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Optimal Green Technology Adoption and Policy Implementation

Jean-Marc Bourgeon*

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Abstract

The importance of network externalities affecting technology diffusion in the greening of the economy is analyzed using a simple dynamic model. The socially optimal path of the economy can be implemented by requiring firms to comply to technical standards. As otherwise firms make investment decisions based on their expectations of the magnitude of shocks affecting network effects, using only incentive-based instruments of regulation (emissions taxes and subsidies for green investments) to green the economy leads to efficiency losses due to economic fluctuations.

Keywords: Growth, sustainability, Technology adoption.

JEL: O33, O44, E37, Q55

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1 Introduction

Decoupling resource use and environmental discharges from economic growth is one of the main challenges facing modern economies. Among these, decarbonisation, i.e. reducing the carbon content of fuels, is a top priority. To achieve this goal, firms must adopt cleaner and often newer technologies, which can be risky. At first, there is a conversion cost: Green technologies are often more costly and less productive than the technologies they use, and may require retraining of their workforces and radical changes in their supply chains. In addition, these investment choices may prove ineffective in the long term because they are too specific.¹ Network externalities and technology spillovers play an important role in determining the optimal technology for firms to adopt. The profitability of a production line depends on technology spillovers, like the known-how of the workforce, and the availability of the inputs and of the maintenance services required by the technology. Since a machine is a durable good, usually used for several years, it is important that the parts needed for its operation and maintenance services are readily available in the future, which is all the more likely as the technology becomes more widespread. There is also the issue of compatibility of the technology with existing and future industry standards, which determines what other machines can be employed. Moreover, the more a technology is used, the higher the research effort of the machine industry sector to improve the technology. Therefore, the optimal investment choice for a firm depends on the decisions made by all firms, and the more firms use a given technology, the more attractive it is. It is however difficult to anticipate these network effects, due to the many external factors that affect the relative performance of technologies. Consider the case of energy-efficient technologies, where profit depends on the price of energy, a highly fluctuating variable. Firms investment, and hence the resulting network effects, depends on their expectations about the magnitude of energy price changes.

Whatever the extend of these network effects, as markets prices do not account for the environmental footprint of the economy, public policies must be designed to guide firms in their investment choices. This can be done either by using incentive-based instruments, like taxes on polluting emissions and subsidies for the purchase of green technology, or by imposing that firms use machines that follow demanding

¹Consider, for example, that a firm decides to replace its fleet of internal combustion vehicles with electric vehicles (EVs). If many companies expect EVs to be used nationwide in the near future and decide to do the same, it is likely that the network of charging stations for EVs will spread widely and that the use of EVs can become very convenient and cheap. If the rest of the economy turns to hydrogen vehicles instead, the few companies that have chosen EVs could be penalized.

environmental standards, which reduces the firms' possible choices but reduces the coordination problem.

In this paper, I analyze the problem of greening the economy using a simple dynamic model that incorporates network externalities and technology spillovers. Firms technology is represented by two variables, one capturing the productive capacity of production lines (similar to the traditional economic definition of capital), the other their pollution intensity. An essential feature of this framework is that a firm equipped at a given date with the most profitable production process does not mean that it is the latest or most innovative one: Network externalities and technological spillovers determine optimal investment choices. However, in the absence of a coordination mechanism, firms must decide on their investment on the basis of their expectations on the extend of shocks affecting network effects. This imperfect assessment of the future economic environment is reflected in an industrial sector made up of firms with heterogeneous technological processes. Shocks affecting network effects are also the cause of economic fluctuations.

The optimal investment policy is derived by solving the problem of a social planner who can decide on the levels of investment in production capacity and pollution intensity of firms' technology, taking into account the intertemporal trade-offs between consumption, environmental quality and (total) investment. This policy corresponds to productive and green investment levels that evolve over time. The implementation of this policy by mean of incentive-based instruments, namely schedules of emissions taxes and of subsidies for green investments, is then considered. The effectiveness of these instruments depends on network effects and thus on the anticipations of firms. The firms' problem of adapting their production process over time is framed as a dynamic global game where firms face a coordination problem guided by private signals on the optimal technological choice for the next period. Firms' optimal investment strategy is derived assuming Markov perfect equilibria. The resulting path followed by the economy is affected by the uncertainty inherent in these network effects, which leads to efficiency losses compared to a policy imposing technological standards on firms. Interestingly, the subsidy scheme for green investments entails negative values passed a certain date, i.e. it becomes a tax scheme, because network effects lead firms to over-invest in green technology compared to the optimal path.

These general results are illustrated in a simple framework where the marginal rate of substitution between environmental and industrial goods is constant, consumer preferences are exponential (CARA) and the production function is Cobb-Douglas. This framework allows for explicit solutions of the optimal policy and its implementation

using incentive-based instruments. It shows that the optimal dynamic of the economy entails 3 sequences, the first one corresponding to a decrease in the stock of productive capital, and thus in GDP, the second one to a stagnant GDP but where investment in clean technologies allows the economy to increase the quality of the environment, and finally, a third sequence corresponding to an increasing GDP along an environmentally neutral path (supposed to be the best that can be obtained with green technologies). The economic fluctuations generated by shocks affecting network effects are derived by assuming rational expectations on the part of consumers. While these shocks are independent, they create economic fluctuations that are driven by the consumers desire to smooth consumption over time. The resulting distribution of the interest rate, and thus of the GDP of the economy, is shown to follow a first-order autoregressive process. Since this path dependence can be significant, the likely trajectories of the economy using emissions taxes and green subsidies can be very inefficient.

There is a abundant literature on growth and sustainability. The literature on endogenous green growth focuses on productivity improvements and frontier innovation. This is the case in the AK paradigm where capital-knowledge accumulates with learning-by-doing (Stokey, 1998), and Lucas-like extensions (Bovenberg & Smulders, 1995), within a framework of product variety (Gerlagh & Kuik, 2007) or within the Schumpeterian growth paradigm of destructive creation and directed technical changes (Acemoglu et al., 2012), where the most productive innovations are adopted by firms as soon as they are discovered. This article focuses on the adoption of existing technologies that have knock-on effects leading to the gradual replacement of old and polluting machines with greener ones. The approach is thus close to the literature on endogenous growth viewed as a process of adoption of existing ideas and mutual imitation between firms, as exposed by Eaton & Kortum (1999); Lucas Jr & Moll (2014); Lucas (2009); Perla & Tonetti (2014). These papers assume that each agent in the economy is endowed with a certain amount of knowledge (“ideas”) and this knowledge evolves through contact with the rest of the population. The approach here is similar for describing the adoption of technologies: although the R&D sector is not spelled out, there is a set of existing technologies whose potential is more or less exploited depending on the proportion of firms that use them. The distribution of technology used among firms changes over time as firms’ incentives to adopt new technologies evolve.² Comparisons of policy instruments to implement an environmental objective have been quite nu-

²There is also a microeconomic literature that investigates the problem of network externalities, initiated by Katz & Shapiro (1985). See Guimaraes & Pereira (2016) for a recent contribution, and Shy (2011) for a survey.

merous since Weitzman (1974). The main focus is on the information available to the regulator on the firms' pollution abatement costs. Few papers consider the importance of firms' expectations in achieving an environmental goal. In a recent paper, Aldy & Armitage (2020) compare an emissions tax with a cap-and-trade instrument in which firms are subject to forecasting errors in the price of pollution allowances on the secondary market, the effectiveness of the abatement technology being the same in both cases. In what follows, the firms' expectations about their economic environment are important because of network effects that determine the effectiveness of the technology.

The remainder of the paper is organized as follows: Section 2 describes the economy and the dynamics of the technology adoption. The social planner's program, the optimal policy and its implementation using incentive-based instruments is presented section 3. These paths are illustrated section 4, in a framework that allows the economy to follow a Gaussian random path. The last section concludes.

2 Technology dynamics

Consider a discrete time economy composed of a continuum of firms, of total mass equal to one, that collectively produce at date t an amount q_t of output, taken as the numeraire and corresponding to the GDP of the economy. The firms' production may come from various technological processes with different environmental impacts. More precisely, we suppose that for each particular production task, firms have the choice among a large set of machines characterized by their productivity and polluting emissions level. These sets of machines evolve overtime, but each period, comparing machines with similar productivity levels, the cheaper they are, the more they pollute. A firm's production line is characterized by machines acquired over time and renewed or replaced when dimmed necessary. Each period, firms' technological processes may thus be very different, and to capture this heterogeneity in a simple way, the process of firm i at date t is described by two parameters, k_{it} and x_{it} , dubbed 'productive capacity' and 'green technology index' (or 'technology mix') respectively. The productive capacity of the technology is similar to the standard economic notion of capital: it corresponds to the input that, once combined with labor, allows the firm to produce items or services. It yields a gross revenue $q_{it} = Q(k_{it}, \ell_{it})$ where ℓ_{it} is the firm's employment level, and Q a production function, homogeneous of degree one and satisfying the Inada conditions. The green technology index captures the firm's effort to diminish the environmental

impact of its production, given by $\iota_{it}q_{it}$ where

$$\iota_{it} = \varphi - \xi x_{it}/q_{it} \quad (1)$$

corresponds to the emission intensity of its production process. The emission intensity diminishes with x_{it} , with a maximum equal to φ if $x_{it} = 0$. Hence, the higher x_{it} , the lower the environmental impact of the firm's production. This impact depends on ξ , a positive parameter that measures the environmental efficiency of green investments.

Each period, part of the technology used by firms is depreciated, due to wear and tear, or because some machines are obsolete or so deteriorated that maintenance costs are too high. We suppose a constant depreciation rate $1 - g$ that affects both the productive capacity and the green technology index. To renew and adapt their production process, firms buy new machines that correspond to an investment in productive capacity and/or in environmental quality. Firm i 's investment in productive capacity, I_{it} , leads to a dynamic of k_{it} given by $k_{it+1} = gk_{it} + I_{it}$. Similarly, firm i 's spending in environmental quality, κ_{it} , modifies its technology mix according to $x_{it+1} = gx_{it} + \kappa_{it}$.³

Depending on their technology choices, firms benefit from (or suffer from the lack of) network effects related to their input suppliers, maintenance services (how easy it is to find specific inputs and parts to service the machines), the know-how of workers, and more generally, the peculiarities of the machines they use. The more widespread is the technology used, the less a firm encounters problems and the easier it is to achieve a production target. Our focus being on network effects related to the environmental characteristics of the machines, we consider that for firm i at date t they can be summarized to a monetary amount G_{it} given by

$$G_{it} = G(x_{it}; \mu_{t-1}, K_{t-1}, \omega_t) = (\mu_{t-1} + \lambda K_{t-1} + \omega_t) x_{it} - x_{it}^2/2 - \bar{G}_t, \quad (2)$$

where $\mu_{t-1} \equiv \int_0^1 x_{it-1} di$ is the adoption of green technology (AGT) index of the economy at the previous period, $K_{t-1} \equiv \int_0^1 \kappa_{jt-1} dj$ the sum of green investments made simultaneously by firms at $t - 1$ for period t , and ω_t a time-independent noise, normally distributed with variance σ_ω^2 , that summarizes the many unmodeled shocks affecting the diffusion of technology.⁴ G_{it} may be positive or negative, and the last term,

³These investments may be negative, i.e. firms may consider resealing some of their machines. Observe that we can have $I_{it} = 0$ and $\kappa_{it} > 0$, in which case firm i buys pollution abatement equipment, i.e. devices that do not produce any items but reduce the polluting emissions of the firm.

⁴These shocks are due to factors external to the economy, such as energy prices that depend on the worldwide demand and supply and can be affected by local conflicts, geopolitical rivalries, or pandemics. To simplify the analysis, these factors are assumed to affect only network effects, not the

\bar{G}_t , ensures that the average network gain is nil.⁵ We suppose that μ_{t-1} is known by firms at date $t - 1$, when they decide to invest for period t , but the realization of ω_t , and thus the sum of their choices K_{t-1} , is unknown. Parameter $\lambda \in [0, 1)$ in (2) captures the immediate technology spillovers. If $\lambda = 0$, the first term of (2) reduces to $\mu_{t-1} + \omega_t$, i.e. the former average mix plus a noise which may be positive or negative. The productivity of firm i depends on the AGT index of the previous period thanks to the diffusion of knowledge and the adaptation of the supply chain that occurred previously. When immediate demand spillovers are at work ($\lambda > 0$), this productivity factor also depends on the most recent investments made by all firms, K_{t-1} , the extreme case $\lambda \rightarrow 1$ corresponding to an immediate adaptation of the workers and the supply chain to new investment choices.⁶ With an average network effect null, i.e. $\int_0^1 G(\mu_{t-1}, K_{t-1}, x_{it}, \omega_t) di = 0$, simple algebra allows us to rewrite (2) as

$$G_{it} = [(x_t^* - \mu_t)^2 + \sigma_{x_t}^2 - (x_{it} - x_t^*)^2]/2, \quad (3)$$

where $\sigma_{x_t}^2 = \int_0^1 (x_{jt} - \mu_t)^2 dj$ is the variance of the technology indexes at date t , and

$$x_t^* \equiv \mu_{t-1} + \lambda K_{t-1} + \omega_t \quad (4)$$

the ideal technology mix at date t , $(x_{it} - x_t^*)^2/2$ corresponding to firm i 's relative loss due to a less effective mix x_{it} . Firms would ideally be equipped with the most efficient mix, but they cannot perfectly assess it because they don't know at date $t - 1$ the realization of the random shock ω_t and the total of green investments made simultaneously by all firms, K_{t-1} . Hence, firms must somehow anticipate the extent of the resulting total green investment when making their own, which leads to an intertemporal coordination problem that is formalized as a succession of global games taking place each period.⁷ This dynamic setup is solved sequentially: our focus is on Markov perfect equilibria where x_t^* is a state variable. In period t , firms must anticipate the realization of the next period ideal technology mix, \tilde{x}_{t+1}^* , a random variable which distribution depends on the other firms behavior. More specifically, firm i infers the other firms decisions

productivity of the technology, which depends of k_{it} .

⁵Since green investments are not productive per se, firms' monetary gains and losses from network effects must cancel out. For the sake of simplicity, it is supposed that they compensate for each other each period.

⁶It is likely that the value of λ is particularly low. As stressed by Battisti (2008), a consistent literature has shown that, even when a clean or a cost-reducing technology is ready available in the market, its spreading takes several years.

⁷These coordination problems with strategic complementarity are known as "beauty contests" (see, e.g., Angeletos & Pavan, 2004, 2007 and Morris & Shin, 2002).

according to its beliefs on the shock affecting the diffusion of technology. We suppose that these beliefs are unbiased idiosyncratic private signals of the shock: they are formed according to $\tilde{\eta}_{it} = \omega_{t+1} + \tilde{v}_{it}$, where ω_{t+1} is the date- $t + 1$ realization of this shock, \tilde{v}_{it} a time-independent noise, normally distributed with variance σ_v^2 , verifying $\mathbb{E}[\tilde{v}_{it}] = \mathbb{E}[\tilde{v}_{it}\tilde{v}_{jt}] = 0$ for all i, j , and $\int_0^1 \tilde{v}_{it} di = 0$.

