it} di = \omega_{t+1}$ . Re-organizing terms gives (8). As idiosyncratic investments depend on signals that are normally distributed,  $x_{it+1}$  is normally distributed around  $\mu_{t+1}$  with variance  $\mathbb{V}[x_{it+1}] = \sigma_v^2 / (1 - \lambda)^2 \equiv \sigma_x^2$ .

## B Proof of Lemma 1

At each date  $t$ , the Bellman equation corresponding to (9) can be written as

$$v(S_{t-1}; e_t) = \max_{s_t} u(R_t + r_{t-1}S_{t-1} - s_t, e_t) + \beta \mathbb{E}_t[v(S_{t-1} + s_t; \tilde{e}_{t+1})]$$

where  $S_t$  and  $s_t$  are the state and the control variables respectively. The first-order equation gives

$$\partial u(c_t, e_t) / \partial c = \beta \mathbb{E}_t [\partial v(S_t; \tilde{e}_{t+1}) / \partial S], \quad (45)$$

and the envelope theorem gives

$$\partial v(S_{t-1}; e_t) / \partial S = r_{t-1} \partial u(c_t, e_t) / \partial c + \beta \mathbb{E}_t [\partial v(S_t; \tilde{e}_{t+1}) / \partial S].$$

Replacing the last term using (45), we get

$$\partial v(S_{t-1}; e_t) / \partial S = (1 + r_{t-1}) \partial u(c_t, e_t) / \partial c.$$

Taking the expectation and replacing in (45) yields (10) where  $1 + r_t$  on the RHS is factorized out of the expected value since the date- $t$  interest rate is a known parameter.

## C Proof of Proposition 2

The planner's program is equivalently stated as

$$\max_{\{\mu_t, k_t, e_t\}_{t>0}} \left\{ \sum_{t=0}^{+\infty} \beta^t u(q(k_t) - k_{t+1} + gk_t - \mu_{t+1} + g\mu_t, e_t) : (14), (15), (16) \right\},$$

given  $e_0, k_0$  and  $\mu_0$ . Neglecting the constraints (15) and (16), and denoting by  $\hat{\lambda}_t$  the multiplier associated to (14), the Lagrangian of this program is given by

$$\mathcal{L} = \sum_{t=0}^{+\infty} \beta^t u(q(k_t) - k_{t+1} + gk_t - \mu_{t+1} + g\mu_t, e_t) - \hat{\lambda}_t (e_{t+1} - \theta e_t - \xi \mu_t + \varphi q(k_t) - \hat{e}).$$

When (16) is not binding, the FOCs are

$$\frac{\partial \mathcal{L}}{\partial k_t} = \beta^t \frac{\partial u_t}{\partial c} (q'(k_t) + g) - \beta^{t-1} \frac{\partial u_{t-1}}{\partial c} - \hat{\lambda}_t \varphi q'(k_t) \leq 0 \perp k_t \geq gk_{t-1}, \quad (46)$$

$$\frac{\partial \mathcal{L}}{\partial \mu_t} = \beta^t \frac{\partial u_t}{\partial c} g - \beta^{t-1} \frac{\partial u_{t-1}}{\partial c} + \hat{\lambda}_t \xi = 0, \quad (47)$$

for all  $t > 0$ , and

$$\frac{\partial \mathcal{L}}{\partial e_t} = \beta^t \frac{\partial u_t}{\partial e} - \hat{\lambda}_{t-1} + \theta \hat{\lambda}_t = 0, \quad (48)$$

for all  $t > 1$ . Eq. (47) gives

$$\hat{\lambda}_t = \left( \beta^{t-1} \frac{\partial u_{t-1}}{\partial c} - \beta^t \frac{\partial u_t}{\partial c} g \right) / \xi$$

and, using (18),

$$\hat{\lambda}_t = \beta^t \frac{\partial u_t}{\partial c} (h_{t-1} - g) / \xi. \quad (49)$$

Substituting in (48) evaluated at  $t + 2$ , yields

$$\begin{aligned} 0 &= \beta^{t+2} \frac{\partial u_{t+2}}{\partial e} - \beta^{t+1} \frac{\partial u_{t+1}}{\partial c} (h_t - g) / \xi + \theta \beta^{t+2} \frac{\partial u_{t+2}}{\partial c} (h_{t+1} - g) / \xi \\ &= \beta^{t+2} \frac{\partial u_{t+2}}{\partial e} - \beta^{t+2} \frac{\partial u_{t+2}}{\partial c} [h_{t+1} (h_t - g) - \theta (h_{t+1} - g)] / \xi, \end{aligned}$$

using (18), hence

$$\xi \frac{\partial u_{t+2} / \partial e}{\partial u_{t+2} / \partial c} = h_{t+1} (h_t - g - \theta) + \theta g$$

for all  $t > 1$  such that (16) is not binding. Using (49) to substitute for  $\hat{\lambda}_{t+1}$  in (46) evaluated at  $t + 1$  gives

$$\begin{aligned} 0 &\geq \beta^{t+1} \frac{\partial u_{t+1}}{\partial c} (q'(k_{t+1}) + g) - \beta^t \frac{\partial u_t}{\partial c} - \beta^{t+1} \frac{\partial u_{t+1}}{\partial c} (h_t - g) q'(k_{t+1}) \varphi / \xi \\ &= \beta^{t+1} \frac{\partial u_{t+1}}{\partial c} [q'(k_{t+1}) - (h_t - g) (1 + q'(k_{t+1}) \varphi / \xi)], \end{aligned}$$



using (18), hence

$$h_t - g \geq \frac{q'(k_{t+1})}{1 + q'(k_{t+1})\varphi/\xi} \perp k_{t+1} \geq gk_t.$$

Assuming (16) is binding for all  $t \geq T$ , the planer's program becomes

$$\max_{\{k_t\}_{t>T}} \left\{ \sum_{t=T}^{+\infty} \beta^t u(q(k_t) - k_{t+1} + gk_t - [q(k_{t+1}) - gq(k_t)]\varphi/\xi, e_t) : e_{t+1} = \theta e_t + \hat{e} \right\},$$

given  $e_T, k_T$ . The FOCs are

$$\frac{\partial \mathcal{L}}{\partial k_t} = \beta^t \frac{\partial u_t}{\partial c} [q'(k_t)(1 + g\varphi/\xi) + g] - \beta^{t-1} \frac{\partial u_{t-1}}{\partial c} [1 + q'(k_t)\varphi/\xi] = 0$$

for all  $t > T$ , which gives

$$h_{t-1} = \frac{g + q'(k_t)(1 + g\varphi/\xi)}{1 + q'(k_t)\varphi/\xi},$$

or, equivalently,

$$h_t - g = \frac{q'(k_{t+1})}{1 + q'(k_{t+1})\varphi/\xi}$$

for all  $t \geq T$ .

