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# Optimal Green Technology Adoption and Policy Implementation

Jean-Marc Bourgeon\*

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## Abstract

The importance of coordination problems in the greening of the economy is analyzed using a global game approach in a simple macro-dynamic model. Two policy options to motivate firms to adopt green technology are examined: one coupling an emissions tax with subsidies, the other coupling an emissions tax with environmental standards. Compared to the First-Best, these policies face coordination problems and their effectiveness depends on shocks affecting network externalities. The resulting path of the economy is stochastic and can deviate significantly from its benchmark. The subsidy policy is more particularly subject to these shocks and its path dependence is stronger than that affecting the environmental standards policy. These results are derived in a simple framework and illustrated by numerical simulations.

**Keywords:** Growth, sustainability, Technology adoption.

**JEL:** O33, O44, E37, Q55

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# 1 Introduction

Decoupling resource use and environmental discharges from economic growth is one of the main challenges facing modern economies. Among these, decarbonization, i.e. reducing carbon dioxide (CO<sub>2</sub>) emissions per unit of GDP, is a top priority. As the International Energy Agency (IEA, 2021) points out, most of the global reductions in CO<sub>2</sub> emissions through 2030 in its model come from technologies readily available today, but by 2050, nearly half of the reductions come from technologies that are currently in the demonstration or prototype stage. In heavy industry and long-distance transportation, the share of emission reductions from technologies still in development today is even higher. This is a significant challenge for decarbonization because, as noted by Battisti (2008), the diffusion of existing cost-effective clean technologies has been notoriously slow and below potential. Indeed, the diffusion of green technologies is hindered by multiple obstacles. At first, there is a conversion cost: Green technologies are often more costly and less productive than the technologies that firms are using, and may require retraining of their workforce and radical changes in their supply chains. Because of network externalities and technological spillovers, optimal investment choices do not necessarily correspond to the latest or most innovative machines.<sup>1</sup> Since a machine is a durable good, usually used for several years, it is important that the parts needed for its operation and maintenance services are readily available in the future, which is all the more likely as the technology becomes more widespread. There is also the issue of compatibility of the technology with existing and future industry standards, which determines what other machines can be employed. Moreover, the more a technology is used, the higher the research effort of the machine industry sector to improve the technology. Therefore, the optimal investment choice for a firm depends on the decisions made by all firms, and the more firms use a given technology, the more attractive it is. All of these phenomena, hereafter referred to as “network effects”, are difficult to anticipate, due to the many external factors that affect the supply chain and the supporting infrastructure of firms.<sup>2</sup>

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<sup>1</sup>Technology spillovers refer to the unintended benefits resulting from the research and development efforts of others. Network externalities refers to a situation in which the value of a technology increases with the number of its users.

<sup>2</sup>Hoevenagel et al. (2007) discusses the problems of green technology adoption in Europe. 99.8% of firms are small and medium-sized enterprises, run by entrepreneurs with significant time and task pressure. Their basic perception of environmental technologies is one of cost and risk. Their staff is not adequately trained and familiar with the installation and operation of new technologies. They are generally uninformed about potential environmental technologies and rely heavily on the advice of their professional entourage, such as suppliers, who often have the same lack of information. A firm’s network relationships influence the adoption of environmental innovations at each stage as it

In this paper, I analyze the issue of designing policies to green the economy when firms face such a coordination problem using an original micro-founded growth model. A large number of firms must adopt cleaner and often newer technologies whose development and diffusion are subject to network effects. Firms' investment decisions are based on their expectations about these effects. Their heterogeneous assessments are reflected in an industrial sector made up of firms with diverse production processes. This diversity is captured by the distribution of two variables, one corresponding to the productive capacity of the production processes (similar to the traditional economic definition of capital), the other to their pollution intensity. The firms' coordination problem in their investment choices is framed as a sequence of global games (Carlsson & Van Damme, 1993) played each period. The dynamic unfold as follows: Each period, given the prevailing interest rate and wage (and the public policy), the knowledge of the current distribution of production processes and private signals about the shock affecting network effects allow firms to determine their optimal investment. Because of idiosyncratic signals leading to different expectations, firms' choices are diverse, resulting in a new distribution of production processes in the next period, when another global game takes place, and so on. The resulting path of the economy is stochastic due to the shocks affecting network effects. The incentives created by environmental policies are also subject to these shocks, and this framework allows us to compare the paths taken by the economy with different policy instruments without making arbitrary assumptions about firms' self-fulfilling beliefs.<sup>3</sup>

Two widely used environmental policies are examined. Both involve an emissions tax that makes firms responsible for the pollution they emit. It influences their choice of machines both on their production capacity and on their pollution intensity characteristics. To better guide firms in their choices, these policy options also incorporate one of two alternative instruments: either a schedule of subsidies for green investments, or a schedule of environmental standards, i.e. restrictions on the pollution intensity of new machines. Although the shocks affecting network effects are independent, both

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gathers information about potential solutions for implementing new technologies, and conversely, a firm's strategic orientation influences the types of external expertise and network relationships that are developed.