Consider the Laissez-faire situation. Denote by $t = 0$ the present period and suppose that firm i is endowed with technology $(k_{i0}, x_{i0}) \geq 0$. Each period t , given w_t and r_t the date- t wage and interest rate, it plans its technological investments and employment level $\{(\kappa_{it}, I_{it}, \ell_{it}), t = 0, 1, 2, \dots\}$ to maximize the expected discounted sum of its profits $\mathbb{E}_t[\sum_{h \geq 0} (\prod_{\tau=0}^{t+h} \delta_\tau)(q_{it+h} + G_{it+h} - w_t \ell_{it+h} - I_{it+h} - \kappa_{it+h})]$ where $\delta_\tau = 1/(1 + r_\tau)$ is the date- τ discount factor.⁸ With a total supply of labor equal to one each period, i.e. $\int_0^1 \ell_{it} di = \ell_t = 1$ for all t , it is shown in the appendix that

Proposition 1 *Firms' equilibrium investment in productive capacity at time t is given by*

$$I_t = q'^{-1}(1 + r_t - g) - gk_t \quad (5)$$

for all i , where $q(k) \equiv Q(k, 1)$ is an increasing and concave function, leading to a total capacity $k_{t+1} = q'^{-1}(1 + r_t - g)$. The equilibrium wage is given by $w_t = q(k_t) - r_{t-1}k_t$. Firm i 's investment strategy in green technology satisfies

$$\kappa_{it} = \mathbb{E}[\tilde{x}_{t+1}^* | \eta_{it}, \mu_t, r_t] - gx_{it} - r_t - 1 + g, \quad (6)$$

leading to

$$\kappa_{it} = \frac{(1 - \lambda g)\mu_t + \eta_{it} - r_t - 1 + g}{1 - \lambda} - gx_{it}. \quad (7)$$

at equilibrium. The resulting firms' technology mixes at $t + 1$ are normally distributed with mean

$$\mu_{t+1} = \frac{(1 - \lambda g)\mu_t + \omega_{t+1} - r_t - 1 + g}{1 - \lambda}, \quad (8)$$

corresponding to the date- $t + 1$ AGT level of the economy, and constant standard deviation $\sigma_{x_t} = \sigma_x \equiv \sigma_v/(1 - \lambda)$ for all t .

Because firms have the same production function Q , the productive capacity of all firms is the same, determined by choices that are guided unambiguously by the interest rate. This is not the case for their spending in green technology (6) that depend on

⁸ $\mathbb{E}_t[\tilde{X}]$ is a shorthand for $\mathbb{E}[\tilde{X} | \mathcal{I}_t]$, i.e. the expectation of the random variable \tilde{X} given the information \mathcal{I}_t available at date t .

individual estimates of the most efficient mix $\mathbb{E} [\tilde{x}_{t+1}^* | \eta_{it}, \mu_t, r_t]$. At equilibrium, firm i adopts the green investment strategy (7) that is a linear function of publicly observable variables (the AGT index μ_t and the prevailing interest rate r_t), its own technology index x_{it} and its private signal η_{it} .⁹ For firms with a low mix and a high estimate, this strategy corresponds to buying machines with a low environmental impact. For those with a high mix and a low estimate, their investment is directed in the opposite direction: they save on new equipment spending by buying less expensive brown technologies. The resulting dynamic of the AGT index (8) is magnified by a factor equal to $(1 - \lambda)^{-1}$: the more reactive is the supply chain, the larger are the effects of the shock and of the cost of capital r_t on the next period AGT index.¹⁰

The expected dynamic of the AGT index under Laissez-faire is positive if $\mu_t > [1 + r_t/(1 - g)]/\lambda$, hence if the technology spillovers are large enough, the depreciation rate large and/or the interest rate low. Indeed, as firms have to renew their machines, they may invest in less polluting ones if they expect that the supply chain and the workforce know-how adapt rapidly. This is however not possible if the interest rate is large and/or the depreciation rate of capital (i.e. the need to renew machines) is low.

Due to the idiosyncratic shocks on beliefs, v_{it} , firms have different expectations on x_{t+1}^* , hence choose machines with similar productive capacities but different environmental impacts. These discrepancies lead to a Gaussian distribution of firms' green indexes around the AGT, given by $x_{it} = \mu_t + v_{it}/(1 - \lambda)$, resulting in an industrial sector that can be thought of as a 'cloud' of firms with a green technology level that is drawn each period from a normal distribution centered on the AGT index μ_t with standard deviation $\sigma_v/(1 - \lambda)$ that increases with λ .

2.1 Environmental dynamic

Production generates pollution that deteriorates the quality and the availability of environmental goods and services provided by Nature. These effects are summarized in the dynamic of the environmental quality (EQ) index e_t , which is given by

$$e_{t+1} = \theta e_t + \hat{e} - \iota_t q_t \tag{9}$$

⁹The proof follows Angeletos & Pavan (2004). Morris & Shin (2002) show that this linear, symmetric, rational-expectations strategy leads to the unique (per period) equilibrium.

¹⁰Observe that (8) is not biased by the lack of information. This is because signals are private and affected by independent idiosyncratic noises. Would firms also shared a public signal, their choices would be distorted in the same direction, and so the resulting dynamic of μ_t , to an extent that depends on the relative signal reliability: the better the public signal's precision, the larger the distortion.

where $\theta \in (0, 1)$ is the environmental inertia rate, \hat{e} the per-period maximum regeneration capacity of the environment, and

$$\iota_t = \int_0^1 q_{it} \iota_{it} di / q_t = \varphi - \xi \mu_t / q_t \quad (10)$$

the emission intensity of the economy at date t , which measures the total damage of firms to the environment per unit of GDP. Without human interference ($\iota_t = 0$), the EQ index is at its pristine level $e_N = \hat{e} / (1 - \theta)$. More generally, we obtain using (10) a dynamic of EQ that follows the linear first-order recursive equation

$$e_{t+1} = \theta e_t + \xi \mu_t - \varphi q_t + \hat{e}. \quad (11)$$

We suppose that green technologies can only reduce emissions, i.e. that they do not allow for direct improvement of EQ. As a result, environmental neutrality is the best society can achieve.

Definition 1 (Environmentally Neutral Path) *The economy has reached at date T an Environmentally Neutral Path (ENP) if for all $t \geq T$, $\iota_t = 0$.*

An ENP is a sustainable situation in which the emission intensity of the economy is nil. Along an ENP, thanks to the natural regeneration capacity of the environment, the average EQ increases and tends toward its pristine level e_N .

2.2 Consumers

Consumers maximize their intertemporal utility by arbitraging between consumption and savings each period. Consumers' well-being comes from consuming both manufactured and environmental goods and services. Environmental goods are freely available and their consumption is subsumed by the EQ index e_t . The consumption of manufactured goods is denoted by c_t . We suppose that consumers do not try to modify the environment through their consumption and saving plans.¹¹ Consumers' per-period preferences are represented by a concave utility function $u(c_t, e_t)$, and their behavior is modeled by considering a representative consumer whose saving and consumption

¹¹This could be because they consider that they are too numerous for their individual behavior to have a significant impact on it.

plans solve

$$\max \mathbb{E}_t \left\{ \sum_{h=0}^{+\infty} \beta^h u(\tilde{c}_{t+h}, \tilde{e}_{t+h}) : \tilde{c}_{t+h} = \tilde{R}_{t+h} + \tilde{r}_{t+h-1} \tilde{S}_{t+h-1} - \tilde{s}_{t+h}, \tilde{s}_{t+h} = \tilde{S}_{t+h} - \tilde{S}_{t+h-1} \right\} \quad (12)$$

each period, where R_t is her date- t revenue, S_{t-1} her savings from the previous period, $r_{t-1}S_{t-1}$ the corresponding date- t capital earnings, s_t the savings adjustment of period t , and β the psychological discount factor. Solving (12), we obtain

Lemma 1 *The consumption rule that solves (12) satisfies*

$$\frac{\partial u(c_t, e_t) / \partial c}{\beta \mathbb{E}_t [\partial u(\tilde{c}_{t+1}, \tilde{e}_{t+1}) / \partial c]} = 1 + r_t \quad (13)$$

at each date t .

Equation (13) corresponds to the Ramsey-Euler rule which states that the expected intertemporal consumption rate of substitution (IRS) is equal to the return of capital each period. It also defines the supply function of capital, while (5) and (8) are the demand side coming from firms. At market equilibrium, aggregate production net of investment must be equal to total consumption of manufactured goods, i.e.

$$c_t = q_t - I_t - K_t = q(k_t) - (k_{t+1} - gk_t) - (\mu_{t+1} - g\mu_t). \quad (14)$$

3 First-Best path of the economy

Consider a benevolent social planner in charge of determining the dynamics of the productive capacity and the AGT index of the economy to maximize the consumer's welfare. We suppose that total investment cannot be negative so that this maximization is constrained by¹²

$$k_{t+1} \geq gk_t, \quad (15)$$

$$\varphi q(k_t) \geq \xi \mu_t, \quad (16)$$

¹²Constraint (15) implies that the social planner cannot forbid the use of machines that are too environmentally damaging. Also, as the policy should result in green technology investments, the constraint $\mu_{t+1} \geq g\mu_t$ is neglected (It is shown that it never binds in the illustrative example).

where the last inequality corresponds to the ENP constraint. The planner's problem is thus to solve

$$\max_{\{\mu_t, k_t, e_t\}_{t>0}} \left\{ \sum_{t=0}^{+\infty} \beta^t u(c_t, e_t) : (11), (14) - (16) \right\} \quad (17)$$

given e_0, k_0 and μ_0 . We suppose $\xi\mu_0 < \varphi q_0$, so that society is not already on a ENP. Denoting

$$h_t = \frac{\partial u(c_t, e_t) / \partial c}{\beta \partial u(c_{t+1}, e_{t+1}) / \partial c} \quad (18)$$

the IRS, it is shown in the appendix that:

Proposition 2 *The optimal path of the economy $\{\mu_t^*, k_t^*, c_t^*, e_t^*\}_{t>0}$ satisfies (11), (14),*

$$q'(k_{t+1}^*) \leq \frac{h_t^* - g}{1 - (h_t^* - g)\varphi/\xi}, \quad (19)$$

for all $t > 0$, with an equality when (15) is not binding and $k_{t+1}^* = gk_t^*$ otherwise, and

$$h_{t+1}^*(h_t^* - g - \theta) + \theta g = \xi \frac{\partial u(c_{t+2}^*, e_{t+2}^*) / \partial e}{\partial u(c_{t+2}^*, e_{t+2}^*) / \partial c}. \quad (20)$$

when constraint (16) is not binding, where h_t^* is the date- t IRS evaluated along the optimal path.

The optimal policy is thus characterized by a sequence of IRS $\{h_t^*\}_{t>0}$ that solves (20). To interpret this condition, first observe that investment in green technology at time t has an impact on the environment after two periods, i.e. at $t + 2$. The IRS over these two periods is given by $h_{t+1}h_t$. Absent stock effects, i.e. $g = \theta = 0$, this condition states that the date- t two-period IRS must be equal to the marginal rate of substitution (MRS) of consumption for EQ at date $t + 2$. Condition (19) indicates that when (15) is not binding, the IRS is larger than the rental rate of capital $g + q'(k_{t+1})$ due to the impact of production on the environment (i.e. unless $\varphi = 0$).

3.1 Policy implementation with incentive instruments

The optimal policy consists in a sequence of technological standards, i.e. limits on the stock of productive capital and on the AGT index of the economy that change overtime. Implementing such a policy at the firms level implies imposing these standards to the production lines of each firm, hence limiting the set of machines they can use. Suppose instead that the government chooses to implement the policy using incentive

instruments, namely an emissions tax scheme $\{\tau_t\}_{t>0}$ and a green technology subsidy scheme $\{z_t\}_{t>0}$. Given this policy, firm i 's per period profit becomes

$$\pi_{it} \equiv Q(k_{it}, \ell_{it}) + G_{it} - w_t \ell_{it} - I_{it} - \kappa_{it}(1 - z_t) - \tau_t (\varphi Q(k_{it}, \ell_{it}) - \xi x_{it})$$

where the last term corresponds to the environmental tax payment, and the term $\kappa_{it}(1 - z_t)$ to the net payment for clean technology investment. It is shown in the appendix that at the optimum of the firms' program, the total stock of productive capital satisfies

$$q'(k_{t+1}) = \frac{1 + r_t - g}{1 - \tau_{t+1}\varphi}, \quad (21)$$

and that the dynamic of the AGT index is given by

$$\mu_{t+1} = \frac{1}{1 - \lambda} [(1 - \lambda g)\mu_t + \omega_{t+1} + \xi\tau_{t+1} + g(1 - z_{t+1}) - (1 + r_t)(1 - z_t)]. \quad (22)$$

Compared to the first-best policy, the shock ω_{t+1} that affects the diffusion of technology makes the path of the AGT index stochastic. This uncertainty affects the environmental quality but also the financial markets through the total demand of capital, hence the equilibrium interest rate. As this uncertainty is detrimental for the consumer, the optimal path of the economy should be revised to account for the cost of this risk (leading to a second-best path). The aim of this section being to assess the impact of using incentive instruments to implement the environmental policy, suppose for the sake of the argument that the social planner wants the economy to follow its first-best path in expectation. To derive the rational expectation equilibrium (REE), suppose also that he can credibly commit to implement the corresponding tax and subsidy levels $\{\tau_t, z_t\}_{t>0}$ as derived at $t = 0$ (it is thus an open loop policy that allows for consistent expectations). It is shown in the appendix that

Proposition 3 *The taxation and subsidy schemes $\{\tau_t, z_t\}_{t>0}$ that implement the first-best path in expectation are given by*

$$\tau_{t+1} = (h_t^* - g)/\xi \quad (23)$$

and

$$z_t = [(1 - \lambda)\mu_{t+1}^* - (1 - \lambda g)\mu_t^* - (h_t^* - \mathbb{E}[h_t]) + gz_{t+1}]/\mathbb{E}[h_t] \quad (24)$$

for all $t > 0$, with $\mathbb{E}[h_t] = g + q'(k_{t+1})(1 - \tau_{t+1}\varphi)$ when $t < t_0$, and $\mathbb{E}[h_t] = h_t^*$ otherwise. We have $\lim_{t \rightarrow \infty} z_t < 0$.

The emissions tax (23) is straightforwardly derived from (19) and (21) in the case the constraint $k_{t+1} \geq gk_t$ is not binding. When $k_{t+1} = gk_t$, which corresponds to an excess of productive capital, firms use all their capital stock nevertheless, leading to a low interest rate at equilibrium, hence $\mathbb{E}[h_t] < h_t^*$. Proposition 3 also shows that the subsidy schedule entails negative values passed a certain date, i.e. it becomes a tax scheme. This is because network effects lead firms to over-invest in green technology compared to the optimal path once the AGT index is high enough. As noted earlier, the expected dynamic of the AGT index under Laissez-faire is positive if $\mu_t > [1 + r_t/(1 - g)]/\lambda$. Hence, from an initial situation $\mu_0 < [1 + (h_0 - 1)/(1 - g)]/\lambda$, i.e. where firms would disinvest in green technology without an environmental policy, we have $\mu_t^* > [1 + (\bar{h} - 1)/(1 - g)]/\lambda$ at some $t > t_0$: even without a governmental intervention, firms' investment in green technology would be greater than $g\mu_t^*$, the level necessary to renew the green capital that is depreciated. This investment could still be lower than the optimal next period investment μ_{t+1}^* , i.e. we could still have $\mu_{t+1}^* > \mu_t^* + [(1 - g)(\lambda\mu_t^* - 1) - (\bar{h} - 1)]/(1 - \lambda)$, in which case a subsidy is still needed. However, firms over-invest when this inequality is reversed, and it is then optimal to tax green investment.¹³

In the following section, we derive the optimal policy and detail the consequences of incentive environmental instruments in a simple illustrative case.