## D Proof of lemma 2

The problem of firm  $i$  is to solve

$$\mathcal{V}(k_{it}, x_{it}, x_t^*) = \max_{I_{it}, \ell_{it}, o_{it}} \pi_{it} + \delta_t \mathbb{E}_t[\mathcal{V}(gk_{it} + I_{it}, gx_{it} + o_{it}, \tilde{x}_{t+1}^*) | \eta_{it}]$$

where

$$\pi_{it} \equiv Q(k_{it}, \ell_{it}) - w_t \ell_{it} - I_{it} - (x_{it} - x_t^*)^2/2 - o_{it}(1 - z_t) - \tau_t(\varphi Q(k_{it}, \ell_{it}) - \xi x_{it}).$$

Using the same steps as those described in the proof of Proposition 1, it comes (21) and

$$o_{it} = \mathbb{E}[\tilde{x}_{t+1}^* | \eta_{it}, \mu_t, r_t] - gx_{it} + \xi \tau_{t+1} + g(1 - z_{t+1}) - (1 + r_t)(1 - z_t).$$

At equilibrium, we get

$$o_{it} = \frac{1}{1 - \lambda} [(1 - \lambda g)\mu_t + \eta_{it} + \xi \tau_{t+1} + g(1 - z_{t+1}) - (1 + r_t)(1 - z_t)] - gx_{it},$$

hence

$$x_{it+1} = gx_{it} + o_{it} = \frac{1}{1-\lambda} [(1-\lambda g)\mu_t + \eta_{it} + \xi\tau_{t+1} + g(1-z_{t+1}) - (1+r_t)(1-z_t)].$$

Summing over all firms and using  $\int_0^1 \eta_{it} di = \omega_{t+1}$  gives (22).

## E Proof of Lemma 3

In that case, we have

$$\pi_{it} = Q(k_{it}, \ell_{it}) - w_t \ell_{it} - I_{it} - (x_{it} - x_t^*)^2/2 - o_{it} - \tau_t (\varphi Q(k_{it}, \ell_{it}) - \xi x_{it})$$

where  $\tilde{x}_{t+1}^*$  is given by (23). It comes

$$o_{it} = o_t(\mu_t, \underline{x}_t, \eta_{it}, x_{it}, r_t) \equiv \mathbb{E} [\tilde{x}_{t+1}^* | \eta_{it}, \mu_t, \underline{x}_t, r_t] - gx_{it} + \xi\tau_{t+1} + g - 1 - r_t. \quad (50)$$

Assuming linearity, i.e.

$$o_t(\mu_t, \eta_{it}, x_{it}, r_t) = \beta_0 \underline{x}_t + \beta_1 \mu_t + \beta_2 \eta_{it} + \beta_3 x_{it} + \beta_4 r_t + \beta_5 + \beta_6 \tau_{t+1}, \quad (51)$$

which leads to

$$\int_0^1 o_{jt} dj = \beta_0 \underline{x}_t + (\beta_1 + \beta_3) \mu_t + \beta_2 \omega_t + \beta_4 r_t + \beta_5 + \beta_6 \tau_{t+1},$$

and substituting in (23), we arrive at

$$\begin{aligned} x_{t+1}^* &= g\mu_t + (1-g)\underline{x}_t + \lambda \int_0^1 o_{jt} dj + \omega_{t+1} \\ &= (1-g + \lambda\beta_0)\underline{x}_t + [g + \lambda(\beta_1 + \beta_3)]\mu_t + \lambda\beta_4 r_t + \lambda\beta_5 + \beta_6 \tau_{t+1} + (1 + \lambda\beta_2)\omega_{t+1}. \end{aligned}$$

Using (50) and  $\mathbb{E}[\tilde{\omega}_{t+1} | \eta_{it}] = \eta_{it}$  we get

$$\begin{aligned} o_t(\mu_t, \eta_{it}, x_{it}, r_t) &= (1-g + \lambda\beta_0)\underline{x}_t + [g + \lambda(\beta_1 + \beta_3)]\mu_t + (\lambda\beta_4 - 1)r_t + \lambda\beta_5 + (\lambda\beta_6 + \xi)\tau_{t+1} \\ &\quad + (1 + \lambda\beta_2)\eta_{it} - gx_{it} + g - 1. \end{aligned}$$

Identifying with (51) yields  $\beta_2 = 1/(1-\lambda)$ ,  $\beta_0 = (1-g)\beta_2$ ,  $\beta_1 = g$ ,  $\beta_3 = -g$ ,

$\beta_4 = -\beta_2$ ,  $\beta_5 = (g - 1)\beta_2$ ,  $\beta_6 = \xi\beta_2$ . It comes

$$o_{it} = g(\mu_t - x_{it}) + \frac{(1 - g)\underline{x}_t + \eta_{it} - 1 - r_t + g + \xi\tau_{t+1}}{1 - \lambda},$$

hence

$$x_{it+1} = gx_{it} + o_{it} = g\mu_t + \frac{(1 - g)\underline{x}_t + \eta_{it} - 1 - r_t + g + \xi\tau_{t+1}}{1 - \lambda}.$$

Summing over  $[0, 1]$  and using  $\int_0^1 \eta_{it} di = \omega_{t+1}$  gives (24). We thus have  $x_{it+1} = \mu_{t+1} + v_{it}/(1 - \lambda)$  which is normally distributed with standard deviation  $\sigma_v/(1 - \lambda) = \sigma_x$ .

## F Proof of Proposition 3

To implement the First-Best, the realized IRS given by (10), that satisfies  $h_t^a = 1 + r_t$ , must be equal to  $h_t^*$ . For the tax policy, we have from (21),  $1 + r_t - g = q'(k_{t+1}^*)(1 - \tau_{t+1}\varphi)$ , while (19) gives  $h_t^* - g \geq q'(k_{t+1}^*)[1 - (h_t^* - g)\varphi/\xi]$ , with an equality when (15) is not binding. Identifying the two equations when  $k_{t+1}^* > gk_t^*$ , i.e. when  $t \geq t_0$ , gives  $\tau_{t+1} = (h_t^* - g)/\xi$ . When  $k_{t+1}^* = gk_t^*$ , substituting  $h_t^*$  for  $1 + r_t$  in (21) defines  $\tau_{t+1}$  for  $t < t_0$ , and we have  $h_t^* - g = q'(gk_t^*)[1 - \tau_{t+1}\varphi] > q'(gk_t^*)[1 - (h_t^* - g)\varphi/\xi]$  implying  $\tau_{t+1} < (h_t^* - g)/\xi$ . Substituting  $h_t^*$  for  $1 + r_t$  in (22) taken in expectation gives

$$\mathbb{E}[\mu_{t+1}] = [(1 - \lambda g)\mu_t + h_t^* z_t - g z_{t+1} + \xi\tau_{t+1} - (h_t^* - g)]/(1 - \lambda).$$

Re-arranging terms gives (26) using  $\mu_t = \mu_t^*$  and  $\mathbb{E}[\mu_{t+1}] = \mu_{t+1}^*$ . (27) is obtained similarly from (24). At the stationary state, denoting with subscript  $\infty$  the values of the variables, we have  $e_\infty = e_N$ ,  $c_\infty = q(k_\infty) - (1 - g)(k_\infty + \mu_\infty)$ ,  $h_\infty = 1/\beta$ ,  $\tau_\infty = (1/\beta - g)\varphi/\xi$  and  $k_\infty = q'^{-1}((1/\beta - g)/(1 - (1/\beta - g)\varphi/\xi))$ . (16) and (22) imply  $\mu_\infty = (\varphi/\xi)q(k_\infty)$  and  $(1 - \lambda)\mu_\infty = (1 - \lambda g)\mu_\infty + \xi\tau_\infty - (1 - z_\infty)(h_\infty - g)$ . Replacing and reorganizing terms yields  $z_\infty = -\lambda(1 - g)(\xi/\varphi)q(k_\infty)/(1/\beta - g) < 0$ .

