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Optimal Green Technology Adoption and Policy Implementation

Jean-Marc Bourgeon*

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Abstract

The importance of coordination problems in the greening of the economy is analyzed using a global game approach in a simple macro-dynamic model. Two policy options to motivate firms to adopt green technology are examined: one coupling an emissions tax with subsidies that allow firms to better adapt to shocks, the other coupling an emissions tax with technology standards that reduce firms' choices. Both policies are affected by coordination problems and their effectiveness depends on firms' expectations. Since the demand for capital depends on shocks affecting network externalities, the interest rate, the path of the economy and its impact on the environment are stochastic. These paths are derived in a simple framework and illustrated by numerical simulations.

Keywords: Growth, sustainability, Technology adoption.

JEL: O33, O44, E37, Q55

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1 Introduction

Decoupling resource use and environmental discharges from economic growth is one of the main challenges facing modern economies. Among these, decarbonisation, i.e. reducing carbon dioxide (CO₂) emissions per unit of GDP, is a top priority. The historical average rate of decarbonisation (1.5%) falls short of the progress required to meet existing climate targets: according to the IPCC (2018) Special Report, under emission pathways consistent with 1.5°C global warming, CO₂ emissions decline by about 45% from 2010 levels by 2030, reaching a net zero level around 2050. To limit global warming to below 2°C, they would need to decline by about 25% by 2030 in most trajectories and reach net zero by about 2070. Non-CO₂ emissions in scenarios that limit global warming to 1.5°C show significant reductions, similar to those in scenarios that limit warming to 2°C. To operate these radical changes, firms must reconsider their production processes. Investments needed are very large and involve all sectors.

As the International Energy Agency (IEA, 2021) points out, most of the global reductions in CO₂ emissions through 2030 in its model come from technologies readily available today, but by 2050, nearly half of the reductions come from technologies that are currently in the demonstration or prototype stage. In heavy industry and long-distance transportation, the share of emission reductions from technologies still in development today is even higher. However, the diffusion of cost-effective clean technologies has been notoriously slow and below potential to date (Battisti, 2008), and investing in green technologies can be a risky undertaking. At first, there is a conversion cost: Green technologies are often more costly and less productive than the technologies that firms are using, and may require retraining of their workforces and radical changes in their supply chains. Because of network externalities and technological spillovers, optimal investment choices do not necessarily correspond to the latest or most innovative machines.¹ The profitability of a production line depends on technology spillovers, like the known-how of the workforce, and the availability of the inputs and of the maintenance services required by the technology. Since a machine is a durable good, usually used for several years, it is important that the parts needed for its operation and maintenance services are readily available in the future, which is all the more likely as the technology becomes more widespread. There is also the issue of compatibility of the technology with existing and future industry standards, which

¹Technology spillovers refer to the unintended benefits resulting from the research and development efforts of others. Network externalities refers to a situation in which the value of a technology increases with the number of its users.

determines what other machines can be employed. Moreover, the more a technology is used, the higher the research effort of the machine industry sector to improve the technology. Therefore, the optimal investment choice for a firm depends on the decisions made by all firms, and the more firms use a given technology, the more attractive it is. It is however difficult to anticipate these network effects, due to the many external factors that affect the supply chain and the supporting infrastructure of firms.

Hoevenagel et al. (2007) highlights the problems of small and medium-sized enterprises (SMEs), which account for 99.8 percent of all enterprises and dominate all sectors in the European Union, with the adoption of environmental technologies. In these companies, ownership and management are often concentrated in the same hands and these entrepreneurs are likely to suffer from significant time and task pressures. They are often risk-averse given the relatively small economic size of their enterprises, and their basic perception of environmental technologies is that of the costs and risks associated with them. They also do not have adequately trained staff who are familiar with the installation and operation of new technologies. Finally, they are generally not well informed about potential environmental technologies and rely heavily on the advice of their professional entourage, such as suppliers, who often have the same lack of information. Hoevenagel et al. (2007) also notes that a firm's network relationships influence the adoption of environmental innovations at each stage, when it gathers information about possible solutions to implementing new technologies, and that, conversely, a firm's strategic orientation influences the types of external competencies and network relationships that are developed.

In this paper, I analyze the issue of designing policies to green the economy using a simple dynamic model that incorporates these features of the coordination problem faced by firms in the European industry. More precisely, I consider an economy composed of a large number of small firms that must adopt cleaner and often newer technologies, in which network externalities and technology spillovers play an important role in determining the optimal technology choice. In the absence of a coordination mechanism, firms must make investment decisions based on their expectations about the extend of shocks to the network of machine and service suppliers. This imperfect assessment of the future economic environment is reflected in an industrial sector made up of firms with heterogeneous technological processes. This diversity is represented by two variables, one capturing the productive capacity of production lines (similar to the traditional economic definition of capital), the other their pollution intensity.

Whatever the extend of network and spillover effects, as market prices do not account for the environmental footprint of the economy, public policies must be designed

to guide firms in their investment choices. The optimal investment policy is derived by solving the problem of a social planner who can decide on the levels of investment in production capacity and pollution intensity of firms, taking into account the intertemporal trade-offs between consumption, environmental quality and (total) investment. Because production processes are imposed to firms by assumption, coordination problems, network and spillover effects are not relevant. The economy follows a first-best path to environmental neutrality which is deterministic.

A government policy that imposes its investment choices on each and every firm at all time, is not, however, a realistic depiction of actual environmental policies. I thus consider in the following a policy implementation where the regulator uses two instruments, one for each characteristic of the firms' production processes. Limitations on production capacity can be achieved through the use of an emission tax, which makes firms responsible for the pollution they emit and induces them also to invest in green technologies. I consider two alternative policies for the second instrument: another incentive policy, namely a schedule of subsidies for green investments, and a schedule of technological standards, i.e. restrictions on the pollution intensity of new machines that are tightened over time. Subsidies allow firms to better adapt to network shocks (depending on their anticipations), while requiring that machines follow demanding environmental standards reduces the firms' choices and the coordination problem. Compared to the first-best path, since investment decisions are left to firms, the effectiveness of these policies depend on firms' expectations about the shocks affecting the network of machine and service suppliers.

The coordination problem faced by firms in adapting their production process over time is framed as a dynamic global game assuming Markov perfect equilibria. Each period, firms' optimal investment strategies are guided by private signals about the future network shock. Idiosyncratic signals lead to different investment choices, but the resulting path of the economy is affected only by the uncertainty of the network effects. In other words, economic fluctuations do not depend on the dispersion of individual signals, but on the volatility of the initial shock and on the responsiveness of the network of suppliers. To assess the effects of these fluctuations on the economy, I then simplify the framework by assuming a constant marginal rate of substitution between environmental and industrial goods, exponential consumer preferences (CARA), and a Cobb-Douglas production function. This setup allows for explicit solutions of the optimal policy and its implementation under the two policy implementations.

The results show that the optimal dynamic of the economy entails three sequences. The first one corresponds to a decrease in the stock of productive capital, and thus

in GDP, if the footprint of the technologies originally employed is too heavy on the environment. This sequence corresponds to a swift catching up of investment in green technologies at the expense of the production capacity of firms. The second sequence corresponds to a stagnation of GDP, with the production capacity of the industrial sector just maintained, while investment in clean technologies is still growing fast to allow environmental quality to increase. Finally, the third sequence corresponds to a growing GDP along an environmentally neutral trajectory (supposed to be the best that can be achieved with green technologies).

To analyze the impact of the economic fluctuations that occur under the two policy options considered, I assume that consumers have rational expectations. While the shocks affecting network externalities are independent, they induce a path dependence driven by the consumers desire to smooth consumption over time. The resulting distribution of the interest rate is shown to follow a first-order autoregressive process under both policy options during the transition period to environmental neutrality. This can be explained as follows: At the beginning of the transition period, assuming that GDP and the interest rate are initially equal to their nominal values for that period, a positive network shock increases the demand for green technology, all the more so if the supporting infrastructure of the economy is responsive due to network and spillover effects. This leads to an increase in the interest rate, which in turn decreases investment in productive capital. During the transition period to environmental neutrality, the tax is such that this investment just allows firms to maintain their productivity. The increase in the interest rate thus reduces the productive capital stock to a level below its nominal level. As the decline in investment reduces the next period GDP, the willingness of consumers to lend capital is reduced, which tends to increase the next period's interest rate even more in the event of a new positive shock. The opposite trend occurs if the initial shock is negative. Given the scale of investment needed to green the economy, long lasting deviations of the economy from the optimal path are thus likely. Simulations show that this path dependence can be significant under the two policies examined, and that technology standards permit a transition to green technologies more likely in line with the regulator's intended trajectory than subsidies do. This is because producers have less flexibility to adapt their green investments to economic conditions when standards rather than subsidies are implemented.

There is a abundant literature on growth and sustainability. The literature on endogenous green growth focuses on productivity improvements and frontier innovation. This is the case in the AK paradigm where capital-knowledge accumulates with learning-by-doing (Stockey, 1998), and Lucas-like extensions (Bovenberg & Smulders,

1995), within a framework of product variety (Gerlagh & Kuik, 2007) or within the Schumpeterian growth paradigm of destructive creation and directed technical changes (Acemoglu et al., 2012), where the most productive innovations are adopted by firms as soon as they are discovered. This article focuses on the adoption of existing technologies that have knock-on effects leading to the gradual replacement of old and polluting machines with greener ones. The approach is thus close to the literature on endogenous growth viewed as a process of adoption of existing ideas and mutual imitation between firms, as exposed by Eaton & Kortum (1999); Lucas Jr & Moll (2014); Lucas (2009); Perla & Tonetti (2014).² The approach here is similar for describing the adoption of technologies: although the R&D sector is not spelled out, there is a set of existing technologies whose potential is more or less exploited depending on the proportion of firms that use them. The distribution of technology used among firms changes over time as firms' incentives to adopt new technologies evolve.³ Comparisons of policy instruments to implement an environmental objective have been quite numerous since Weitzman (1974). The main focus is on the information available to the regulator on the firms' pollution abatement costs. Few papers consider the importance of firms' expectations in achieving an environmental goal. In a recent paper, Aldy & Armitage (2020) compare an emissions tax with a cap-and-trade instrument in which firms are subject to forecasting errors in the price of pollution allowances on the secondary market, the effectiveness of the abatement technology being the same in both cases. In what follows, the firms' expectations about their economic environment are important because of network externalities that determine the effectiveness of the technology.

The remainder of the paper is organized as follows: Section 2 describes the dynamics of technology adoption, the environment and the supply of capital. Each period, total demand for capital must meet supply provided by consumers, leading to the equilibrium interest rate. Since the demand for capital depends on network shocks, the interest rate is also stochastic. Its distribution depends on the expectations of the (representative) consumer, whose preferences are affected by the quality of the environment, and by his desire to smooth his consumption over time. Section 3 presents the social planner's program, the optimal policy, and its implementation using an environmental tax supplemented by a subsidy scheme or a technological standard scheme. Both implementations are affected by network and spillover effects and are defined such

²These papers assume that each agent in the economy is endowed with a certain amount of knowledge ("ideas") and this knowledge evolves through contact with the rest of the population.

³There is also a microeconomics literature that investigates the problem of network externalities, initiated by Katz & Shapiro (1985). See Guimaraes & Pereira (2016) for a recent contribution, and Shy (2011) for a survey.

that the expected trajectory of the economy matches the deterministic first-best path. In section 4, consumer preferences and the production function are specified to better characterize how these network effects affect the path of the economy. Assuming a constant marginal rate of substitution between consumption and environmental quality, the intertemporal rate of substitution is constant during the transition to environmental neutrality. The optimal path is derived by assuming a Cobb-Douglas production function and an exponential utility function (CARA). During the transition path to environmental neutrality, assuming that consumers' expectations are rational, the distribution of the interest rate follows a first-order stochastic process and the economy a Gaussian random walk under the two policy options. These paths are illustrated with numerical simulations presented at the end of the section. The last section concludes.

2 Technology adoption, the environment and capital markets

Consider a discrete time economy composed of a continuum of firms, of total mass equal to one, that collectively produce at date t an amount q_t of output, taken as the numeraire and corresponding to the GDP. The firms' production may come from various technological processes with different environmental impacts. More precisely, I suppose that for each particular production task, firms have the choice among a large set of machines characterized by their productivity and polluting emissions level. These sets of machines evolve over time, but each period, comparing machines with similar productivity levels, the cheaper they are, the more they pollute. A firm's production line is characterized by machines acquired over time and renewed or replaced when deemed necessary. Each period, firms' technological processes may thus be very different, and to capture this heterogeneity in a simple way, the process of firm i at date t is described by two parameters, k_{it} and x_{it} , dubbed 'productive capacity' and 'green technology index' (or 'technology mix') respectively. The productive capacity of the technology is similar to the standard economic notion of capital: it corresponds to the input that, once combined with labor, allows the firm to produce items or services. It yields a gross revenue $q_{it} = Q(k_{it}, \ell_{it})$ where ℓ_{it} is the firm's employment level, and Q a production function, homogeneous of degree one and satisfying the Inada conditions. The green technology index captures the firm's effort to diminish the environmental

impact of its production, given by $\iota_{it}q_{it}$ where

$$\iota_{it} = \max\{0, \varphi - \xi x_{it}/q_{it}\} \quad (1)$$

corresponds to the emission intensity of its production process. It is positive because I suppose that green technologies can only reduce the firm's emissions. The emission intensity diminishes with x_{it} , with a maximum equal to φ if $x_{it} = 0$. Hence, the higher x_{it} , the lower the environmental impact of the firm's production. This impact depends on ξ , a positive parameter that measures the environmental efficiency of green investments.

Each period, part of the machines used by firms is depreciated, due to wear and tear, or because some of them are obsolete or so deteriorated that maintenance costs are too high. I suppose a constant depreciation rate $1 - g$ that affects both the productive capacity and the green technology index. To renew and adapt their production process, firms buy new machines that correspond to an investment in productive capacity and/or in environmental quality. Firm i 's investment in productive capacity, I_{it} , leads to a dynamic of k_{it} given by $k_{it+1} = gk_{it} + I_{it}$. Similarly, firm i 's spending in environmental quality, κ_{it} , modifies its technology mix according to $x_{it+1} = gx_{it} + \kappa_{it}$.⁴

Depending on their technology choices, firms benefit from (or suffer from the lack of) network externalities related to their input suppliers, maintenance services (how easy it is to find specific inputs and parts to service the machines), the know-how of workers, and more generally, the peculiarities of the machines they use. The more widespread is the technology used, the less a firm encounters problems and the easier it is to achieve a production target.

As an illustration, consider the problem of a firm having to renew (part of) its fleet of vehicles. Electric vehicles (EVs) are an option, but their main drawback (in addition to being pricey) is a limited operating range due to the battery capacity. To reduce the dependency on charging stations, different types of EVs have emerged with different impacts on the environment: in addition to electric vehicles that are entirely battery-powered, there are several so-called hybrid vehicles, such as plug-in hybrids that operate on a battery with the help of an internal combustion engine (ICE) or hybrids that have their battery charged by an ICE. Similarly, fuel cell electric vehicles that produce electricity by combining hydrogen from their tanks with oxygen from

⁴These investments may be negative, i.e. firms may consider resealing some of their machines. Observe that we can have $I_{it} = 0$ and $\kappa_{it} > 0$, in which case firm i buys pollution abatement equipment, i.e. devices that do not produce any items but reduce the polluting emissions of the firm.

the air have recently emerged, but the hydrogen distribution network is very limited. This multiplicity of options makes it difficult for users to reach a decision, even though some of these technologies require a large number of users to make the charging station networks and all other supporting services profitable, which hinders their development in initial stages.

Hence, depending on the environmental characteristics of the machines chosen, network and spillover effects are more or less significant. These effects are not investment costs (due to the price of the machines), but losses or savings related to the ease or difficulty of their use given the existing supporting infrastructure of the economy (e.g., delays due to maintenance services, charging times and location of charging stations in the case of EVs). I suppose that under the Laissez-faire situation, these monetary amounts are given by⁵

$$G_{it} = G(x_{it}; \mu_{t-1}, K_{t-1}, \omega_t) = (\mu_{t-1} + \lambda K_{t-1} + \omega_t) x_{it} - x_{it}^2/2 - \bar{G}_t, \quad (2)$$

for firm i at date t , where $\mu_{t-1} \equiv \int_0^1 x_{it-1} di$ is the green technology index (GTI) of the economy at the previous period, $K_{t-1} \equiv \int_0^1 \kappa_{jt-1} dj$ the sum of green investments made simultaneously by firms at $t-1$ for period t , and the realization ω_t of a time-independent normally distributed noise, with variance σ_ω^2 , that summarizes the many unmodeled shocks affecting the diffusion of technology.⁶ G_{it} may be positive or negative, and the last term, \bar{G}_t , ensures that the average network gain is nil.⁷

GTI level μ_{t-1} reflects the set of the most commonly used machines at $t-1$. Through knowledge diffusion and supply chain adaptation over time, they are well served by the existing service network. Date- t network externalities depend on μ_{t-1} because only part of the capital stock is renewed in each period, and thus the remaining part still requires a supporting infrastructure close to the existing one, which leads firms to make conservative investments (e.g. ICE or hybrid vehicles may reveal the most convenient

⁵This expression is used for analytical convenience as it leads to close-form solutions of the coordination game described below. It can be considered as a second-order approximation (around zero) of a more general expression.

⁶These shocks may come from the service network, input suppliers or machines producers, reflecting the expectations of these operators on technology development. I assume that these shocks are due to factors external to the economy, and to streamline the analysis, that they affect only network effects and not firms' production level $Q(k_{it}, \ell_{it})$, $i \in [0, 1]$.

⁷Since green investments are not productive per se, firms' monetary gains and losses from network externalities must cancel out. For the sake of simplicity, I suppose that they offset each other in each period. Of course, that does not mean that firms must neglect their signal about ω_t . On the contrary, by following the optimal strategy that depends on their private information, as described below, they maximize their expected profit from network effects, a profit which is null on average.

choices because they rely on the existing network of gas and service stations).⁸ μ_{t-1} is known by all firms at date $t - 1$, when they decide to invest for period t . Changes in the relative attractiveness of green technologies from one period to the next (e.g., due to technological improvements or exogenous changes in the supporting infrastructure) are summarized by ω_t , the realization of the shock $\tilde{\omega}_t$ which is unknown at $t - 1$, when firms make their investments. The adaptation of the service network depends on this shock and on choices made by firms at $t - 1$, a sum equal to K_{t-1} of investments that are made simultaneously. This adaptation may be more or less rapid, depending on the extent of immediate technology spillovers which is captured by parameter $\lambda \in [0, 1)$. If $\lambda = 0$, the first term of (2) reduces to $\mu_{t-1} + \omega_t$, i.e. the former average mix plus a noise which may be positive or negative. This noise is not amplified or reduced by the firms' investments because firms do not expect that workers or the supporting infrastructure will adapt rapidly. When immediate demand spillovers are at work ($\lambda > 0$), the first term of (2) also depends on the most recent investments made by all firms, K_{t-1} . The economic infrastructure evolves more rapidly the larger λ is, the extreme case where λ is close to 1 corresponding to an immediate adaptation of the workers and the supporting infrastructure to new investment choices.⁹

Without shocks, firms would not face any uncertainty about the choice of machines since μ_{t-1} is known. Coordination problems arise from uncertainty about the shock affecting the supporting infrastructure and its responsiveness to firms' investments. Firms must anticipate their optimal technology mix that will result from the choices made simultaneously by other firms according to their expectations.