4 Illustrative example

In this section, we first characterize the optimal path of the economy assuming that the MRS is constant. It is then completely derived assuming exponential (CARA) preferences and a Cobb-Douglas production function. Finally, the properties of the policy implementation are analyzed.

To characterize the optimal policy, assume that the consumption of manufactured and environmental good and services can be subsumed in a 'global wealth index' $y_t \equiv c_t + pe_t$ where p is the constant value of the environment, so that the consumer's MRS is the same whatever the GDP of the economy, equal to p .¹⁴ With a constant MRS,

¹³Because the first-best policy imposes standards in the long term, the public intervention is long lasting and results in taxing both polluting emissions and green investments. The government may consider alternatively a policy without public intervention in the long run, perhaps as soon as an ENP is reached. The resulting over-investment in green technology allows for greater productive investments than the optimal ones, and thus an higher GDP level in the long run, but because consumption of manufactured goods is reduced in the meantime, it is detrimental for the consumer welfare.

¹⁴Our results easily generalize to the case where p can take several discrete values depending on the consumption and EQ levels, e.g. $p \in \{p_i, i = 1, \dots, n\}$, $p_{i+1} < p_i$, with $p = p_i$ if $e_t/c_t \in [m_i, m_{i+1})$,

(20) simplifies to

$$h_{t+1}^* h_t^* - h_{t+1}^* (g + \theta) + \theta g = p\xi, \quad (25)$$

which must hold for all t as long as (16) is not binding. Without stock effects, i.e. $g = \theta = 0$, we would have $h_{t+1}^* h_t^* = p\xi$ for all t , and thus $h_t^* = \sqrt{p\xi}$ for all t (as h_t must be positive, and ruling out cyclical solutions that are suboptimal since u is concave). If $\beta > 1/\sqrt{p\xi}$, i.e. if consumers are not too impatient, this corresponds to an increasing path of the global wealth index y_t . More generally, a constant IRS \bar{h} that solves (25) is a root of $P(h) = 0$ where

$$P(h) \equiv p\xi - (h - g)(h - \theta). \quad (26)$$

Using (19), we obtain that the optimal sequence of productive capital $\{k_t\}_{t>0}$ satisfies $k_{t+1} = \max\{gk_t, \bar{k}\}$ where

$$\bar{k} \equiv q'^{-1} \left(\frac{\bar{h} - g}{1 - (\bar{h} - g)\varphi/\xi} \right). \quad (27)$$

The following lemma gives the principal properties of the optimal path of the economy when the MRS is constant.

Lemma 2 *When the ENP constraint (16) is not binding, the IRS is constant, given by $h_t^* = \bar{h} = (g + \theta + \sqrt{(g - \theta)^2 + 4p\xi})/2$. $\bar{h} > 1/\beta$ iff $p\xi > (1/\beta - \theta)(1/\beta - g)$, and $\bar{k} > 0$ iff $p < \xi/\varphi^2 + (g - \theta)/\varphi$. Moreover, we must have $\xi/\varphi > 1/\beta - (3g - \theta)/2$ to have both $\bar{h} > 1$ and $\bar{k} > 0$. If $gk_0 > \bar{k}$, $k_{t+1}^* = gk_t^*$ for all $t < t_0 \equiv \ln(\bar{k}/k_0)/\ln g$. $k_t^* = \bar{k}$ for all $t_0 \leq t \leq T + 1$ where T is given by $\mu_T^* = q(\bar{k})\varphi/\xi$.*

The optimal path of the economy is thus characterized by a constant IRS as long as environmental neutrality is not reached. This IRS corresponds to the largest root of (26). Compared to the case $\theta = g = 0$, stock effects relax the conditions on parameters p , ξ and β to have an increasing path of the global wealth index: rather than $p\xi > 1/\beta^2$, we must have $p\xi > (1/\beta - \theta)(1/\beta - g)$. The condition $p < \xi/\varphi^2 + (g - \theta)/\varphi$ is deduced from (15) that is indefinitely binding otherwise: if p is too large, the environmental quality would completely supplant consumption: the stock of capital would always decrease overtime, i.e. $k_{t+1} = gk_t$ for all t . To have an increasing path of the global wealth index and positive consumption (and production) of industrial goods, we also must have $\xi/\varphi > 1/\beta - (3g - \theta)/2$, i.e. green technologies that are sufficiently effective in reducing CO2 emissions.

with $m_1 = 0$ and $m_{n+1} = +\infty$.

Lemma 2 also states that the optimal dynamic of the economy may go through up to 3 sequences depending on the parameter values (and at least two if $\xi\mu_0 \ll \varphi q_0$). If $t_0 > 1$, the first sequence, $1 \leq t < t_0$, corresponds to a progressive decrease in the productive capital stock to \bar{k} since the depreciated capital is not replaced. This is the case when p , the social value of the environment, is large since \bar{h} increases with p and \bar{k} given by (27) is a decreasing function of \bar{h} . Hence, while the IRS is constant, the stock of capital may have to adapt over several period before reaching \bar{k} and the GDP progressively decreases (while it takes only one period if k_0 is not too large, i.e. $\bar{k} < k_0 < \bar{k}/g$). The second sequence, $t_0 \leq t < T$ corresponds to a stagnant GDP, equal to $q(\bar{k})$, where the investment in productive capital allows firms to maintain the stock to \bar{k} , while the investment in clean technology allows the economy to increase its AGT and EQ indexes. Finally, the third sequence, $t > T$, corresponds to an increasing GDP along an environmentally neutral path: the increase in GDP is proportional to the increase in clean technology (the factor of proportionality being equal to ξ/φ).¹⁵

In the following proposition, the optimal dynamic of the economy is derived assuming a Cobb-Douglas production function $q_t = Ak_t^\alpha$ and exponential (CARA) consumer's preferences, i.e. $u(c_t, e_t) = -e^{-\gamma(c_t + pe_t)}$.

Proposition 4 *Assume $\xi/\varphi \geq 1 - g$. With constant MRS, exponential preferences and a Cobb-Douglas production function, the optimal sequence of the EQ index $\{e_t^*\}_{t>1}$ is given by*

$$e_t^* = e_1 + \left(\frac{g^\alpha - g^{\alpha t}}{1 - g^\alpha} - \left(\frac{g^\alpha}{\bar{h}} \right)^{t_0-1} \frac{\bar{h}^t - \bar{h}}{\bar{h} - 1} \right) \nu_0 + (t-1)\nu_1 - \frac{(1-\theta)(1-g)\nu_1}{\xi p \bar{h}^T} \frac{\bar{h}^t - \bar{h}}{\bar{h} - 1} \quad (28)$$

for all $1 < t \leq T$, and $e_t^* = e_N - \theta^{t-T}(e_N - e_T^*)$ for all $t > T$, where $e_1 = \theta e_0 + \xi\mu_0 - \varphi q_0 + \hat{e}$, $\nu_1 = \ln(\beta\bar{h})^{\xi/\gamma}/P(1) > 0$, $\nu_0 = [\xi + \varphi(g - g^\alpha)](1 - g^\alpha)q_0/P(g^\alpha) > 0$ for all $1 < t < t_0$, $\nu_0 = 0$ for all $t_0 \leq t \leq T$, and where T is deduced from $e_T^* = e_N - \ln(\beta\bar{h})^{1/(1-\theta)p\gamma}$. The value of the AGT index over $\{1, \dots, T\}$ is deduced from EQ using (11) with $q_t = g^{\alpha t}q_0$ for all $t < t_0$ and $q_t = q(\bar{k})$ for all $t_0 \leq t \leq T + 2$. EQ increases and consumption decreases at decreasing rates for all $1 < t \leq T$. The total welfare reached over $\{0, \dots, T\}$ is given by $W(T) = u_0(\bar{h} - 1/\bar{h}^{T-1})/(\bar{h} - 1)$.

Because the IRS is constant as long as environmental neutrality is not reached, the

¹⁵From this result, we can infer that if p takes several discrete values $p \in \{p_i, i = 1, \dots, n\}$, $p_{i+1} < p_i$, as mentioned footnote 14, the optimal solution entails decreasing optimal IRS \bar{h}_i , and thus increasing levels of productive capital \bar{k}_i and GDP levels when the economy as not yet reached an ENP. In term of industrial production, such a policy is thus very restrictive at first, and then progressively more permissive as the AGT and EQ indexes are improving.

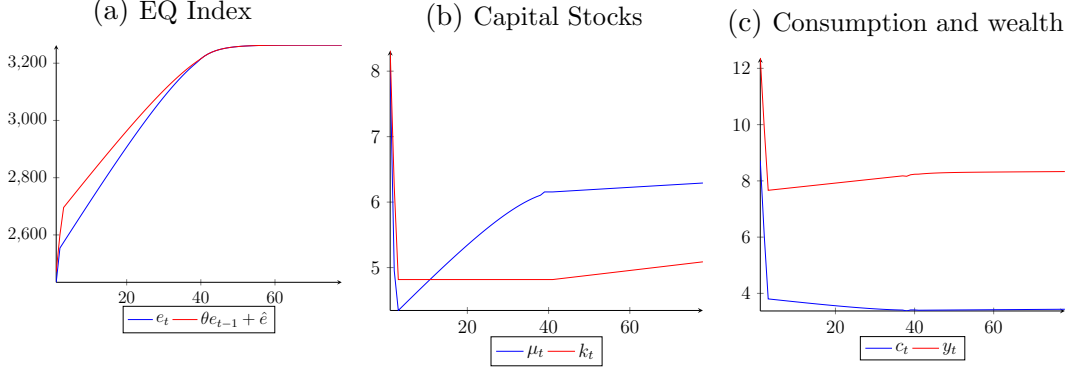


Figure 1: Optimal dynamic ($\xi = \varphi, \theta = 0.8, g = 3/4, \alpha = 2/5, p\xi = .0975, \beta = .985, e_0 = 2273.5, \mu_0 = 8.02, k_0 = 11.9$).

increase in global wealth is constant: $y_{t+1} - y_t = c_{t+1} - c_t + p(e_{t+1} - e_t) = \ln(\beta\bar{h})^{1/\gamma}$. However, because EQ is rapidly increasing, consumption decreases. This is due to the investment in the green technology since investment in productive capital is either null or constant during this period. Hence, during the first two sequences, when the GDP decreases ($1 \leq t < t_0$) and when it is stabilized at level $q(\bar{k})$ ($t_0 \leq t \leq T$), consumption of industrial goods decreases.

These results are illustrated Fig. 1 where $\xi = \varphi, \theta = 4/5, g = 3/4$ and $p\xi = .0975$, leading to $\bar{h} = 1.089$. The initial values of the state variables are $e_1 = 2434.38, k_0 = 11.9$ and $\mu_0 = 8.02$. The optimal EQ levels (28) are computed using $\alpha = 2/5$ and $v_0 = 616.11$ and $v_1 = 20.66$.¹⁶

Fig. 1a shows the dynamic of EQ and the corresponding ENP levels. EQ increases steeply the first two periods (when the stock of capital and thus the GDP decreases), and still rapidly until $t = T$, when the ENP constraint is reached. The corresponding level $e_T = 3203.46$ is lower than $e_N = 3263$, and EQ increases but at a very low rate after T . Fig. 1b shows the sharp decrease in the productive capital but also in AGT over the first 2 periods (from $k_0 = 11.9$ to $\bar{k} = 4.82$ and $\mu_0 = 8.02$ to $\mu_T = 4.34$ respectively). Then, the productive capital stays at \bar{k} until T , while AGT sharply increases. Both capital stocks increase afterward, at the slow rate permitted by the ENP constraint (the long term level of capital, which determines $\mu_\infty = q_\infty = 7.77$, is $k_\infty = 8.61$.) Fig. 1c shows the decrease in consumption, very sharp at first (from $c_1 = 8.72$ to $c_3 = 3.8$), then slower until $t = 41$. It increases afterward, at a very slow pace. Total wealth decreases the first two periods, but increases afterward. This simulations also show the limit of the assumption of a constant MRS: the decreases

¹⁶See Annex H for a detail of the model calibration.

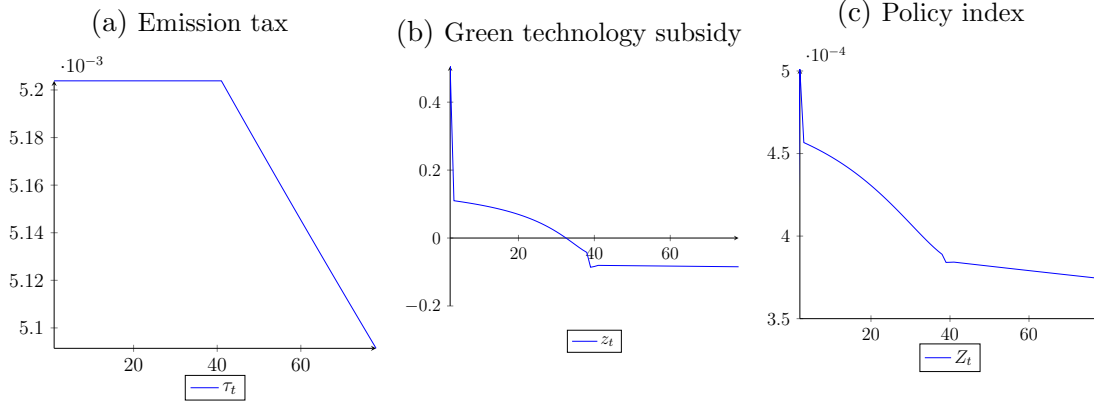


Figure 2: Tax and subsidy schemes ($\lambda = 0.02$).

in GDP and consumption are very sharp due to the perfect substitutability between industrial and environmental goods.

The policy schemes (23) and (24) are illustrated Fig. 4 assuming a spillover coefficient $\lambda = .02$. The emissions tax is constant at .0052 over the period $1 < t < T$ and then decreases progressively. The subsidy scheme is around 10% at $t = 3$, and decrease to becomes negative at $t = 33$.