## G Proof of Lemma 4

(i). The discriminant of  $P(h) = 0$  is  $\Delta \equiv (g - \theta)^2 + 4p\xi > 0$ , and the equation admits two roots,  $\bar{h} = (g + \theta + \sqrt{\Delta})/2 > \max\{g, \theta\}$  and  $\underline{h} = (g + \theta - \sqrt{\Delta})/2 < \min\{g, \theta\}$ . We have  $\underline{h} > 0$  iff  $\xi p < \theta g$ . The First-Best policy corresponds to a sequence of IRS with a subsequence defined by (28) as long as the ENP constraint (16) is not binding, i.e.  $t < T$  where  $T$  is the first period (16) binds. The sequence  $\{h_t^*\}_{t=0}^{T-1}$  is either degenerate,

i.e.  $h_t^* = \underline{h}$  or  $h_t^* = \bar{h}$  for all  $t \in \{0, \dots, T-1\}$ , or  $h_0^* \notin \{\underline{h}, \bar{h}\}$ , and, reorganizing (28),

$$h_t^* = (\xi p - \theta g) / (h_{t-1}^* - g - \theta) \quad (52)$$

for all  $t = 1, \dots, T-1$ . If  $T$  is large, this sequence eventually converges to a root of  $P(h) = 0$  that we denote by  $h_\infty$ . We can derived this non-degenerate sequence as follows. Defining  $v_t = (h_t^* - h_\#)^{-1}$ ,  $h_\# \in \{\underline{h}, \bar{h}\}$ , we have  $h_t^* = 1/v_t + h_\#$  and (52) becomes

$$\frac{1}{v_{t+1}} + h_\# = \frac{p\xi - \theta g}{1/v_t + h_\# - g - \theta},$$

which gives

$$\frac{1}{v_{t+1}} = \frac{-h_\#}{1 + v_t(h_\# - g - \theta)},$$

using (29). We thus have

$$v_{t+1} = v_t(g + \theta - h_\#) / h_\# - 1/h_\# \equiv v_t b_1 - b_0$$

with  $b_1 = \underline{h}/\bar{h}$  if  $h_\# = \bar{h}$  and  $b_1 = \bar{h}/\underline{h}$  if  $h_\# = \underline{h}$ . With an initial value  $v_0$  at  $t = t_0$ , the solution of this recurrence equation is given by

$$\begin{aligned} v_t &= v_0 b_1^t - b_0(1 - b_1^t) / (1 - b_1) \\ &= [v_0 + b_0 / (1 - b_1)] b_1^t - b_0 / (1 - b_1). \end{aligned} \quad (53)$$

where

$$\frac{b_0}{1 - b_1} = \frac{1/h_\#}{1 + (h_\# - g - \theta)/h_\#} = \frac{1}{2h_\# - g - \theta}.$$

If  $|b_1| < 1$ , i.e. if  $h_\# = \bar{h}$ ,  $v_t$  converges toward  $v_\infty = -b_0 / (1 - b_1) = 1 / (g + \theta - 2\bar{h}) = (h_\infty - \bar{h})^{-1}$ , hence  $h_t^*$  converges toward  $h_\infty = g + \theta - \bar{h} = \underline{h}$ . If  $|b_1| > 1$ , i.e. if  $h_\# = \underline{h}$ ,  $v_t$  diverges and thus  $h_\infty = \underline{h}$ . Hence, the recursion does not converge to  $\bar{h}$  (unless in the degenerate case  $h_0 = \bar{h}$ ), while it converges to  $\underline{h}$  from any initial value  $h_0 \neq \bar{h}$ . This cannot be optimal if  $\xi p \geq \theta g$  since  $\underline{h} \leq 0$ . If  $\xi p < \theta g$ , as  $\underline{h} < \min\{g, \theta\} \leq g$ , there is no stock of productive capital that can satisfy (19) when  $h_t^* \rightarrow \underline{h}$ . The optimal solution is thus  $h_t^* = \bar{h}$  for all  $t \in \{0, \dots, T-1\}$ .

**Derivation of H1 and H2.** The path of  $y_t$  is increasing if  $\bar{h} > 1/\beta$ . From the concavity of  $P$ , as  $1/\beta > \max\{g, \theta\}$ , we have  $\bar{h} > 1/\beta$  iff  $P(1/\beta) > 0$ , i.e. iff  $p\xi > (1/\beta - \theta)(1/\beta - g)$ . Also, we have to verify that (15) is not binding indefinitely. From (19), using  $\lim_{k \rightarrow 0} q'(k) = +\infty$ , this is the case if  $\bar{h}$  is lower than  $g + \xi/\varphi$ , i.e.

$p < \xi/\varphi^2 + (g - \theta)/\varphi$ . The conditions  $P(1/\beta) > 0$  and  $p < \xi/\varphi^2 + (g - \theta)/\varphi$  imply that  $(\xi/\varphi)^2 + (g - \theta)\xi/\varphi > p\xi > (1/\beta - \theta)(1/\beta - g)$  which is assumption H2. H2 is possible only if  $F(\xi/\varphi) > 0$ , where  $F(x) = x^2 + (g - \theta)x - (1/\beta - \theta)(1/\beta - g)$  is a second degree polynomial. The discriminant of  $F(x) = 0$  is given by  $[2/\beta - (g + \theta)]^2$ , implying that  $F(x) = (x - \underline{x})(x - \bar{x})$  where  $\underline{x} < 0 < \bar{x}$  are the two real roots of  $F(x) = 0$ .  $F(x)$  is positive if  $x < \underline{x}$  or  $x > \bar{x}$ , and since  $\xi/\varphi > 0$ , we must have  $\xi/\varphi > \bar{x} = 1/\beta - (3g - \theta)/2$  (assumption H1).

(ii). If  $gk_0 > \bar{k}$ , (15) is binding as long as  $g^t k_0 > \bar{k}$ , i.e.  $t \leq t_0$  given by  $g^{t_0} k_0 = \bar{k}$ , hence  $t_0 = \ln(\bar{k}/k_0)/\ln g$ .

(iii). For  $t > t_0$ ,  $k_{t+1} = \bar{k}$  until (16) is binding, i.e. until  $T$  given by  $\mu_T^* = q(k_{T+1})\varphi/\xi$ . Using (28) for  $t = T - 1$  yields  $h_T = \bar{h}$ , hence  $k_{T+1} = \bar{k}$ .

Finally, the path of the economy for  $t > T$  is defined recursively by (14), (11), (16), (18) and (19) that holds for all  $t > t_0$ . Given the initial values  $e_T$ ,  $k_{T+1} = k_T = \bar{k}$ , the IRS can be written as  $h_t = h(k_t, k_{t+1}, k_{t+2})$  and (19) gives the implicit equation  $q'(k_{t+1})[1 - (h(k_t, k_{t+1}, k_{t+2}) - g)\varphi/\xi] - h(k_t, k_{t+1}, k_{t+2}) + g = 0$  defining  $k_{t+2}$  for all  $t \geq T$ .

## H Proof of Proposition 4

With exponential preferences  $h_t = e^{\gamma[c_{t+1} - c_t + p(e_{t+1} - e_t)]}/\beta$ .  $h_t = \bar{h}$  gives

$$c_{t+1} - c_t + p(e_{t+1} - e_t) = \ln(\beta\bar{h})^{1/\gamma} \quad (54)$$

for all  $t \leq T$ . Multiplying both sides by  $\xi$  and using (14) and (11) leads to

$$\begin{aligned} \ln(\beta\bar{h})^{\xi/\gamma} &= \xi[q_{t+1} - q_t - k_{t+2} + (1 + g)k_{t+1} - gk_t] - \varphi[q_{t+2} - (1 + g)q_{t+1} + gq_t] \\ &\quad - (e_{t+3} - e_{t+2}) + (\theta + g)(e_{t+2} - e_{t+1}) + (\xi p - g\theta)(e_{t+1} - e_t) \end{aligned} \quad (55)$$

for all  $t \leq T$ . For  $t < t_0$ , we have  $k_t = g^t k_0$ , and using  $q(k) = Ak^\alpha$ ,  $q_t = g^{\alpha t} q_0$ . For  $t_0 \leq t \leq T + 1$ , we have  $k_t = \bar{k}$  and  $q_t = q(\bar{k}) \equiv \bar{q}$ . (55) can thus be expressed as

$$m_{t+2} = (\theta + g)m_{t+1} + (\xi p - g\theta)m_t - \kappa g^{\alpha t} - \ln(\beta\bar{h})^{\xi/\gamma} \quad (56)$$

for all  $t \leq T - 1$ , where  $m_t \equiv e_{t+1} - e_t$  and

$$\kappa = \begin{cases} [\xi + \varphi(g - g^\alpha)](1 - g^\alpha)q_0 & t < t_0 \\ 0 & t_0 \leq t \leq T - 1 \end{cases} .$$

For  $t \geq T$ , as  $\varphi q(k_t) = \xi \mu_t$ , we have  $e_{t+1} = \theta e_t + \hat{e}$  implying  $m_{t+1} = \theta m_t$ .