<sup>3</sup>Unlike other approaches that often lead to multiple equilibria, global games yield a single equilibrium by taking into account the intrinsic uncertainty affecting the economy. Multiplicity of equilibria in coordination games is due to the strong assumptions that the economic fundamentals are common knowledge and that agents are certain about each other's behavior in equilibrium thanks to an appropriate set of beliefs. By introducing a noisy signal about the actual state of the world and some uncertainty about what other agent knows, global games lead to a unique equilibrium for each underlying state of the world without resorting to arbitrary assumptions about self-fulfilling beliefs (see Morris & Shin, 2000, 2003).

policies present a path dependency induced by the consumers' desire to smooth their consumption over time, which impacts the interest rate. The main results are that the tax-subsidy policy is more particularly subject to the variations of the interest rate, and the path dependence of the economy under this policy is likely to be stronger than that under the tax-standards policy. Interestingly, because network externalities generate excessive demand for prevailing technologies, optimal subsidies become taxes once green technologies are widespread, in order to curb over-investment.

To examine more closely the fluctuations of the economy under these policies, I consider their implementation of a deterministic path. This benchmark path corresponds to the socially optimal path assuming that production processes are imposed on firms and are not subject to network shocks. This analysis is done using an illustrative example that allows explicit solutions of the First-Best policy and its two implementations, as well as explicit confidence intervals for the stochastic paths of the economy. The corresponding distribution of the interest rate follows a first-order autoregressive process during the transition period to environmental neutrality under the two policy options. Given the scale of investment needed to green the economy, long lasting deviations of the economy from the First-Best path are thus likely. Simulations show that this path dependence can be significant under the two policies examined, and that environmental standards permit a transition to green technologies more likely in line with the regulator's intended path than subsidies do.

There is an abundant literature on growth and sustainability. The literature on endogenous green growth focuses on productivity improvements and frontier innovation. This is the case in the AK paradigm where capital-knowledge accumulates with learning-by-doing (Stokey, 1998), and Lucas-like extensions (Bovenberg & Smulders, 1995), within a framework of product variety (Gerlagh & Kuik, 2007) or within the Schumpeterian growth paradigm of destructive creation and directed technical changes (Acemoglu et al., 2012), where the most productive innovations are adopted by firms as soon as they are discovered. This article focuses on the adoption of existing technologies that have knock-on effects leading to the gradual replacement of old and polluting machines with greener ones. The approach is thus close to the literature on endogenous growth viewed as a process of adoption of existing ideas and mutual imitation between firms, as exposed by Eaton & Kortum (1999); Lucas Jr & Moll (2014); Lucas (2009); Perla & Tonetti (2014).<sup>4</sup> The approach here is similar for describing the adoption of technologies: although the R&D sector is not spelled out, there is a set of existing

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<sup>4</sup>These papers assume that each agent in the economy is endowed with a certain amount of knowledge ("ideas") and this knowledge evolves through contact with the rest of the population.

technologies whose potential is more or less exploited depending on the proportion of firms that use them. The distribution of production processes changes over time as firms' incentives to adopt new technologies evolve.<sup>5</sup> Comparisons of policy instruments to implement an environmental objective have been quite numerous since Weitzman (1974). The main focus is on the information available to the regulator on the firms' pollution abatement costs. Few papers consider the importance of firms' expectations in achieving an environmental goal. Aldy & Armitage (2020) compare an emissions tax with a cap-and-trade instrument in which firms are subject to forecasting errors in the price of pollution allowances on the secondary market, the effectiveness of the abatement technology being the same in both cases. In what follows, the firms' expectations about their economic environment are important because of network externalities that determine the effectiveness of the technology.