4.1 Policy implementation

Given the tax and subsidy schemes (23) and (24), it is possible to characterize the dynamic of the economy for $t \in \{t_0, \dots, T\}$ assuming rational expectations. Indeed, the interest rate is then given by $\tilde{r}_t = \bar{r}(1 + \tilde{\varepsilon}_t)$ where $\bar{r} = \bar{h} - 1$ is the expected interest rate, and $\tilde{\varepsilon}_t$ is a zero-mean random shock whose distribution depends on the present and past realizations of ω_t . The productive capacity and total production \tilde{k}_t and \tilde{q}_t are also randomly distributed around their stationary values \bar{k} and $q(\bar{k})$. Using linear approximations, the path of global wealth index y_t can be approximated by a Gaussian random walk at the REE, and the supply function of capital (29) by¹⁷

$$r_t = \psi + \gamma(\mathbb{E}_t[\tilde{y}_{t+1}] - y_t) - \gamma^2 \mathbb{V}_t[\tilde{y}_{t+1}]/2 \quad (29)$$

where $\psi = -\ln \beta$ is the intrinsic discount factor. Expression (29) exhibits the familiar effects that determine the rental price of capital: the intrinsic preference for an immediate consumption ψ , the economic trend of the global wealth index that also encourages

¹⁷This expression is derived using $\mathbb{E}[e^{-\gamma \tilde{y}}] = e^{-\gamma(\mathbb{E}[\tilde{y}] - \gamma \mathbb{V}[\tilde{y}]/2)}$ when \tilde{y} is normally distributed, and the approximation $1 + r_t \approx e^{r_t}$.

immediate consumption if it is positive, and a precautionary effect that operates in the opposite direction and corresponds to a risk premium due to the uncertainty affecting the economy.¹⁸

The following proposition details the resulting paths of the AGT and EQ indexes.

Proposition 5 *Given $\{\tau_t, z_t\}_{t>0}$, under a REE with constant MRS, exponential preferences and a Cobb-Douglas production function, the interest rate for $t \in \{t_0, \dots, T\}$ is approximated by $\tilde{r}_t = \bar{r}(1 + \tilde{\varepsilon}_t)$ where*

$$\tilde{\varepsilon}_t = \chi \tilde{\omega}_{t+1} + \rho \varepsilon_{t-1}, \quad (30)$$

with $\rho > 0$. $\tilde{\varepsilon}_t$ is normally distributed, with $\mathbb{E}[\tilde{\varepsilon}_t | \varepsilon_{t_0}] = \rho^{t-t_0} \varepsilon_{t_0}$ and $\mathbb{V}[\tilde{\varepsilon}_t | \varepsilon_{t_0}] = \chi^2 \sigma_\omega^2 (1 - \rho^{2(t-t_0)+1}) / (1 - \rho^2)$. Assuming $\lambda \leq 1/[1 + (1-g)/(1-\theta)]$ and $p \leq g + (1-\lambda g)/\gamma$, the paths of e_t , μ_t and y_t can be approximated by Gaussian random walks deduced from (14), (11) and

$$\mu_{t+1} = a_1 \mu_t + a_2 e_t + a_3 + Z_t + b_1 \varepsilon_t + b_2 \varepsilon_{t-1} \quad (31)$$

where

$$Z_t = a_0 \sum_{i=0}^{+\infty} (a_0 \gamma)^i (\xi \tau_{t+1+i} + h_{t+i} z_{t+i} - g z_{t+1+i}). \quad (32)$$

Moreover, we have $\chi = 1/[(1-\lambda)b_1 + \bar{r}]$, with $0 < a_1 < g + (1-\lambda)/\gamma$, $0 < a_2 < p$ and $0 < a_0 < 1/\gamma$.

The equilibrium dynamic of the AGT index (31) follows a linear first-order recursive equation that includes a forward looking term Z_t given by (32). This policy index is an exponential smoothing of the future tax and subsidy levels that is illustrated Fig. 2c. While z_t is negative from $t = 33$, Z_t is positive for all t because τ_t is positive for all t . Eq. (30) shows that the distribution of the shocks affecting the interest rate, and thus the GDP of the economy, follows a first-order autoregressive process. Hence, the uncorrelated shocks affecting the network externalities generate path-dependent fluctuations. This business cycle is not due to changes in the environmental policy: we suppose that the regulator commits to a policy that is perfectly anticipated by the agents. Rather, it is due to the consumer's desire to smooth her consumption overtime. The effect on the path of the economy of the date- t network shock ω_t attenuates overtime if $\rho < 1$ (i.e. if this process is stationary). In that case, the variance $\mathbb{V}[\tilde{\varepsilon}_t | \varepsilon_{t_0}]$ converges to $\chi^2 \sigma_\omega^2 / (1 - \rho^2)$, and the future states of the economy are

¹⁸This simple expression is due to the exponential preferences assumption that correspond to CARA preferences under uncertainty, γ being the coefficient of absolute risk aversion.

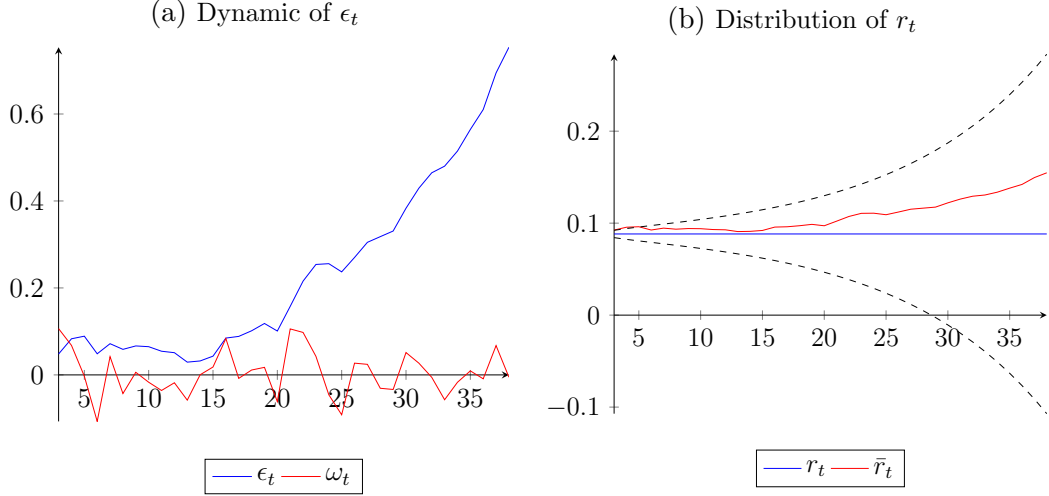


Figure 3: Shocks and the interest rate. Dashed lines correspond to the 95% CI. ($\lambda = .02, \sigma_\omega = .05, \chi = .452, \rho = 1.088$).

likely to be close to their first-best levels if σ_ω is small. The use of incentive-based instruments is then not too detrimental, particularly if ρ is small. However, as shown in the simulations, ρ can be very large, leading to probable paths of the economy that are very ineffective. This is illustrated Fig. 3a. The shocks ω_t are distributed almost evenly around 0 with a low variability ($\sigma_\omega = .05$), but the first two values, which are positive, determine the increasing trend of ϵ_t which diverges rapidly from $\epsilon_0 = 0$, due to a large autocorrelation ($\rho = 1.088$).

The confidence interval of the path of the economy when $t_0 \leq t \leq T$ can be approximated from (11) and (31) using the recursion

$$\tilde{Y}_t = B_t Y_{t-1} + H \tilde{v}_t \quad (33)$$

where $\tilde{Y}_t = (\tilde{\mu}_t, \tilde{e}_t, \tilde{q}_t, 1, \tilde{\epsilon}_t, \tilde{\epsilon}_{t-1})'$ is the column vector of state values (with the constant) and of the interest rate shock of date t and $t - 1$, B_t is the time-dependent transition matrix

$$B_t = \begin{bmatrix} a_1 & a_2 & 0 & a_3 + Z_t & b_1 & b_2 \\ \xi & \theta & -\varphi & \hat{e} & 0 & 0 \\ 0 & 0 & 0 & \bar{q} & q'(\bar{k})/q''(\bar{k}) & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, H = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \chi\sigma_\omega \\ 0 \end{bmatrix},$$

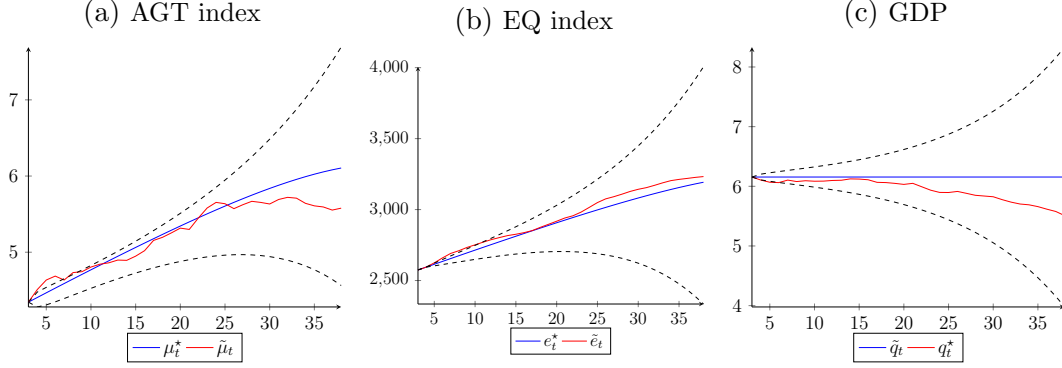


Figure 4: Policy implementation with taxes and subsidies. Dashed lines correspond to the 95% CI.

and $\tilde{\nu}_t$ is an independent standardized Gaussian variable. \tilde{Y}_t follows a Gaussian random walk with $\mathbb{E}[\tilde{Y}_t] = (\Pi_{i=0}^{t-t_0} B_i) X_{t_0}$ and $\mathbb{V}[\tilde{Y}_t] = \sum_{i=0}^{t-t_0} (\Pi_{j=0}^i B_j) H H' (\Pi_{j=0}^i B_j)'$. The transition matrix B_t is time-dependent because of the policy index Z_t defined by (32).

Fig. 3b and 4 depict the confidence intervals of the economic variables around their expected values. Due to the autocorrelation of ϵ_t that is larger than 1, these intervals increase rapidly.

5 Conclusion

This paper analyses the optimal path of the economy towards environmental neutrality. This path, which can be achieved through a binding policy of technical standards, is compared to the one obtained by using economic instruments of regulation, namely emissions taxes and green investment subsidies. These instruments are less effective than standards because of the network effects that affect diffusion of technologies. Because investment choices are ultimately made by private agents who react to shocks affecting their economic environment according to their anticipations, and because the efficiency of technologies is the result of their choices, the economic instruments of public policy can only steer the economy imperfectly. Although highly stylized, this model shows that shocks that affect network effects cause economic fluctuations that can push society away from the optimal path. There are, of course, many limitations to this approach. In particular, it is very doubtful that governments will be able to determine which technologies will prove most effective in the coming decades. At best, the definition of technological standards should be limited to setting limit values for the environmental quality indices that machines must meet, leaving it to firms to choose the technologies best suited to achieving their objectives. But this model illustrates

the problems faced by governments that rely on environmental taxes and subsidies to guide agents' choices (as for example in France, where these policies have also led to significant social unrests). Emission standard policies such as those implemented by the European Union, which force car manufacturers to move towards lower emissions, appear to be much more effective to guide society toward environmental neutrality than economic incentives given to car buyers.

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Appendix

A Proof of Proposition 1

As firm i 's profit is separable in revenues from productive capacity and network effects from its green index, the firm's investment problem can be split into two independent programs. Applying the principle of optimality, the investment strategy in productive capacity can be derived by using the Bellman equation

$$\mathcal{W}(k_{it}) = \max_{I_{it}, \ell_{it}} Q(k_{it}, \ell_{it}) - w\ell_{it} - I_{it} + \delta_t \mathcal{W}(gk_{it} + I_{it}). \quad (34)$$

Maximizing (34) with respect to ℓ_{it} gives

$$\partial Q(k_{it}, \ell_{it}) / \partial \ell = w_t,$$

while the first-order condition with respect to I_{it} leads to

$$-1 + \delta_t \mathcal{W}'(gk_{it} + I_{it}) = 0. \quad (35)$$

The envelop condition yields

$$\mathcal{W}'(k_{it}) = \partial Q(k_{it}, \ell_{it}) / \partial k + g\delta_t \mathcal{W}'(gk_{it} + I_{it})$$

implying $\mathcal{W}'(k_{it}) = \partial Q(k_{it}, \ell_{it}) / \partial k + g$. Plugging this expression in (35) evaluated for period $t + 1$ yields

$$\partial Q(gk_{it} + I_{it}, \ell_{it+1}) / \partial k = 1 - g + r_t.$$

As Q is homogeneous of degree 1, we thus get $q'(k_{it+1}/\ell_{it+1}) = 1 - g + r_t$ where $q(k) \equiv Q(k, 1)$ is an increasing and concave function. Inverting, it comes that $k_{it+1}/\ell_{it+1} = q'^{-1}(r_t + 1 - g)$, and using $\int_0^1 \ell_{it} di = \ell_t = 1$, $k_{it}/\ell_{it} = k_t/\ell_t = k_t$ for all i and t , with $k_t = q'^{-1}(r_{t-1} + 1 - g)$. Using $w_t = \partial Q(k_{it}, \ell_{it}) / \partial \ell = d[\ell_{it} q(k_{it}/\ell_{it})] / d\ell_{it} = q(k_{it}/\ell_{it}) - q'(k_{it}/\ell_{it})k_{it}/\ell_{it}$ yields $w_t = q_t - r_{t-1}k_t$.

Neglecting the constants in (3), the Bellman equation corresponding to the technology mix is given by

$$\mathcal{V}(x_{it}; x_t^*) = \max_{\kappa_{it}} -(x_{it} - x_t^*)^2 / 2 - \kappa_{it} + \delta_t \mathbb{E}[\mathcal{V}(gx_{it} + \kappa_{it}; \tilde{x}_{t+1}^*) | \eta_{it}, \mu_t, r_t]. \quad (36)$$

Maximizing (36) with respect to κ_{it} leads to

$$-1 + \delta_t \mathbb{E} [\partial \mathcal{V}(gx_{it} + \kappa_{it}; \tilde{x}_{t+1}^*) / \partial x | \eta_{it}, \mu_t, r_t] = 0. \quad (37)$$

The envelop condition yields

$$\partial \mathcal{V}(x_{it}; x_t^*) / \partial x = x_t^* - x_{it} + g \delta_t \mathbb{E} [\partial \mathcal{V}(gx_{it} + \kappa_{it}; \tilde{x}_{t+1}^*) / \partial x | \eta_{it}, \mu_t, r_t] \quad (38)$$

implying $\partial \mathcal{V}(x_{it}; x_t^*) / \partial x = x_t^* - x_{it} + g$. Plugging this expression in (37) evaluated in expectation for period $t + 1$ yields

$$1 + r_t = \mathbb{E} [\tilde{x}_{t+1}^* - (gx_{it} + \kappa_{it}) + g | \eta_{it}, \mu_t, r_t] = \mathbb{E} [\tilde{x}_{t+1}^* | \eta_{it}, \mu_t, r_t] - (gx_{it} + \kappa_{it}) + g,$$

which gives (6).