It is easily shown that the ideal technology mix at date t is¹⁰

$$x_t^* \equiv \mu_{t-1} + \lambda K_{t-1} + \omega_t. \quad (3)$$

⁸It is different when firms are constrained by technological standards that prevent them from replacing old machines with new ones that have the same negative impact on the environment. See Section 3.2 for details.

⁹It is likely that the value of λ is low. As stressed by Battisti (2008), a consistent literature has shown that, even when a clean or a cost-reducing technology is readily available in the market, its spreading takes several years.

¹⁰As ω_t is normally distributed, x_t^* can take negative values, and all the more probably if μ_t is small. I will neglect this possibility in the following by assuming that μ_t remains far enough from 0. In the Laissez-faire situation, this implies that an environmental catastrophe eventually occurs while μ_t is positive, hence an initial state of environmental quality (at $t = 0$) which is low.

Indeed, using the fact that the average network effect is zero, it comes

$$\int_0^1 G(\mu_{t-1}, K_{t-1}, x_{it}, \omega_t) di = \int_0^1 (x_t^* x_{it} - x_{it}^2/2) di - \bar{G}_t = 0$$

which leads to

$$\bar{G}_t = x_t^* \mu_t - (\sigma_{x_t}^2 + \mu_t^2)/2$$

where $\sigma_{x_t}^2 \equiv \int_0^1 (x_{jt} - \mu_t)^2 dj$ is the variance of the technology indexes at date t . Substituting in (2) and collecting terms gives

$$G_{it} = [(x_t^* - \mu_t)^2 + \sigma_{x_t}^2 - (x_{it} - x_t^*)^2]/2, \quad (4)$$

where $(x_{it} - x_t^*)^2/2$ corresponds to firm i 's loss due a mix x_{it} different from the ideal one x_t^* . Observe that if all firms were equipped with the ideal mix, i.e. $x_{it} = x_t^*$ for all i , we would have $G_{it} = 0$ for all i , meaning that no firm would gain or lose from network effects due to perfect coordination. However, firms cannot perfectly assess the ideal mix x_t^* because they don't know at date $t - 1$ the realization of the random shock ω_t and the total of green investments K_{t-1} made simultaneously by all firms. Hence, firms must somehow anticipate the extent of the resulting total green investment when making their own, which leads to an intertemporal coordination problem that is formalized as a succession of global games taking place each period.¹¹ In the following, this dynamic setup is solved sequentially, focusing on Markov perfect equilibria where x_t^* is a state variable. In period t , firms must anticipate the realization of the next period ideal technology mix, \tilde{x}_{t+1}^* , a random variable which distribution depends on the other firms behavior. More specifically, firm i infers the other firms decisions according to its beliefs on the shock affecting the diffusion of technology. I suppose that these beliefs are unbiased idiosyncratic private signals of the shock: they are formed according to $\tilde{\eta}_{it} = \omega_{t+1} + \tilde{v}_{it}$, where ω_{t+1} is the date- $t + 1$ realization of this shock, \tilde{v}_{it} a time-independent noise, normally distributed with variance σ_v^2 , verifying $\mathbb{E}[\tilde{v}_{it}] = \mathbb{E}[\tilde{v}_{it}\tilde{v}_{jt}] = 0$ for all i, j , and $\int_0^1 v_{it} di = 0$.

Consider the Laissez-faire situation from date $t = 0$ (the present period) assuming that all firm i is endowed with technology $(k_{i0}, x_{i0}) > 0$, and thus that $\mu_0 > 0$. Each following period t , given w_t and r_t the date- t wage and interest rate, firm i plans its technological investments and employment level $\{(\kappa_{it}, I_{it}, \ell_{it}), t = 0, 1, 2, \dots\}$ to maximize the expected discounted sum of its profits $\mathbb{E}_t[\sum_{h \geq 0} (\prod_{\tau=1}^h \delta_{t+\tau})(q_{it+h} + G_{it+h} -$

¹¹These coordination problems with strategic complementarity are known as “beauty contests” (see, e.g., Angeletos & Pavan, 2004, 2007 and Morris & Shin, 2002).

$w_t \ell_{it+h} - I_{it+h} - \kappa_{it+h})]$ where $\delta_t = 1/(1 + r_t)$ is the date- t discount factor.¹² With a total supply of labor equal to one each period, i.e. $\int_0^1 \ell_{it} di = \ell_t = 1$ for all t , it is shown in the appendix that

Proposition 1 *Firms' equilibrium investment in productive capacity at time t is given by*

$$I_{it} = q'^{-1}(1 + r_t - g) - gk_{it} \quad (5)$$

where $q(k) \equiv Q(k, 1)$ is an increasing and concave function, leading to $k_{it+1} = k_{t+1} = q'^{-1}(1 + r_t - g)$ for all i . The equilibrium wage is given by $w_t = q(k_t) - r_{t-1}k_t$. Assuming μ_t large enough, firm i 's investment strategy in green technology satisfies

$$\kappa_{it} = \mathbb{E} [\tilde{x}_{t+1}^* | \eta_{it}, \mu_t, r_t] - gx_{it} - 1 - r_t + g, \quad (6)$$

leading to

$$\kappa_{it} = g(\mu_t - x_{it}) + \frac{(1 - g)(\mu_t - 1) + \eta_{it} - r_t}{1 - \lambda} \quad (7)$$

at equilibrium. The resulting firms' technology mixes at $t + 1$ are normally distributed with mean

$$\mu_{t+1} = g\mu_t + \frac{(1 - g)(\mu_t - 1) + \omega_{t+1} - r_t}{1 - \lambda}, \quad (8)$$

corresponding to the date- $t + 1$ GTI, and standard deviation $\sigma_{x_t} = \sigma_x \equiv \sigma_v / (1 - \lambda)$ for all t .

Because firms have the same production function Q , the productive capacity of all firms is the same, determined by choices that are guided unambiguously by the interest rate. This is not the case for their spending in green technology (6) that depend on individual estimates of the most efficient mix $\mathbb{E} [\tilde{x}_{t+1}^* | \eta_{it}, \mu_t, r_t]$. The trade-offs that the firm is facing when investing are the following. The date- t borrowing cost of one monetary unit is $1 + r_t$. Invested in productive capital, its return is equal to the sum of the marginal productivity of capital, $q'(k_{t+1})$, and g , the remaining value of capital (the rest being depreciated). With investment I_t , the date- $t + 1$ capital stock is equal to $k_{t+1} = gk_t + I_t$, hence an optimal productive investment that solves $q'(gk_t + I_t) + g = 1 + r_t$, a deterministic equation. (6) is derived similarly: a monetary unit invested in green technology has a marginal return equal to the sum of the marginal network gain $dG_{it+1}/dx_{it+1} = \tilde{x}_{t+1}^* - x_{it+1}$ and g . With investment κ_{it} , the firm date- $t + 1$ green index is given by $x_{it+1} = gx_{it} + \kappa_{it}$. Firm i 's investment rule is thus to

¹² $\mathbb{E}_t[\tilde{X}]$ is a shorthand for $\mathbb{E}[\tilde{X} | \mathcal{I}_t]$, i.e. the expectation of the random variable \tilde{X} given the information \mathcal{I}_t available at date t .

equalize $1 + r_t$ to the expected return $\mathbb{E} [\tilde{x}_{t+1}^* | \eta_{it}, \mu_t, r_t] - (gx_{it} + \kappa_{it}) + g$, since \tilde{x}_{t+1}^* is unknown. Firm i 's expectation takes into account the fact all other firms are behaving the same way and form similar expectations given their signals. At equilibrium, firm i adopts the green investment strategy (7) that is a linear function of publicly observable variables (the GTI level μ_t and the prevailing interest rate r_t), its own technology index x_{it} , and its private signal η_{it} .¹³ The first term in (7), $g(\mu_t - x_{it})$, indicates that firm i makes-up for the difference between its green index and that of the economy on its undepreciated capital stock. The second term shows that to replace old capital, the firm considers the marginal net return of investing at the GTI level given its appraisal of the network shock, $(1 - g)\mu_t + g - (1 + r_t) + \eta_{it}$, multiplied by the magnification effect of the network externality, $1/(1 - \lambda)$, since all firms operate the same way. For firms with a low mix and a high signal, (7) corresponds to buying machines with a low environmental impact. For those with a high mix and a low signal, their investment is directed in the opposite direction: they save on new equipment spending by buying less expensive brown technologies. Thanks to unbiased idiosyncratic signals, while firms' green indexes are distributed normally around the GTI level, the network shock ω_t is correctly taken into account by firms on average as shown by (8).¹⁴ This equation also shows that, as the other relevant variables, this shock is magnified by $1/(1 - \lambda)$, a factor that is larger the more responsive the supporting infrastructure. Hence, the more reactive is the supporting infrastructure, the larger are the effects of the network shock and of the cost of capital r_t on the next period GTI. However, the interest rate is determined on financial markets and its equilibrium value depends on the total demand of capital, and thus on ω_t as discussed below.

The expected dynamic of the GTI under Laissez-faire (8) can be negative, in which case firms acquire increasingly dirty technologies, or positive if μ_t is large enough, greater than $[1 + r_t/(1 - g)]/\lambda$, i.e. if the technology spillovers are sufficiently large, the depreciation rate large and/or the interest rate low. Indeed, as firms have to renew their machines, they may invest in less polluting ones if they expect that the supporting infrastructure and the workforce know-how adapt rapidly. This is however very unlikely if the initial GTI level μ_0 is low. Indeed, firms will only purchase green

¹³The proof follows Angeletos & Pavan (2004). Morris & Shin (2002) show that this linear, symmetric, rational-expectations strategy leads to the unique (per period) equilibrium.

¹⁴Observe that (8) is not biased by the lack of information. This is because the signals are private and affected by independent idiosyncratic noises. If firms also shared a public signal, their choices would be distorted in the same direction, as would the resulting dynamics of μ_t , to an extent that depends on the relative reliability of the public signal: the more precise the signal, the larger the distortion.

technologies in the absence of governmental incentives if these technologies are already the most widely used in the economy, and thus the most easily exploitable.

Due to the idiosyncratic shocks on believes, v_{it} , firms have different expectations on x_{t+1}^* , hence choose machines with similar productive capacities but different environmental impacts. These discrepancies lead to a Gaussian distribution of firms' green indexes around the GTI level, given by $x_{it} = \mu_t + v_{it}/(1 - \lambda)$, resulting in an industrial sector that can be thought of as a 'cloud' of firms with green technology levels that are drawn each period from a normal distribution centered on GTI level μ_t with standard deviation $\sigma_v/(1 - \lambda)$ which is all the greater as λ is large.

2.1 Environmental dynamic

Production generates pollution that deteriorates the quality and the availability of environmental goods and services provided by Nature. These effects are summarized in the dynamic of the environmental quality (EQ) index e_t , which is given by

$$e_{t+1} = \theta e_t + \hat{e} - \iota_t q_t \quad (9)$$

where $\theta \in (0, 1)$ is the environmental inertia rate, \hat{e} the per-period maximum regeneration capacity of the environment, and

$$\iota_t = \int_0^1 q_{it} \iota_{it} di / q_t \quad (10)$$

the emission intensity of the economy at date t , which measures the total environmental damage per unit of GDP. Without human interference ($\iota_t = 0$), the EQ index is at its pristine level $e_N = \hat{e}/(1 - \theta)$. More generally, it comes using $\iota_t \approx \varphi - \xi \mu_t / q_t$ that the dynamic of EQ follows approximatively the linear first-order recursive equation

$$e_{t+1} = \theta e_t + \xi \mu_t - \varphi q_t + \hat{e}. \quad (11)$$

Since green technologies can only reduce emissions and do not allow for direct improvement of EQ, environmental neutrality is the best society can achieve.

Definition 1 (Environmentally Neutral Path) *The economy has reached at date T an Environmentally Neutral Path (ENP) if for all $t \geq T$, $\iota_t = 0$.*

An ENP is a sustainable situation in which the emission intensity of the economy is nil. Along an ENP, thanks to the natural regeneration capacity of the environment,

EQ increases and tends toward its pristine level e_N .

2.2 Consumers and capital markets

Capital supply comes from consumers who maximize their intertemporal utility by arbitraging between consumption and savings each period. Consumers derive well-being from manufactured and environmental goods and services. Environmental goods are freely available and their consumption is subsumed by e_t . The consumption of manufactured goods is denoted by c_t . I suppose that consumers do not try to modify the environment through their consumption and saving plans.¹⁵ Consumers' per-period preferences are represented by a concave utility function $u(c_t, e_t)$, and their behavior is modeled by considering a representative consumer whose saving and consumption plans solve

$$\max \mathbb{E}_t \left\{ \sum_{h=0}^{+\infty} \beta^h u(\tilde{c}_{t+h}, \tilde{e}_{t+h}) : \tilde{c}_{t+h} = \tilde{R}_{t+h} + \tilde{r}_{t+h-1} \tilde{S}_{t+h-1} - \tilde{s}_{t+h}, \tilde{s}_{t+h} = \tilde{S}_{t+h} - \tilde{S}_{t+h-1} \right\} \quad (12)$$

each period, where R_t is her date- t revenue, S_{t-1} her savings from the previous period, $r_{t-1}S_{t-1}$ the corresponding date- t capital earnings, s_t the savings adjustment of period t , and β the psychological discount factor. Solving (12), we obtain

Lemma 1 *The consumption rule that solves (12) satisfies*

$$\frac{\partial u(c_t, e_t) / \partial c}{\beta \mathbb{E}_t [\partial u(\tilde{c}_{t+1}, \tilde{e}_{t+1}) / \partial c]} = 1 + r_t \quad (13)$$

at each date t .

Equation (13) corresponds to the Ramsey-Euler rule which states that the expected intertemporal consumption rate of substitution (IRS) is equal to the return of capital each period. It also defines the supply function of capital, while (5) and (8) are the demand side coming from firms. At equilibrium, aggregate production net of investment must be equal to total consumption of manufactured goods, i.e.

$$c_t = q_t - I_t - K_t = q(k_t) - (k_{t+1} - gk_t) - (\mu_{t+1} - g\mu_t). \quad (14)$$

¹⁵This could be because they consider that they are too numerous for their individual behavior to have a significant impact on it.

Hence, at equilibrium on financial markets, i.e. when the total demand for capital by firms meets the supply provided by consumers, the interest rate is affected by the network shocks. Its distribution over time depends on accumulated capital, on the consumer's expectations whose preferences are affected by the quality of the environment, and on her desire to smooth consumption over time. Since productive investment depends on the interest rate, GDP is also affected by these fluctuations and thus follows a random path. As shown below, these properties of the general equilibrium under Laissez-Faire are also present when an environmental regulation is implemented as long as network effects are present.

3 First-best path of the economy and policy implementation

Because market prices do not account for the environmental footprint of the economy, public policies must be designed to guide firms in their investment choices. In this section, I first consider the problem of a benevolent social planner who can force firms to adopt the socially optimal set of machines in each period. Since production processes are imposed on firms, coordination problems and network effects are irrelevant. In the absence of frictions and shocks, the economy follows a deterministic path to environmental neutrality that corresponds to the first-best.¹⁶

I then examine the problem of policy implementation. The optimal regulation requires at least two instruments. Two policy options are considered that both involve the use of an emission tax to limit the productive capacity of firms, but differ in their instrument promoting green technologies: subsidies or technological standards. Subsidies allow firms to better adapt to network shocks (depending on their expectations), while requiring that machines follow demanding environmental standards reduces the firms' choices and the coordination problem. However, compared to the first-best path, both policies are affected by network and spillover effects and their effectiveness depends on firms' expectations. Both policy schemes are defined such that the expected trajectory of the economy matches the first-best path.

¹⁶While the supporting infrastructure can adapt well in advance to the policy, shocks may still exist, requiring the social planner to revise his or her policy plan each period. I neglect these shocks in the determination of the optimal policy because they do not cause a coordination problem.

3.1 first-best path of the economy

Consider a social planner who determines the dynamics of the productive capacity and of the GTI of the economy to maximize consumer welfare given their impact on consumption and the environment. I suppose that total investment cannot be negative so that this maximization is constrained by¹⁷

$$k_{t+1} \geq gk_t, \quad (15)$$

$$\varphi q(k_t) \geq \xi \mu_t, \quad (16)$$

where the last inequality corresponds to the ENP constraint. In absence of shocks, the planner's problem is to solve

$$\max_{\{\mu_t, k_t, e_t\}_{t>0}} \left\{ \sum_{t=0}^{+\infty} \beta^t u(c_t, e_t) : (11), (14) - (16) \right\} \quad (17)$$

given e_0, k_0 and μ_0 . I suppose $\xi \mu_0 < \varphi q(k_0)$, so that society is not already on a ENP. Denoting

$$h_t = \frac{\partial u(c_t, e_t) / \partial c}{\beta \partial u(c_{t+1}, e_{t+1}) / \partial c} \quad (18)$$

the date- t IRS, it is shown in the appendix that:

Proposition 2 *The optimal path of the economy $\{\mu_t^*, k_t^*, c_t^*, e_t^*\}_{t>0}$ satisfies (11), (14),*

$$q'(k_{t+1}^*) \leq \frac{h_t^* - g}{1 - (h_t^* - g)\varphi/\xi}, \quad (19)$$

for all $t > 0$, with an equality when (15) is not binding and $k_{t+1}^* = gk_t^*$ otherwise, and

$$h_{t+1}^*(h_t^* - g - \theta) + \theta g = \xi \frac{\partial u(c_{t+2}^*, e_{t+2}^*) / \partial e}{\partial u(c_{t+2}^*, e_{t+2}^*) / \partial c}. \quad (20)$$

when constraint (16) is not binding and $\mu_t^* = q(k_t^*)\varphi/\xi$ otherwise, where h_t^* is the date- t IRS evaluated along the optimal path.

The optimal policy is thus characterized by a sequence of IRS $\{h_t^*\}_{t>0}$ that solves (20) when neither (15) nor (16) is binding. To interpret this condition, first observe

¹⁷Constraint (15) implies that the social planner cannot forbid the use of machines already bought that are too environmentally damaging. Also, as the policy should result in green technology investments, the constraint $\mu_{t+1} \geq g\mu_t$ is neglected (It is shown that it never binds in the illustrative example).

that investment in green technology at time t has an impact on the environment after two periods, i.e. at $t + 2$. The IRS over these two periods is given by $h_{t+1}h_t$. Absent stock effects, i.e. $g = \theta = 0$, (20) states that the optimal date- t two-period IRS must be equal to the marginal rate of substitution (MRS) of consumption for EQ at date $t + 2$. The regenerative capacity of the environment ($\theta > 0$) and the durability of capital ($g > 0$) allow for larger IRSs, i.e. larger increases in consumption over time. Condition (19) indicates that when (15) is not binding, the IRS is larger than the rental rate of capital $g + q'(k_{t+1})$ due to the impact of production on the environment (i.e. unless $\varphi = 0$).