The solution of (56) is  $m_t = n_t + v_t$  where  $n_t$  and  $v_t$  are the solutions of the corresponding homogeneous and particular equations. The characteristic equation of the homogeneous equation is  $P(x) = 0$ . As  $\underline{h}$  is irrelevant,  $n_t = \phi \bar{h}^t$  where  $\phi$  is a constant. The particular solution of (56) is given by  $v_t = g^{\alpha t} \nu_0 + \nu_1$  where  $\nu_0$  and  $\nu_1$  solve

$$g^{\alpha(t+2)} \nu_0 + \nu_1 = (\theta + g)(\nu_0 g^{\alpha(t+1)} + \nu_1) + (\xi p - g\theta)(\nu_0 g^{\alpha t} + \nu_1) - \kappa g^{\alpha t} - \ln(\beta \bar{h})^{\xi/\gamma}$$

for all  $t \leq T - 1$ , which gives  $\nu_0 = \kappa/P(g^\alpha)$  and  $\nu_1 = \ln(\beta \bar{h})^{\xi/\gamma}/P(1)$ . As  $P$  is concave with  $P(g) = p\xi$  and  $P(1/\beta) > 0$ , we have  $P(g^\alpha) > 0$  and  $P(1) > 0$ , hence  $\nu_1 > 0$  since  $P(1/\beta) > 0$  implies  $\bar{h} > 1/\beta$ . We also have  $\nu_0 \geq 0$  if  $\xi/\varphi \geq 1 - g > g^\alpha - g$ . The solution of (56) is thus

$$m_t = \phi \bar{h}^t + g^{\alpha t} \nu_0 + \nu_1, \quad (57)$$

where  $\phi$  is specific to each sub-sequence and is derived from their initial conditions, i.e. from (56) evaluated at  $t = t_0 - 1$ , (55) evaluated at  $t = T$ , and from (14) at  $t = 1$ .

Using (55) at  $t = T$ , yields

$$\begin{aligned} \ln(\beta \bar{h})^{\xi/\gamma} &= \xi(\bar{k} - k_{T+2}) - \varphi(q(k_{T+2}) - \bar{q}) - m_{T+2} + (\theta + g)m_{T+1} + (\xi p - g\theta)m_T \\ &= \xi \bar{k} + \varphi \bar{q} - [\xi k_{T+2} + \varphi q(k_{T+2})] - \theta^2 m_T + \theta(\theta + g)m_T + (\xi p - g\theta)m_T \\ &= \xi \bar{k} + \varphi \bar{q} - [\xi k_{T+2} + \varphi q(k_{T+2})] + \xi p m_T, \end{aligned}$$

where the first line comes from  $k_t = \bar{k}$  and  $q_t = \bar{q}$  for  $t_0 \leq t \leq T + 1$ , and the second from  $m_{t+1} = \theta m_t$  for  $t \geq T$ . Identifying each term with (54) multiplied by  $\xi$  on both sides, it comes

$$\xi(c_{T+1} - c_T) = \xi \bar{k} + \varphi \bar{q} - [\xi k_{T+2} + \varphi q(k_{T+2})] \equiv \kappa_1. \quad (58)$$

We thus get, using (57) where  $\nu_0 = 0$  and  $\ln(\beta \bar{h})^{\xi/\gamma} = \nu_1 P(1)$ ,

$$\phi \bar{h}^T + \nu_1 = [\nu_1 P(1) - \kappa_1]/\xi p.$$

It comes

$$\phi = \nu_1(P(1)/\xi p - 1)/\bar{h}^T - \kappa_1/(\xi p \bar{h}^T) = -[(1 - \theta)(1 - g)\nu_1 + \kappa_1]/(\xi p \bar{h}^T)$$

for all  $t_0 \leq t \leq T$ . Substituting into (57) and using  $\nu_0 = 0$  for all  $t_0 \leq t \leq T$  gives

$$m_t = \nu_1 - [(1 - \theta)(1 - g)\nu_1 + \kappa_1]/(\xi p \bar{h}^{T-t})$$

for all  $t_0 \leq t \leq T$ . Using (54) evaluated at  $T$  and (58) we get

$$c_{T+1} - c_T + pm_T = \kappa_1/\xi + p\nu_1 - [(1 - \theta)(1 - g)\nu_1 + \kappa_1]/\xi = P(1)\nu_1/\xi = \ln(\beta \bar{h})^{1/\gamma},$$

and it comes

$$h_T = e^{\gamma[c_{T+1} - c_T + p(e_{T+1} - e_T)]}/\beta = \bar{h} = g + q'(k_{T+2})/[1 + q'(k_{T+2})\varphi/\xi].$$

We thus have  $k_{T+2} = \bar{k}$ , and from and (58)  $c_{T+1} - c_T = 0 = \kappa_1$ , implying

$$m_t = \nu_1 - [(1 - \theta)(1 - g)\nu_1]/(\xi p \bar{h}^{T-t})$$

for all  $t_0 \leq t \leq T$ . As  $m_t \geq P(1)\nu_1/\xi p = \ln(\beta \bar{h})^{1/p\gamma} > 0$ ,  $e_{t+1} > e_t$  for all  $t_0 \leq t \leq T$ .

Using (56) for  $t = t_0 - 1$ , it comes

$$\begin{aligned} m_{t_0+1} &= \nu_1 - [(1 - \theta)(1 - g)\nu_1]/(\xi p \bar{h}^{(T-t_0-1)}) \\ &= (\theta + g)\{\nu_1 - [(1 - \theta)(1 - g)\nu_1]/(\xi p \bar{h}^{(T-t_0)})\} + (\xi p - \theta g)(\phi \bar{h}^{t_0-1} + g^{\alpha(t_0-1)}\nu_0 + \nu_1) - \nu_1 P(1). \end{aligned}$$

As  $P(1) = \xi p - \theta g - 1 + \theta + g$ , this equation simplifies to

$$\begin{aligned} [(1 - \theta)(1 - g)\nu_1]/(\xi p \bar{h}^{(T-t_0-1)}) &= (\theta + g)[(1 - \theta)(1 - g)\nu_1]/(\xi p \bar{h}^{(T-t_0)}) \\ &\quad - (\xi p - \theta g)(\phi \bar{h}^{t_0-1} + g^{\alpha(t_0-1)}\nu_0). \end{aligned}$$