Following Angeletos & Pavan (2004), the resulting investment strategy is linear in the firm's observable, i.e. it is given by

$$\kappa(\mu_t, \eta_{jt}, x_{jt}, r_t) = \beta_1 \mu_t + \beta_2 \eta_{jt} + \beta_3 x_{jt} + \beta_4 r_t + \beta_5 \quad (39)$$

where the coefficients $\beta_k, k = 1, \dots, 5$, are derived as follows. On average, as $\int_0^1 \eta_{jt} dj = \omega_{t+1}$ and $\int_0^1 x_{jt} dj = \mu_t$, we have

$$\int_0^1 \kappa(\mu_t, \eta_t, \varepsilon_{jt}, x_{jt}, r_t) dj = (\beta_1 + \beta_3) \mu_t + \beta_2 \omega_{t+1} + \beta_4 r_t + \beta_5$$

and thus, from (4),

$$x_{t+1}^* = \mu_t + \lambda \int_0^1 \kappa_{it} di + \omega_{t+1} = [1 + \lambda(\beta_1 + \beta_3)] \mu_t + (1 + \lambda\beta_2) \omega_{t+1} + \lambda\beta_4 r_t + \lambda\beta_5.$$

Using (6) and $\mathbb{E}[\tilde{\omega}_{t+1} | \eta_{it}] = \eta_{it}$, we get

$$\begin{aligned} \kappa_{it} &= \mathbb{E}[\tilde{x}_{t+1}^* | \eta_{it}, \mu_t, r_t] - 1 + g - gx_{it} - r_t \\ &= [1 + \lambda(\beta_1 + \beta_3)] \mu_t + (1 + \lambda\beta_2) \eta_{it} - (1 - \lambda\beta_4) r_t + \lambda\beta_5 - 1 + g - gx_{it}. \end{aligned}$$

Identifying with (39) yields $\beta_3 = -g$, $\beta_1 = 1 + \lambda(\beta_1 + \beta_3) = (1 - \lambda g)/(1 - \lambda)$, $\beta_2 = 1 + \lambda\beta_2 = 1/(1 - \lambda)$, $\beta_4 = -1 + \lambda\beta_4 = -1/(1 - \lambda)$, $\beta_5 = -(1 - g)/(1 - \lambda)$, hence

(7). Consequently,

$$x_{it+1} = gx_{it} + \kappa_{it} = \frac{(1 - \lambda g)\mu_t + \eta_{it} + g - (1 + r_t)}{1 - \lambda},$$

and

$$\mu_{t+1} = \frac{(1 - \lambda g)\mu_t + \omega_{t+1} + g - (1 + r_t)}{1 - \lambda} = \mu_t + \frac{(1 - g)(\lambda\mu_t - 1) + \omega_{t+1} - r_t}{1 - \lambda}$$

using $\int_0^1 \eta_{it} di = \omega_{t+1}$. As idiosyncratic investments depend on signals that are normally distributed, x_{it+1} is normally distributed around μ_{t+1} with variance $\mathbb{V}[x_{it+1}] = \sigma_v^2/(1 - \lambda)^2 \equiv \sigma_x^2$.

B Proof of Lemma 1

At each date t , the Bellman equation corresponding to (12) can be written as

$$v(S_{t-1}; e_t) = \max_{s_t} u(r_{t-1}S_{t-1} + R_t - s_t, e_t) + \beta \mathbb{E}_t[v(S_{t-1} + s_t; \tilde{e}_{t+1})]$$

where S_t and s_t are the state and the control variables respectively. The first-order equation is given by

$$\partial u(c_t, e_t)/\partial c = \beta \mathbb{E}_t[\partial v(S_t; \tilde{e}_{t+1})/\partial S] \quad (40)$$

and the envelope theorem gives

$$\partial v(S_{t-1}; e_t)/\partial S = r_{t-1} \partial u(c_t, e_t)/\partial c + \beta \mathbb{E}_t[\partial v(S_t; \tilde{e}_{t+1})/\partial S].$$

Replacing the last term using (40), we get

$$\partial v(S_{t-1}; e_t)/\partial S = (1 + r_{t-1}) \partial u(c_t, e_t)/\partial c.$$

Taking the expectation and replacing in (40) yields (13) where $1 + r_t$ on the RHS is factorized out of the expected value since the date- t interest rate is a known parameter.

C Proof of Proposition 2

The planner's program is equivalently stated as

$$\max_{\{\mu_t, k_t, e_t\}_{t>0}} \left\{ \sum_{t=0}^{+\infty} \beta^t u(q(k_t) - k_{t+1} + gk_t - \mu_{t+1} + g\mu_t, e_t) : e_{t+1} = \theta e_t + \xi \mu_t - \varphi q_t + \hat{e}; (15), (16) \right\},$$

given e_0, k_0 and μ_0 . Neglecting the constraints (15) and (16), and denoting by λ_t the multiplier associated to the date- t equation of the dynamic of EQ, the Lagrangian of this program is given by

$$\mathcal{L} = \sum_{t=0}^{+\infty} \beta^t u(q(k_t) - k_{t+1} + gk_t - \mu_{t+1} + g\mu_t, e_t) - \lambda_t (e_{t+1} - \theta e_t - \xi \mu_t + \varphi q(k_t) - \hat{e}).$$

When (16) is not binding, the FOCs are

$$\frac{\partial \mathcal{L}}{\partial k_t} = \beta^t \frac{\partial u_t}{\partial c} (q'(k_t) + g) - \beta^{t-1} \frac{\partial u_{t-1}}{\partial c} - \lambda_t \varphi q'(k_t) \leq 0 \perp k_t \geq gk_{t-1}, \quad (41)$$

$$\frac{\partial \mathcal{L}}{\partial \mu_t} = \beta^t \frac{\partial u_t}{\partial c} g - \beta^{t-1} \frac{\partial u_{t-1}}{\partial c} + \lambda_t \xi = 0, \quad (42)$$

for all $t > 0$, and

$$\frac{\partial \mathcal{L}}{\partial e_t} = \beta^t \frac{\partial u_t}{\partial e} - \lambda_{t-1} + \theta \lambda_t = 0, \quad (43)$$

for all $t > 1$. Eq. (42) gives

$$\lambda_t = \left(\beta^{t-1} \frac{\partial u_{t-1}}{\partial c} - \beta^t \frac{\partial u_t}{\partial c} g \right) / \xi$$

and, using (18),

$$\lambda_t = \beta^t \frac{\partial u_t}{\partial c} (h_{t-1} - g) / \xi. \quad (44)$$

Substituting in (43) evaluated at $t + 2$, yields

$$\begin{aligned} 0 &= \beta^{t+2} \frac{\partial u_{t+2}}{\partial e} - \beta^{t+1} \frac{\partial u_{t+1}}{\partial c} (h_t - g) / \xi + \theta \beta^{t+2} \frac{\partial u_{t+2}}{\partial c} (h_{t+1} - g) / \xi \\ &= \beta^{t+2} \frac{\partial u_{t+2}}{\partial e} - \beta^{t+2} \frac{\partial u_{t+2}}{\partial c} [h_{t+1} (h_t - g) - \theta (h_{t+1} - g)] / \xi, \end{aligned}$$

using (18), hence

$$\xi \frac{\partial u_{t+2} / \partial e}{\partial u_{t+2} / \partial c} = h_{t+1} (h_t - g - \theta) + \theta g$$

for all $t > 1$ such that (16) is not binding. Using (44) to substitute for λ_{t+1} in (41) evaluated at $t + 1$ gives

$$\begin{aligned} 0 &\geq \beta^{t+1} \frac{\partial u_{t+1}}{\partial c} (q'(k_{t+1}) + g) - \beta^t \frac{\partial u_t}{\partial c} - \beta^{t+1} \frac{\partial u_{t+1}}{\partial c} (h_t - g) q'(k_{t+1}) \varphi / \xi \\ &= \beta^{t+1} \frac{\partial u_{t+1}}{\partial c} [q'(k_{t+1}) - (h_t - g)(1 + q'(k_{t+1}) \varphi / \xi)], \end{aligned}$$

using (18), hence

$$h_t - g \geq \frac{q'(k_{t+1})}{1 + q'(k_{t+1}) \varphi / \xi} \perp k_{t+1} \geq gk_t.$$

Assuming (16) is binding for all $t \geq T$, the planer's program becomes

$$\max_{\{k_t\}_{t>T}} \left\{ \sum_{t=T}^{+\infty} \beta^t u(q(k_t) - k_{t+1} + gk_t - [q(k_{t+1}) - gq(k_t)] \varphi / \xi, e_t) : e_{t+1} = \theta e_t + \hat{e} \right\},$$

given e_T, k_T . The FOCs are

$$\frac{\partial \mathcal{L}}{\partial k_t} = \beta^t \frac{\partial u_t}{\partial c} [q'(k_t)(1 + g\varphi/\xi) + g] - \beta^{t-1} \frac{\partial u_{t-1}}{\partial c} [1 + q'(k_t) \varphi / \xi] = 0$$

for all $t > T$, which gives

$$h_{t-1} = \frac{g + q'(k_t)(1 + g\varphi/\xi)}{1 + q'(k_t) \varphi / \xi},$$

or, equivalently,

$$h_t - g = \frac{q'(k_{t+1})}{1 + q'(k_{t+1}) \varphi / \xi}$$

for all $t \geq T$.

D Proof of Proposition 3

The problem of firm i is to solve

$$\mathcal{V}(k_{it}, x_{it}; x_t^*) = \max_{I_{it}, \ell_{it}, \kappa_{it}} \pi_{it} + \delta_t \mathbb{E}_t[\mathcal{V}(gk_{it} + I_{it}, gx_{it} + \kappa_{it}; \tilde{x}_{t+1}^*) | \eta_{it}]$$

where

$$\pi_{it} \equiv Q(k_{it}, \ell_{it}) - w_t \ell_{it} - I_{it} - (x_{it} - x_t^*)^2 / 2 - \kappa_{it}(1 - z_t) - \tau_t (\varphi Q(k_{it}, \ell_{it}) - \xi x_{it}).$$

Using the same steps as those described above, it comes (21) and

$$\kappa_{it} = \mathbb{E}[\tilde{x}_{t+1}^* | \eta_{it}, \mu_t, r_t] - gx_{it} + \xi\tau_{t+1} + g(1 - z_{t+1}) - (1 + r_t)(1 - z_t).$$

At equilibrium, we get

$$\kappa_{it} = \frac{1}{1 - \lambda} [(1 - \lambda g)\mu_t + \eta_{it} + \xi\tau_{t+1} + g(1 - z_{t+1}) - (1 + r_t)(1 - z_t)] - gx_{it},$$

hence

$$x_{it+1} = gx_{it} + \kappa_{it} = \frac{1}{1 - \lambda} [(1 - \lambda g)\mu_t + \eta_{it} + \xi\tau_{t+1} + g(1 - z_{t+1}) - (1 + r_t)(1 - z_t)].$$

Summing over all firms and using $\int_0^1 \eta_{it} di = \omega_{t+1}$ gives (22).

From (21), $1 + r_t - g = q'(k_{t+1})(1 - \tau_{t+1}\varphi)$, while (19) gives $h_t^* - g \geq q'(k_{t+1})[1 - (h_t^* - g)\varphi/\xi]$, with an equality when (15) is not binding. From (29), the realized IRS satisfies $h_t = 1 + r_t$ and we have $h_t = h_t^*$ when $k_{t+1} > gk_t$. Identifying gives (23). When $k_{t+1} = gk_t$, setting $\tau_{t+1} = (h_t^* - g)/\xi$ implies $h_t^* - g > g + q'(k_{t+1})(1 - \tau_{t+1}\varphi) = 1 + r_t - g$, hence $1 + r_t < h_t^*$: due to an excess of capital, the rental price of capital at equilibrium is lower than the first-best IRS. Substituting (23) for τ_{t+1} in the expectation of (22) gives

$$\mathbb{E}[\mu_{t+1}] = \frac{1}{1 - \lambda} [(1 - \lambda g)\mu_t + h_t^* - \mathbb{E}[h_t] - gz_{t+1} + \mathbb{E}[h_t]z_t].$$

Assuming $\mu_t = \mu_t^*$, we have $\mathbb{E}[\mu_{t+1}] = \mu_{t+1}^*$ if

$$z_t = [(1 - \lambda)\mu_{t+1}^* - (1 - \lambda g)\mu_t^* - (h_t^* - \mathbb{E}[h_t]) + gz_{t+1}]/\mathbb{E}[h_t].$$

At the stationary state, denoting with subscript ∞ the values of the variables, we have $e_\infty = e_N$, $c_\infty = q(k_\infty) - (1 - g)(k_\infty + \mu_\infty)$, $h_\infty = 1/\beta$, $\tau_\infty = (1/\beta - g)\varphi/\xi$ and $k_\infty = q'^{-1}((1/\beta - g)/(1 - (1/\beta - g)\varphi/\xi))$. (16) and (22) imply $\mu_\infty = (\varphi/\xi)q(k_\infty)$ and $(1 - \lambda)\mu_\infty = (1 - \lambda g)\mu_\infty + \xi\tau_\infty - (1 - z_\infty)(h_\infty - g)$. Replacing and reorganizing terms yields $z_\infty = -\lambda(1 - g)(\xi/\varphi)q(k_\infty)/(1/\beta - g) < 0$.