3.2 Policy implementation

The first-best path is derived assuming that it is possible to dictate their production processes to firms. This is hardly possible (let alone desirable) in practice. To implement this path, actual regulations must entail restrictions on the productive capacity of firms (since the IRS must be larger than the rental rate of capital at the optimum) and provisions to improve the GTI of the economy. Restrictions on production capacity can be achieved through the use of an environmental (carbon) tax. This policy reduces investment in production capacity and also provides an incentive for firms to invest in green technologies. To further promote the use of green technologies, the government may supplement this environmental tax with a subsidy program for the acquisition of environmentally friendly machinery, or with environmental standards that new machinery must meet. In both cases, firms having some leeway in their investment choices, coordination problems are at work due to the shocks affecting network externalities.

Subsidy programs

Suppose that the government implements its environmental policy using only incentive instruments, namely an emissions tax scheme $\{\tau_t\}_{t>0}$ and a green technology subsidy scheme $\{z_t\}_{t>0}$. Given this policy, firm i 's per period profit becomes

$$\pi_{it} \equiv Q(k_{it}, \ell_{it}) + G_{it} - w_t \ell_{it} - I_{it} - \kappa_{it}(1 - z_t) - \tau_t(\varphi Q(k_{it}, \ell_{it}) - \xi x_{it})$$

where the last term corresponds to the environmental tax payment, and the term $\kappa_{it}(1 - z_t)$ to the net payment for green technology investment. It is shown in the appendix that

Lemma 2 *Under a tax-subsidy scheme $\{\tau_t, z_t\}_{t>0}$, firms' investment in productive capacity at date t satisfies*

$$q'(k_{t+1}) = \frac{1 + r_t - g}{1 - \tau_{t+1}\varphi}, \quad (21)$$

and the dynamic of GTI is given by

$$\mu_{t+1} = g\mu_t + \frac{(1 - g)(\mu_t - 1) + \omega_{t+1} - r_t + \xi\tau_{t+1} + (1 + r_t)z_t - gz_{t+1}}{1 - \lambda}. \quad (22)$$

The impact of the environmental tax on productive capital appears in the denominator of (21): the higher the tax level, the lower the denominator and thus the higher the marginal productivity of k_{t+1} , i.e. the lower its level. As expected, the environmental tax also has a positive impact on the dynamic of GTI (22). The difference with the Laissez-faire dynamic (8) appears in the numerator, given by the additional terms $\xi\tau_{t+1} + (1 + r_t)z_t - gz_{t+1}$. Interestingly, the subsidy policy has countervailing effects: on the one hand, the current subsidy level z_t impacts positively GTI, but that of the next period decreases it: indeed, the anticipation of a large subsidy in the next period encourages firms to postpone their green investments.

Environmental standards

As an alternative to the subsidy policy, consider that the government decides to complement the environmental tax program with environmental standards, i.e. policy instruments that constrain investment choices. These standards limit the set of machines that can be used or that are allowed to be offered by machine suppliers on the market. They correspond to restrictions on the pollution intensity of the machines that are tightened over time.¹⁸ Denote by $\underline{\kappa}_t$ the date- t green index targeted for new machines by the government with a technological standard policy. Absent network effects, firms would renew their old machines (a proportion $1 - g$ of their stocks) with new ones that just meet the standard. The corresponding dynamic of firm i 's green index would thus be given by $x_{it} = gx_{it-1} + (1 - g)\underline{\kappa}_t$, and summing over all firms, GTI would evolve according to $g\mu_{t-1} + (1 - g)\underline{\kappa}_t$. Taking into account networks effects that induce firms to buy the same machines, they have to anticipate an ideal technology mix given by

$$x_t^* = g\mu_{t-1} + (1 - g)\underline{\kappa}_t + \lambda K_{t-1} + \omega_t. \quad (23)$$

¹⁸This is the case in the European Union which imposes emission standards for vehicles that have evolved in stages of 4 to 5 years (from the Euro 1 standard in 1992 to the Euro 6 standard currently in force for light-duty vehicles).

Compared to (3), the index of the most used machines in the last period, μ_{t-1} , is replaced by $g\mu_{t-1} + (1-g)\underline{\kappa}_t$ in (23) due to the governmental restrictions on new machines that prohibit the replacement of old machines with ones that have the same environmental impact. The remaining terms in (23) are the same as in (3): the ideal mix depends on the shock that will affect the service network, ω_t , and on its sensitivity λ to total green investment K_{t-1} .

Lemma 3 *With a tax-standard policy $\{\tau_t, \underline{\kappa}_t\}_{t>0}$, firms' investments satisfy (21) and*

$$\kappa_{it} = g(\mu_t - x_{it}) + \frac{(1-g)(\underline{\kappa}_t - 1) + \eta_{it} - r_t + \xi\tau_{t+1}}{1-\lambda}.$$

The resulting firms' technology mixes at $t+1$ are normally distributed with mean

$$\mu_{t+1} = g\mu_t + \frac{(1-g)(\underline{\kappa}_t - 1) + \omega_{t+1} - r_t + \xi\tau_{t+1}}{1-\lambda}, \quad (24)$$

and standard deviation σ_x for all t .

Hence, the environmental tax is the same under the two policy options and produces the same positive effect on the GTI dynamics (24) and (22). The main difference between these dynamics is that the term $(1-g)(\mu_t - 1)/(1-\lambda)$ in (22) is replaced by $(1-g)(\underline{\kappa}_t - 1)/(1-\lambda)$ in (24), which anchors the GTI level to its first-best path under the standard policy (assuming, of course, that $\underline{\kappa}_t$ is adequately chosen). Therefore, although they have the same first term $g\mu_t$, the path dependence of GTI can be expected to be reduced under the standard policy relative to the subsidy policy, i.e., the standard policy should be a more effective instrument than the subsidy policy at driving the greening of the economy.

Both (22) and (24) show that whatever the implementation chosen by the government, the shock ω_{t+1} affects the diffusion of technology, rendering the path of GTI stochastic. Under both policy regimes, this uncertainty affects EQ but also financial markets through the total demand of capital, hence the interest rate. As this uncertainty is detrimental for the consumer, the optimal path of the economy should be revised to account for the cost of this risk (leading to second-best paths). The aim of this section being to assess and compare the impact of network and spillover effects on the path of the economy using different policy options, suppose for the sake of argument that the social planner wants the economy to follow its first-best path in expectation whatever the policy option that is implemented.

To derive the rational expectation equilibrium (REE), suppose also that he can credibly commit to implement the corresponding tax-subsidy levels $\{\tau_t, z_t\}_{t>0}$ or tax-standard levels $\{\tau_t, \underline{\kappa}_t\}_{t>0}$ as derived at $t = 0$ (they are thus open loop policies allowing economic agents to have consistent expectations).¹⁹ It is shown in the appendix that

Proposition 3 *The tax-subsidy and the tax-standard schemes that implement the first-best path in expectation are given by*

$$\tau_{t+1} = (h_t^* - g)/\xi, \quad (25)$$

$$z_t = [(1 - \lambda)\mu_{t+1}^* - (1 - \lambda g)\mu_t^* - (h_t^* - \mathbb{E}[h_t]) + gz_{t+1}]/\mathbb{E}[h_t], \quad (26)$$

and

$$\underline{\kappa}_t = \frac{(1 - \lambda)(\mu_{t+1}^* - g\mu_t^*) - (h_t^* - \mathbb{E}[h_t])}{1 - g}, \quad (27)$$

for all $t > 0$, with $\mathbb{E}[h_t] = g + q'(k_{t+1})(1 - \tau_{t+1}\varphi)$ when $t < t_0$, and $\mathbb{E}[h_t] = h_t^*$ otherwise. We have $\lim_{t \rightarrow \infty} z_t < 0$.

The emissions tax (25) is directly deduced from (19) and (21) in the case the constraint $k_{t+1} \geq gk_t$ is not binding. When the stock of productive capital is large, firms do not renew the part that is depreciated, leading to $k_{t+1} = gk_t$. Compared to the optimal path, it corresponds to an excess of productive capital and a low interest rate at equilibrium, hence $\mathbb{E}[h_t] < h_t^*$. This discrepancy appears in (26) and (27), indicating that the subsidies and the standards are negatively affected when $k_{t+1} = gk_t$. Proposition 3 also shows that the subsidy schedule entails negative values passed a certain date, i.e. it becomes a tax scheme. This is due to the herd behavior of firms generated by the network effects: firms over-invest in green technology compared to the optimal path once GTI is high enough. As noted earlier, the expected dynamic of GTI under Laissez-faire is positive if $\mu_t > [1 + r_t/(1 - g)]/\lambda$. With environmental taxes as given by (25) and absent a complementary subsidy or standard policy, this condition becomes $\mu_t > [h_t^* - (1 + r_t)]/[(1 - g)\lambda]$, i.e. $\mu_t > 0$ since $\mathbb{E}[r_t] = h_t^* - 1$. Hence, firms' investment in green technology could be greater than $g\mu_t$, the level necessary to renew the green capital that is depreciated, without further governmental intervention than the environmental tax. However, the resulting investment could still be lower than the optimal next period level μ_{t+1}^* . More generally, it is different from it, and

¹⁹An open-loop policy is designed before the regulatory period and is not revised thereafter: the levels of taxes and subsidies or standards that have been chosen for each date by the regulator are implemented regardless of the actual path followed by the economy.

a policy complementing the environmental tax is necessary to guide firms along the optimal path. As GTI increases, firms are very likely to over-invest after a while, and it is then optimal to tax green investment.²⁰

Propositions 3 and 2 define the first-best path and policy schemes that drive the economy along paths whose expectation matches the optimal path. As these paths are random walks, it is necessary to specify the model to examine their properties in more detail. This is done in the next section.

4 Illustrative example

To assess the extent of the fluctuations affecting the economy, I consider in this section a framework that allows for explicit solutions of the optimal policy and its implementations. First, Lemma 4 shows that when MRS is constant, the optimal dynamic of the economy entails at most 3 sequences. The first corresponds to a decrease in the stock of productive capital, and thus in GDP. This sequence is likely to occur if the footprint of the technologies originally employed is too heavy on the environment. It corresponds to a swift catching up of investments in green technologies at the expense of the productive capacity of firms. The second sequence corresponds to a stagnation of GDP, with the productive capacity of the industrial sector just maintained, while investment in clean technologies is still growing fast to allow EQ to increase. Finally, the third sequence corresponds to a growing GDP along an ENP.

The optimal path is then fully specified Proposition 4 assuming a CARA utility function and a Cobb-Douglas production function. The policy implementations are examined under these assumptions and assuming rational expectations on the part of consumers. Proposition 5 shows that during the transition period to environmental neutrality, the distribution of the interest rate can be approximated by a first-order autoregressive process and the path of the economy by Gaussian random walks under the two policy options. Their parameters are specified and compared with simulations.

²⁰Because the first-best policy imposes restrictions in the long term, the public intervention is long lasting and results in taxing both polluting emissions and green investments. The government may consider alternatively a policy without public intervention in the long run, perhaps as soon as an ENP is reached. The resulting over-investment in green technology allows for larger productive investments than optimal levels, and thus an higher GDP level in the long run. However, because consumption of manufactured goods is reduced in the meantime, it is detrimental for consumers. The government may also consider that, for some political reason, the first-best environmental tax is too harsh on immediate consumption, and take advantage of network effects to impose at the beginning of its intervention lower environmental tax levels and larger green subsidies than those prescribed by the first-best policy.

4.1 Constant MRS and the optimal policy

To characterize the optimal policy, assume that the consumption of manufactured and environmental goods and services can be subsumed in a ‘global wealth index’ $y_t \equiv c_t + pe_t$ where p is the constant value of the environment, so that the consumer’s MRS is the same whatever the GDP, equal to p .²¹ With a constant MRS, (20) simplifies to

$$h_{t+1}^* h_t^* - h_{t+1}^*(g + \theta) + \theta g = p\xi, \quad (28)$$

which must hold for all t as long as (16) is not binding. Without stock effects, i.e. $g = \theta = 0$, we would have $h_{t+1}^* h_t^* = p\xi$ for all t , and thus $h_t^* = \sqrt{p\xi} \equiv \bar{h}$, a constant (as h_t must be positive, and ruling out cyclical solutions that are suboptimal since u is concave). If $\beta > 1/\sqrt{p\xi}$, i.e. if consumers are not too impatient, this corresponds to $\bar{h} > 1$, hence an increasing path of the global wealth index y_t . More generally, a constant IRS that solves (28) is a root of $P(\bar{h}) = 0$ where

$$P(h) \equiv p\xi - (h - g)(h - \theta). \quad (29)$$

The corresponding optimal sequence of productive capital $\{k_t\}_{t>0}$ satisfies $k_{t+1} = \max\{gk_t, \bar{k}\}$ where, from (19),

$$\bar{k} \equiv q'^{-1} \left(\frac{\bar{h} - g}{1 - (\bar{h} - g)\varphi/\xi} \right). \quad (30)$$

The following lemma gives the principal properties of the optimal path of the economy when the MRS is constant.

Lemma 4 *When the ENP constraint (16) is not binding, the IRS is constant, given by $h_t^* = \bar{h} = (g + \theta + \sqrt{(g - \theta)^2 + 4p\xi})/2$. $\bar{h} > 1/\beta$ iff $p\xi > (1/\beta - \theta)(1/\beta - g)$, and $\bar{k} > 0$ iff $p < \xi/\varphi^2 + (g - \theta)/\varphi$. Moreover, we must have $\xi/\varphi > 1/\beta - (3g - \theta)/2$ to have both $\bar{h} > 1$ and $\bar{k} > 0$. If $gk_0 > \bar{k}$, $k_{t+1}^* = gk_t^*$ for all $t < t_0 \equiv \ln(\bar{k}/k_0)/\ln g$. $k_t^* = \bar{k}$ for all $t_0 \leq t \leq T + 1$ where T is given by $\mu_T^* = q(\bar{k})\varphi/\xi$.*

The optimal path of the economy is thus characterized by a constant IRS as long as environmental neutrality is not reached. This IRS corresponds to the largest root of (29). Compared to the case $\theta = g = 0$, stock effects relax the conditions on parameters p , ξ and β to have an increasing path of the global wealth index: rather than $p\xi > 1/\beta^2$,

²¹The following results easily generalize to the case where p takes several discrete values depending on the relative levels of consumption and EQ.

we must have $p\xi > (1/\beta - \theta)(1/\beta - g)$. The condition $p < \xi/\varphi^2 + (g - \theta)/\varphi$ is deduced from (15) that is indefinitely binding otherwise: if p is too large, EQ would completely supplant consumption: the stock of capital would always decrease over time, i.e. $k_{t+1} = gk_t$ for all t . To have an increasing path of the global wealth index and positive consumption (and production) of industrial goods, we also must have $\xi/\varphi > 1/\beta - (3g - \theta)/2$, i.e. green technologies that are sufficiently effective in reducing polluting emissions.

Lemma 4 also states that the optimal dynamic of the economy may go through up to 3 sequences depending on the parameter values (and at least two if $\xi\mu_0 \ll \varphi q_0$). If $t_0 > 1$, the first sequence, $1 \leq t < t_0$, corresponds to a progressive decrease in the productive capital stock to \bar{k} since the depreciated capital is not replaced. This is the case when p , the social value of the environment, is large (since \bar{h} increases with p and \bar{k} given by (30) is a decreasing function of \bar{h}). Hence, while the IRS is constant, the stock of capital may have to adapt over several period before reaching \bar{k} and the GDP progressively decreases (while it takes only one period if k_0 is not too large, i.e. $\bar{k} < k_0 < \bar{k}/g$). The second sequence, $t_0 \leq t < T$ corresponds to a stagnant GDP, equal to $q(\bar{k})$, where the investment in productive capital allows firms to maintain the stock to \bar{k} , while the investment in clean technology allows the economy to increase GTI and EQ. Finally, the third sequence, $t > T$, corresponds to an increasing GDP along an ENP: the increase in GDP is proportional to the increase in clean technology (the factor of proportionality being equal to ξ/φ).²²

In the following proposition, the optimal dynamic of the economy is derived assuming a Cobb-Douglas production function $q_t = Ak_t^\alpha$ and CARA (exponential) consumer's utility, i.e. $u(c_t, e_t) = -e^{-\gamma(c_t + pe_t)}$.

Proposition 4 *Assume $\xi/\varphi \geq 1 - g$. With constant MRS, exponential utility function and a Cobb-Douglas production function, the optimal sequence of the EQ index $\{e_t^*\}_{t>1}$ is given by*

$$e_t^* = e_1 + \left(\frac{g^\alpha - g^{\alpha t}}{1 - g^\alpha} - \left(\frac{g^\alpha}{\bar{h}} \right)^{t_0-1} \frac{\bar{h}^t - \bar{h}}{\bar{h} - 1} \right) \nu_0 + (t - 1)\nu_1 - \frac{(1 - \theta)(1 - g)\nu_1}{\xi p \bar{h}^T} \frac{\bar{h}^t - \bar{h}}{\bar{h} - 1}$$

²²From this result, we can infer that if p decreases with e_t/c_t by discrete steps as mentioned footnote 21, the optimal solution entails decreasing optimal IRS \bar{h}_i , and thus increasing levels of productive capital \bar{k}_i and GDP levels when the economy as not yet reached an ENP. In term of industrial production, such a policy is thus very restrictive at first, and then progressively more permissive as GTI and EQ improve.

Table 1: Calibration parameters and equilibrium values

q_0	Pop	r_0	g	α	E_0	ι_0	ξ	φ	μ_0	A
85.91	7.7	0.06	$4/5$	$1/3$	37.1	0.432	0.84	0.84	41.74	17.92
CO ₂₀	CO _{2N}	TP	θ	\hat{e}	e_0	e_N	p	γ	ψ	
407.4	280	450	0.985	19.81	331	1,320.9	$15/1000$	0.2	0.015	
k_0	\bar{h}	\bar{k}	\bar{q}	$\bar{\tau}$	t_0	T	k_∞	q_∞	λ	σ_ω
110.14	1.038	83.69	78.39	0.283	2	60	101.77	83.68	0.005	$1/2$

Date $t = 0$ is 2018. GDP and capital stocks (q_t , k_t , μ_t) are expressed in US\$ trillions, world population in billions, CO₂ concentrations (Tipping Point TP, CO₂₀, CO_{2N}) in ppm, 2018 emissions E_0 and carbon budgets (e_0 , e_N , \hat{e}) in Gt of CO₂ (see footnote 23 for unit definitions). Emission intensities ι_0 and φ in kg of CO₂ for US\$1. φ corresponds to the emission intensity for 1960 (as estimated by the World Bank).

for all $1 < t < t_0$,

$$e_t^* = e_{t_0-1}^* + (t - t_0 + 1)\nu_1 - \frac{(1 - \theta)(1 - g)\nu_1}{\xi p \bar{h}^T} \frac{\bar{h}^t - \bar{h}^{t_0-1}}{\bar{h} - 1} \quad (31)$$

for all $t_0 \leq t \leq T$, and

$$e_t^* = e_N - \theta^{t-T}(e_N - e_T^*)$$

for all $t > T$, where $e_1 = \theta e_0 + \xi \mu_0 - \varphi q_0 + \hat{e}$, $\nu_1 = \ln(\beta \bar{h})^{\xi/\gamma} / P(1) > 0$, $\nu_0 = [\xi + \varphi(g - g^\alpha)](1 - g^\alpha)q_0 / P(g^\alpha) > 0$, and where T is deduced from $e_T^* = e_N - \ln(\beta \bar{h})^{1/(1-\theta)p\gamma}$. The value of GTI over $\{1, \dots, T\}$ is deduced from (31) and (11) using $q_t = g^{\alpha t} q_0$ for all $t < t_0$ and $q_t = q(\bar{k})$ for all $t_0 \leq t \leq T + 2$. EQ increases and consumption decreases at decreasing rates for all $t \in \{1, \dots, T\}$. The total welfare reached over $\{0, \dots, T\}$ is given by $W(T) = u_0(\bar{h} - 1/\bar{h}^{T-1})/(\bar{h} - 1)$.