Multiplying by  $\xi p \bar{h}^{T-t_0+1}$  yields

$$[(1 - \theta)(1 - g)\nu_1]\bar{h}^2 = (\theta + g)[(1 - \theta)(1 - g)\nu_1]\bar{h} - (\xi p - \theta g)(\phi \bar{h}^{t_0-1} + g^{\alpha(t_0-1)}\nu_0)\xi p \bar{h}^{T-t_0+1},$$

and re-organizing terms,

$$[(1 - \theta)(1 - g)\nu_1][\bar{h}^2 - (\theta + g)\bar{h}] = -(\xi p - \theta g)(\phi \bar{h}^{t_0-1} + g^{\alpha(t_0-1)}\nu_0)\xi p \bar{h}^{T-t_0+1},$$

where, as  $P(\bar{h}) = 0$ ,  $p\xi - \theta g = \bar{h}^2 - (\theta + g)\bar{h}$ . Simplifying, we get

$$(1 - \theta)(1 - g)\nu_1 = -[\phi + (g^\alpha/\bar{h})^{t_0-1}\nu_0]\xi p \bar{h}^T,$$

which gives

$$\phi = - \left( \frac{g^\alpha}{\bar{h}} \right)^{t_0-1} \nu_0 - \frac{(1-\theta)(1-g)\nu_1}{\xi p \bar{h}^T}.$$

Substituting into (57) yields

$$m_t = \nu_0 g^{\alpha t} [1 - (g^\alpha/\bar{h})^{t_0-1-t}] + \nu_1 - [(1-\theta)(1-g)\nu_1]/(\xi p \bar{h}^{T-t})$$

for all  $0 < t < t_0$ . As  $m_t > 0$ ,  $e_{t+1} > e_t$  for all  $0 < t < t_0$ . (31) is deduced from

$$\begin{aligned} \sum_{\tau=1}^{t-1} m_\tau &= e_t - e_1 = \sum_{\tau=1}^{t-1} (\phi \bar{h}^\tau + g^{\alpha\tau} \nu_0 + \nu_1) = \phi \frac{\bar{h}^t - \bar{h}}{\bar{h} - 1} + \nu_0 \frac{g^\alpha - g^{\alpha t}}{1 - g^\alpha} + (t-1)\nu_1 \\ &= \nu_0 \left( \frac{g^\alpha - g^{\alpha t}}{1 - g^\alpha} - \left( \frac{g^\alpha}{\bar{h}} \right)^{t_0-1} \frac{\bar{h}^t - \bar{h}}{\bar{h} - 1} \right) + \nu_1 (t-1) - \frac{(1-\theta)(1-g)\nu_1}{\xi p \bar{h}^T} \frac{\bar{h}^t - \bar{h}}{\bar{h} - 1} \end{aligned}$$

for all  $1 < t < t_0$ , where  $e_1 = \theta e_0 + \xi \mu_0 - \varphi q_0 + \hat{e}$ , and from

$$\begin{aligned} \sum_{\tau=t_0-1}^{t-1} m_\tau &= e_t - e_{t_0-1} = \phi \sum_{\tau=t_0-1}^{t-1} \bar{h}^\tau + (t-t_0)\nu_1 = \phi \frac{\bar{h}^t - \bar{h}^{t_0-1}}{\bar{h} - 1} + (t-t_0+1)\nu_1 \\ &= \nu_1 (t-t_0+1) - \frac{(1-\theta)(1-g)\nu_1}{\xi p \bar{h}^T} \frac{\bar{h}^t - \bar{h}^{t_0-1}}{\bar{h} - 1}, \end{aligned}$$

for all  $t_0 \leq t \leq T$ . Using  $m_T = \ln(\beta \bar{h})^{1/(\rho\gamma)}$  and  $m_T = e_{T+1} - e_T = \hat{e} - (1-\theta)e_T$ , it comes that  $T$  is solution of  $e_T = e_N - \ln(\beta \bar{h})^{1/(1-\theta)\rho\gamma}$ . As

$$\begin{aligned} m_t - m_{t-1} &= \nu_0 (g^{\alpha t} - g^{\alpha(t_0-1)})/\bar{h}^{t_0-1-t} - g^{\alpha(t-1)} + g^{\alpha(t_0-1)}/\bar{h}^{t_0-t} - \frac{\nu_1(1-\theta)(1-g)}{\xi p \bar{h}^{T-t+1}} (\bar{h} - 1) \\ &= -\nu_0 [(1/g^\alpha - 1)g^{\alpha t} + (\bar{h} - 1)g^{\alpha(t_0-1)}/\bar{h}^{t_0-t}] - \frac{\nu_1(1-\theta)(1-g)}{\xi p \bar{h}^{T-t+1}} (\bar{h} - 1), \end{aligned}$$

where  $\nu_0 > 0$  if  $t < t_0$  and  $\nu_0 = 0$  if  $t_0 \leq t \leq T$ ,  $m_t - m_{t-1} < 0$  for all  $t \leq T$ : the sequence  $\{e_t\}_{2 \leq t \leq T}$  increases at a decreasing rate. From (14), the GTI index is deduced from these expressions using  $\mu_t = (e_{t+1} - \theta e_t + \varphi q_t - \hat{e})/\xi$ . From (54),

$$c_{t+1} - c_t = \ln(\beta \bar{h})^{1/\gamma} - p m_t$$



for all  $t \leq T$  where

$$\begin{aligned} m_t &= \nu_0 g^{\alpha t} [1 - (g^\alpha / \bar{h})^{t_0 - 1 - t}] + \nu_1 [1 - (1 - \theta)(1 - g) / (\xi p \bar{h}^{T-t})] \\ &= \nu_0 g^{\alpha t} [1 - (g^\alpha / \bar{h})^{t_0 - 1 - t}] + \ln(\beta \bar{h})^{1/\gamma} [\xi p - (1 - \theta)(1 - g) / \bar{h}^{T-t}] / [pP(1)] \end{aligned}$$

for all  $1 < t \leq T$ . We thus have

$$\begin{aligned} c_{t+1} - c_t &= -\nu_0 g^{\alpha t} [1 - (g^\alpha / \bar{h})^{t_0 - 1 - t}] + \ln(\beta \bar{h})^{1/\gamma} [P(1) - (\xi p - (1 - \theta)(1 - g) / \bar{h}^{T-t})] / P(1) \\ &= -\nu_0 g^{\alpha t} [1 - (g^\alpha / \bar{h})^{t_0 - 1 - t}] - \ln(\beta \bar{h})^{1/\gamma} (1 - \theta)(1 - g)(1 - 1/\bar{h}^{T-t}) / P(1) \end{aligned}$$

which is negative and increasing, with a maximum equal to 0 at  $t = T$ . As  $c_{t+1} - c_t - (c_t - c_{t-1}) = -p(m_t - m_{t-1}) > 0$ , the consumption sequence decreases at a decreasing rate.

## I Proof of Proposition 5

For ease of notation, superscript  $\ell$  is omitted and results are derived in the tax-subsidy policy case ( $\ell = sub$ ). Variations from these results in the tax-standard policy case are specified in the text. Also, subscript  $t$  is used as a shorthand for coefficients function of  $z_t$ , e.g.  $a_{3t} \equiv a_3^{sub}(z_t)$ . I first characterize the distribution of  $\tilde{y}_{t+1}$  given the information available in period  $t$ ,  $t_0 < t \leq T$ , a random variable denoted by  $\tilde{y}_{t+1|t}$ . Define

$$\hat{k}(r) \equiv q^{-1}((1 + r - g)/(1 - (\bar{h} - g)\varphi/\xi)),$$

and  $\hat{q}(r) \equiv q(\hat{k}(r))$ . We have  $\bar{k} = \hat{k}(\bar{r})$ , and  $\bar{q} \equiv \hat{q}(\bar{r})$  denotes the corresponding GDP.