E Proof of Lemma 2

The discriminant of $P(h) = 0$ is $\Delta \equiv (g + \theta)^2 - 4(\theta g - \xi p) = (g - \theta)^2 + 4p\xi > 0$, and the equation admits two roots, $\bar{h} = (g + \theta + A)/2 > \max\{g, \theta\}$ and $\underline{h} = (g + \theta - A)/2 < \min\{g, \theta\}$, where $A = \sqrt{\Delta}$. We have $\underline{h} > 0$ iff $\xi p < \theta g$. The optimal policy corresponds

to a sequence of IRS with a subsequence defined by (25) as long as the ENP constraint (16) is not binding, i.e. $t < T - 1$ where T is the first period (16) binds. The sequence $\{h_t^*\}_{t=0}^{T-1}$ is either degenerate, i.e. $h_t^* \in \{\underline{h}, \bar{h}\}$ for all $t \in \{0, \dots, T - 1\}$, or $h_0^* \notin \{\underline{h}, \bar{h}\}$, and, reorganizing (25),

$$h_t^* = (\xi p - \theta g)/(h_{t-1}^* - g - \theta) \quad (45)$$

for all $t = 1, \dots, T - 1$. If T is large, this subsequence eventually converges to a root of $P(h) = 0$ that we denote by h_∞ . We can derived the non degenerate subsequence as follows. Defining $v_t = (h_t^* - h_\#)^{-1}$, $h_\# \in \{\underline{h}, \bar{h}\}$, we have $h_t^* = 1/v_t + h_\#$ and (45) becomes

$$\frac{1}{v_{t+1}} + h_\# = \frac{p\xi - \theta g}{1/v_t + h_\# - g - \theta},$$

which gives

$$\frac{1}{v_{t+1}} = \frac{-h_\#}{1 + v_t(h_\# - g - \theta)},$$

using (26). We thus have

$$v_{t+1} = v_t(g + \theta - h_\#)/h_\# - 1/h_\# \equiv v_t b_1 - b_0$$

with $b_1 = \underline{h}/\bar{h}$ if $h_\# = \bar{h}$ and $b_1 = \bar{h}/\underline{h}$ otherwise. With an initial value v_0 at $t = t_0$, the solution of this recurrence equation is given by

$$\begin{aligned} v_t &= v_0 b_1^t - b_0(1 - b_1^t)/(1 - b_1) \\ &= [v_0 + b_0/(1 - b_1)]b_1^t - b_0/(1 - b_1). \end{aligned} \quad (46)$$

where

$$\frac{b_0}{1 - b_1} = \frac{1/h_\#}{1 + (h_\# - g - \theta)/h_\#} = \frac{1}{2h_\# - g - \theta}.$$

If $|b_1| < 1$, i.e. if $h_\# = \bar{h}$, v_t converges toward $v_\infty = -b_0/(1 - b_1) = 1/(g + \theta - 2\bar{h}) = (h_\infty - \bar{h})^{-1}$, hence h_t^* converges toward $h_\infty = g + \theta - \bar{h} = \underline{h}$. If $|b_1| > 1$, i.e. if $h_\# = \underline{h}$, v_t diverges and thus $h_\infty = \underline{h}$. Hence, the recursion does not converge to \bar{h} (unless in the degenerate case $h_0 = \bar{h}$), while it converges to \underline{h} from any initial value $h_0 \neq \bar{h}$. This cannot be optimal if $\xi p \geq \theta g$ since $\underline{h} \leq 0$. If $\xi p < \theta g$, as $\underline{h} < \min\{g, \theta\} \leq g$, there is no stock of productive capital that can satisfy (19) when $h_t^* \rightarrow \underline{h}$. We thus cannot have $h_0^* < \underline{h}$. If $h_t^* > \underline{h}$, re-expressing (46) using $h_\# = \underline{h}$ and $b_1 = \bar{h}/\underline{h} > 1$, the sequence

$\{h_t^*\}_{t=0}^{T-1}$ is given by

$$h_t^* = \frac{A(h_0^* - \underline{h})}{[A - (h_0^* - \underline{h})](\bar{h}/\underline{h})^t + h_0^* - \underline{h}} + \underline{h}$$

where $h_0^* < A + \underline{h}$ to have $h_t^* > \underline{h}$ for all t . As this sequence is decreasing, tending to $\underline{h} < g$, the stock of productive capital increases and eventually diverges as h_t^* converges to \underline{h} . We would thus have a decreasing path of global wealth with an investment of productive capital tending to $q(k_t^*)$, hence c_t and e_t tending to 0, which is obviously not optimal. The optimal solution is thus $h_t^* = \bar{h}$ for all $t \in \{0, \dots, T-1\}$.

From the concavity of P , we have $\bar{h} > 1/\beta$ iff $P(1/\beta) > 0$, i.e. iff $p\xi > (1/\beta - \theta)(1/\beta - g)$. From (19), using $\lim_{k \rightarrow 0} q'(k) = +\infty$, \bar{h} corresponds to the solution of the (17) if it is lower than $g + \xi/\varphi$, otherwise (15) would be binding indefinitely. The condition $\bar{h} < g + \xi/\varphi$ simplifies to $p < \xi/\varphi^2 + (g - \theta)/\varphi$. The conditions $P(1/\beta) > 0$ and $p < \xi/\varphi^2 + (g - \theta)/\varphi$ imply that $(\xi/\varphi)^2 + (g - \theta)\xi/\varphi > p\xi > (1/\beta - \theta)(1/\beta - g)$. We thus must have $F(\xi/\varphi) > 0$, where $F(x) = x^2 + (g - \theta)x - (1/\beta - \theta)(1/\beta - g)$ is a second degree polynomial. The discriminant is given by

$$\begin{aligned} \Delta &= (g - \theta)^2 + 4(1/\beta - \theta)(1/\beta - g) = (g + \theta)^2 - 4g\theta + 4[1/\beta^2 - (g + \theta)/\beta + g\theta] \\ &= [2/\beta - (g + \theta)]^2, \end{aligned}$$

implying that $F(x) = (x - \underline{x})(x - \bar{x})$ where $\underline{x} < 0 < \bar{x}$ are the two real roots of $F(x) = 0$. $F(x)$ is positive if $x < \underline{x}$ or $x > \bar{x}$, and since $\xi/\varphi > 0$, we must have $\xi/\varphi > \bar{x} = 1/\beta - (3g - \theta)/2$.

If $gk_0 > \bar{k}$, (15) is binding as long as $g^t k_0 > \bar{k}$, i.e. $t \leq t_0$ given by $g^{t_0} k_0 = \bar{k}$, hence $t_0 = \ln(\bar{k}/k_0)/\ln g$. For $t > t_0$, $k_{t+1} = \bar{k}$ until the ENP constraint (16) is binding, i.e. until T given by $\mu_T^* = q(k_{T+1})\varphi/\xi$. Using (25) for $t = T - 1$ yields $h_T = \bar{h}$, hence $k_{T+1} = \bar{k}$. The path of the economy for $t > T$ is defined recursively by (11), (14), (16), (18) and (19) that holds for all $t > t_0$. Given the initial values e_T , $k_{T+1} = k_T = \bar{k}$, the IRS can be written as $h_t = h(k_t, k_{t+1}, k_{t+2})$ and (19) gives the implicit equation $q'(k_{t+1})[1 - (h(k_t, k_{t+1}, k_{t+2}) - g)\varphi/\xi] - h(k_t, k_{t+1}, k_{t+2}) + g = 0$ defining k_{t+2} for all $t \geq T$.

F Proof of Proposition 4

With exponential preferences $h_t = e^{\gamma[c_{t+1} - c_t + p(e_{t+1} - e_t)]} / \beta$. $h_t = \bar{h}$ gives

$$c_{t+1} - c_t + p(e_{t+1} - e_t) = \ln(\beta\bar{h})^{1/\gamma} \quad (47)$$

for all $t \leq T$. Using (11) to get $\mu_t = (e_{t+1} - \theta e_t + \varphi q_t - \hat{e}) / \xi$ and (14), (47) can be expressed as

$$\begin{aligned} \ln(\beta\bar{h})^{\xi/\gamma} &= \xi[q_{t+1} - q_t - k_{t+2} + (1+g)k_{t+1} - gk_t] - \varphi[q_{t+2} - (1+g)q_{t+1} + gq_t] \\ &\quad - (e_{t+3} - e_{t+2}) + (\theta + g)(e_{t+2} - e_{t+1}) + (\xi p - g\theta)(e_{t+1} - e_t) \end{aligned} \quad (48)$$

for all $t \leq T$. For $t < t_0$, we have $k_t = g^t k_0$, and using $q(k) = Ak^\alpha$, $q_t = g^{\alpha t} q_0$. For $t_0 \leq t \leq T+1$, we have $k_t = \bar{k}$ and $q_t = q(\bar{k}) \equiv \bar{q}$. (48) can thus be expressed as

$$m_{t+2} = (\theta + g)m_{t+1} + (\xi p - g\theta)m_t - \kappa g^{\alpha t} - \ln(\beta\bar{h})^{\xi/\gamma} \quad (49)$$

for all $t \leq T-1$, where $m_t \equiv e_{t+1} - e_t$ and

$$\kappa = \begin{cases} [\xi + \varphi(g - g^\alpha)](1 - g^\alpha)q_0 & t < t_0 \\ 0 & t_0 \leq t \leq T-1 \end{cases}.$$

The solution of (49) is $m_t = n_t + v_t$ where n_t and v_t are the solutions of the corresponding homogeneous and particular equations. The characteristic equation of the homogeneous equation is $P(x) = 0$. As \bar{h} is irrelevant, $n_t = \phi \bar{h}^t$ where ϕ is a constant. The particular solution of (49) is given by $v_t = g^{\alpha t} \nu_0 + \nu_1$ where ν_0 and ν_1 solve

$$g^{\alpha(t+2)} \nu_0 + \nu_1 = (\theta + g)(\nu_0 g^{\alpha(t+1)} + \nu_1) + (\xi p - g\theta)(\nu_0 g^{\alpha t} + \nu_1) - \kappa g^{\alpha t} - \ln(\beta\bar{h})^{\xi/\gamma}$$

for all $t \leq T-1$, which gives $\nu_0 = \kappa / P(g^\alpha)$ and $\nu_1 = \ln(\beta\bar{h})^{\xi/\gamma} / P(1)$. As P is concave with $P(g) = p\xi$ and $P(1/\beta) > 0$, we have $P(g^\alpha) > 0$ and $P(1) > 0$, hence $\nu_1 > 0$ since $P(1/\beta) > 0$ implies $\bar{h} > 1/\beta$, and $\nu_0 \geq 0$ if $\xi/\varphi \geq 1 - g > g^\alpha - g$. The solution of (49) is thus $m_t = n_t + v_t = \phi \bar{h}^t + g^{\alpha t} \nu_0 + \nu_1$, leading to

$$e_{t+1} = e_t + \phi \bar{h}^t + g^{\alpha t} \nu_0 + \nu_1,$$

where ϕ is specific to each sub-sequence and is derived from their initial conditions, i.e. from (49) evaluated at $t = t_0 - 1$, (48) evaluated at $t = T$, and from (11) at $t = 1$. For

$t \geq T$, as $\varphi q(k_t) = \xi \mu_t$, we have $e_{t+1} = \theta e_t + \hat{e}$ that implies $m_{t+1} = \theta m_t$ for all $t \geq T$. Using (48) at $t = T$, yields

$$\begin{aligned} \ln(\beta \bar{h})^{\xi/\gamma} &= \xi(\bar{k} - k_{T+2}) - \varphi(q(k_{T+2}) - \bar{q}) - m_{T+2} + (\theta + g)m_{T+1} + (\xi p - g\theta)m_T \\ &= \xi \bar{k} + \varphi \bar{q} - [\xi k_{T+2} + \varphi q(k_{T+2})] - \theta^2 m_T + \theta(\theta + g)m_T + (\xi p - g\theta)m_T \\ &= \xi \bar{k} + \varphi \bar{q} - [\xi k_{T+2} + \varphi q(k_{T+2})] + \xi p m_T, \end{aligned}$$

hence, using $\ln(\beta \bar{h})^{\xi/\gamma} = \nu_1 P(1)$,

$$\phi \bar{h}^T + \nu_1 = [\nu_1 P(1) + \kappa_1]/\xi p,$$

where $\kappa_1 \equiv \xi k_{T+2} + \varphi q(k_{T+2}) - \xi \bar{k} - \varphi \bar{q}$, which gives

$$\phi = \nu_1 (P(1)/\xi p - 1)/\bar{h}^T + \kappa_1/(\xi p \bar{h}^T) = -[(1 - \theta)(1 - g)\nu_1 - \kappa_1]/(\xi p \bar{h}^T)$$

for all $t_0 \leq t \leq T$. We thus have

$$m_t = \nu_1 - [(1 - \theta)(1 - g)\nu_1 - \kappa_1]/(\xi p \bar{h}^{T-t})$$

for all $t_0 \leq t \leq T$. Using $e_{T+1} = \theta e_T + \hat{e} = e_T + m_T$, it comes

$$\begin{aligned} \hat{e} - (1 - \theta)e_T &= \nu_1 - [(1 - \theta)(1 - g)\nu_1 - \kappa_1]/(\xi p) = [P(1)\nu_1 + \kappa_1]/(\xi p) \\ &= [\ln(\beta \bar{h})^{\xi/\gamma} + \kappa_1]/(\xi p) = \ln(\beta \bar{h})^{1/(p\gamma)} + \kappa_1/(\xi p). \end{aligned}$$

As $c_{T+1} - c_T + p(e_{T+1} - e_T) = -\kappa_1 + p[\hat{e} - (1 - \theta)e_T]$, we have

$$h_T = e^{\gamma[p(\hat{e} - (1 - \theta)e_T) - \kappa_1]}/\beta = \bar{h} e^{\gamma \kappa_1 (1/\xi - 1)} = g + q'(k_{T+2})/[1 + q'(k_{T+2})\varphi/\xi].$$

k_{T+2} thus solves

$$\bar{h} e^{\gamma(1 - \xi)[k_{T+2} + q(k_{T+2})\varphi/\xi - \bar{k} - \bar{q}\varphi/\xi]} = g + q'(k_{T+2})/[1 + q'(k_{T+2})\varphi/\xi].$$

As the LHT increases with k_{T+2} while the RHT decreases with k_{T+2} , the unique solution is $k_{T+2} = \bar{k}$ which implies $\kappa_1 = 0$, hence $m_t = \nu_1 - [(1 - \theta)(1 - g)\nu_1]/(\xi p \bar{h}^{T-t})$ for all $t_0 \leq t \leq T$. As $m_t \geq P(1)\nu_1/\xi p = \ln(\beta \bar{h})^{1/p\gamma} > 0$, $e_{t+1} > e_t$ for all $t_0 \leq t \leq T$.

Using (49) for $t = t_0 - 1$, it comes

$$\begin{aligned} \nu_1 - [(1 - \theta)(1 - g)\nu_1]/(\xi p \bar{h}^{(T-t_0-1)}) &= (\theta + g)\{\nu_1 - [(1 - \theta)(1 - g)\nu_1]/(\xi p \bar{h}^{(T-t_0)})\} \\ &\quad + (\xi p - \theta g)(\phi \bar{h}^{t_0-1} + g^{\alpha(t_0-1)}\nu_0 + \nu_1) - \nu_1 P(1). \end{aligned}$$

As $P(1) = \xi p - \theta g - 1 + \theta + g$, this equation simplifies to

$$\begin{aligned} [(1 - \theta)(1 - g)\nu_1]/(\xi p \bar{h}^{(T-t_0-1)}) &= (\theta + g)[(1 - \theta)(1 - g)\nu_1]/(\xi p \bar{h}^{(T-t_0)}) \\ &\quad - (\xi p - \theta g)(\phi \bar{h}^{t_0-1} + g^{\alpha(t_0-1)}\nu_0). \end{aligned}$$