The remainder of the paper is organized as follows: Section 2 describes the dynamics of technology adoption under Laissez-Faire and the two policy options. Section 3 presents the environmental dynamic, the First-Best path of the economy and the policy schemes that implement it in expectation. An illustrative example is detailed section 4. The last section concludes. All proofs are in the Appendix.

## 2 Technology adoption

Consider a discrete time economy composed of a continuum of firms, of total mass equal to one, that collectively produce at date  $t$  an amount  $q_t = \int_0^1 q_{it} di$  of output corresponding to the GDP. Firms' output may come from various production processes with different environmental impacts. More precisely, I suppose that for each particular production task, firms have the choice among a large set of machines characterized by their productivity and polluting emissions level. These sets of machines evolve over time, but each period, comparing machines with the same productivity level, the cheaper they are, the more they pollute. Firm  $i$ 's investment at date  $t$  is therefore made up of spending on production capacity and environmental quality, noted respectively  $I_{it}$  and  $o_{it}$ , representing a total expenditure on new machines equal to  $I_{it} + o_{it}$ , and the higher  $o_{it}$  is for a given  $I_{it}$ , the less polluting the machines are.<sup>6</sup> Assuming a depreciation rate  $1 - g$  of the machines, firm  $i$ 's production process at date  $t$  is described by vector

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<sup>5</sup>There is also a microeconomics literature that investigates the problem of network externalities, initiated by Katz & Shapiro (1985). See Guimaraes & Pereira (2016) for a recent contribution, and Shy (2011) for a survey.

<sup>6</sup>We may have  $I_{it} = 0$  and  $o_{it} > 0$ , in which case firm  $i$  buys pollution abatement equipment, i.e. devices that do not produce any items but reduce the polluting emissions of the firm.

$(k_{it}, x_{it})$  whose dynamic is given by

$$\begin{bmatrix} k_{it+1} \\ x_{it+1} \end{bmatrix} = g \begin{bmatrix} k_{it} \\ x_{it} \end{bmatrix} + \begin{bmatrix} I_{it} \\ O_{it} \end{bmatrix}. \quad (1)$$

The productive capacity of firm  $i$ 's production process at date  $t$ ,  $k_{it}$ , is similar to the standard economic notion of capital: it is a monetary measure of the durable goods that allow the firm to produce items or services when combined with labor. It yields a gross revenue  $q_{it} = Q(k_{it}, \ell_{it})$  where  $\ell_{it}$  is the firm's employment level, and  $Q$  a production function, homogeneous of degree one and satisfying the Inada conditions.<sup>7</sup> The environmental quality of the firm  $i$ 's production process,  $x_{it}$ , also dubbed 'technology mix', is a monetary measure of its effort to diminish the environmental impact of its production, given by  $\iota_{it}q_{it}$  where

$$\iota_{it} = \varphi - \xi x_{it}/q_{it} \quad (2)$$

is its emission intensity of the production process. As green technologies are often the most recent and innovative,  $x_{it}$  is also a monetary measure of the degree of innovation of firm  $i$ 's production process, which should be compared with that of other firms in the economy, given by  $\mu_t \equiv \int_0^1 x_{jt} dj$ , dubbed the 'green technology index' (GTI) of the economy at date  $t$ .<sup>8</sup> A very low  $x_{it}$  compared to  $\mu_t$  indicates that firm  $i$ 's production process is polluting and most probably dated. A high index, on the other hand, indicates that its production process is sound and most probably uses recent technologies.

Depending on their choice of machines over time, firms benefit from (or suffer from the absence of) network effects linked to their input suppliers, maintenance services (ease of finding inputs and specific parts to maintain machines), workers' know-how and, more generally, the particularities of their production process. The more widespread the technology used by a firm, the fewer logistical problems it encounters, and the easier it is to reach a production target. A firm must therefore anticipate the other firms' choice of machines when making its own choice, which leads to an intertemporal coordination problem that is formalized in the following as a succession of global games taking place each period.<sup>9</sup> More formally, denoting by  $\mathcal{O}_t \equiv \int_0^1 o_{jt} dj$

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<sup>7</sup>We thus have  $k_{it} > 0$  for all  $i$  and  $t$ .

<sup>8</sup>Of course, there are also technical improvements in traditional production processes, but these are mostly focused on increasing productivity and thus lowering the relative price of the machines that benefit from them.