Given the information available in period  $t$ , the prevailing interest rate  $r_t$  (and thus the realization  $\varepsilon_t$  of  $\tilde{\varepsilon}_t$ ) is known, and both  $k_{t+1} = \hat{k}(r_t)$  and  $q_{t+1} = \hat{q}(r_{t+1})$  are known. From (14),  $e_{t+1}$  is also known since it depends on variables of the previous period, that are function of  $\varepsilon_{t-1}$ . Hence, from (11), we get

$$\tilde{y}_{t+1|t} = \tilde{c}_{t+1|t} + pe_{t+1} = q_{t+1} - \tilde{k}_{t+2|t} + gk_{t+1} - (\tilde{\mu}_{t+2|t} - g\mu_{t+1}) + pe_{t+1}, \quad (59)$$

where  $\tilde{k}_{t+2}$  and  $\tilde{\mu}_{t+2}$  are unknown since they depend on the realization of  $\tilde{\varepsilon}_{t+1}$ : we have  $\tilde{k}_{t+2} = \hat{k}(\tilde{r}_{t+1})$  and, from (34),

$$\tilde{\mu}_{t+2|t} = a_1\mu_{t+1} + a_2e_{t+1} + a_{3t} + Z_{t+1} + b_{1t}\tilde{\varepsilon}_{t+1} + b_{2t}\varepsilon_t. \quad (60)$$

Replacing, we get

$$\tilde{y}_{t+1|t} = q_{t+1} - \tilde{k}_{t+2|t} + gk_{t+1} + (g - a_1)\mu_{t+1} + (p - a_2)e_{t+1} - a_{3t} - Z_{t+1} - b_{1t}\tilde{\varepsilon}_{t+1} - b_2\varepsilon_t.$$

Using the linear approximation  $k_t \approx \bar{k} + \hat{k}'(\bar{r})\bar{r}\varepsilon_{t-1}$  where  $\hat{k}'(\bar{r}) = 1/[(1 - \tau\varphi)q''(\bar{k})]$ , we get using  $\mathbb{E}_t[\tilde{k}_{t+2}] \approx \bar{k} + \hat{k}'(\bar{r})\bar{r}\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t]$  and (33),

$$\begin{aligned} \tilde{y}_{t+1|t} - \mathbb{E}_t[\tilde{y}_{t+1}] &\approx -(\tilde{k}_{t+2|t} - \mathbb{E}[\tilde{k}_{t+2}|\varepsilon_t]) - b_{1t}(\tilde{\varepsilon}_{t+1} - \mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t]) \\ &= -(\hat{k}'(\bar{r})\bar{r} + b_{1t})(\tilde{\varepsilon}_{t+1} - \mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t]) \\ &= -(\hat{k}'(\bar{r})\bar{r} + b_{1t})\chi_t\tilde{\omega}_{t+2} \end{aligned}$$

when  $t_0 < t \leq T$ . Hence,  $\tilde{y}_{t+1|t}$  is approximately normally distributed with variance  $\mathbb{V}_t[\tilde{y}_{t+1}] = (\hat{k}'(\bar{r})\bar{r} + b_{1t})^2\chi_t^2\sigma_\omega^2 \equiv \sigma_{y_{t+1|t}}^2$  when  $t_0 < t \leq T$ . For the tax-standard policy, as  $b_1^{std}$  and  $\chi^{std}$  do not depend on  $t$  (as shown below),  $\sigma_{y_{t+1|t}}^2$  is a constant.

The coefficients  $a_1, a_2, a_{3t}, b_{1t}, b_2$ , and  $Z_t$  in (34), and  $\chi_t$  and  $\rho_t$  in (33), are derived as follows. Using (14), (11), (59), (60) and  $\mathbb{E}_t[\tilde{k}_{t+2}] \approx \bar{k} + \hat{k}'(\bar{r})\bar{r}\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t]$ , it comes

$$\begin{aligned} \mathbb{E}_t[\tilde{y}_{t+1}] - y_t &\approx q_{t+1} - q_t - \bar{k} - \hat{k}'(\bar{r})\bar{r}\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] + k_{t+1}(1 + g) - gk_t - \mathbb{E}_t[\tilde{\mu}_{t+2}] + (1 + g)\mu_{t+1} \\ &\quad - g\mu_t + p(e_{t+1} - e_t) \\ &= q_{t+1} - q_t - \bar{k} - \hat{k}'(\bar{r})\bar{r}\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] + k_{t+1}(1 + g) - gk_t \\ &\quad - (a_1\mu_{t+1} + a_2e_{t+1} + a_{3t} + Z_{t+1} + b_{1t}\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] + b_2\varepsilon_t) + (1 + g)\mu_{t+1} \\ &\quad - \mu_t g + p(e_{t+1} - e_t) \\ &= q_{t+1} - q_t - \bar{k} - \hat{k}'(\bar{r})\bar{r}\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] + k_{t+1}(1 + g) - gk_t - a_{3t} - Z_{t+1} \\ &\quad - b_{1t}\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] - b_2\varepsilon_t + (1 + g - a_1)\mu_{t+1} - \mu_t g + (p - a_2)e_{t+1} - pe_t \\ &= q_{t+1} - q_t - \bar{k} - (b_{1t} + \hat{k}'(\bar{r})\bar{r})\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] + k_{t+1}(1 + g) - gk_t - a_{3t} - Z_{t+1} \\ &\quad - b_2\varepsilon_t + (1 + g - a_1)\mu_{t+1} - \mu_t g + (p - a_2)(\theta e_t + \xi\mu_t - \varphi q_t + \hat{e}) - pe_t \\ &= q_{t+1} - q_t[1 + \varphi(p - a_2)] - \bar{k} - (b_{1t} + \hat{k}'(\bar{r})\bar{r})\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] + k_{t+1}(1 + g) - gk_t \\ &\quad + (1 + g - a_1)\mu_{t+1} - \mu_t[g - \xi(p - a_2)] + [\theta(p - a_2) - p]e_t + (p - a_2)\hat{e} - a_{3t} \\ &\quad - Z_{t+1} - b_2\varepsilon_t. \end{aligned}$$

Using  $(1 + r_t)z_t = (\bar{h} + \bar{r}\varepsilon_t)z_t$  in (22) and (32), we get

$$(1 - \lambda)(g\mu_t - \mu_{t+1}) + (1 - g)(\mu_t - 1) + \omega_{t+1} + \xi\tau_{t+1} + (\bar{h} + \bar{r}\varepsilon_t)z_t - gz_{t+1} \approx r_t \approx \psi - \gamma^2\sigma_{y_{t+1}}^2/2 + \gamma\{\mathbb{E}_t[\tilde{y}_{t+1}] - y_t\}$$

which gives, denoting  $a_0 \equiv 1/[1 - \lambda + \gamma(1 + g - a_1)]$ ,

$$\begin{aligned} \mu_{t+1}/a_0 &\approx [1 - \lambda g + \gamma g - \gamma \xi(p - a_2)]\mu_t - \gamma[\theta(p - a_2) - p]e_t + \omega_{t+1} + \xi\tau_{t+1} + (\bar{h} + \bar{r}\varepsilon_t)z_t - gz_{t+1} \\ &\quad - (\psi - \gamma^2\sigma_{y_{t+1}}^2/2) - \gamma(p - a_2)\hat{e} + \gamma a_3 + g - 1 + \gamma Z_{t+1} + \gamma(b_1 + \hat{k}'(\bar{r})\bar{r})\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] \\ &\quad + \gamma b_2\varepsilon_t - \gamma q_{t+1} + \gamma q_t[1 + \varphi(p - a_2)] + \gamma\bar{k} - \gamma k_{t+1}(1 + g) + \gamma g k_t. \end{aligned}$$