Multiplying by $\xi p \bar{h}^{T-t_0+1}$ yields

$$[(1 - \theta)(1 - g)\nu_1]\bar{h}^2 = (\theta + g)[(1 - \theta)(1 - g)\nu_1]\bar{h} - (\xi p - \theta g)(\phi \bar{h}^{t_0-1} + g^{\alpha(t_0-1)}\nu_0)\xi p \bar{h}^{T-t_0+1},$$

and re-organizing,

$$[(1 - \theta)(1 - g)\nu_1][\bar{h} - (\theta + g)\bar{h}] = -(\xi p - \theta g)(\phi \bar{h}^{t_0-1} + g^{\alpha(t_0-1)}\nu_0)\xi p \bar{h}^{T-t_0+1},$$

where, as $P(\bar{h}) = 0$, $p\xi - \theta g = \bar{h}^2 - (\theta + g)\bar{h}$. Simplifying, we get

$$(1 - \theta)(1 - g)\nu_1 = -[\phi + (g^\alpha/\bar{h})^{t_0-1}\nu_0]\xi p \bar{h}^T,$$

which gives

$$\phi = -\left(\frac{g^\alpha}{\bar{h}}\right)^{t_0-1}\nu_0 - \frac{(1 - \theta)(1 - g)\nu_1}{\xi p \bar{h}^T},$$

hence

$$m_t = \nu_0 g^{\alpha t} [1 - (g^\alpha/\bar{h})^{t_0-1-t}] + \nu_1 - [(1 - \theta)(1 - g)\nu_1]/(\xi p \bar{h}^{T-t})$$

for all $0 < t < t_0$. As $m_t > 0$, $e_{t+1} > e_t$ for all $0 < t < t_0$. (28) is deduced from

$$\begin{aligned} \sum_{\tau=1}^{t-1} m_\tau &= e_t - e_1 = \sum_{\tau=1}^{t-1} (\phi \bar{h}^\tau + g^{\alpha\tau}\nu_0 + \nu_1) = \phi \frac{\bar{h}^t - \bar{h}}{\bar{h} - 1} + \nu_0 \frac{g^\alpha - g^{\alpha t}}{1 - g^\alpha} + (t - 1)\nu_1 \\ &= \nu_0 \left(\frac{g^\alpha - g^{\alpha t}}{1 - g^\alpha} - \left(\frac{g^\alpha}{\bar{h}}\right)^{t_0-1} \frac{\bar{h}^t - \bar{h}}{\bar{h} - 1} \right) + \nu_1(t - 1) - \frac{(1 - \theta)(1 - g)\nu_1}{\xi p \bar{h}^T} \frac{\bar{h}^t - \bar{h}}{\bar{h} - 1} \end{aligned}$$

for all $1 < t < t_0$, where $e_1 = \theta e_0 + \xi \mu_0 - \varphi q_0 + \hat{e}$, and from

$$\begin{aligned} \sum_{\tau=t_0-1}^{t-1} m_\tau &= e_t - e_{t_0-1} = \phi \sum_{\tau=t_0-1}^{t-1} \bar{h}^\tau + (t - t_0)\nu_1 = \phi \frac{\bar{h}^t - \bar{h}^{t_0-1}}{\bar{h} - 1} + (t - t_0 + 1)\nu_1 \\ &= \nu_1(t - t_0 + 1) - \frac{(1 - \theta)(1 - g)\nu_1}{\xi p \bar{h}^T} \frac{\bar{h}^t - \bar{h}^{t_0-1}}{\bar{h} - 1}, \end{aligned}$$

for all $t_0 \leq t \leq T$. T is solution of $(1 - \theta)e_T = \hat{e} - \ln(\beta \bar{h})^{1/p\gamma}$, hence $e_T = e_N - \ln(\beta \bar{h})^{1/(1-\theta)p\gamma}$. As

$$\begin{aligned} m_t - m_{t-1} &= \nu_0(g^{\alpha t} - g^{\alpha(t_0-1)}/\bar{h}^{t_0-1-t} - g^{\alpha(t-1)} + g^{\alpha(t_0-1)}/\bar{h}^{t_0-t}) - \frac{\nu_1(1 - \theta)(1 - g)}{\xi p \bar{h}^{T-t+1}}(\bar{h} - 1) \\ &= -\nu_0[(1/g^\alpha - 1)g^{\alpha t} + (\bar{h} - 1)g^{\alpha(t_0-1)}/\bar{h}^{t_0-t}] - \frac{\nu_1(1 - \theta)(1 - g)}{\xi p \bar{h}^{T-t+1}}(\bar{h} - 1), \end{aligned}$$

where $\nu_0 > 0$ if $t < t_0$ and $\nu_0 = 0$ if $t_0 \leq t \leq T$, $m_t - m_{t-1} < 0$ for all $t \leq T$: the sequence $\{e_t\}_{2 \leq t \leq T}$ increases at a decreasing rate. From (11), the AGT index is deduced from these expressions using $\mu_t = (e_{t+1} - \theta e_t + \varphi q_t - \hat{e})/\xi$. From (47),

$$c_{t+1} - c_t = \ln(\beta \bar{h})^{1/\gamma} - p m_t$$

for all $t \leq T$ where

$$\begin{aligned} m_t &= \nu_0 g^{\alpha t} [1 - (g^\alpha/\bar{h})^{t_0-1-t}] + \nu_1 [1 - (1 - \theta)(1 - g)/(\xi p \bar{h}^{T-t})] \\ &= \nu_0 g^{\alpha t} [1 - (g^\alpha/\bar{h})^{t_0-1-t}] + \ln(\beta \bar{h})^{1/\gamma} [\xi p - (1 - \theta)(1 - g)/\bar{h}^{T-t}]/[pP(1)] \end{aligned}$$

for all $1 < t \leq T$. We thus have

$$\begin{aligned} c_{t+1} - c_t &= -\nu_0 g^{\alpha t} [1 - (g^\alpha/\bar{h})^{t_0-1-t}] + \ln(\beta \bar{h})^{1/\gamma} [P(1) - (\xi p - (1 - \theta)(1 - g)/\bar{h}^{T-t})]/P(1) \\ &= -\nu_0 g^{\alpha t} [1 - (g^\alpha/\bar{h})^{t_0-1-t}] - \ln(\beta \bar{h})^{1/\gamma} (1 - \theta)(1 - g)(1 - 1/\bar{h}^{T-t})/P(1) \end{aligned}$$

which is negative and increasing, with a maximum equal to 0 at $t = T$. As $c_{t+1} - c_t - (c_t - c_{t-1}) = -p(m_t - m_{t-1}) > 0$, the consumption sequence is decreasing at a decreasing rate. With u CARA, we have $h_t = u_t/\beta u_{t+1}$ and thus $u_{t+1} = u_t/(\beta h_t) = u_0 \beta^{-t} \prod_{\tau=1}^t 1/h_\tau$. As $h_t = \bar{h}$ for all $t < T$ at the optimum, we get $u_{t+1} = u_0(\bar{h}\beta)^{-t}$ and

$$\sum_{t=0}^{T-1} \beta^t u_t = u_0 \sum_{t=0}^{T-1} (1/\bar{h})^t = u_0 \frac{1 - 1/\bar{h}^T}{1 - 1/\bar{h}} = u_0 \frac{\bar{h} - 1/\bar{h}^{T-1}}{\bar{h} - 1}.$$

G Proof of Proposition 5

We first characterize the distribution of \tilde{y}_{t+1} given the information available in period t , $t_0 < t \leq T$, that we denote by $\tilde{y}_{t+1|t}$. Given the information available in period t , the prevailing interest rate r_t (and thus the realization ε_t of $\tilde{\varepsilon}_t$) is known, and thus both $\tilde{k}_{t+1} = q^{-1}((1 + r_t - g)/(1 - \tau_{t+1}\varphi))$ and $q_{t+1} = q(\tilde{k}_{t+1})$ are known. From (11), e_{t+1} is also known since it depends on date t variables that are function of ε_{t-1} . Hence, from (14), we get

$$\tilde{y}_{t+1|t} = \tilde{c}_{t+1|t} + pe_{t+1} = q_{t+1} - \tilde{k}_{t+2|t} + gk_{t+1} - (\tilde{\mu}_{t+2|t} - g\mu_{t+1}) + pe_{t+1}, \quad (50)$$

where \tilde{k}_{t+2} and $\tilde{\mu}_{t+2}$ are unknown since they depend on the realization of $\tilde{\varepsilon}_{t+1}$: we have $\tilde{k}_{t+2} = q^{-1}((1 + \tilde{r}_{t+1} - g)/(1 - \tau_{t+2}\varphi))$ and, from (31),

$$\tilde{\mu}_{t+2|t} = a_1\mu_{t+1} + a_2e_{t+1} + a_3 + Z_{t+1} + b_1\tilde{\varepsilon}_{t+1} + b_2\varepsilon_t. \quad (51)$$

Replacing, we get

$$\tilde{y}_{t+1|t} = q_{t+1} - \tilde{k}_{t+2|t} + gk_{t+1} + (g - a_1)\mu_{t+1} + (p - a_2)e_{t+1} - a_3 - Z_{t+1} - b_1\tilde{\varepsilon}_{t+1} - b_2\varepsilon_t.$$

Using the linear approximation $k_t \approx \bar{k} + \bar{k}'(\bar{r})\bar{r}\varepsilon_{t-1}$ where $\bar{k}'(\bar{r}) \equiv d\bar{k}/d\bar{r} = 1/[(1 - \tau\varphi)q''(\bar{k})] < 0$, we get using $\mathbb{E}_t[\tilde{k}_{t+2}] \approx \bar{k} + \bar{k}'(\bar{r})\bar{r}\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t]$ when $t_0 < t \leq T$, and (30) that

$$\begin{aligned} \tilde{y}_{t+1|t} - \mathbb{E}_t[\tilde{y}_{t+1}] &\approx -(\tilde{k}_{t+2|t} - \mathbb{E}[\tilde{k}_{t+2}|\varepsilon_t]) - b_1(\tilde{\varepsilon}_{t+1} - \mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t]) \\ &= -(\bar{k}'(\bar{r})\bar{r} + b_1)(\tilde{\varepsilon}_{t+1} - \mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t]) \\ &= -(\bar{k}'(\bar{r})\bar{r} + b_1)\chi\tilde{\omega}_{t+2} \end{aligned}$$

when $t_0 < t \leq T$. Hence, $\tilde{y}_{t+1|t}$ when $t_0 < t \leq T$ is approximatively normally distributed with variance $\mathbb{V}_t[\tilde{y}_{t+1}] = (\bar{k}'(\bar{r})\bar{r} + b_1)^2\chi^2\sigma_\omega^2 \equiv \sigma_{y_{t+1}}^2$.

The coefficients $\{a_j\}_{j=1,\dots,3}$, b_1 , b_2 , and Z_t in (31), and χ and ρ in (30), are derived

as follows. Using (50), (51), (11), (14) and $\mathbb{E}_t[\tilde{k}_{t+2}] = \bar{k} + \bar{k}'(\bar{r})\bar{r}\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t]$, we get

$$\begin{aligned}
\mathbb{E}_t[\tilde{y}_{t+1}] - y_t &\approx q_{t+1} - q_t - \bar{k} - \bar{k}'(\bar{r})\bar{r}\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] + k_{t+1}(1+g) - gk_t - \mathbb{E}_t[\tilde{\mu}_{t+2}] + (1+g)\mu_{t+1} - g\mu_t \\
&\quad + p(e_{t+1} - e_t) \\
&= q_{t+1} - q_t - \bar{k} - \bar{k}'(\bar{r})\bar{r}\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] + k_{t+1}(1+g) - gk_t \\
&\quad - (a_1\mu_{t+1} + a_2e_{t+1} + a_3 + Z_{t+1} + b_1\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] + b_2\varepsilon_t) + (1+g)\mu_{t+1} - \mu_t g + p(e_{t+1} - e_t) \\
&= q_{t+1} - q_t - \bar{k} - \bar{k}'(\bar{r})\bar{r}\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] + k_{t+1}(1+g) - gk_t - a_3 - Z_{t+1} \\
&\quad - b_1\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] - b_2\varepsilon_t + (1+g-a_1)\mu_{t+1} - \mu_t g + (p-a_2)e_{t+1} - pe_t \\
&= q_{t+1} - q_t - \bar{k} - (b_1 + \bar{k}'(\bar{r})\bar{r})\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] + k_{t+1}(1+g) - gk_t - a_3 - Z_{t+1} \\
&\quad - b_2\varepsilon_t + (1+g-a_1)\mu_{t+1} - \mu_t g + (p-a_2)(\theta e_t + \xi\mu_t - \varphi q_t + \hat{e}) - pe_t \\
&= q_{t+1} - q_t [1 + \varphi(p-a_2)] - \bar{k} - (b_1 + \bar{k}'(\bar{r})\bar{r})\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] + k_{t+1}(1+g) - gk_t \\
&\quad + (1+g-a_1)\mu_{t+1} - \mu_t [g - \xi(p-a_2)] + [\theta(p-a_2) - p]e_t + (p-a_2)\hat{e} - a_3 \\
&\quad - Z_{t+1} - b_2\varepsilon_t.
\end{aligned}$$

Approximating $(1+r_t)(1-z_t) \approx 1+r_t - \bar{h}z_t$ in (22) and using (29), we get

$$r_t \approx \psi - \gamma^2 \sigma_{y+1}^2 / 2 + \gamma \{ \mathbb{E}_t[\tilde{y}_{t+1}] - y_t \} \approx (1-\lambda g)\mu_t + \omega_{t+1} + \xi\tau_{t+1} + g(1-z_{t+1}) - 1 + \bar{h}z_t - (1-\lambda)\mu_{t+1}$$

which gives, denoting $a_0 \equiv 1/[1-\lambda+\gamma(1+g-a_1)]$,

$$\begin{aligned}
\mu_{t+1}/a_0 &\approx [1-\lambda g + \gamma g - \gamma\xi(p-a_2)]\mu_t - \gamma[\theta(p-a_2) - p]e_t + \omega_{t+1} + \xi\tau_{t+1} + \bar{h}z_t - gz_{t+1} \\
&\quad - (\psi - \gamma^2 \sigma_{y+1}^2 / 2) - \gamma(p-a_2)\hat{e} + \gamma a_3 + g - 1 + \gamma Z_{t+1} + \gamma(b_1 + \bar{k}'(\bar{r})\bar{r})\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] \\
&\quad + \gamma b_2\varepsilon_t - \gamma q_{t+1} + \gamma q_t [1 + \varphi(p-a_2)] + \gamma\bar{k} - \gamma k_{t+1}(1+g) + \gamma gk_t.
\end{aligned}$$

Using $k_{t+1} \approx \bar{k} + \bar{k}'(\bar{r})\bar{r}\varepsilon_t$ and $q_{t+1} \approx \bar{q} + \bar{q}'(\bar{r})\bar{r}\varepsilon_t$ where $\bar{q}'(\bar{r}) \equiv d\bar{q}/d\bar{r} = q'(\bar{k})\bar{k}'(\bar{r}) < 0$ for t and $t-1$, we get

$$-\gamma q_{t+1} + \gamma q_t [1 + \varphi(p-a_2)] \approx -\gamma \bar{q}'(\bar{r})\bar{r}\varepsilon_t + \gamma \bar{q}\varphi(p-a_2) + \gamma \bar{q}'(\bar{r})\bar{r}\varepsilon_{t-1} [1 + \varphi(p-a_2)]$$

and

$$\gamma\bar{k} - \gamma k_{t+1}(1+g) + \gamma gk_t \approx -\gamma\bar{k}'(\bar{r})\bar{r}\varepsilon_t(1+g) + \gamma g\bar{k}'(\bar{r})\bar{r}\varepsilon_{t-1}.$$

Substituting and, from (30), using $\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] = \rho\varepsilon_t$, we arrive at

$$\begin{aligned}\mu_{t+1} &\approx a_0[1 - \lambda g + \gamma g - \gamma\xi(p - a_2)]\mu_t - a_0\gamma[\theta(p - a_2) - p]e_t + a_0\omega_{t+1} \\ &\quad - a_0[\psi - \gamma^2\sigma_{y_{t+1}}^2/2 + \gamma(p - a_2)(\hat{e} - \bar{q}\varphi) - \gamma a_3 + 1 - g] + a_0[\xi\tau_{t+1} + \bar{h}z_t - gz_{t+1} + \gamma Z_{t+1}] \\ &\quad + a_0\gamma(\bar{q}'(\bar{r})[1 + \varphi(p - a_2)] + g\bar{k}')\bar{r}\varepsilon_{t-1} - a_0\gamma[\bar{q}'(\bar{r})\bar{r} + \bar{k}'(\bar{r})\bar{r}(1 + g) - b_2 - (b_1 + \bar{k}'(\bar{r})\bar{r})\rho]\varepsilon_t\end{aligned}\quad (52)$$