<sup>9</sup>These coordination problems with strategic complementarity are known as "beauty contests" (see,

the sum of green investments made simultaneously by firms at date  $t$ , network effects correspond for firm  $i$  to monetary amounts given by

$$G_{it} = G(x_{it}; \mu_{t-1}, \mathcal{O}_{t-1}, \omega_t), \quad (3)$$

where  $G(\cdot)$  is a concave function of  $x_{it}$  that increases with  $\mu_{t-1}$  and  $\mathcal{O}_{t-1}$ , and  $\omega_t$  the realization of a time-independent normally distributed noise,  $\tilde{\omega}_t$  with  $\mathbb{E}[\tilde{\omega}_t] = 0$  and standard deviation  $\sigma_\omega$ , that summarizes the many unmodeled shocks affecting the development and the diffusion of technology.<sup>10</sup>  $G(\cdot)$  increases with  $\mu_{t-1}$  because only part of the capital stock is renewed each period, and  $\mu_{t-1}$  reflects the set of the most commonly used machines at  $t-1$ . It also increases with  $\mathcal{O}_{t-1}$  since the service network adapts to the investments made by all firms. However, this adaptation may be more or less rapid, depending on the reactivity of the network of suppliers. To fix ideas, I assume that<sup>11</sup>

$$G(x_{it}; \mu_{t-1}, \mathcal{O}_{t-1}, \omega_t) = (\mu_{t-1} + \lambda \mathcal{O}_{t-1} + \omega_t) x_{it} - x_{it}^2/2, \quad (4)$$

where  $\lambda \in [0, 1)$  captures the extent of immediate network effects. If  $\lambda = 0$ , the first term in (4) reduces to  $\mu_{t-1} + \omega_t$ , i.e. the former average mix plus a noise which may be positive or negative. When immediate network effects are at work ( $\lambda > 0$ ), the first term in (4) also depends on the most recent green investments made by all firms,  $\mathcal{O}_{t-1}$ . The economic infrastructure evolves more rapidly the larger  $\lambda$  is, the extreme case where  $\lambda$  is close to 1 corresponding to an immediate adaptation of the workers and the supporting infrastructure to new investment choices.<sup>12</sup> Using (3) and (4), we can define the ‘ideal’ technology mix at date  $t$  as

$$x_t^* \equiv \mu_{t-1} + \lambda \mathcal{O}_{t-1} + \omega_t \quad (5)$$

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e.g., Angeletos & Pavan, 2004, 2007 and Morris & Shin, 2002).

<sup>10</sup>These shocks may come from the service network, input suppliers or machines producers, reflecting the expectations of these operators on technology development. To streamline the analysis, I assume that these shocks do not directly affect the firms’ production level  $q_{it}$ .

<sup>11</sup>This formulation can be considered as a second-order approximation of a more general expression where the term  $-x_{it}^2/2$  ensures that  $G(\cdot)$  is strictly concave with respect to  $x_{it}$  and thus that there is a unique investment level in green technology for firm  $i$  at date  $t-1$ . It leads to close-form solutions and more importantly, it preserves the structure of the intertemporal coordination game between firms from one period to the next (the distribution of  $x_{it}$  at equilibrium remains Gaussian with mean  $\mu_t$  and a constant standard deviation).

<sup>12</sup>It is likely that the value of  $\lambda$  is low. As stressed by Battisti (2008), a consistent literature has shown that, even when a clean or a cost-reducing technology is readily available in the market, its spreading takes several years.



since we have

$$G_{it} = x_t^{*2} - (x_t^* - x_{it})^2/2 \quad (6)$$

where  $(x_t^* - x_{it})^2/2$  corresponds to firm  $i$ 's loss due to a mix  $x_{it}$  different from the ideal one.<sup>13</sup> Firm  $i$  must anticipate at  $t - 1$  its optimal date- $t$  technology mix  $x_{it}$ , but it cannot perfectly assess the ideal mix  $x_t^*$  because it doesn't know the realization of  $\tilde{\omega}_t$  and the total of green investments  $\mathcal{O}_{t-1}$  made simultaneously by all firms.