Because the IRS is constant as long as environmental neutrality is not reached, the increase in global wealth is constant: $y_{t+1} - y_t = c_{t+1} - c_t + p(e_{t+1} - e_t) = \ln(\beta \bar{h})^{1/\gamma}$. However, while EQ is increasing, consumption decreases. This is due to the investment in green technology since the investment in productive capital is either null or constant during this period. Hence, during the first two sequences, when the GDP decreases ($1 \leq t < t_0$) and when it is stabilized at level $q(\bar{k})$ ($t_0 \leq t \leq T$), consumption of industrial goods decreases.

These results are illustrated Fig. 1 using the calibration parameters and the resulting equilibrium values presented Table 1. The world GDP and population in 2018 (the reference year corresponding to $t = 0$) are $q_0 = \text{US\$}85.91$ trillions and $\text{Pop} = 7.7$ billions. p is set at $\text{US\$}15/\text{tCO}_2$, the psychological discount rate $\psi = 1.5\%$, hence $\beta \approx .985$, and

the intertemporal elasticity parameter $\gamma = .2$. Computations are made using the per-capita global wealth level (y_t/Pop). The initial capital stock is derived from the interest rate by $k_0 = \alpha q_0 / (1 + r_0 - g)$ where $r_0 = 6\%$. e_t is defined as a global “carbon budget” at date t (expressed in Gt of CO_2 in the atmosphere), i.e. the difference between a tipping point (TP) and the level of GHG at date t expressed in CO_2 equivalent.²³ TP is set at 450 ppm (3496.5 Gt CO_2), and the initial concentration level CO_{20} is 407.4 ppm (3165.5 Gt CO_2), hence an initial EQ index $e_0 = 331$ Gt CO_2 .²⁴ Accordingly, given a pre-industrial level $\text{CO}_{2N}=280$ ppm (2176 Gt CO_2), the pre-industrial budget is $e_N = 1321$ Gt CO_2 . θ is set at .985, leading to $\hat{e} = (1 - \theta)e_N = 19.81$ Gt CO_2 . The initial emission intensity ι_0 corresponds to the ratio $\text{CO}_{20}/q_0 = 431.5$ g $\text{CO}_2/\text{US\$}$, leading to an initial GTI index $\mu_0 = q_0(\phi - \iota_0)/\xi = \text{US\$}41.74$ trillions. Finally, given q_0 and μ_0 , it comes $e_1 = \theta e_0 + \hat{e} - \varphi q_0 + \xi \mu_0 = 310$ Gt CO_2 . Given these parameters, the de-growth sequence takes two periods (from $t = 1$ to $t = t_0 = 2$), and economic stagnation lasts until $t = T = 60$. During this second sequence, IRS is $\bar{h} = 1.038$.

Panel 1a shows the dynamic of EQ that increases rapidly until $t = T$, when the ENP is reached. Then EQ increases at a lower rate toward e_N . Panel 1b shows the sharp decrease in the productive capital and the sharp increase in GTI from period 0 to period 1 (from $k_0 = 110.14$ to $\bar{k} = 83.69$ and $\mu_0 = 41.74$ to $\mu_1 = 71.6$ respectively). Then, the productive capital stays at \bar{k} until $t = T$, while GTI increases. Both capital stocks increase afterward, at the slow rate permitted by the ENP constraint (the long term level of capital, which determines $\mu_\infty = q_\infty = 83.68$, is $k_\infty = 101.77$). Panel 1c shows the decrease in consumption, very sharp at first (from $c_0 = 55.4$ to $c_3 = 47.14$), then slower until period T . It increases afterward, at a very slow pace. Total wealth also decreases the first two periods, but increases sharply afterward. This simulations also show the limit of the assumption of a constant MRS: the decreases in GDP and consumption are very sharp due to the perfect substitutability between industrial and environmental goods.²⁵

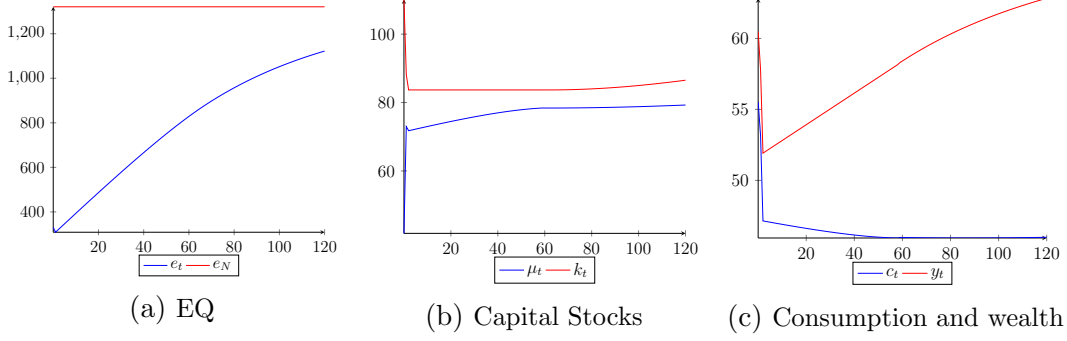


Figure 1: Optimal dynamic.

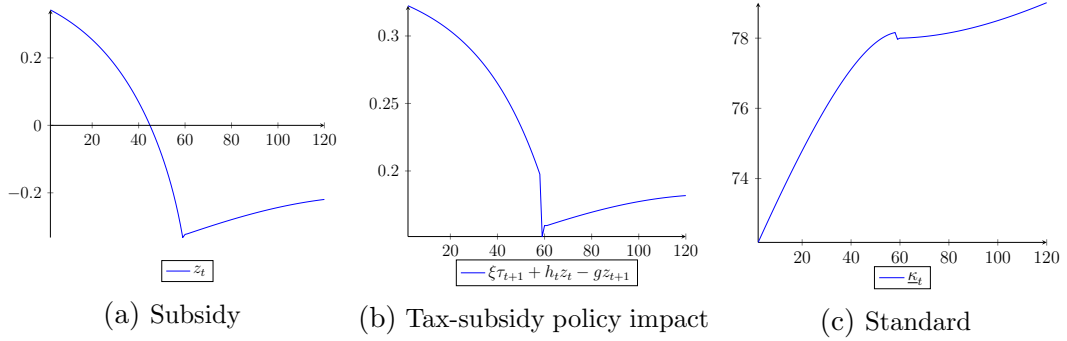


Figure 2: Tax and subsidy schemes ($\lambda = 0.01$).

4.2 Policy implementation

The policy schemes (25)–(27) given Proposition 3 that implement the first-best path in expectation are illustrated Fig. 2 assuming a spillover coefficient $\lambda = .005$. For both policy, the emissions tax is constant at $\bar{\tau} = .283$ (i.e. 283 US\$/t CO₂) over the period $t_0 < t < T$ (it then slowly decreases). The subsidy scheme (panel 2a) is around 34% at $t_0 = 2$ and decreases to become negative at $t = 45$. As shown panel 2b, the resulting net impact of the tax-subsidy policy on GTI, i.e. the term $\xi\tau_{t+1} + h_t z_t - g z_{t+1}$ in (22), is decreasing but positive over this period (it stays positive and increases slightly thereafter). Simulations show that the subsidy scheme is extremely sensitive to λ (the schedule is positive for only a decade when $\lambda = .01$, and entire negative when $\lambda \geq .02$). The standard policy depicted panel 2c increases rapidly until environmental neutrality

²³Units are either gigaton (Gt shorthand), i.e. 10^9 (billions) metric tons, or part per million (ppm shorthand), which refers to the atmospheric concentration. Each ppm represents approximately 2.13 Gt of carbon in the atmosphere as a whole, equivalent to 7.77 Gt of CO₂.

²⁴According to the IPCC Fifth Assessment Report, 450 ppm lead to a temperature increase of approximately 2°C.

²⁵The assumption of a constant MRS simplifies the dynamics of the transition period which can be derived explicitly. As indicated footnote 22, we may expect by relaxing this assumption that the productive capital increases during the transition period to an ENP.

is reached, then slowly along the ENP.

Assuming rational expectations, it is possible to be more specific about the dynamic of the economy for $t \in \{t_0, \dots, T\}$ under these policy implementations. Indeed, over this period the interest rate satisfies $\tilde{r}_t = \bar{r}(1 + \tilde{\varepsilon}_t)$ where $\bar{r} = \bar{h} - 1$ is the expected interest rate and $\tilde{\varepsilon}_t$ is a zero-mean random shock whose distribution depends on the present and past realizations of $\tilde{\omega}_t$. The productive capacity and total production \tilde{k}_t and \tilde{y}_t are also randomly distributed around their stationary values \bar{k} and $q(\bar{k})$. Using linear approximations, the path of global wealth index y_t can be approximated by a Gaussian random walk at the rational expectations equilibrium (REE), and the supply function of capital (13) by²⁶

$$r_t = \psi + \gamma(\mathbb{E}_t[\tilde{y}_{t+1}] - y_t) - \gamma^2 \mathbb{V}_t[\tilde{y}_{t+1}]/2 \quad (32)$$

where $\psi = -\ln \beta$ is the intrinsic discount factor. Expression (32) exhibits the familiar effects that determine the rental price of capital: the intrinsic preference for an immediate consumption ψ , the economic trend of the global wealth index that also encourages immediate consumption if it is positive, and a precautionary effect that operates in the opposite direction and corresponds to a risk premium due to the uncertainty affecting the economy.²⁷

Proposition 5 *Assuming a REE with constant MRS, a CARA utility function and a Cobb-Douglas production function, the interest rate under either the tax-subsidy ($\ell = sub$) or the tax-standard ($\ell = std$) policy is approximated by $\tilde{r}_t^\ell = \bar{r}(1 + \tilde{\varepsilon}_t^\ell)$ for $t \in \{t_0, \dots, T\}$, where*

$$\begin{aligned} \tilde{\varepsilon}_t^{sub} &= \chi^{sub}(z_t)\tilde{\omega}_{t+1} + \rho^{sub}(z_t)\varepsilon_{t-1}^{sub}, \\ \tilde{\varepsilon}_t^{std} &= \chi^{std}\tilde{\omega}_{t+1} + \rho^{std}\varepsilon_{t-1}^{std}, \end{aligned} \quad (33)$$

are normally distributed. The paths of $\tilde{\varepsilon}_t^\ell$, $\tilde{\mu}_t^\ell$ and \tilde{y}_t^ℓ under policy $\ell \in \{sub, std\}$, can be approximated by Gaussian random walks deduced from (11), (14) and

$$\mu_{t+1}^{sub} = a_1^{sub}\mu_t^{sub} + a_2^{sub}e_t^{sub} + a_3^{sub}(z_t) + Z_t^{sub} + b_1^{sub}(z_t)\varepsilon_t^{sub} + b_2^{sub}\varepsilon_{t-1}^{sub} \quad (34)$$

²⁶This expression is derived by approximating $1 + r_t \approx e^{r_t}$, and using $\mathbb{E}[e^{-\gamma\tilde{y}}] = e^{-\gamma(\mathbb{E}[\tilde{y}] - \gamma\mathbb{V}[\tilde{y}]/2)}$ when \tilde{y} is normally distributed.

²⁷This simple expression is due to the CARA preferences, γ being the coefficient of absolute risk aversion.

with

$$Z_t^{sub} = a_0^{sub} \sum_{i=0}^{+\infty} (a_0^{sub} \gamma)^i (\xi \tau_{t+1+i} + h_{t+i} z_{t+i} - g z_{t+1+i}), \quad (35)$$

and

$$\mu_{t+1}^{std} = a_1^{std} \mu_t^{std} + a_2^{std} e_t^{std} + a_3^{std} + Z_t^{std} + b_1^{std} \varepsilon_t^{std} + b_2^{std} \varepsilon_{t-1}^{std} \quad (36)$$

with

$$Z_t^{std} = a_0^{std} \sum_{i=0}^{+\infty} (a_0^{std} \gamma)^i (\xi \tau_{t+1+i} + (1-g) \underline{\kappa}_{t+i}). \quad (37)$$

We have $0 < a_0^\ell < 1/\gamma$, $0 < a_2^\ell < p$, and $0 < a_1^\ell < g + (1-\lambda)/\gamma$ assuming $\lambda \leq 1/[1 + (1-g)/(1-\theta)]$ and $g(1 + (1-\lambda)/\gamma) \geq p\xi$.

The network externality shock $\tilde{\omega}_t$ generates the stochastic distributions (33) that depend on the policy implemented. These distributions of the shocks affecting the interest rate follow first-order autoregressive processes, with parameters that are constant in the tax-standard policy case while they are functions of the subsidy level z_t in the tax-subsidy case. In both cases, the uncorrelated shocks affecting the network externalities generate path-dependent fluctuations. This business cycle is not due to changes in the environmental policy: the regulator is supposed to commit to a policy that is perfectly anticipated by the agents. Rather, it is due to the consumer's desire to smooth her consumption over time. This auto-correlation can be explained as follows: At the beginning of the transition period, assuming that the GDP and the interest rate are initially equal to their nominal values for that period, a positive shock increases the demand for green technology, all the more so if the supporting infrastructure is responsive (λ is large). This leads to an increase in the interest rate, which in turn decreases investment in productive capital. During the transition period, the environmental tax is such that this investment just allows firms to maintain their productivity. The increase in the interest rate thus reduces the productive capital stock to a level below its nominal level. As the decline in investment reduces the next period's GDP, the willingness of consumers to lend capital is reduced, which tends to increase the next period's interest rate even more in the event of a new positive shock. The opposite trend occurs if the initial shock is negative.

The equilibrium dynamic of GTI given by (34) and (36) are linear first-order recursive equations with constant parameter in the latter, and some parameters (a_3^{sub} and b_1^{sub}) function of the subsidy level z_t in the former. Both include forward looking term Z_t^ℓ given by either (35) or (37) depending on the policy $\ell \in \{sub, std\}$. These policy indexes are exponential smoothing of future tax and subsidy levels or future tax and

Table 2: REE coefficients

a_0^ℓ	a_1^ℓ	a_2^ℓ	\bar{a}_3^ℓ	\bar{b}_1^ℓ	b_2^ℓ	$\bar{\rho}^\ell$	$\bar{\chi}^\ell$	$\bar{\sigma}_{y_{t+1} t}^\ell$
0.866	0.998	4.696 ₋₅	-8.045 ₋₂	28.849	-27.878	0.965	0.035	1.536 ₋₁
0.836	0.798	4.507 ₋₅	-7.757 ₋₂	23.081	-22.293	0.964	0.043	6.651 ₋₂

Values and average values of the coefficients of (33)–(37) under the tax-subsidy ($\ell = sub$, first row) and the tax-standard ($\ell = std$, second row) policies, with $\bar{a}_3^{sub} \equiv \sum_{t=t_0}^T a_3^{sub}(z_t)/(T-t_0+1)$, $\bar{a}_3^{std} \equiv a_3^{std}$ and similarly for \bar{b}_1^ℓ , $\bar{\rho}^\ell$, $\bar{\chi}^\ell$ and $\bar{\sigma}_{y_{t+1}|t}^\ell$. The last column corresponds to the average value of the one-period-ahead standard deviation of the aggregate wealth. Subscripts correspond to exponents (i.e. $2.3_{-2} = 2.3 \cdot 10^{-2}$).

standard levels respectively.²⁸

The simulations of policy implementations show that $\rho_1^{sub}(z_t)$ and $\chi^{sub}(z_t)$ are slightly increasing while $a_3^{sub}(z_t)$, $b_1^{sub}(z_t)$ and $\sigma_{y_{t+1}|t}^{sub}$ are slightly decreasing. They all stay very close to their average values reported Table 2. Coefficients a_0^ℓ determine the weights associated to future policy levels in (35) or (37). We have $a_1^{sub} > a_1^{std}$ which reflects an higher path dependency under the tax-subsidy policy than under the tax-standard policy. The importance of shocks is reflected by \bar{b}_1^ℓ and b_2^ℓ that have large absolute values of opposite signs, allowing the consumer to adjust her anticipations of GTI to the business cycle fluctuations. As a result, the volatility of aggregate wealth $\sigma_{y_{t+1}|t}^\ell$ from period to period is small, but larger under the tax-subsidy policy than under the tax-standard policy (more than twice as much on average), reflecting the greater latitude given to producers in their green investment choices in the former case compared to the latter. The autocorrelation coefficient of $\tilde{\varepsilon}_t^\ell$ is slightly larger under the tax-subsidy policy than under the tax-standard policy, i.e. $\bar{\rho}^{sub} > \rho^{std}$, while it is the reverse for the impact of the immediate network shock: we have $\chi^{std} > \bar{\chi}^{sub}$.

The resulting distributions of $\tilde{\varepsilon}_t$ and \tilde{r}_t are illustrated Fig. 3. In panel 3a, the shocks $\tilde{\omega}_t$ are randomly distributed around 0, while $\tilde{\varepsilon}_t^\ell$ fluctuates somehow smoothly and stay below $\varepsilon_0^\ell = 0$ under both policies. The amplitude of its path under the tax-standard policy is often larger than under the tax-subsidy policy due to $\chi^{std} > \bar{\chi}^{sub}$. The large autocorrelation coefficient ρ^ℓ compared to the low impact of the innovation χ^ℓ explain this paths dependency of $\tilde{\varepsilon}_t^\ell$ under both policies. As a result, the interest rate departs from its nominal value $\bar{r} = \bar{h} - 1 = 3.8\%$, following the same variations as ε_t (panel 3b). On these graphs, the dashed curves above and below the horizontal lines located at $\varepsilon_{t_0} = 0$ and \bar{r} delineate the 95% confidence interval (CI) deduced from the

²⁸Under the assumption of open-loop policies, the paths (35) and (37) of Z_t^ℓ and the values of the parameters that are functions of z_t in (33) and (36) are deterministic.