Identifying the non-stochastic terms with those of (31) and simplifying gives

$$\begin{aligned}a_1 &= \frac{1 - \lambda g + \gamma[g + \xi(a_2 - p)]}{1 - \lambda + \gamma(1 + g - a_1)}, \quad a_2 = \frac{\gamma p(1 - \theta)}{1 - \lambda + \gamma(1 + g - a_1 - \theta)}, \\ a_3 &= -\frac{\psi - \gamma^2\sigma_{y_{t+1}}^2/2 - \gamma(p - a_2)(\varphi\bar{q} - \hat{e}) + 1 - g}{1 - \lambda + \gamma(g - a_1)},\end{aligned}\quad (53)$$

and

$$Z_t = a_0(\xi\tau_{t+1} + \bar{h}z_t - gz_{t+1} + \gamma Z_{t+1}) = a_0 \sum_{i=0}^{+\infty} (a_0\gamma)^i (\xi\tau_{t+1+i} + h_{t+i}z_{t+i} - gz_{t+1+i}). \quad (54)$$

The parameters concerning the stochastic terms are derived as follows. When $\varepsilon_{t-1} = \omega_{t+1} = 0$, we have $\varepsilon_t = 0$ implying $r_t = \bar{r}$. Denoting by $\mu_{t+1}|_{r_t=\bar{r}}$ the value of μ_{t+1} in such a case, we get from (22), using $(1 + r_t)(1 - z_t) \approx 1 + r_t - \bar{h}z_t$,

$$\mu_{t+1} - \mu_{t+1}|_{r_t=\bar{r}} \approx \frac{\omega_{t+1} + \bar{r} - r_t}{1 - \lambda} = \frac{\omega_{t+1} - \bar{r}\varepsilon_t}{1 - \lambda},$$

and from (52),

$$\begin{aligned}\mu_{t+1} - \mu_{t+1}|_{r_t=\bar{r}} &\approx a_0\{\omega_{t+1} + \gamma(\bar{q}'(\bar{r})[1 + \varphi(p - a_2)] + g\bar{k}')\bar{r}\varepsilon_{t-1} \\ &\quad - \gamma[\bar{q}'(\bar{r})\bar{r} + \bar{k}'(\bar{r})\bar{r}(1 + g) - b_2 - (b_1 + \bar{k}'(\bar{r})\bar{r})\rho]\varepsilon_t\}.\end{aligned}\quad (55)$$

Equalizing gives

$$\begin{aligned}\omega_{t+1} &= \frac{\bar{r} - (1 - \lambda)a_0\gamma[\bar{q}'(\bar{r})\bar{r} + \bar{k}'(\bar{r})\bar{r}(1 + g) - b_2 - (b_1 + \bar{k}'(\bar{r})\bar{r})\rho]}{1 - (1 - \lambda)a_0}\varepsilon_t \\ &\quad + (1 - \lambda)\gamma a_0 \frac{\bar{q}'(\bar{r})[1 + \varphi(p - a_2)] + g\bar{k}'(\bar{r})}{1 - (1 - \lambda)a_0}\bar{r}\varepsilon_{t-1}.\end{aligned}\quad (56)$$

Substituting in (55) yields

$$\begin{aligned} \mu_{t+1} - \mu_{t+1|r_t=\bar{r}} &\approx \gamma a_0 \frac{\bar{q}'(\bar{r})[1 + \varphi(p - a_2)] + g\bar{k}'}{1 - (1 - \lambda)a_0} \bar{r}\varepsilon_{t-1} \\ &+ a_0 \frac{\bar{r} - \gamma[\bar{q}'(\bar{r})\bar{r} + \bar{k}'(\bar{r})\bar{r}(1 + g) - b_2 - (b_1 + \bar{k}'(\bar{r})\bar{r})\rho]}{1 - (1 - \lambda)a_0} \varepsilon_t. \end{aligned}$$

Identifying with the stochastic terms of (31) gives, using $1 - (1 - \lambda)a_0 = \gamma a_0(1 + g - a_1)$,

$$b_2 = \frac{\bar{q}'(\bar{r})[1 + \varphi(p - a_2)] + g\bar{k}'(\bar{r})\bar{r}}{1 + g - a_1} \quad (57)$$

$$b_1 = \frac{\bar{r} - \gamma[\bar{q}'(\bar{r})\bar{r} + \bar{k}'(\bar{r})\bar{r}(1 + g - \rho) - b_2 - \rho b_1]}{\gamma(1 + g - a_1)}. \quad (58)$$

As (56) can be written as $\omega_{t+1} = [(1 - \lambda)b_1 + \bar{r}] \varepsilon_t + b_2(1 - \lambda)\varepsilon_{t-1}$, we get $\chi = [(1 - \lambda)b_1 + \bar{r}]^{-1}$ and

$$\rho = \frac{-b_2(1 - \lambda)}{(1 - \lambda)b_1 + \bar{r}}. \quad (59)$$

Reorganizing terms to get $b_2 = -\rho[b_1 + \bar{r}/(1 - \lambda)]$ and substituting in (58), we arrive at

$$b_1 = \frac{1 - \gamma[\bar{q}'(\bar{r}) + \bar{k}'(\bar{r})(1 + g - \rho) + \rho/(1 - \lambda)]}{\gamma(1 + g - a_1)} \bar{r}.$$

From (57) and (59), we obtain that ρ solves

$$\rho = \frac{-(1 - \lambda)\gamma a_0 \{\bar{q}'(\bar{r})[1 + \varphi(p - a_2)] + g\bar{k}'(\bar{r})\}}{1 - \gamma a_0 \{(1 - \lambda)[\bar{q}'(\bar{r}) + \bar{k}'(\bar{r})(1 + g)] + \rho [1 - \bar{k}'(\bar{r})(1 - \lambda)]\}}. \quad (60)$$

The first two equations of (53) form a system involving only coefficients a_1 and a_2 that can be solved separately from the others. More precisely, defining $a_1^0 \equiv g + (1 - \lambda)/\gamma$, we can express (53) as

$$\begin{aligned} a_1 &= \frac{a_1^0 + \lambda(1 - g)/\gamma + \xi(a_2 - p)}{a_1^0 - a_1 + 1}, a_2 = \frac{p(1 - \theta)}{a_1^0 - a_1 + 1 - \theta}, \\ a_3 &= -\frac{\psi - \gamma^2 \sigma_{y+1}^2 / 2 - \gamma(p - a_2)(\varphi\bar{q} - \hat{e}) + 1 - g}{\gamma(a_1^0 - a_1)}, a_0 = \frac{1}{\gamma(a_1^0 - a_1 + 1)} \end{aligned} \quad (61)$$

From the expression of a_2 , we get

$$a_2 - p = -p \frac{a_1^0 - a_1}{a_1^0 - a_1 + 1 - \theta}, \quad (62)$$

which, plugged into the expression of a_1 , gives

$$(a_1^0 - a_1) \left(a_1 - 1 + \frac{\xi p}{a_1^0 - a_1 + 1 - \theta} \right) = \frac{\lambda(1-g)}{\gamma}$$

that can be expressed as $Q(a_1^0 - a_1) = 0$ where

$$Q(x) \equiv x^3 + (2 - \theta - a_1^0)x^2 + [(1 - \theta)(1 - a_1^0) - \xi p + \lambda(1 - g)/\gamma]x + \lambda(1 - \theta)(1 - g)/\gamma$$

is a third degree polynomial. As Z_t given by (54) converges if $\gamma a_0 < 1$, i.e. if $a_1 < a_1^0$, only positive roots are relevant. As $P(1) > 0$, we have

$$\begin{aligned} (1 - \theta)(1 - a_1^0) - \xi p + \lambda(1 - g)/\gamma &< (1 - \theta)(1 - a_1^0) - (1 - \theta)(1 - g) + \lambda(1 - g)/\gamma \\ &= (1 - \theta)(g - a_1^0) + \lambda(1 - g)/\gamma \\ &= -(1 - \theta)(1 - \lambda)/\gamma + \lambda(1 - g)/\gamma \end{aligned}$$

and a sufficient condition for the third coefficient to be negative is $\lambda < 1/[1 + (1 - g)/(1 - \theta)]$. Under this condition, we have $Q'(0) < 0$, and since $Q(0) > 0$, there is at most two positive roots, the smallest one corresponding to the largest value of a_1 . From (61) and (62), $a_1 > 0$ if $a_1^0 + \lambda(1 - g)/\gamma + \xi(a_2 - p) = g + (1 - \lambda g)/\gamma - p\xi(a_1^0 - a_1)/(a_1^0 - a_1 + 1 - \theta) > 0$. Since $a_1^0 > a_1$, we must have $g + (1 - \lambda g)/\gamma > p$. Equation (60) can be written as $\zeta(\rho) = 0$ where

$$\begin{aligned} \zeta(x) &= -x^2 [1 - \bar{k}'(\bar{r})(1 - \lambda)] \gamma a_0 + x \{1 - \gamma a_0(1 - \lambda)[\bar{q}'(\bar{r}) + \bar{k}'(\bar{r})(1 + g)]\} \\ &\quad + (1 - \lambda)\gamma a_0 \{ \bar{q}'(\bar{r})[1 + \varphi(p - a_2)] + g\bar{k}'(\bar{r}) \}. \end{aligned}$$

Differentiating gives $\zeta'(x) = -2x [1 - \bar{k}'(\bar{r})(1 - \lambda)] \gamma a_0 + 1 - \gamma a_0(1 - \lambda)[\bar{q}'(\bar{r}) + \bar{k}'(\bar{r})(1 + g)]$. As ζ is concave, with $\zeta(0) < 0$ and $\zeta'(0) > 0$, it admits at most two positive roots. As $\bar{k}'(\bar{r}) = -\bar{k}^{2-\alpha}/[(1 - (\bar{h} - g)\varphi/\xi)A\alpha(1 - \alpha)]$ and $\bar{q}'(\bar{r}) = -\bar{k}/[(1 - (\bar{h} - g)\varphi/\xi)(1 - \alpha)]$ where $\bar{k} = \alpha A(1/(\bar{h} - g) - \varphi/\xi)^{1/(1-\alpha)}$ which decreases with \bar{h} and tends to 0 when $\bar{h} \rightarrow g + \xi/\varphi$, we have $\lim_{\bar{h} \rightarrow g + \xi/\varphi} \zeta(x) = x(1 - \gamma a_0 x)$ and thus $\zeta(x)$ admits positive roots if \bar{h} is sufficiently large.

Solving the recursion of (30), it comes

$$\tilde{\varepsilon}_t = \chi \tilde{\omega}_{t+1} + \rho \tilde{\varepsilon}_{t-1} = \chi \sum_{k=0}^{t-t_0} \rho^k \tilde{\omega}_{t+1-k} + \rho^{t-t_0} \varepsilon_{t_0},$$

hence $\mathbb{E}[\tilde{\varepsilon}_t|\varepsilon_{t_0}] = \rho^{t-t_0}\varepsilon_{t_0}$ and

$$\mathbb{V}[\tilde{\varepsilon}_t|\varepsilon_{t_0}] = \mathbb{E}[(\tilde{\varepsilon}_t - \mathbb{E}[\tilde{\varepsilon}_t|\varepsilon_{t_0}])^2|\varepsilon_{t_0}] = \mathbb{E}\left[\left(\chi \sum_{k=0}^{t-t_0} \rho^k \tilde{\omega}_{t+1-k}\right)^2\right] = \chi^2 \sum_{k=0}^{t-t_0} \rho^{2k} \mathbb{E}[\tilde{\omega}]^2$$

using independence, which gives $\mathbb{V}[\tilde{\varepsilon}_t|\varepsilon_{t_0}] = \chi^2 \sigma_\omega^2 (1 - \rho^{2(t-t_0+1)}) / (1 - \rho^2)$.

H Calibration of the model

The world GDP and population in 2018 (the reference year corresponding to $t = 0$) are $q_0 = 8.59$ trillions US\$ and $Pop = 7.7$ billions. The consumer's MRS p is set at \$15/tCO₂, the psychological discount rate $\psi = 1.5\%$, hence $\beta \approx .985$, and the intertemporal elasticity parameter $\gamma = 7$. As the intertemporal elasticity of consumption is inversely related to the consumption level, computations are made using the per capita global wealth level ($y_t/Pop \times 10^4$).

The capital share of GDP is set at 40% (i.e. $\alpha = 2/5$) and the capital depreciation at 25% (i.e., $g = 3/4$). With a Cobb-Douglas production function, the capital stock is derived from the interest rate by $k_0 = \alpha q_0 / (1 + r_0 - g)$ where $r_0 = 6\%$, which gives $k_0 = 11.9$ trillions US\$. The corresponding scale parameter is $A = 3.28$.

The EQ index e_t is defined as a global ‘‘carbon budget’’ at date t , i.e. the difference between a tipping point level of CO₂ in the atmosphere, ℓ_M , and the level of GHG at date t , ℓ_t , expressed in CO₂ equivalent: $e_t = \ell_M - \ell_t$.¹⁹ The world emissions of CO₂ in 2018 is 37.1 Gt CO₂, leading to emission intensity $\iota_0 = 37.1 / 8.59 = 0.36\text{kg/US\$}$. The maximum emission intensity φ is set to 6.5kg/US\$ and the efficiency ratio of green technology is equal to 1, i.e. $\xi = \varphi$. The resulting initial level of the AGT index is $\mu_0 = q_0(\varphi - \iota_0)/\xi = 8.02$. The standard deviation of the shocks affecting the diffusion of technology is $\sigma_\omega = 0.05$.

It is assumed that EQ has reached its long term equilibrium $e_N = \ell_M - \hat{\ell}$ in the preindustrial period, where $\hat{\ell} = 2176$ Gt CO₂ (280 ppm) and $\ell_M = 5439$ Gt CO₂ (700 ppm), which gives $e_N = 3264$ Gt CO₂. Assuming that $\theta = 4/5$ yields $\hat{e} = (1 - \theta)e_N = 652$ Gt CO₂. The GHG level in 2018 is 3165.5 Gt CO₂ (407.4 ppm) hence an initial EQ index $e_0 = 2273.5$ Gt CO₂. Given q_0 and μ_0 derived above, we get

¹⁹The unit used is the gigaton or Gt shorthand, i.e. 10⁹ metric tons. Theses levels are also commonly expressed in atmospheric concentration, the unit being the part per million or ppm shorthand, i.e. 0.01%. Each ppm represents approximately 2.13 Gt of carbon in the atmosphere as a whole, equivalent to 7.77 Gt of CO₂.

$$e_1 = \theta e_0 + \hat{e} - \varphi q_0 + \xi \mu_0 = 2434.38 \text{ Gt CO}_2.$$