This dynamic setup is solved sequentially, focusing on Markov perfect equilibria where  $x_t^*$  is a state variable. In period  $t - 1$ , firms learn the distribution of  $(k_{it-1}, x_{it-1})$  and must anticipate the realization of the next period ideal technology mix,  $\tilde{x}_t^*$ , a random variable which distribution depends on  $\tilde{\omega}_t$  and the behavior of all firms. Firms beliefs are unbiased idiosyncratic private signals of the shock: they are formed according to  $\tilde{\eta}_{it-1} = \tilde{\omega}_t + \tilde{v}_{it-1}$ , where  $\tilde{v}_{it-1}$  is a time-independent noise, normally distributed with variance  $\sigma_v^2$ , verifying  $\mathbb{E}[\tilde{v}_{it-1}] = \mathbb{E}[\tilde{v}_{it-1}\tilde{\omega}_t] = \mathbb{E}[\tilde{v}_{it-1}\tilde{v}_{jt-1}] = 0$  for all  $i, j$ , and  $\int_0^1 v_{it-1} di = 0$ .<sup>14</sup> Given its signal, firm  $i$ 's expectation of the next period shock is given by  $\mathbb{E}[\tilde{\omega}_t | \eta_{it-1}] = \eta_{it-1} / (1 + \sigma_v^2 / \sigma_\omega^2)$  where  $1 / (1 + \sigma_v^2 / \sigma_\omega^2)$  is the signal-to-noise ratio in this inference.

## 2.1 Laissez-Faire

Consider the Laissez-Faire situation from date  $t = 0$  (the present period) assuming that firm  $i$  is endowed with technology  $(k_{i0}, x_{i0}) > 0$ . Each following period  $t$ , given  $w_t$  and  $r_t$ , the date- $t$  wage and interest rate, firm  $i$  plans its investments and employment levels  $\{(o_{it}, I_{it}, \ell_{it}), t = 0, 1, 2, \dots\}$  to maximize the expected discounted sum of its profits  $\mathbb{E}_t[\sum_{h \geq 0} (\prod_{\tau=1}^h \delta_{t+\tau})(q_{it+h} + G_{it+h} - w_t \ell_{it+h} - I_{it+h} - o_{it+h})]$  where  $\delta_t = 1 / (1 + r_t)$  is the date- $t$  discount factor.<sup>15</sup> With a total supply of labor equal to one each period, i.e.  $\int_0^1 \ell_{it} di = \ell_t = 1$  for all  $t$ , it is shown in the appendix that

**Proposition 1** *Under Laissez-Faire, firms' equilibrium investment in productive ca-*

<sup>13</sup>As  $\tilde{\omega}_t$  is normally distributed,  $x_t^*$  can take negative values, and all the more probably if  $\mu_t$  is small. I will neglect this possibility in the following by assuming that  $\mu_t$  remains far enough from 0.

<sup>14</sup>I thus assume that the law of large numbers applies to a continuum of i.i.d. random variables. While Judd (1985) points out that this is not true when working within the usual probability space on realizations of a continuum of draws, Uhlig (1996) shows that it suffices to redefine all integrals over the continuum as  $L_2$ -Riemann integrals. See also Sun (2006) for a framework in which pairwise independence ensures that all law of large numbers and sample distribution equalities hold exactly.

<sup>15</sup> $\mathbb{E}_t[\tilde{X}]$  is a shorthand for  $\mathbb{E}[\tilde{X} | \mathcal{I}_t]$ , i.e. the expectation of the random variable  $\tilde{X}$  given the information  $\mathcal{I}_t$  available at date  $t$ .

capacity at time  $t$  is given by

$$I_{it} = q'^{-1}(1 + r_t - g) - gk_{it} \quad (7)$$

where  $q(k) \equiv Q(k, 1)$ , leading to  $k_{it+1} = k_{t+1} = q'^{-1}(1 + r_t - g)$  for all  $i$ . The equilibrium wage is given by  $w_t = q(k_t) - (1 + r_{t-1} - g)k_t$ . Firm  $i$ 's investment strategy in green technology satisfies

$$o_{it} = \mathbb{E} [\tilde{x}_{t+1}^* | \eta_{it}, \mu_t, r_t] - gx_{it} + g - 1 - r_t, \quad (8)$$

leading to

$$o_{it} = g(\mu_t - x_{it}) + \frac{(1 - g)\mu_t + g - 1 - r_t}{1 - \lambda} + \frac{\eta_{it}}{1 - \lambda + \sigma_v^2/\sigma_\omega^2} \quad (9)$$

at equilibrium. The resulting firms' technology mixes at  $t + 1$  are normally distributed with mean

$$\mu_{t+1} = g\mu_t + \frac{(1 - g)\mu_t + g - 1 - r_t}{1 - \lambda} + \frac{\omega_{t+1}}{1 - \lambda + \sigma_v^2/\sigma_\omega^2}, \quad (10)$$

and standard deviation  $\sigma_{x_t} = \sigma_x \equiv \sigma_v/(1 - \lambda + \sigma_v^2/\sigma_\omega^2)$  for all  $t$ .