Using  $k_{t+1} \approx \bar{k} + \hat{k}'(\bar{r})\bar{r}\varepsilon_t$  and  $q_{t+1} \approx \bar{q} + \hat{q}'(\bar{r})\bar{r}\varepsilon_t$  where  $\hat{q}'(\bar{r}) = q'(\bar{k})\hat{k}'(\bar{r})$  for  $t$  and  $t - 1$ , the last terms can be approximated by

$$-\gamma q_{t+1} + \gamma q_t[1 + \varphi(p - a_2)] \approx -\gamma\hat{q}'(\bar{r})\bar{r}\varepsilon_t + \gamma\bar{q}\varphi(p - a_2) + \gamma\hat{q}'(\bar{r})\bar{r}\varepsilon_{t-1}[1 + \varphi(p - a_2)]$$

and

$$\gamma\bar{k} - \gamma k_{t+1}(1 + g) + \gamma g k_t \approx -\gamma\hat{k}'(\bar{r})\bar{r}\varepsilon_t(1 + g) + \gamma g\hat{k}'(\bar{r})\bar{r}\varepsilon_{t-1}.$$

Substituting and, from (33), using  $\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] = \rho_t\varepsilon_t$ , we arrive at

$$\begin{aligned} \mu_{t+1} &\approx a_0[1 - \lambda g + \gamma g - \gamma \xi(p - a_2)]\mu_t - a_0\gamma[\theta(p - a_2) - p]e_t + a_0\omega_{t+1} \\ &\quad - a_0[\psi - \gamma^2\sigma_{y_{t+1}|t}^2/2 + \gamma(p - a_2)(\hat{e} - \bar{q}\varphi) - \gamma a_{3t} + 1 - g] + a_0[\xi\tau_{t+1} + \bar{h}z_t - gz_{t+1} + \gamma Z_{t+1}] \\ &\quad + a_0\gamma(\hat{q}'(\bar{r})[1 + \varphi(p - a_2)] + g\hat{k}'(\bar{r}))\bar{r}\varepsilon_{t-1} \\ &\quad - a_0\gamma[\hat{q}'(\bar{r})\bar{r} + \hat{k}'(\bar{r})\bar{r}(1 + g) - b_2 - \bar{r}z_t/\gamma - (b_{1t} + \hat{k}'(\bar{r})\bar{r})\rho_t]\varepsilon_t \end{aligned} \quad (61)$$

Identifying the non-stochastic terms with those of (34) and simplifying gives

$$\begin{aligned} a_1 &= \frac{1 - \lambda g + \gamma[g + \xi(a_2 - p)]}{1 - \lambda + \gamma(1 + g - a_1)}, a_2 = \frac{\gamma p(1 - \theta)}{1 - \lambda + \gamma(1 + g - a_1 - \theta)}, \\ a_{3t} &= -\frac{\psi - \gamma^2\sigma_{y_{t+1}|t}^2/2 - \gamma(p - a_2)(\varphi\bar{q} - \hat{e}) + 1 - g}{1 - \lambda + \gamma(g - a_1)}, \end{aligned} \quad (62)$$

and  $Z_t = a_0(\xi\tau_{t+1} + \bar{h}z_t - gz_{t+1} + \gamma Z_{t+1})$ . Solving the recursion gives (35).

For the standard policy, using (24) and (32) to get

$$(1 - \lambda)(g\mu_t - \mu_{t+1}) + (1 - g)(\underline{x}_t - 1) + \omega_{t+1} + \xi\tau_{t+1} = r_t \approx \psi - \gamma^2\sigma_{y_{t+1}}^2/2 + \gamma\{\mathbb{E}_t[\tilde{y}_{t+1}] - y_t\},$$

it comes following the same steps that  $a_2^{std}$  and  $a_3^{std}$  are given also by (62) (but where  $\sigma_{y_{t+1}|t}^2$  is a constant in the latter) while we have

$$a_1^{std} = \frac{(1 - \lambda)g + \gamma[g + \xi(a_2^{std} - p)]}{1 - \lambda + \gamma(1 + g - a_1^{std})}$$

and  $Z_t^{std} = a_0^{std}(\xi\tau_{t+1} + (1-g)x_t + \gamma Z_{t+1}^{std})$ , with  $a_0^{std} = 1/[1 - \lambda + \gamma(1 + g - a_1^{std})]$ . Solving the recursion gives (37).

The parameters of the stochastic terms are derived as follows. When  $\varepsilon_{t-1} = \omega_{t+1} = 0$ , we have  $\varepsilon_t = 0$  implying  $r_t = \bar{r}$ . Denoting by  $\mu_{t+1|r_t=\bar{r}}$  the value of  $\mu_{t+1}$  in such a case, we get from (22),

$$\mu_{t+1} - \mu_{t+1|r_t=\bar{r}} \approx \frac{\omega_{t+1} + (1+r_t - \bar{h})z_t + \bar{r} - r_t}{1-\lambda} = \frac{\omega_{t+1} - (1-z_t)\bar{r}\varepsilon_t}{1-\lambda},$$

using  $(1+r_t)z_t \approx (\bar{h} + \bar{r}\varepsilon_t)z_t$ , and from (61),

$$\begin{aligned} \mu_{t+1} - \mu_{t+1|r_t=\bar{r}} &\approx a_0\{\omega_{t+1} + \gamma[\hat{q}'(\bar{r})[1 + \varphi(p - a_2)] + g\hat{k}'(\bar{r})]\bar{r}\varepsilon_{t-1} \\ &\quad - \gamma[\hat{q}'(\bar{r})\bar{r} + \hat{k}'(\bar{r})\bar{r}(1+g) - b_2 - \bar{r}z_t/\gamma - (b_{1t} + \hat{k}'(\bar{r})\bar{r})\rho_t]\varepsilon_t\}. \end{aligned} \quad (63)$$

Equalizing gives

$$\omega_{t+1} = \frac{\bar{r}(1-z_t) - (1-\lambda)a_0\gamma[\hat{q}'(\bar{r})\bar{r} + \hat{k}'(\bar{r})\bar{r}(1+g) - b_2 - \bar{r}z_t/\gamma - (b_{1t} + \hat{k}'(\bar{r})\bar{r})\rho_t]}{1 - (1-\lambda)a_0}\varepsilon_t \quad (64)$$

$$+ (1-\lambda)\gamma a_0 \frac{\hat{q}'(\bar{r})[1 + \varphi(p - a_2)] + g\hat{k}'(\bar{r})}{1 - (1-\lambda)a_0}\bar{r}\varepsilon_{t-1}.$$

Substituting in (63) yields

$$\begin{aligned} \mu_{t+1} - \mu_{t+1|r_t=\bar{r}} &\approx \gamma a_0 \frac{\hat{q}'(\bar{r})[1 + \varphi(p - a_2)] + g\hat{k}'(\bar{r})}{1 - (1-\lambda)a_0}\bar{r}\varepsilon_{t-1} \\ &\quad + a_0 \frac{\bar{r} - \gamma[\hat{q}'(\bar{r})\bar{r} + \hat{k}'(\bar{r})\bar{r}(1+g) - b_2 - (b_{1t} + \hat{k}'(\bar{r})\bar{r})\rho_t]}{1 - (1-\lambda)a_0}\varepsilon_t. \end{aligned}$$

Identifying with the stochastic terms of (34) gives, using  $1 - (1-\lambda)a_0 = \gamma a_0(1 + g - a_1)$ ,

$$b_2 = \frac{\hat{q}'(\bar{r})[1 + \varphi(p - a_2)] + g\hat{k}'(\bar{r})}{1 + g - a_1}\bar{r} \quad (65)$$

$$b_{1t} = \frac{\bar{r} - \gamma[\hat{q}'(\bar{r})\bar{r} + \hat{k}'(\bar{r})\bar{r}(1+g - \rho_t) - b_2 - \rho b_{1t}]}{\gamma(1 + g - a_1)}. \quad (66)$$