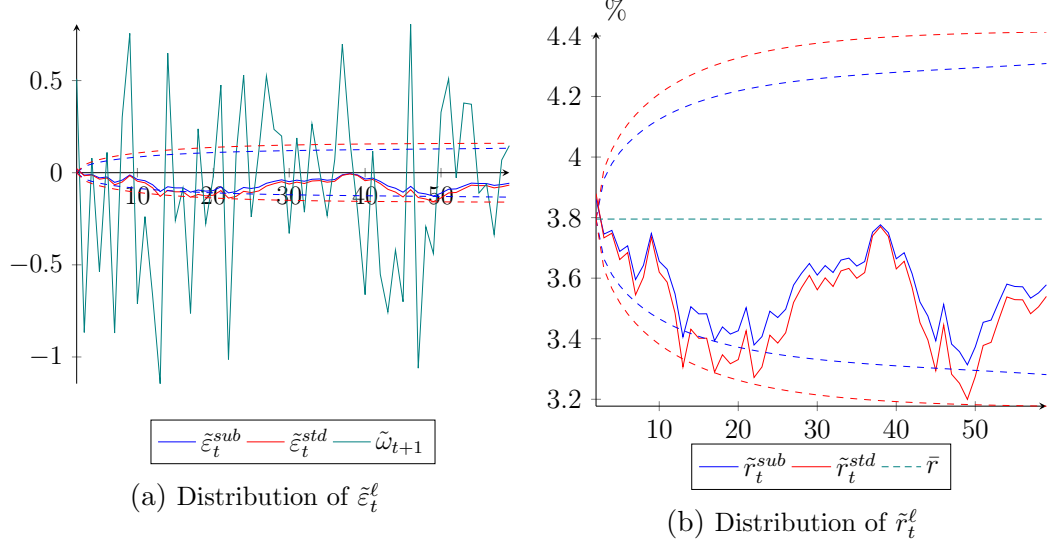


Figure 3: Shocks and the interest rate (95% CIs are delineated by the dashed black lines).

Gaussian distributions $\mathcal{N}(0, \mathbb{V}[\tilde{\varepsilon}_t^\ell])$ (panel 3a) and $\mathcal{N}(\bar{r}, \bar{r}^2 \mathbb{V}[\tilde{\varepsilon}_t^\ell])$ (panel 3b). Because $\chi^{std} > \bar{\chi}^{sub}$, the CI is larger at first under the tax-standard policy than under the tax-subsidy policy. This effect is progressively attenuated due to the difference in the autocorrelation of the shocks affecting the interest rate under the two policy (the one of the subsidy policy is larger on average and increasing).

The resulting impact on GTI and EQ is illustrated Fig. 4. Panel 4a shows large fluctuations of μ_t^ℓ due to the network effects. While $\tilde{\mu}_t^{std}$ fluctuates along the optimal path μ^* , the fluctuations of $\tilde{\mu}_t^{sub}$ do not show this positive trend. The consequence of these larger fluctuations on EQ is apparent Panel 4b: the stochastic path \tilde{e}_t^{std} is much closer to the optimal path e_t^* than \tilde{e}_t^{sub} .

It is also possible to derive approximations of the CIs for the paths of $\tilde{\mu}_t^\ell$, \tilde{e}_t^ℓ and \tilde{q}_t^ℓ over $\{t_0, \dots, T\}$ using (11) and (34). These equations give the recursive expression

$$\tilde{Y}_t^\ell = B_t^\ell Y_{t-1}^\ell + H_t^\ell \tilde{\nu}_t \quad (38)$$

where $\tilde{Y}_t^\ell = (\tilde{\mu}_t^\ell, \tilde{e}_t^\ell, \tilde{q}_t^\ell, 1, \tilde{\varepsilon}_t^\ell, \tilde{\varepsilon}_{t-1}^\ell)'$ is the column vector of state values (with the con-

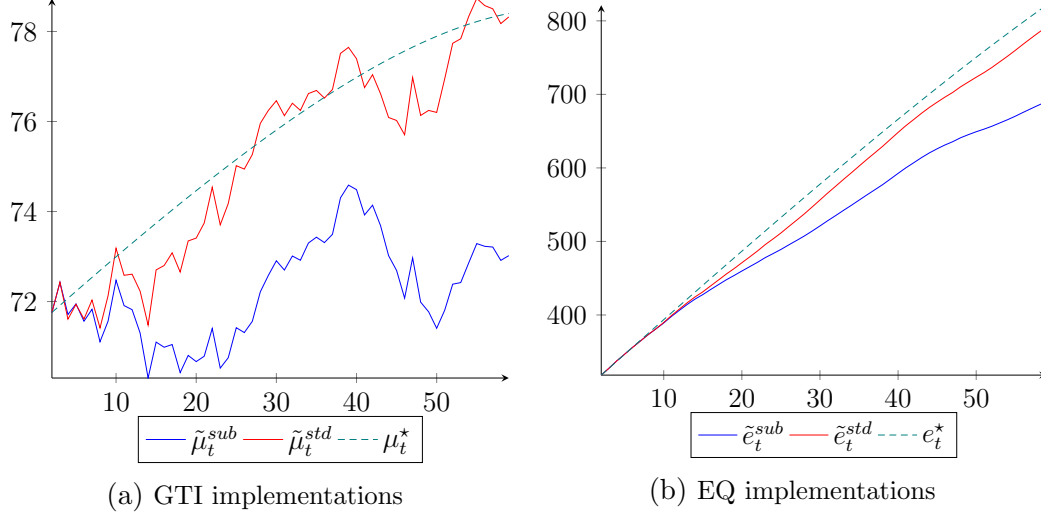


Figure 4: GTI and EQ under the two implementations.

stant) and of the interest rate shock of date t and $t - 1$,

$$B_t^\ell = \begin{bmatrix} a_1^\ell & a_2^\ell & 0 & a_{3t}^\ell + Z_t^\ell & b_{1t}^\ell & b_2^\ell \\ \xi & \theta & -\varphi & \hat{e} & 0 & 0 \\ 0 & 0 & 0 & \bar{q} & q'(\bar{k})/q''(\bar{k}) & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_t^\ell & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, H_t^\ell = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \chi_t^\ell \sigma_\omega \\ 0 \end{bmatrix},$$

and \tilde{v}_t is an independent standardized Gaussian variable. \tilde{Y}_t^ℓ follows a Gaussian random walk with $\mathbb{E}[\tilde{Y}_t^\ell] = (\Pi_{i=0}^{t-t_0} B_i^\ell) X_{t_0}$ and $\mathbb{V}[\tilde{Y}_t^\ell] = \sum_{i=0}^{t-t_0} (\Pi_{j=0}^i B_j^\ell) H_\ell H_\ell' (\Pi_{j=0}^i B_j^\ell)'$. The transition matrix B_t^ℓ and the column vector H_t^ℓ are time-dependent because of the policy index Z_t^ℓ that changes over time according to (35) or (37), and also because coefficients a_{3t}^{sub} , b_{1t}^{sub} , ρ_t^{sub} and χ_t^{sub} are functions of the subsidy schedule z_t .

Fig. 5 depicts the results derived from (38) and shows a striking difference in the magnitude of the confidence intervals (panels 5a and 5b): The CIs for the tax-subsidy policy delineated by the dashed blue curves are much larger than those for the tax-standard policy (dashed red curves). This is not the case for the GDP shown in panel 5c, which also present realized GDP paths under the two implementations that are (relatively) close. This is because investment in productive capacity is not directly affected by network effects, but only through the variations of the interest rate. The amplification effects of network shocks that affects EQ (which depends on μ_t) are thus absent for GDP (which depends on k_t). The differences in GDP are thus the mirror

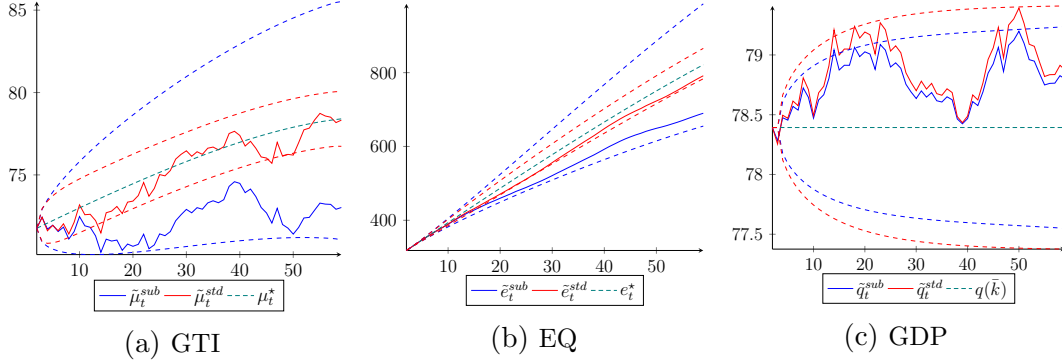


Figure 5: GTI, EQ and GDP under the two policies (95% CIs are delineated by the dashed blue (red) lines under the tax-subsidy (tax-standard) implementation).

image of the differences in interest rate levels.

5 Conclusion

This paper analyzes the effect of uncertainty on the adoption of green technologies. Governments design policies to drive their economies toward environmental neutrality, but investment choices are ultimately made by private agents who respond to policies according to their own expectations about the future states of the economy. The effectiveness of green technologies is partly the result of their choices, through complex network and spillover effects. As a result, regulatory instruments can only imperfectly guide economies on their paths to environmental neutrality. Although highly stylized, the model presented in this paper shows that shocks affecting network externalities cause economic fluctuations that can move society away from the optimal path as envisioned by the authorities. Technology standards permit a transition to green technologies more in line with the regulator’s intended trajectory than subsidies do. This is because producers have less flexibility to adapt their green investments to economic conditions when standards rather than subsidies are implemented. The latter instrument allows firms to better react to changes on financial markets, for better or for worse in terms of the resulting environmental quality.

The optimal path of the economy is derived assuming that governments know the technologies that make such a path feasible, which is unrealistic. Environmental policies are based on scenarios that depend heavily on assumptions about how technologies will evolve –both in terms of performance and cost– in the distant future. As noted by the IEA (2021), its model’s forecasts based on already existing technologies have a time horizon of about a decade. Forecasts beyond 2030 (and up to 2070) rely on

assumptions about the evolution of green technology. These assumptions are based on the opinion of experts who cannot have complete knowledge of all the technologies that will be deployed, but only indications of those that are under development, at the research stage or in demonstration projects. The likely evolution of these technologies, in terms of cost and performance, is subject to many uncertainties. Determining the appropriate policy is therefore a very difficult task. Technology standards, by imposing minimum environmental quality characteristics that machines must meet, seem to be the most appropriate instrument for achieving binding objectives such as those imposed by international environmental agreements. And indeed, emission standards policies such as those implemented by the European Union for vehicles, which oblige car manufacturers to produce low-emission models, seem to be more effective in guiding society towards environmental neutrality than have been the economic incentives given to car buyers to date.

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Appendix

A Proof of Proposition 1

As firm i 's profit is separable in revenues from productive capacity and network effects from its green index, the firm's investment problem can be split into two independent programs. Applying the principle of optimality, the investment strategy in productive capacity can be derived by using the Bellman equation

$$\mathcal{W}(k_{it}) = \max_{I_{it}, \ell_{it}} Q(k_{it}, \ell_{it}) - w\ell_{it} - I_{it} + \delta_t \mathcal{W}(gk_{it} + I_{it}). \quad (39)$$

Maximizing (39) with respect to ℓ_{it} gives

$$\partial Q(k_{it}, \ell_{it}) / \partial \ell = w_t,$$

while the first-order condition with respect to I_{it} leads to

$$-1 + \delta_t \mathcal{W}'(gk_{it} + I_{it}) = 0. \quad (40)$$

The envelop condition yields

$$\mathcal{W}'(k_{it}) = \partial Q(k_{it}, \ell_{it}) / \partial k + g\delta_t \mathcal{W}'(gk_{it} + I_{it})$$

implying $\mathcal{W}'(k_{it}) = \partial Q(k_{it}, \ell_{it}) / \partial k + g$. Plugging this expression in (40) evaluated for period $t + 1$ yields

$$\partial Q(gk_{it} + I_{it}, \ell_{it+1}) / \partial k = 1 + r_t - g.$$

As Q is homogeneous of degree 1, we thus get $q'(k_{it+1}/\ell_{it+1}) = 1 + r_t - g$ where $q(k) \equiv Q(k, 1)$ is an increasing and concave function. Inverting, it comes that $k_{it+1}/\ell_{it+1} = q'^{-1}(1 + r_t - g)$, and using $\int_0^1 \ell_{it} di = \ell_t = 1$, $k_{it}/\ell_{it} = k_t/\ell_t = k_t$ for all i and t , with $k_t = q'^{-1}(1 + r_{t-1} - g)$. Using $w_t = \partial Q(k_{it}, \ell_{it}) / \partial \ell = d[\ell_{it} q(k_{it}/\ell_{it})] / d\ell_{it} = q(k_{it}/\ell_{it}) - q'(k_{it}/\ell_{it})k_{it}/\ell_{it}$ yields $w_t = q_t - r_{t-1}k_t$.

Neglecting the constants in (4), the Bellman equation corresponding to the technology mix is given by

$$\mathcal{V}(x_{it}; x_t^*) = \max_{\kappa_{it}} -(x_{it} - x_t^*)^2 / 2 - \kappa_{it} + \delta_t \mathbb{E}[\mathcal{V}(gx_{it} + \kappa_{it}; \tilde{x}_{t+1}^*) | \eta_{it}, \mu_t, r_t]. \quad (41)$$

Maximizing (41) with respect to κ_{it} leads to

$$-1 + \delta_t \mathbb{E} [\partial \mathcal{V}(gx_{it} + \kappa_{it}; \tilde{x}_{t+1}^*) / \partial x | \eta_{it}, \mu_t, r_t] = 0. \quad (42)$$

The envelop condition yields

$$\partial \mathcal{V}(x_{it}; x_t^*) / \partial x = x_t^* - x_{it} + g \delta_t \mathbb{E} [\partial \mathcal{V}(gx_{it} + \kappa_{it}; \tilde{x}_{t+1}^*) / \partial x | \eta_{it}, \mu_t, r_t] \quad (43)$$

implying $\partial \mathcal{V}(x_{it}; x_t^*) / \partial x = x_t^* - x_{it} + g$. Plugging this expression in (42) evaluated in expectation for period $t + 1$ yields

$$1 + r_t = \mathbb{E} [\tilde{x}_{t+1}^* - (gx_{it} + \kappa_{it}) + g | \eta_{it}, \mu_t, r_t] = \mathbb{E} [\tilde{x}_{t+1}^* | \eta_{it}, \mu_t, r_t] - (gx_{it} + \kappa_{it}) + g,$$

which gives (6). Following Angeletos & Pavan (2004), the resulting investment strategy is linear in the variables observed by the firm at date t , i.e. it is given by

$$\kappa(\mu_t, \eta_{jt}, x_{jt}, r_t) = \beta_1 \mu_t + \beta_2 \eta_{jt} + \beta_3 x_{jt} + \beta_4 r_t + \beta_5 \quad (44)$$

where the coefficients $\beta_k, k = 1, \dots, 5$, are derived as follows. On average, as $\int_0^1 \eta_{jt} dj = \omega_{t+1}$ and $\int_0^1 x_{jt} dj = \mu_t$, we have

$$\int_0^1 \kappa(\mu_t, \eta_t, \varepsilon_{jt}, x_{jt}, r_t) dj = (\beta_1 + \beta_3) \mu_t + \beta_2 \omega_{t+1} + \beta_4 r_t + \beta_5$$

and thus, from (3),

$$x_{t+1}^* = \mu_t + \lambda \int_0^1 \kappa_{it} di + \omega_{t+1} = [1 + \lambda(\beta_1 + \beta_3)] \mu_t + (1 + \lambda\beta_2) \omega_{t+1} + \lambda\beta_4 r_t + \lambda\beta_5.$$

Using (6) and $\mathbb{E}[\tilde{\omega}_{t+1} | \eta_{it}] = \eta_{it}$, we get

$$\begin{aligned} \kappa_{it} &= \mathbb{E}[\tilde{x}_{t+1}^* | \eta_{it}, \mu_t, r_t] - 1 + g - gx_{it} - r_t \\ &= [1 + \lambda(\beta_1 + \beta_3)] \mu_t + (1 + \lambda\beta_2) \eta_{it} - (1 - \lambda\beta_4) r_t + \lambda\beta_5 - 1 + g - gx_{it}. \end{aligned}$$

Identifying with (44) yields $\beta_3 = -g$, $\beta_1 = 1 + \lambda(\beta_1 + \beta_3) = (1 - \lambda g)/(1 - \lambda)$, $\beta_2 = 1 + \lambda\beta_2 = 1/(1 - \lambda)$, $\beta_4 = -1 + \lambda\beta_4 = -1/(1 - \lambda)$, $\beta_5 = -(1 - g)/(1 - \lambda)$, hence (7). Consequently,

$$x_{it+1} = gx_{it} + \kappa_{it} = \frac{(1 - \lambda g) \mu_t + \eta_{it} + g - 1 - r_t}{1 - \lambda},$$

and, integrating,

$$\mu_{t+1} = \frac{(1 - \lambda g)\mu_t + \omega_{t+1} + g - (1 + r_t)}{1 - \lambda}$$

using $\int_0^1 \eta_{it} di = \omega_{t+1}$. Re-organizing terms gives (8). As idiosyncratic investments depend on signals that are normally distributed, x_{it+1} is normally distributed around μ_{t+1} with variance $\mathbb{V}[x_{it+1}] = \sigma_v^2 / (1 - \lambda)^2 \equiv \sigma_x^2$.

B Proof of Lemma 1

At each date t , the Bellman equation corresponding to (12) can be written as

$$v(S_{t-1}; e_t) = \max_{s_t} u(R_t + r_{t-1}S_{t-1} - s_t, e_t) + \beta \mathbb{E}_t[v(S_{t-1} + s_t; \tilde{e}_{t+1})]$$

where S_t and s_t are the state and the control variables respectively. The first-order equation gives

$$\partial u(c_t, e_t) / \partial c = \beta \mathbb{E}_t [\partial v(S_t; \tilde{e}_{t+1}) / \partial S], \quad (45)$$

and the envelope theorem gives

$$\partial v(S_{t-1}; e_t) / \partial S = r_{t-1} \partial u(c_t, e_t) / \partial c + \beta \mathbb{E}_t [\partial v(S_t; \tilde{e}_{t+1}) / \partial S].$$

Replacing the last term using (45), we get

$$\partial v(S_{t-1}; e_t) / \partial S = (1 + r_{t-1}) \partial u(c_t, e_t) / \partial c.$$

Taking the expectation and replacing in (45) yields (13) where $1 + r_t$ on the RHS is factorized out of the expected value since the date- t interest rate is a known parameter.