Because firms have the same production function  $Q$ , the productive capacity of all firms is the same, determined by choices that are guided unambiguously by the interest rate. This is not the case for their spending on green technology (8) which depends on their estimates of the most efficient mix  $\mathbb{E} [\tilde{x}_{t+1}^* | \eta_{it}, \mu_t, r_t]$  and thus on their individual signals  $\eta_{it}$ . The trade-offs that firms are facing when investing are the following. The date- $t$  borrowing cost of one monetary unit is  $1 + r_t$ . Invested in productive capital, its return is equal to the sum of the marginal productivity of capital,  $q'(k_{t+1})$ , and  $g$ , the remaining value of capital (the rest being depreciated). With investment  $I_t$ , the date- $t + 1$  productive capital stock is equal to  $k_{t+1} = gk_t + I_t$ , hence an optimal productive investment that solves  $q'(gk_t + I_t) + g = 1 + r_t$ . (8) is derived similarly: a monetary unit invested in green technology has a marginal return equal to the sum of the marginal network gain  $dG_{it+1}/dx_{it+1} = \tilde{x}_{t+1}^* - x_{it+1}$  and  $g$ . With investment  $o_{it}$ , the firm date- $t + 1$  green index is given by  $x_{it+1} = gx_{it} + o_{it}$ . Firm  $i$ 's investment rule is thus to equalize  $1 + r_t$  to the expected return  $\mathbb{E} [\tilde{x}_{t+1}^* | \eta_{it}, \mu_t, r_t] - (gx_{it} + o_{it}) + g$  given its signal  $\eta_{it}$ . At equilibrium, firm  $i$  adopts the green investment strategy (9) that is a linear function of the publicly observable variables  $\mu_t$  and  $r_t$ , its own technology

index  $x_{it}$ , and its private signal  $\eta_{it}$ .<sup>16</sup> The first term in (9),  $g(\mu_t - x_{it})$ , indicates that firm  $i$  makes-up for the difference between its green index and that of the economy on its undepreciated capital stock. The two other terms correspond to its estimate of the average green investment. It is affected by the magnification effect of the network externality,  $1/(1 - \lambda)$ , since all firms operate the same way.

Firms' green indexes are distributed normally around the GTI given by (10). This equation shows that the more reactive is the supporting infrastructure, the larger are the effects of  $\omega_t$  and  $r_t$  on the next period GTI. The expected dynamic given by (10) can be negative, i.e.  $\mathbb{E}[\mu_{t+1}] < \mu_t$ , in which case firms acquire increasingly dirty technologies, or positive if  $\mu_t$  is large enough, greater than  $[1 + r_t/(1 - g)]/\lambda$ , i.e. if the technology spillovers are sufficiently large, the depreciation rate large and/or the interest rate low. Indeed, as firms have to renew their machines, they may invest in less polluting ones if they expect that the supporting infrastructure and the workforce know-how adapt rapidly. This is however very unlikely if the initial GTI  $\mu_0$  is low. Indeed, firms will only purchase green technologies in the absence of governmental incentives if these technologies are already the most widely used in the economy, and thus the most easily exploitable.

Due to the idiosyncratic shocks on beliefs,  $v_{it}$ , firms have different expectations on  $x_{it}^*$ , hence choose machines with similar productive capacities but different environmental impacts. These discrepancies lead to a Gaussian distribution of firms' green indexes around the GTI, given by  $x_{it} = \mu_t + v_{it}/(1 - \lambda + \sigma_v^2/\sigma_\omega^2)$ . Hence, the industry sector can be viewed as a "cloud" of firms whose green technology levels are drawn in each period from a normal distribution centered on  $\mu_t$  with standard deviation  $\sigma_v/(1 - \lambda + \sigma_v^2/\sigma_\omega^2)$  which is all the greater as  $\lambda$  is large.

## 2.2 Environmental policies

Consider two environmental policy options involving the use of an emissions tax to limit firms' production capacity, but differing in their instrument for promoting green technologies: subsidies or technology standards.<sup>17</sup>

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<sup>16</sup>The proof follows Angeletos & Pavan (2004). Morris & Shin (2002) show that this linear, symmetric, rational-expectations strategy leads to the unique (per period) equilibrium.