As (64) can be written as  $\omega_{t+1} = [(1-\lambda)b_1 + \bar{r}(1-z_t)]\varepsilon_t + b_2(1-\lambda)\varepsilon_{t-1}$ , we get

from (33) that  $\chi_t = [(1 - \lambda)b_1 + \bar{r}(1 - z_t)]^{-1}$  and

$$\rho_t = \frac{-b_2(1 - \lambda)}{(1 - \lambda)b_{1t} + \bar{r}(1 - z_t)}. \quad (67)$$

Reorganizing terms to get  $b_2 = -\rho_t [b_{1t} + \bar{r}(1 - z_t)/(1 - \lambda)]$  and substituting in (66), we arrive at

$$b_{1t} = \frac{1 - \gamma[\hat{q}'(\bar{r}) + \hat{k}'(\bar{r})(1 + g - \rho_t) + \rho_t(1 - z_t)/(1 - \lambda)]}{\gamma(1 + g - a_1)} \bar{r}.$$

From (65) and (67), we obtain that  $\rho_t$  solves

$$\rho_t = \frac{-(1 - \lambda)\gamma a_0 \{ \hat{q}'(\bar{r})[1 + \varphi(p - a_2)] + g\hat{k}'(\bar{r}) \}}{1 - \gamma a_0 \left\{ (1 - \lambda)[\hat{q}'(\bar{r}) + \hat{k}'(\bar{r})(1 + g)] + \rho_t [1 - z_t - \hat{k}'(\bar{r})(1 - \lambda)] \right\}}. \quad (68)$$

For the standard policy, the same steps give identical formulae but with  $z_t$  replaced by 0. As a result, parameters  $b_1^{std}$ ,  $\rho^{std}$  and  $\chi^{std}$  are constant.

The first two equations of (62) form a system involving only coefficients  $a_1$  and  $a_2$  that can be solved separately from the others. More precisely, defining  $a_1^0 \equiv g + (1 - \lambda)/\gamma$ , we can express (62) as

$$\begin{aligned} a_1 &= \frac{a_1^0 + \lambda(1 - g)/\gamma + \xi(a_2 - p)}{a_1^0 - a_1 + 1}, a_2 = \frac{p(1 - \theta)}{a_1^0 - a_1 + 1 - \theta}, \\ a_{3t} &= -\frac{\psi - \gamma^2 \sigma_{y_{t+1|t}}^2 / 2 - \gamma(p - a_2)(\varphi \bar{q} - \hat{e}) + 1 - g}{\gamma(a_1^0 - a_1)}, a_0 = \frac{1}{\gamma(a_1^0 - a_1 + 1)} \end{aligned} \quad (69)$$

From the expression of  $a_2$ , we get

$$a_2 - p = -p \frac{a_1^0 - a_1}{a_1^0 - a_1 + 1 - \theta}, \quad (70)$$

which, plugged into the expression of  $a_1^0 - a_1$ , gives

$$(a_1^0 - a_1) \left( a_1 - 1 + \frac{\xi p}{a_1^0 - a_1 + 1 - \theta} \right) = \frac{\lambda(1 - g)}{\gamma}$$

that can be expressed as  $Q(a_1^0 - a_1) = 0$  where

$$Q(x) \equiv x^3 + (2 - \theta - a_1^0)x^2 + [(1 - \theta)(1 - a_1^0) - \xi p + \lambda(1 - g)/\gamma]x + \lambda(1 - \theta)(1 - g)/\gamma$$

is a third degree polynomial. As  $Z_t$  given by (35) converges if  $\gamma a_0 < 1$ , i.e. if  $a_1 < a_1^0$ , only positive roots are relevant. From (69) and (70),  $a_1 > 0$  if  $a_1^0 + \lambda(1-g)/\gamma + \xi(a_2 - p) = g + (1-\lambda g)/\gamma - p\xi(a_1^0 - a_1)/(a_1^0 - a_1 + 1 - \theta) > 0$ . Since  $a_1^0 > a_1$ , it suffices to have  $g + (1-\lambda g)/\gamma \geq p\xi$ . Under the tax-standard policy,  $a_1^{std} > 0$  if  $a_1^0 - (1-g)(1-\lambda)/\gamma + \xi(a_2 - p)g = g(1 + (1-\lambda)/\gamma) - p\xi(a_1^0 - a_1)/(a_1^0 - a_1 + 1 - \theta) > 0$ . A sufficient condition is thus  $g(1 + (1-\lambda)/\gamma) \geq p\xi$  which is more stringent than the tax-subsidy one. Both are satisfied if  $g \geq p\xi$ .

Equation (68) can be written as  $\zeta_t(\rho) = -A_t\rho^2 + B\rho - C = 0$ , where  $A_t \equiv [1 - z_t - \hat{k}'(\bar{r})(1-\lambda)]\gamma a_0$ ,  $B \equiv 1 - (1-\lambda)\gamma a_0[\hat{q}'(\bar{r}) + \hat{k}'(\bar{r})(1+g)]$ , and  $C \equiv -(1-\lambda)\gamma a_0\{\hat{q}'(\bar{r})[1 + \varphi(p - a_2)] + g\hat{k}'(\bar{r})\}$  are positive coefficients. As  $\zeta(0) < 0$  and  $\zeta'(0) > 0$ ,  $\zeta$  admits two positive roots provided that  $\Delta_t \equiv B^2 - 4A_tC > 0$ , the smallest root being given by  $(B - \sqrt{\Delta_t})/(2A_t)$ . In the tax-standard policy case, formulae are identical but with  $z_t$  replace by 0, implying that  $\rho^{std}$  is a constant.

Solving the recursion of (33), it comes

$$\tilde{\varepsilon}_t = \chi_t \tilde{\omega}_{t+1} + \rho_t \tilde{\varepsilon}_{t-1} = \sum_{k=0}^{t-t_0} \chi_{t-k} \tilde{\omega}_{t+1-k} \prod_{i=0}^{k-1} \rho_{t-i} + \varepsilon_{t_0} \prod_{k=0}^{t_0} \rho_{t-k},$$

hence  $\mathbb{E}[\tilde{\varepsilon}_t | \varepsilon_{t_0}] = \varepsilon_{t_0} \prod_{k=0}^{t_0} \rho_{t-k}$  and

$$\mathbb{V}[\tilde{\varepsilon}_t | \varepsilon_{t_0}] = \mathbb{E}[(\tilde{\varepsilon}_t - \mathbb{E}[\tilde{\varepsilon}_t | \varepsilon_{t_0}])^2 | \varepsilon_{t_0}] = \mathbb{E} \left[ \left( \sum_{k=0}^{t-t_0} \chi_{t-k} \prod_{i=0}^{k-1} \rho_{t-i} \right)^2 \right] \mathbb{E}[\tilde{\omega}]^2$$

using independence. In the tax-standard policy case, as  $\rho_t = \rho^{std}$  and  $\chi_t = \chi^{std}$ , it comes  $\mathbb{E}[\tilde{\varepsilon}_t | \varepsilon_{t_0}] = \varepsilon_{t_0} (\rho^{std})^{t-t_0}$  and  $\mathbb{V}[\tilde{\varepsilon}_t | \varepsilon_{t_0}] = (\chi^{std} \sigma_\omega)^2 (1 - (\rho^{std})^{2(t-t_0)+1}) / (1 - (\rho^{std})^2)$ .