C Proof of Proposition 2

The planner's program is equivalently stated as

$$\max_{\{\mu_t, k_t, e_t\}_{t>0}} \left\{ \sum_{t=0}^{+\infty} \beta^t u(q(k_t) - k_{t+1} + gk_t - \mu_{t+1} + g\mu_t, e_t) : (11), (15), (16) \right\},$$

given e_0, k_0 and μ_0 . Neglecting the constraints (15) and (16), and denoting by $\hat{\lambda}_t$ the multiplier associated to (11), the Lagrangian of this program is given by

$$\mathcal{L} = \sum_{t=0}^{+\infty} \beta^t u(q(k_t) - k_{t+1} + gk_t - \mu_{t+1} + g\mu_t, e_t) - \hat{\lambda}_t (e_{t+1} - \theta e_t - \xi \mu_t + \varphi q(k_t) - \hat{e}).$$

When (16) is not binding, the FOCs are

$$\frac{\partial \mathcal{L}}{\partial k_t} = \beta^t \frac{\partial u_t}{\partial c} (q'(k_t) + g) - \beta^{t-1} \frac{\partial u_{t-1}}{\partial c} - \hat{\lambda}_t \varphi q'(k_t) \leq 0 \perp k_t \geq gk_{t-1}, \quad (46)$$

$$\frac{\partial \mathcal{L}}{\partial \mu_t} = \beta^t \frac{\partial u_t}{\partial c} g - \beta^{t-1} \frac{\partial u_{t-1}}{\partial c} + \hat{\lambda}_t \xi = 0, \quad (47)$$

for all $t > 0$, and

$$\frac{\partial \mathcal{L}}{\partial e_t} = \beta^t \frac{\partial u_t}{\partial e} - \hat{\lambda}_{t-1} + \theta \hat{\lambda}_t = 0, \quad (48)$$

for all $t > 1$. Eq. (47) gives

$$\hat{\lambda}_t = \left(\beta^{t-1} \frac{\partial u_{t-1}}{\partial c} - \beta^t \frac{\partial u_t}{\partial c} g \right) / \xi$$

and, using (18),

$$\hat{\lambda}_t = \beta^t \frac{\partial u_t}{\partial c} (h_{t-1} - g) / \xi. \quad (49)$$

Substituting in (48) evaluated at $t + 2$, yields

$$\begin{aligned} 0 &= \beta^{t+2} \frac{\partial u_{t+2}}{\partial e} - \beta^{t+1} \frac{\partial u_{t+1}}{\partial c} (h_t - g) / \xi + \theta \beta^{t+2} \frac{\partial u_{t+2}}{\partial c} (h_{t+1} - g) / \xi \\ &= \beta^{t+2} \frac{\partial u_{t+2}}{\partial e} - \beta^{t+2} \frac{\partial u_{t+2}}{\partial c} [h_{t+1} (h_t - g) - \theta (h_{t+1} - g)] / \xi, \end{aligned}$$

using (18), hence

$$\xi \frac{\partial u_{t+2} / \partial e}{\partial u_{t+2} / \partial c} = h_{t+1} (h_t - g - \theta) + \theta g$$

for all $t > 1$ such that (16) is not binding. Using (49) to substitute for $\hat{\lambda}_{t+1}$ in (46) evaluated at $t + 1$ gives

$$\begin{aligned} 0 &\geq \beta^{t+1} \frac{\partial u_{t+1}}{\partial c} (q'(k_{t+1}) + g) - \beta^t \frac{\partial u_t}{\partial c} - \beta^{t+1} \frac{\partial u_{t+1}}{\partial c} (h_t - g) q'(k_{t+1}) \varphi / \xi \\ &= \beta^{t+1} \frac{\partial u_{t+1}}{\partial c} [q'(k_{t+1}) - (h_t - g) (1 + q'(k_{t+1}) \varphi / \xi)], \end{aligned}$$

using (18), hence

$$h_t - g \geq \frac{q'(k_{t+1})}{1 + q'(k_{t+1})\varphi/\xi} \perp k_{t+1} \geq gk_t.$$

Assuming (16) is binding for all $t \geq T$, the planer's program becomes

$$\max_{\{k_t\}_{t>T}} \left\{ \sum_{t=T}^{+\infty} \beta^t u(q(k_t) - k_{t+1} + gk_t - [q(k_{t+1}) - gq(k_t)]\varphi/\xi, e_t) : e_{t+1} = \theta e_t + \hat{e} \right\},$$

given e_T, k_T . The FOCs are

$$\frac{\partial \mathcal{L}}{\partial k_t} = \beta^t \frac{\partial u_t}{\partial c} [q'(k_t)(1 + g\varphi/\xi) + g] - \beta^{t-1} \frac{\partial u_{t-1}}{\partial c} [1 + q'(k_t)\varphi/\xi] = 0$$

for all $t > T$, which gives

$$h_{t-1} = \frac{g + q'(k_t)(1 + g\varphi/\xi)}{1 + q'(k_t)\varphi/\xi},$$

or, equivalently,

$$h_t - g = \frac{q'(k_{t+1})}{1 + q'(k_{t+1})\varphi/\xi}$$

for all $t \geq T$.

D Proof of lemma 2

The problem of firm i is to solve

$$\mathcal{V}(k_{it}, x_{it}; x_t^*) = \max_{I_{it}, \ell_{it}, \kappa_{it}} \pi_{it} + \delta_t \mathbb{E}_t[\mathcal{V}(gk_{it} + I_{it}, gx_{it} + \kappa_{it}; \tilde{x}_{t+1}^*) | \eta_{it}]$$

where

$$\pi_{it} \equiv Q(k_{it}, \ell_{it}) - w_t \ell_{it} - I_{it} - (x_{it} - x_t^*)^2/2 - \kappa_{it}(1 - z_t) - \tau_t (\varphi Q(k_{it}, \ell_{it}) - \xi x_{it}).$$

Using the same steps as those described in the proof of Proposition 1, it comes (21) and

$$\kappa_{it} = \mathbb{E}[\tilde{x}_{t+1}^* | \eta_{it}, \mu_t, r_t] - gx_{it} + \xi \tau_{t+1} + g(1 - z_{t+1}) - (1 + r_t)(1 - z_t).$$

At equilibrium, we get

$$\kappa_{it} = \frac{1}{1 - \lambda} [(1 - \lambda g)\mu_t + \eta_{it} + \xi \tau_{t+1} + g(1 - z_{t+1}) - (1 + r_t)(1 - z_t)] - gx_{it},$$

hence

$$x_{it+1} = gx_{it} + \kappa_{it} = \frac{1}{1-\lambda} [(1-\lambda g)\mu_t + \eta_{it} + \xi\tau_{t+1} + g(1-z_{t+1}) - (1+r_t)(1-z_t)].$$

Summing over all firms and using $\int_0^1 \eta_{it} di = \omega_{t+1}$ gives (22).

E Proof of Lemma 3

In that case, we have

$$\pi_{it} = Q(k_{it}, \ell_{it}) - w_t \ell_{it} - I_{it} - (x_{it} - x_t^*)^2/2 - \kappa_{it} - \tau_t (\varphi Q(k_{it}, \ell_{it}) - \xi x_{it})$$

where \tilde{x}_{t+1}^* is given by (23). It comes

$$\kappa_{it} = \kappa_t(\mu_t, \underline{\kappa}_t, \eta_{it}, x_{it}, r_t) \equiv \mathbb{E} [\tilde{x}_{t+1}^* | \eta_{it}, \mu_t, \underline{\kappa}_t, r_t] - gx_{it} + \xi\tau_{t+1} + g - 1 - r_t. \quad (50)$$

Assuming linearity, i.e.

$$\kappa_t(\mu_t, \eta_{it}, x_{it}, r_t) = \beta_0 \underline{\kappa}_t + \beta_1 \mu_t + \beta_2 \eta_{it} + \beta_3 x_{it} + \beta_4 r_t + \beta_5 + \beta_6 \tau_{t+1}, \quad (51)$$

which leads to

$$\int_0^1 \kappa_{jt} dj = \beta_0 \underline{\kappa}_t + (\beta_1 + \beta_3) \mu_t + \beta_2 \omega_t + \beta_4 r_t + \beta_5 + \beta_6 \tau_{t+1},$$

and substituting in (23), we arrive at

$$\begin{aligned} x_{t+1}^* &= g\mu_t + (1-g)\underline{\kappa}_t + \lambda \int_0^1 \kappa_{jt} dj + \omega_{t+1} \\ &= (1-g + \lambda\beta_0)\underline{\kappa}_t + [g + \lambda(\beta_1 + \beta_3)]\mu_t + \lambda\beta_4 r_t + \lambda\beta_5 + \beta_6 \tau_{t+1} + (1 + \lambda\beta_2)\omega_{t+1}. \end{aligned}$$

Using (50) and $\mathbb{E}[\tilde{\omega}_{t+1} | \eta_{it}] = \eta_{it}$ we get

$$\begin{aligned} \kappa_t(\mu_t, \eta_{it}, x_{it}, r_t) &= (1-g + \lambda\beta_0)\underline{\kappa}_t + [g + \lambda(\beta_1 + \beta_3)]\mu_t + (\lambda\beta_4 - 1)r_t + \lambda\beta_5 + (\lambda\beta_6 + \xi)\tau_{t+1} \\ &\quad + (1 + \lambda\beta_2)\eta_{it} - gx_{it} + g - 1. \end{aligned}$$

Identifying with (51) yields $\beta_2 = 1/(1-\lambda)$, $\beta_0 = (1-g)\beta_2$, $\beta_1 = g$, $\beta_3 = -g$,

$\beta_4 = -\beta_2$, $\beta_5 = (g - 1)\beta_2$, $\beta_6 = \xi\beta_2$. It comes

$$\kappa_{it} = g(\mu_t - x_{it}) + \frac{(1 - g)\underline{\kappa}_t + \eta_{it} - 1 - r_t + g + \xi\tau_{t+1}}{1 - \lambda},$$

hence

$$x_{it+1} = gx_{it} + \kappa_{it} = g\mu_t + \frac{(1 - g)\underline{\kappa}_t + \eta_{it} - 1 - r_t + g + \xi\tau_{t+1}}{1 - \lambda}.$$

Summing over $[0, 1]$ and using $\int_0^1 \eta_{it} di = \omega_{t+1}$ gives (24). We thus have $x_{it+1} = \mu_{t+1} + v_{it}/(1 - \lambda)$ which is normally distributed with standard deviation $\sigma_v/(1 - \lambda) = \sigma_x$.

F Proof of Proposition 3

For the tax/subsidy policy, we have from (21), $1 + r_t - g = q'(k_{t+1})(1 - \tau_{t+1}\varphi)$, while (19) gives $h_t^* - g \geq q'(k_{t+1})[1 - (h_t^* - g)\varphi/\xi]$, with an equality when (15) is not binding. From (13), the realized IRS satisfies $h_t = 1 + r_t$ and we have $h_t = h_t^*$ when $k_{t+1} > gk_t$. Identifying gives (25). When $k_{t+1} = gk_t$, setting $\tau_{t+1} = (h_t^* - g)/\xi$ implies $h_t^* - g > g + q'(k_{t+1})(1 - \tau_{t+1}\varphi) = 1 + r_t - g$, hence $1 + r_t < h_t^*$: due to an excess of capital, the rental price of capital at equilibrium is lower than the first-best IRS. Substituting (25) for τ_{t+1} in the expectation of (22) gives

$$\mathbb{E}[\mu_{t+1}] = [(1 - \lambda g)\mu_t + h_t^* - gz_{t+1} - \mathbb{E}[h_t](1 - z_t)]/(1 - \lambda).$$

Re-arranging terms gives (26) using $\mu_t = \mu_t^*$ and $\mathbb{E}[\mu_{t+1}] = \mu_{t+1}^*$. (27) is obtained similarly from (24). At the stationary state, denoting with subscript ∞ the values of the variables, we have $e_\infty = e_N$, $c_\infty = q(k_\infty) - (1 - g)(k_\infty + \mu_\infty)$, $h_\infty = 1/\beta$, $\tau_\infty = (1/\beta - g)\varphi/\xi$ and $k_\infty = q'^{-1}((1/\beta - g)/(1 - (1/\beta - g)\varphi/\xi))$. (16) and (22) imply $\mu_\infty = (\varphi/\xi)q(k_\infty)$ and $(1 - \lambda)\mu_\infty = (1 - \lambda g)\mu_\infty + \xi\tau_\infty - (1 - z_\infty)(h_\infty - g)$. Replacing and reorganizing terms yields $z_\infty = -\lambda(1 - g)(\xi/\varphi)q(k_\infty)/(1/\beta - g) < 0$.

G Proof of Lemma 4

The discriminant of $P(h) = 0$ is $\Delta \equiv (g - \theta)^2 + 4p\xi > 0$, and the equation admits two roots, $\bar{h} = (g + \theta + \sqrt{\Delta})/2 > \max\{g, \theta\}$ and $\underline{h} = (g + \theta - \sqrt{\Delta})/2 < \min\{g, \theta\}$. We have $\underline{h} > 0$ iff $\xi p < \theta g$. The optimal policy corresponds to a sequence of IRS with a subsequence defined by (28) as long as the ENP constraint (16) is not binding, i.e. $t < T$ where T is the first period (16) binds. The sequence $\{h_t^*\}_{t=0}^{T-1}$ is either degenerate,

i.e. $h_t^* \in \{\underline{h}, \bar{h}\}$ for all $t \in \{0, \dots, T-1\}$, or $h_0^* \notin \{\underline{h}, \bar{h}\}$, and, reorganizing (28),

$$h_t^* = (\xi p - \theta g) / (h_{t-1}^* - g - \theta) \quad (52)$$

for all $t = 1, \dots, T-1$. If T is large, this sequence eventually converges to a root of $P(h) = 0$ that we denote by h_∞ . We can derived the non degenerate sequence as follows. Defining $v_t = (h_t^* - h_\#)^{-1}$, $h_\# \in \{\underline{h}, \bar{h}\}$, we have $h_t^* = 1/v_t + h_\#$ and (52) becomes

$$\frac{1}{v_{t+1}} + h_\# = \frac{p\xi - \theta g}{1/v_t + h_\# - g - \theta},$$

which gives

$$\frac{1}{v_{t+1}} = \frac{-h_\#}{1 + v_t(h_\# - g - \theta)},$$

using (29). We thus have

$$v_{t+1} = v_t(g + \theta - h_\#) / h_\# - 1/h_\# \equiv v_t b_1 - b_0$$

with $b_1 = \underline{h}/\bar{h}$ if $h_\# = \bar{h}$ and $b_1 = \bar{h}/\underline{h}$ otherwise. With an initial value v_0 at $t = t_0$, the solution of this recurrence equation is given by

$$\begin{aligned} v_t &= v_0 b_1^t - b_0(1 - b_1^t) / (1 - b_1) \\ &= [v_0 + b_0 / (1 - b_1)] b_1^t - b_0 / (1 - b_1). \end{aligned} \quad (53)$$

where

$$\frac{b_0}{1 - b_1} = \frac{1/h_\#}{1 + (h_\# - g - \theta)/h_\#} = \frac{1}{2h_\# - g - \theta}.$$

If $|b_1| < 1$, i.e. if $h_\# = \bar{h}$, v_t converges toward $v_\infty = -b_0 / (1 - b_1) = 1 / (g + \theta - 2\bar{h}) = (h_\infty - \bar{h})^{-1}$, hence h_t^* converges toward $h_\infty = g + \theta - \bar{h} = \underline{h}$. If $|b_1| > 1$, i.e. if $h_\# = \underline{h}$, v_t diverges and thus $h_\infty = \underline{h}$. Hence, the recursion does not converge to \bar{h} (unless in the degenerate case $h_0 = \bar{h}$), while it converges to \underline{h} from any initial value $h_0 \neq \bar{h}$. This cannot be optimal if $\xi p \geq \theta g$ since $\underline{h} \leq 0$. If $\xi p < \theta g$, as $\underline{h} < \min\{g, \theta\} \leq g$, there is no stock of productive capital that can satisfy (19) when $h_t^* \rightarrow \underline{h}$. We thus cannot have $h_0^* < \underline{h}$. If $h_t^* > \underline{h}$, re-expressing (53) using $h_\# = \underline{h}$ and $b_1 = \bar{h}/\underline{h} > 1$, the sequence $\{h_t^*\}_{t=0}^{T-1}$ is given by

$$h_t^* = \frac{\sqrt{\Delta}(h_0^* - \underline{h})}{[\sqrt{\Delta} - (h_0^* - \underline{h})](\bar{h}/\underline{h})^t + h_0^* - \underline{h}} + \underline{h}$$

where $h_0^* < \sqrt{\Delta} + \underline{h}$ to have $h_t^* > \underline{h}$ for all t . As this sequence is decreasing, tending to $\underline{h} < g$, the stock of productive capital increases and eventually diverges as h_t^* converges to \underline{h} . We would thus have a decreasing path of global wealth with an investment of productive capital tending to $q(k_t^*)$, hence c_t and e_t tending to 0, which is obviously not optimal. The optimal solution is thus $h_t^* = \bar{h}$ for all $t \in \{0, \dots, T-1\}$.

From the concavity of P , as $1/\beta > \max\{g, \theta\}$, we have $\bar{h} > 1/\beta$ iff $P(1/\beta) > 0$, i.e. iff $p\xi > (1/\beta - \theta)(1/\beta - g)$. From (19), using $\lim_{k \rightarrow 0} q'(k) = +\infty$, \bar{h} corresponds to the solution of the (17) if it is lower than $g + \xi/\varphi$, otherwise (15) would be binding indefinitely. The condition $\bar{h} < g + \xi/\varphi$ simplifies to $p < \xi/\varphi^2 + (g - \theta)/\varphi$. The conditions $P(1/\beta) > 0$ and $p < \xi/\varphi^2 + (g - \theta)/\varphi$ imply that $(\xi/\varphi)^2 + (g - \theta)\xi/\varphi > p\xi > (1/\beta - \theta)(1/\beta - g)$. We thus must have $F(\xi/\varphi) > 0$, where $F(x) = x^2 + (g - \theta)x - (1/\beta - \theta)(1/\beta - g)$ is a second degree polynomial. The discriminant of $F(x) = 0$ is given by $[2/\beta - (g + \theta)]^2$, implying that $F(x) = (x - \underline{x})(x - \bar{x})$ where $\underline{x} < 0 < \bar{x}$ are the two real roots of $F(x) = 0$. $F(x)$ is positive if $x < \underline{x}$ or $x > \bar{x}$, and since $\xi/\varphi > 0$, we must have $\xi/\varphi > \bar{x} = 1/\beta - (3g - \theta)/2$.

If $gk_0 > \bar{k}$, (15) is binding as long as $g^t k_0 > \bar{k}$, i.e. $t \leq t_0$ given by $g^{t_0} k_0 = \bar{k}$, hence $t_0 = \ln(\bar{k}/k_0)/\ln g$. For $t > t_0$, $k_{t+1} = \bar{k}$ until (16) is binding, i.e. until T given by $\mu_T^* = q(k_{T+1})\varphi/\xi$. Using (28) for $t = T - 1$ yields $h_T = \bar{h}$, hence $k_{T+1} = \bar{k}$. The path of the economy for $t > T$ is defined recursively by (11), (14), (16), (18) and (19) that holds for all $t > t_0$. Given the initial values e_T , $k_{T+1} = k_T = \bar{k}$, the IRS can be written as $h_t = h(k_t, k_{t+1}, k_{t+2})$ and (19) gives the implicit equation $q'(k_{t+1})[1 - (h(k_t, k_{t+1}, k_{t+2}) - g)\varphi/\xi] - h(k_t, k_{t+1}, k_{t+2}) + g = 0$ defining k_{t+2} for all $t \geq T$.

H Proof of Proposition 4

With exponential preferences $h_t = e^{\gamma[c_{t+1} - c_t + p(e_{t+1} - e_t)]}/\beta$. $h_t = \bar{h}$ gives

$$c_{t+1} - c_t + p(e_{t+1} - e_t) = \ln(\beta\bar{h})^{1/\gamma} \quad (54)$$

for all $t \leq T$. Multiplying both sides by ξ and using (11) and (14) leads to

$$\begin{aligned} \ln(\beta\bar{h})^{\xi/\gamma} &= \xi[q_{t+1} - q_t - k_{t+2} + (1 + g)k_{t+1} - gk_t] - \varphi[q_{t+2} - (1 + g)q_{t+1} + gq_t] \\ &\quad - (e_{t+3} - e_{t+2}) + (\theta + g)(e_{t+2} - e_{t+1}) + (\xi p - g\theta)(e_{t+1} - e_t) \end{aligned} \quad (55)$$

for all $t \leq T$. For $t < t_0$, we have $k_t = g^t k_0$, and using $q(k) = Ak^\alpha$, $q_t = g^{\alpha t} q_0$. For $t_0 \leq t \leq T + 1$, we have $k_t = \bar{k}$ and $q_t = q(\bar{k}) \equiv \bar{q}$. (55) can thus be expressed as

$$m_{t+2} = (\theta + g)m_{t+1} + (\xi p - g\theta)m_t - \hat{\kappa}g^{\alpha t} - \ln(\beta\bar{h})^{\xi/\gamma} \quad (56)$$

for all $t \leq T - 1$, where $m_t \equiv e_{t+1} - e_t$ and

$$\hat{\kappa} = \begin{cases} [\xi + \varphi(g - g^\alpha)](1 - g^\alpha)q_0 & t < t_0 \\ 0 & t_0 \leq t \leq T - 1 \end{cases}.$$

For $t \geq T$, as $\varphi q(k_t) = \xi\mu_t$, we have $e_{t+1} = \theta e_t + \hat{e}$ implying $m_{t+1} = \theta m_t$.

The solution of (56) is $m_t = n_t + v_t$ where n_t and v_t are the solutions of the corresponding homogeneous and particular equations. The characteristic equation of the homogeneous equation is $P(x) = 0$. As \bar{h} is irrelevant, $n_t = \phi\bar{h}^t$ where ϕ is a constant. The particular solution of (56) is given by $v_t = g^{\alpha t}\nu_0 + \nu_1$ where ν_0 and ν_1 solve

$$g^{\alpha(t+2)}\nu_0 + \nu_1 = (\theta + g)(\nu_0 g^{\alpha(t+1)} + \nu_1) + (\xi p - g\theta)(\nu_0 g^{\alpha t} + \nu_1) - \hat{\kappa}g^{\alpha t} - \ln(\beta\bar{h})^{\xi/\gamma}$$

for all $t \leq T - 1$, which gives $\nu_0 = \hat{\kappa}/P(g^\alpha)$ and $\nu_1 = \ln(\beta\bar{h})^{\xi/\gamma}/P(1)$. As P is concave with $P(g) = p\xi$ and $P(1/\beta) > 0$, we have $P(g^\alpha) > 0$ and $P(1) > 0$, hence $\nu_1 > 0$ since $P(1/\beta) > 0$ implies $\bar{h} > 1/\beta$. We also have $\nu_0 \geq 0$ if $\xi/\varphi \geq 1 - g > g^\alpha - g$. The solution of (56) is thus

$$m_t = \phi\bar{h}^t + g^{\alpha t}\nu_0 + \nu_1, \quad (57)$$

where ϕ is specific to each sub-sequence and is derived from their initial conditions, i.e. from (56) evaluated at $t = t_0 - 1$, (55) evaluated at $t = T$, and from (11) at $t = 1$.

Using (55) at $t = T$, yields

$$\begin{aligned} \ln(\beta\bar{h})^{\xi/\gamma} &= \xi(\bar{k} - k_{T+2}) - \varphi(q(k_{T+2}) - \bar{q}) - m_{T+2} + (\theta + g)m_{T+1} + (\xi p - g\theta)m_T \\ &= \xi\bar{k} + \varphi\bar{q} - [\xi k_{T+2} + \varphi q(k_{T+2})] - \theta^2 m_T + \theta(\theta + g)m_T + (\xi p - g\theta)m_T \\ &= \xi\bar{k} + \varphi\bar{q} - [\xi k_{T+2} + \varphi q(k_{T+2})] + \xi p m_T. \end{aligned}$$

Identifying each term with (54) multiplied by ξ on both sides, it comes

$$\xi(c_{T+1} - c_T) = \xi\bar{k} + \varphi\bar{q} - [\xi k_{T+2} + \varphi q(k_{T+2})] \equiv \hat{\kappa}_1. \quad (58)$$

We thus get, using (57) and $\ln(\beta\bar{h})^{\xi/\gamma} = \nu_1 P(1)$,

$$\phi\bar{h}^T + \nu_1 = [\nu_1 P(1) - \hat{\kappa}_1]/\xi p.$$

It comes

$$\phi = \nu_1(P(1)/\xi p - 1)/\bar{h}^T - \hat{\kappa}_1/(\xi p\bar{h}^T) = -[(1-\theta)(1-g)\nu_1 + \hat{\kappa}_1]/(\xi p\bar{h}^T)$$

for all $t_0 \leq t \leq T$. Substituting into (57) and using $\nu_0 = 0$ for all $t_0 \leq t \leq T$ gives

$$m_t = \nu_1 - [(1-\theta)(1-g)\nu_1 + \hat{\kappa}_1]/(\xi p\bar{h}^{T-t})$$

for all $t_0 \leq t \leq T$. Using

$$c_{T+1} - c_T + pm_T = \hat{\kappa}_1/\xi + p\nu_1 - [(1-\theta)(1-g)\nu_1 + \hat{\kappa}_1]/\xi = P(1)\nu_1/\xi = \ln(\beta\bar{h})^{1/\gamma},$$

it comes

$$h_T = e^{\gamma[c_{T+1} - c_T + p(e_{T+1} - e_T)]}/\beta = \bar{h} = g + q'(k_{T+2})/[1 + q'(k_{T+2})\varphi/\xi],$$

hence $k_{T+2} = \bar{k}$, implying $c_{T+1} - c_T = 0 = \hat{\kappa}_1$, and thus

$$m_t = \nu_1 - [(1-\theta)(1-g)\nu_1]/(\xi p\bar{h}^{T-t})$$

for all $t_0 \leq t \leq T$. As $m_t \geq P(1)\nu_1/\xi p = \ln(\beta\bar{h})^{1/p\gamma} > 0$, $e_{t+1} > e_t$ for all $t_0 \leq t \leq T$.

Using (56) for $t = t_0 - 1$, it comes

$$\begin{aligned} m_{t_0+1} &= \nu_1 - [(1-\theta)(1-g)\nu_1]/(\xi p\bar{h}^{(T-t_0-1)}) \\ &= (\theta + g)\{\nu_1 - [(1-\theta)(1-g)\nu_1]/(\xi p\bar{h}^{(T-t_0)})\} + (\xi p - \theta g)(\phi\bar{h}^{t_0-1} + g^{\alpha(t_0-1)}\nu_0 + \nu_1) - \nu_1 P(1). \end{aligned}$$

As $P(1) = \xi p - \theta g - 1 + \theta + g$, this equation simplifies to

$$\begin{aligned} [(1-\theta)(1-g)\nu_1]/(\xi p\bar{h}^{(T-t_0-1)}) &= (\theta + g)[(1-\theta)(1-g)\nu_1]/(\xi p\bar{h}^{(T-t_0)}) \\ &\quad - (\xi p - \theta g)(\phi\bar{h}^{t_0-1} + g^{\alpha(t_0-1)}\nu_0). \end{aligned}$$

Multiplying by $\xi p\bar{h}^{T-t_0+1}$ yields

$$[(1-\theta)(1-g)\nu_1]\bar{h}^2 = (\theta + g)[(1-\theta)(1-g)\nu_1]\bar{h} - (\xi p - \theta g)(\phi\bar{h}^{t_0-1} + g^{\alpha(t_0-1)}\nu_0)\xi p\bar{h}^{T-t_0+1},$$

and re-organizing terms,

$$[(1-\theta)(1-g)\nu_1][\bar{h}^2 - (\theta+g)\bar{h}] = -(\xi p - \theta g)(\phi\bar{h}^{t_0-1} + g^{\alpha(t_0-1)}\nu_0)\xi p\bar{h}^{T-t_0+1},$$

where, as $P(\bar{h}) = 0$, $p\xi - \theta g = \bar{h}^2 - (\theta+g)\bar{h}$. Simplifying, we get

$$(1-\theta)(1-g)\nu_1 = -[\phi + (g^\alpha/\bar{h})^{t_0-1}\nu_0]\xi p\bar{h}^T,$$

which gives

$$\phi = -\left(\frac{g^\alpha}{\bar{h}}\right)^{t_0-1}\nu_0 - \frac{(1-\theta)(1-g)\nu_1}{\xi p\bar{h}^T}.$$

Substituting into (57) yields

$$m_t = \nu_0 g^{\alpha t} [1 - (g^\alpha/\bar{h})^{t_0-1-t}] + \nu_1 - [(1-\theta)(1-g)\nu_1]/(\xi p\bar{h}^{T-t})$$

for all $0 < t < t_0$. As $m_t > 0$, $e_{t+1} > e_t$ for all $0 < t < t_0$. (31) is deduced from

$$\begin{aligned} \sum_{\tau=1}^{t-1} m_\tau &= e_t - e_1 = \sum_{\tau=1}^{t-1} (\phi\bar{h}^\tau + g^{\alpha\tau}\nu_0 + \nu_1) = \phi \frac{\bar{h}^t - \bar{h}}{\bar{h} - 1} + \nu_0 \frac{g^\alpha - g^{\alpha t}}{1 - g^\alpha} + (t-1)\nu_1 \\ &= \nu_0 \left(\frac{g^\alpha - g^{\alpha t}}{1 - g^\alpha} - \left(\frac{g^\alpha}{\bar{h}}\right)^{t_0-1} \frac{\bar{h}^t - \bar{h}}{\bar{h} - 1} \right) + \nu_1(t-1) - \frac{(1-\theta)(1-g)\nu_1}{\xi p\bar{h}^T} \frac{\bar{h}^t - \bar{h}}{\bar{h} - 1} \end{aligned}$$

for all $1 < t < t_0$, where $e_1 = \theta e_0 + \xi\mu_0 - \varphi q_0 + \hat{e}$, and from

$$\begin{aligned} \sum_{\tau=t_0-1}^{t-1} m_\tau &= e_t - e_{t_0-1} = \phi \sum_{\tau=t_0-1}^{t-1} \bar{h}^\tau + (t-t_0)\nu_1 = \phi \frac{\bar{h}^t - \bar{h}^{t_0-1}}{\bar{h} - 1} + (t-t_0+1)\nu_1 \\ &= \nu_1(t-t_0+1) - \frac{(1-\theta)(1-g)\nu_1}{\xi p\bar{h}^T} \frac{\bar{h}^t - \bar{h}^{t_0-1}}{\bar{h} - 1}, \end{aligned}$$

for all $t_0 \leq t \leq T$. Using $m_T = \ln(\beta\bar{h})^{1/(\rho\gamma)}$ and $m_T = e_{T+1} - e_T = \hat{e} - (1-\theta)e_T$, it comes that T is solution of $e_T = e_N - \ln(\beta\bar{h})^{1/(1-\theta)\rho\gamma}$. As

$$\begin{aligned} m_t - m_{t-1} &= \nu_0(g^{\alpha t} - g^{\alpha(t_0-1)}/\bar{h}^{t_0-1-t} - g^{\alpha(t-1)} + g^{\alpha(t_0-1)}/\bar{h}^{t_0-t}) - \frac{\nu_1(1-\theta)(1-g)}{\xi p\bar{h}^{T-t+1}}(\bar{h} - 1) \\ &= -\nu_0[(1/g^\alpha - 1)g^{\alpha t} + (\bar{h} - 1)g^{\alpha(t_0-1)}/\bar{h}^{t_0-t}] - \frac{\nu_1(1-\theta)(1-g)}{\xi p\bar{h}^{T-t+1}}(\bar{h} - 1), \end{aligned}$$

where $\nu_0 > 0$ if $t < t_0$ and $\nu_0 = 0$ if $t_0 \leq t \leq T$, $m_t - m_{t-1} < 0$ for all $t \leq T$: the sequence $\{e_t\}_{2 \leq t \leq T}$ increases at a decreasing rate. From (11), the GTI index is deduced

from these expressions using $\mu_t = (e_{t+1} - \theta e_t + \varphi q_t - \hat{e})/\xi$. From (54),

$$c_{t+1} - c_t = \ln(\beta \bar{h})^{1/\gamma} - p m_t$$

for all $t \leq T$ where

$$\begin{aligned} m_t &= \nu_0 g^{\alpha t} [1 - (g^\alpha / \bar{h})^{t_0 - 1 - t}] + \nu_1 [1 - (1 - \theta)(1 - g) / (\xi p \bar{h}^{T-t})] \\ &= \nu_0 g^{\alpha t} [1 - (g^\alpha / \bar{h})^{t_0 - 1 - t}] + \ln(\beta \bar{h})^{1/\gamma} [\xi p - (1 - \theta)(1 - g) / \bar{h}^{T-t}] / [p P(1)] \end{aligned}$$

for all $1 < t \leq T$. We thus have

$$\begin{aligned} c_{t+1} - c_t &= -\nu_0 g^{\alpha t} [1 - (g^\alpha / \bar{h})^{t_0 - 1 - t}] + \ln(\beta \bar{h})^{1/\gamma} [P(1) - (\xi p - (1 - \theta)(1 - g) / \bar{h}^{T-t})] / P(1) \\ &= -\nu_0 g^{\alpha t} [1 - (g^\alpha / \bar{h})^{t_0 - 1 - t}] - \ln(\beta \bar{h})^{1/\gamma} (1 - \theta)(1 - g)(1 - 1/\bar{h}^{T-t}) / P(1) \end{aligned}$$

which is negative and increasing, with a maximum equal to 0 at $t = T$. As $c_{t+1} - c_t - (c_t - c_{t-1}) = -p(m_t - m_{t-1}) > 0$, the consumption sequence decreases at a decreasing rate. With u CARA, we have $h_t = u_t / \beta u_{t+1}$ and thus $u_{t+1} = u_t / (\beta h_t) = u_0 \beta^{-t} \prod_{\tau=1}^t 1/h_\tau$. As $h_t = \bar{h}$ for all $t < T$ at the optimum, we get $u_{t+1} = u_0 (\bar{h} \beta)^{-t}$ and

$$\sum_{t=0}^{T-1} \beta^t u_t = u_0 \sum_{t=0}^{T-1} (1/\bar{h})^t = u_0 \frac{1 - 1/\bar{h}^T}{1 - 1/\bar{h}} = u_0 \frac{\bar{h} - 1/\bar{h}^{T-1}}{\bar{h} - 1}.$$

I Proof of Proposition 5

For ease of notation, superscript ℓ is removed and results are derived in the tax-subsidy policy case ($\ell = sub$). Variations from these results in the tax-standard policy case are specified in the text. Also, subscript t is used as a shorthand for coefficients function of z_t , e.g. $a_{3t} \equiv a_3^{sub}(z_t)$. I first characterize the distribution of \tilde{y}_{t+1} given the information available in period t , $t_0 < t \leq T$, a random variable denoted by $\tilde{y}_{t+1|t}$. Define

$$\hat{k}(r) \equiv q^{t-1} ((1 + r - g) / (1 - (\bar{h} - g)\varphi/\xi)),$$

and $\hat{q}(r) \equiv q(\hat{k}(r))$. We have $\bar{k} = \hat{k}(\bar{r})$, and $\bar{q} \equiv \hat{q}(\bar{r})$ denotes the corresponding GDP level in the following.

Given the information available in period t , the prevailing interest rate r_t (and thus the realization ε_t of $\tilde{\varepsilon}_t$) is known, and both $k_{t+1} = \hat{k}(r_t)$ and $q_{t+1} = \hat{q}(r_{t+1})$ are known. From (11), e_{t+1} is also known since it depends on variables of the previous period, that

are function of ε_{t-1} . Hence, from (14), we get

$$\tilde{y}_{t+1|t} = \tilde{c}_{t+1|t} + pe_{t+1} = q_{t+1} - \tilde{k}_{t+2|t} + gk_{t+1} - (\tilde{\mu}_{t+2|t} - g\mu_{t+1}) + pe_{t+1}, \quad (59)$$

where \tilde{k}_{t+2} and $\tilde{\mu}_{t+2}$ are unknown since they depend on the realization of $\tilde{\varepsilon}_{t+1}$: we have $\tilde{k}_{t+2} = \hat{k}(\tilde{r}_{t+1})$ and, from (34),

$$\tilde{\mu}_{t+2|t} = a_1\mu_{t+1} + a_2e_{t+1} + a_3t + Z_{t+1} + b_{1t}\tilde{\varepsilon}_{t+1} + b_2\varepsilon_t. \quad (60)$$

Replacing, we get

$$\tilde{y}_{t+1|t} = q_{t+1} - \tilde{k}_{t+2|t} + gk_{t+1} + (g - a_1)\mu_{t+1} + (p - a_2)e_{t+1} - a_3t - Z_{t+1} - b_{1t}\tilde{\varepsilon}_{t+1} - b_2\varepsilon_t.$$

Using the linear approximation $k_t \approx \bar{k} + \hat{k}'(\bar{r})\bar{r}\varepsilon_{t-1}$ where $\hat{k}'(\bar{r}) = 1/[(1 - \tau\varphi)q''(\bar{k})]$, we get using $\mathbb{E}_t[\tilde{k}_{t+2}] \approx \bar{k} + \hat{k}'(\bar{r})\bar{r}\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t]$ and (33) that

$$\begin{aligned} \tilde{y}_{t+1|t} - \mathbb{E}_t[\tilde{y}_{t+1}] &\approx -(\tilde{k}_{t+2|t} - \mathbb{E}[\tilde{k}_{t+2}|\varepsilon_t]) - b_{1t}(\tilde{\varepsilon}_{t+1} - \mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t]) \\ &= -(\hat{k}'(\bar{r})\bar{r} + b_{1t})(\tilde{\varepsilon}_{t+1} - \mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t]) \\ &= -(\hat{k}'(\bar{r})\bar{r} + b_{1t})\chi_t\tilde{\omega}_{t+2} \end{aligned}$$

when $t_0 < t \leq T$. Hence, $\tilde{y}_{t+1|t}$ when $t_0 < t \leq T$ is approximately normally distributed with variance $\mathbb{V}_t[\tilde{y}_{t+1}] = (\hat{k}'(\bar{r})\bar{r} + b_{1t})^2\chi_t^2\sigma_\omega^2 \equiv \sigma_{y_{t+1|t}}^2$. For the tax-standard policy, as b_1^{std} and χ^{std} do not depend on t (as shown below), $\sigma_{y_{t+1|t}}^2$ is a constant.

The coefficients $a_1, a_2, a_3, b_{1t}, b_2$, and Z_t in (34), and χ_t and ρ_t in (33), are derived as follows. Using (11), (14), (59), (60) and $\mathbb{E}_t[\tilde{k}_{t+2}] \approx \bar{k} + \hat{k}'(\bar{r})\bar{r}\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t]$, we get

$$\begin{aligned}
\mathbb{E}_t[\tilde{y}_{t+1}] - y_t &\approx q_{t+1} - q_t - \bar{k} - \hat{k}'(\bar{r})\bar{r}\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] + k_{t+1}(1+g) - gk_t - \mathbb{E}_t[\tilde{\mu}_{t+2}] + (1+g)\mu_{t+1} \\
&\quad - g\mu_t + p(e_{t+1} - e_t) \\
&= q_{t+1} - q_t - \bar{k} - \hat{k}'(\bar{r})\bar{r}\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] + k_{t+1}(1+g) - gk_t \\
&\quad - (a_1\mu_{t+1} + a_2e_{t+1} + a_3t + Z_{t+1} + b_{1t}\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] + b_2\varepsilon_t) + (1+g)\mu_{t+1} \\
&\quad - \mu_t g + p(e_{t+1} - e_t) \\
&= q_{t+1} - q_t - \bar{k} - \hat{k}'(\bar{r})\bar{r}\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] + k_{t+1}(1+g) - gk_t - a_3t - Z_{t+1} \\
&\quad - b_{1t}\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] - b_2\varepsilon_t + (1+g-a_1)\mu_{t+1} - \mu_t g + (p-a_2)e_{t+1} - pe_t \\
&= q_{t+1} - q_t - \bar{k} - (b_{1t} + \hat{k}'(\bar{r})\bar{r})\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] + k_{t+1}(1+g) - gk_t - a_3t - Z_{t+1} \\
&\quad - b_2\varepsilon_t + (1+g-a_1)\mu_{t+1} - \mu_t g + (p-a_2)(\theta e_t + \xi\mu_t - \varphi q_t + \hat{e}) - pe_t \\
&= q_{t+1} - q_t[1 + \varphi(p-a_2)] - \bar{k} - (b_{1t} + \hat{k}'(\bar{r})\bar{r})\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] + k_{t+1}(1+g) - gk_t \\
&\quad + (1+g-a_1)\mu_{t+1} - \mu_t[g - \xi(p-a_2)] + [\theta(p-a_2) - p]e_t + (p-a_2)\hat{e} - a_3t \\
&\quad - Z_{t+1} - b_2\varepsilon_t.
\end{aligned}$$

Using $(1+r_t)z_t = (\bar{h} + \bar{r}\varepsilon_t)z_t$ in (22) and (32), we get

$$(1-\lambda)(g\mu_t - \mu_{t+1}) + (1-g)(\mu_t - 1) + \omega_{t+1} + \xi\tau_{t+1} + (\bar{h} + \bar{r}\varepsilon_t)z_t - gz_{t+1} \approx r_t \approx \psi - \gamma^2\sigma_{y_{t+1}}^2/2 + \gamma\{\mathbb{E}_t[\tilde{y}_{t+1}] - y_t\}$$

which gives, denoting $a_0 \equiv 1/[1 - \lambda + \gamma(1 + g - a_1)]$,

$$\begin{aligned}
\mu_{t+1}/a_0 &\approx [1 - \lambda g + \gamma g - \gamma\xi(p-a_2)]\mu_t - \gamma[\theta(p-a_2) - p]e_t + \omega_{t+1} + \xi\tau_{t+1} + (\bar{h} + \bar{r}\varepsilon_t)z_t - gz_{t+1} \\
&\quad - (\psi - \gamma^2\sigma_{y_{t+1}}^2/2) - \gamma(p-a_2)\hat{e} + \gamma a_3 + g - 1 + \gamma Z_{t+1} + \gamma(b_{1t} + \hat{k}'(\bar{r})\bar{r})\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] \\
&\quad + \gamma b_2\varepsilon_t - \gamma q_{t+1} + \gamma q_t[1 + \varphi(p-a_2)] + \gamma\bar{k} - \gamma k_{t+1}(1+g) + \gamma gk_t.
\end{aligned}$$

Using $k_{t+1} \approx \bar{k} + \hat{k}'(\bar{r})\bar{r}\varepsilon_t$ and $q_{t+1} \approx \bar{q} + \hat{q}'(\bar{r})\bar{r}\varepsilon_t$ where $\hat{q}'(\bar{r}) = q'(\bar{k})\hat{k}'(\bar{r})$ for t and $t-1$, we get

$$-\gamma q_{t+1} + \gamma q_t[1 + \varphi(p-a_2)] \approx -\gamma\hat{q}'(\bar{r})\bar{r}\varepsilon_t + \gamma\bar{q}\varphi(p-a_2) + \gamma\hat{q}'(\bar{r})\bar{r}\varepsilon_{t-1}[1 + \varphi(p-a_2)]$$

and

$$\gamma\bar{k} - \gamma k_{t+1}(1+g) + \gamma gk_t \approx -\gamma\hat{k}'(\bar{r})\bar{r}\varepsilon_t(1+g) + \gamma g\hat{k}'(\bar{r})\bar{r}\varepsilon_{t-1}.$$

Substituting and, from (33), using $\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] = \rho_t \varepsilon_t$, we arrive at

$$\begin{aligned}
\mu_{t+1} &\approx a_0[1 - \lambda g + \gamma g - \gamma \xi(p - a_2)]\mu_t - a_0\gamma[\theta(p - a_2) - p]e_t + a_0\omega_{t+1} \\
&- a_0[\psi - \gamma^2\sigma_{y_{t+1}|t}^2/2 + \gamma(p - a_2)(\hat{e} - \bar{q}\varphi) - \gamma a_{3t} + 1 - g] + a_0[\xi\tau_{t+1} + \bar{h}z_t - gz_{t+1} + \gamma Z_{t+1}] \\
&+ a_0\gamma[\hat{q}'(\bar{r})[1 + \varphi(p - a_2)] + g\hat{k}'(\bar{r})]\bar{r}\varepsilon_{t-1} \\
&- a_0\gamma[\hat{q}'(\bar{r})\bar{r} + \hat{k}'(\bar{r})\bar{r}(1 + g) - b_2 - \bar{r}z_t/\gamma - (b_{1t} + \hat{k}'(\bar{r})\bar{r})\rho_t]\varepsilon_t
\end{aligned} \tag{61}$$

Identifying the non-stochastic terms with those of (34) and simplifying gives

$$\begin{aligned}
a_1 &= \frac{1 - \lambda g + \gamma[g + \xi(a_2 - p)]}{1 - \lambda + \gamma(1 + g - a_1)}, \quad a_2 = \frac{\gamma p(1 - \theta)}{1 - \lambda + \gamma(1 + g - a_1 - \theta)}, \\
a_{3t} &= -\frac{\psi - \gamma^2\sigma_{y_{t+1}|t}^2/2 - \gamma(p - a_2)(\varphi\bar{q} - \hat{e}) + 1 - g}{1 - \lambda + \gamma(g - a_1)},
\end{aligned} \tag{62}$$

and $Z_t = a_0(\xi\tau_{t+1} + \bar{h}z_t - gz_{t+1} + \gamma Z_{t+1})$. Solving the recursion gives (35).

For the standard policy, using (24) and (32) to get

$$(1 - \lambda)(g\mu_t - \mu_{t+1}) + (1 - g)(\underline{\kappa}_t - 1) + \omega_{t+1} + \xi\tau_{t+1} = r_t \approx \psi - \gamma^2\sigma_{y_{t+1}}^2/2 + \gamma\{\mathbb{E}_t[\tilde{y}_{t+1}] - y_t\},$$

it comes following the same steps that a_2^{std} and a_3^{std} are given also by (62) (but where $\sigma_{y_{t+1}|t}^2$ is a constant in the latter) while we have

$$a_1^{std} = \frac{(1 - \lambda)g + \gamma[g + \xi(a_2^{std} - p)]}{1 - \lambda + \gamma(1 + g - a_1^{std})}$$

and $Z_t^{std} = a_0^{std}(\xi\tau_{t+1} + (1 - g)\underline{\kappa}_t + \gamma Z_{t+1}^{std})$, with $a_0^{std} = 1/[1 - \lambda + \gamma(1 + g - a_1^{std})]$. Solving the recursion gives (37).

The parameters concerning the stochastic terms under both policies are derived as follows. When $\varepsilon_{t-1} = \omega_{t+1} = 0$, we have $\varepsilon_t = 0$ implying $r_t = \bar{r}$. Denoting by $\mu_{t+1|r_t=\bar{r}}$ the value of μ_{t+1} in such a case, we get from (22),

$$\mu_{t+1} - \mu_{t+1|r_t=\bar{r}} \approx \frac{\omega_{t+1} + (1 + r_t - \bar{h})z_t + \bar{r} - r_t}{1 - \lambda} = \frac{\omega_{t+1} - (1 - z_t)\bar{r}\varepsilon_t}{1 - \lambda},$$

using $(1 + r_t)z_t \approx (\bar{h} + \bar{r}\varepsilon_t)z_t$, and from (61),

$$\begin{aligned}
\mu_{t+1} - \mu_{t+1|r_t=\bar{r}} &\approx a_0\{\omega_{t+1} + \gamma[\hat{q}'(\bar{r})[1 + \varphi(p - a_2)] + g\hat{k}'(\bar{r})]\bar{r}\varepsilon_{t-1} \\
&- \gamma[\hat{q}'(\bar{r})\bar{r} + \hat{k}'(\bar{r})\bar{r}(1 + g) - b_2 - \bar{r}z_t/\gamma - (b_{1t} + \hat{k}'(\bar{r})\bar{r})\rho_t]\varepsilon_t\}.
\end{aligned} \tag{63}$$

Equalizing gives

$$\begin{aligned}\omega_{t+1} &= \frac{\bar{r}(1 - z_t) - (1 - \lambda)a_0\gamma[\hat{q}'(\bar{r})\bar{r} + \hat{k}'(\bar{r})\bar{r}(1 + g) - b_2 - \bar{r}z_t/\gamma - (b_{1t} + \hat{k}'(\bar{r})\bar{r})\rho_t]}{1 - (1 - \lambda)a_0}\varepsilon_t \\ &+ (1 - \lambda)\gamma a_0 \frac{\hat{q}'(\bar{r})[1 + \varphi(p - a_2)] + g\hat{k}'(\bar{r})}{1 - (1 - \lambda)a_0}\bar{r}\varepsilon_{t-1}.\end{aligned}\quad (64)$$

Substituting in (63) yields

$$\begin{aligned}\mu_{t+1} - \mu_{t+1|r_t=\bar{r}} &\approx \gamma a_0 \frac{\hat{q}'(\bar{r})[1 + \varphi(p - a_2)] + g\hat{k}'(\bar{r})}{1 - (1 - \lambda)a_0}\bar{r}\varepsilon_{t-1} \\ &+ a_0 \frac{\bar{r} - \gamma[\hat{q}'(\bar{r})\bar{r} + \hat{k}'(\bar{r})\bar{r}(1 + g) - b_2 - (b_{1t} + \hat{k}'(\bar{r})\bar{r})\rho_t]}{1 - (1 - \lambda)a_0}\varepsilon_t.\end{aligned}$$

Identifying with the stochastic terms of (34) gives, using $1 - (1 - \lambda)a_0 = \gamma a_0(1 + g - a_1)$,

$$b_2 = \frac{\hat{q}'(\bar{r})[1 + \varphi(p - a_2)] + g\hat{k}'(\bar{r})}{1 + g - a_1}\bar{r} \quad (65)$$

$$b_{1t} = \frac{\bar{r} - \gamma[\hat{q}'(\bar{r})\bar{r} + \hat{k}'(\bar{r})\bar{r}(1 + g - \rho_t) - b_2 - \rho b_{1t}]}{\gamma(1 + g - a_1)}. \quad (66)$$

As (64) can be written as $\omega_{t+1} = [(1 - \lambda)b_1 + \bar{r}(1 - z_t)]\varepsilon_t + b_2(1 - \lambda)\varepsilon_{t-1}$, we get from (33) that $\chi_t = [(1 - \lambda)b_1 + \bar{r}(1 - z_t)]^{-1}$ and

$$\rho_t = \frac{-b_2(1 - \lambda)}{(1 - \lambda)b_{1t} + \bar{r}(1 - z_t)}. \quad (67)$$

Reorganizing terms to get $b_2 = -\rho_t[b_{1t} + \bar{r}(1 - z_t)/(1 - \lambda)]$ and substituting in (66), we arrive at

$$b_{1t} = \frac{1 - \gamma[\hat{q}'(\bar{r}) + \hat{k}'(\bar{r})(1 + g - \rho_t) + \rho_t(1 - z_t)/(1 - \lambda)]}{\gamma(1 + g - a_1)}\bar{r}.$$

From (65) and (67), we obtain that ρ_t solves

$$\rho_t = \frac{-(1 - \lambda)\gamma a_0\{\hat{q}'(\bar{r})[1 + \varphi(p - a_2)] + g\hat{k}'(\bar{r})\}}{1 - \gamma a_0 \left\{ (1 - \lambda)[\hat{q}'(\bar{r}) + \hat{k}'(\bar{r})(1 + g)] + \rho_t \left[1 - z_t - \hat{k}'(\bar{r})(1 - \lambda) \right] \right\}}. \quad (68)$$

For the standard policy, the same steps give identical formulae but with z_t replaced

by 0. As a result, parameters b_1^{std} , ρ^{std} and χ^{std} are constant.

The first two equations of (62) form a system involving only coefficients a_1 and a_2 that can be solved separately from the others. More precisely, defining $a_1^0 \equiv g + (1 - \lambda)/\gamma$, we can express (62) as

$$\begin{aligned} a_1 &= \frac{a_1^0 + \lambda(1-g)/\gamma + \xi(a_2 - p)}{a_1^0 - a_1 + 1}, a_2 = \frac{p(1-\theta)}{a_1^0 - a_1 + 1 - \theta}, \\ a_{3t} &= -\frac{\psi - \gamma^2 \sigma_{y_{t+1|t}}^2 / 2 - \gamma(p - a_2)(\varphi\bar{q} - \hat{e}) + 1 - g}{\gamma(a_1^0 - a_1)}, a_0 = \frac{1}{\gamma(a_1^0 - a_1 + 1)} \end{aligned} \quad (69)$$

From the expression of a_2 , we get

$$a_2 - p = -p \frac{a_1^0 - a_1}{a_1^0 - a_1 + 1 - \theta}, \quad (70)$$

which, plugged into the expression of $a_1^0 - a_1$, gives

$$(a_1^0 - a_1) \left(a_1 - 1 + \frac{\xi p}{a_1^0 - a_1 + 1 - \theta} \right) = \frac{\lambda(1-g)}{\gamma}$$

that can be expressed as $Q(a_1^0 - a_1) = 0$ where

$$Q(x) \equiv x^3 + (2 - \theta - a_1^0)x^2 + [(1 - \theta)(1 - a_1^0) - \xi p + \lambda(1 - g)/\gamma]x + \lambda(1 - \theta)(1 - g)/\gamma$$

is a third degree polynomial. As Z_t given by (35) converges if $\gamma a_0 < 1$, i.e. if $a_1 < a_1^0$, only positive roots are relevant. As $P(1) > 0$, the third coefficient of Q satisfies

$$\begin{aligned} (1 - \theta)(1 - a_1^0) - \xi p + \lambda(1 - g)/\gamma &< (1 - \theta)(1 - a_1^0) - (1 - \theta)(1 - g) + \lambda(1 - g)/\gamma \\ &= (1 - \theta)(g - a_1^0) + \lambda(1 - g)/\gamma \\ &= -(1 - \theta)(1 - \lambda)/\gamma + \lambda(1 - g)/\gamma, \end{aligned}$$

and thus a sufficient condition for this coefficient to be negative is $\lambda < 1/[1 + (1 - g)/(1 - \theta)]$. Under this condition, we have $Q'(0) < 0$, and since $Q(0) > 0$, there is at most two positive roots, the smallest one corresponding to the largest value of a_1 . From (69) and (70), $a_1 > 0$ if $a_1^0 + \lambda(1 - g)/\gamma + \xi(a_2 - p) = g + (1 - \lambda g)/\gamma - p\xi(a_1^0 - a_1)/(a_1^0 - a_1 + 1 - \theta) > 0$. Since $a_1^0 > a_1$, it suffices to have $g + (1 - \lambda g)/\gamma \geq p\xi$. Under the tax-standard policy, the term $\lambda(1 - g)/\gamma$ in the previous equations becomes $-(1 - g)(1 - \lambda)/\gamma$. We have $Q'(0) < 0$, and $Q(0) < 0$, hence only one positive root, and $a_1^{std} > 0$ if $a_1^0 - (1 - g)(1 - \lambda)/\gamma + \xi(a_2 - p)g = g(1 + (1 - \lambda)/\gamma) - p\xi(a_1^0 - a_1)/(a_1^0 - a_1 + 1 - \theta) > 0$. A sufficient

condition is thus $g(1 + (1 - \lambda)/\gamma) \geq p\xi$ which is more stringent than the tax-subsidy one.

Equation (68) can be written as $\zeta_t(\rho) = -A_t\rho^2 + B\rho - C = 0$, where $A_t \equiv [1 - z_t - \hat{k}'(\bar{r})(1 - \lambda)]\gamma a_0$, $B \equiv 1 - (1 - \lambda)\gamma a_0[\hat{q}'(\bar{r}) + \hat{k}'(\bar{r})(1 + g)]$, and $C \equiv -(1 - \lambda)\gamma a_0\{\hat{q}'(\bar{r})[1 + \varphi(p - a_2)] + g\hat{k}'(\bar{r})\}$ are positive coefficients. As $\zeta(0) < 0$ and $\zeta'(0) > 0$, ζ admits two positive roots provided that $\Delta_t \equiv B^2 - 4A_tC > 0$, the smallest root being given by $(B - \sqrt{\Delta_t})/(2A_t)$. In the tax-standard policy case, formulae are identical but with z_t replace by 0, implying that ρ^{std} is a constant.

Solving the recursion of (33), it comes

$$\tilde{\varepsilon}_t = \chi_t \tilde{\omega}_{t+1} + \rho_t \tilde{\varepsilon}_{t-1} = \sum_{k=0}^{t-t_0} \chi_{t-k} \tilde{\omega}_{t+1-k} \prod_{i=0}^{k-1} \rho_{t-i} + \varepsilon_{t_0} \prod_{k=0}^{t_0} \rho_{t-k},$$

hence $\mathbb{E}[\tilde{\varepsilon}_t | \varepsilon_{t_0}] = \varepsilon_{t_0} \prod_{k=0}^{t_0} \rho_{t-k}$ and

$$\mathbb{V}[\tilde{\varepsilon}_t | \varepsilon_{t_0}] = \mathbb{E}[(\tilde{\varepsilon}_t - \mathbb{E}[\tilde{\varepsilon}_t | \varepsilon_{t_0}])^2 | \varepsilon_{t_0}] = \mathbb{E} \left[\left(\sum_{k=0}^{t-t_0} \chi_{t-k} \prod_{i=0}^{k-1} \rho_{t-i} \right)^2 \right] \mathbb{E}[\tilde{\omega}^2]$$

using independence. In the tax-standard policy case, as $\rho_t = \rho^{std}$ and $\chi_t = \chi^{std}$, it comes $\mathbb{E}[\tilde{\varepsilon}_t | \varepsilon_{t_0}] = \varepsilon_{t_0} (\rho^{std})^{t-t_0}$ and $\mathbb{V}[\tilde{\varepsilon}_t | \varepsilon_{t_0}] = (\chi^{std} \sigma_\omega)^2 (1 - (\rho^{std})^{2(t-t_0)+1}) / (1 - (\rho^{std})^2)$.