<sup>17</sup>I assume that the social planner can credibly commit to these policies so that economic agents form consistent expectations. They are thus open loop policies, i.e. policies designed before the regulatory period and not revised thereafter: the levels of taxes and subsidies or standards that have been chosen for each period by the regulator are implemented regardless of the actual path followed by the economy.

## Subsidy programs

Suppose that the government implements an environmental policy using an emissions tax scheme  $\{\tau_t\}_{t>0}$  and a green technology subsidy scheme  $\{z_t\}_{t>0}$ . Given this policy, firm  $i$ 's per period profit becomes

$$\pi_{it} \equiv Q(k_{it}, \ell_{it}) + G_{it} - w_t \ell_{it} - I_{it} - o_{it}(1 - z_t) - \tau_t (\varphi Q(k_{it}, \ell_{it}) - \xi x_{it})$$

where the last term corresponds to the environmental tax payment, and the term  $o_{it}(1 - z_t)$  to the net payment for green technology investment. It is shown in the appendix that

**Proposition 2** *Under a tax-subsidy scheme  $\{\tau_t, z_t\}_{t>0}$ , firms' investment in productive capacity at date  $t$  satisfies*

$$q'(k_{t+1}) = \frac{1 + r_t - g}{1 - \tau_{t+1}\varphi}. \quad (11)$$

*Their technology mixes at  $t + 1$  are normally distributed with mean*

$$\mu_{t+1} = g\mu_t + \frac{(1 - g)\mu_t + g - 1 - r_t}{1 - \lambda} + \frac{\omega_t}{1 - \lambda + \sigma_v^2/\sigma_\omega^2} + \frac{\xi\tau_{t+1} + (1 + r_t)z_t - gz_{t+1}}{1 - \lambda} \quad (12)$$

*and standard deviation  $\sigma_x$  for all  $t$ .*

The impact of the environmental tax on productive capital appears in the denominator of (11): the higher the tax level, the lower the denominator and thus the higher the marginal productivity of  $k_{t+1}$ , i.e. the lower its level. Compared to (10), the GTI dynamic (12) presents an additional term  $(\xi\tau_{t+1} + (1 + r_t)z_t - gz_{t+1})/(1 - \lambda)$ . While this term increases with  $\tau_{t+1}$ , it appears that the subsidy policy has countervailing effects on the dynamic: on the one hand, the current subsidy level  $z_t$  impacts it positively, but that of the next period decreases it. Indeed, the anticipation of a large subsidy in the next period encourages firms to postpone their green investments.

## Environmental standards

Consider now environmental standards, i.e. policy instruments that constrain investment choices. These standards limit the set of machines that can be used or that are allowed to be offered by machine suppliers on the market. They correspond to restrictions on the pollution intensity of the machines that are tightened over time.<sup>18</sup>

<sup>18</sup>This is the case in the European Union which imposes emission standards for vehicles that have evolved in stages of 4 to 5 years (from the Euro 1 standard in 1992 to the Euro 6 standard currently

Denote by  $\underline{x}_t$  the green index targeted for new machines by the government at date  $t$  with a standard policy. Absent network effects, firms would renew their old machines (a proportion  $1 - g$  of their stocks) with new ones that just meet the standard. The corresponding dynamic of firm  $i$ 's technology mix would thus be given by  $x_{it} = gx_{it-1} + (1 - g)\underline{x}_t$ , and summing over all firms, GTI would evolve according to  $\mu_t = g\mu_{t-1} + (1 - g)\underline{x}_t$ . Taking into account networks effects that induce firms to buy the same machines, they have to anticipate an ideal technology mix given by

$$x_t^* = g\mu_{t-1} + (1 - g)\underline{x}_t + \lambda\mathcal{O}_{t-1} + \omega_t. \quad (13)$$

Compared to (5), the index of the most used machines in the last period,  $\mu_{t-1}$ , is replaced by  $g\mu_{t-1} + (1 - g)\underline{x}_t$  in (13) due to the governmental restrictions on new machines. The remaining terms in (13) are the same as in (5): the ideal mix depends on the shock that will affect the service network,  $\omega_t$ , and on its sensitivity  $\lambda$  to total green investment  $\mathcal{O}_{t-1}$ .

**Proposition 3** *With a tax-standard policy  $\{\tau_t, \underline{x}_t\}_{t>0}$ , firms' investments satisfy (11) and*

$$o_{it} = g(\mu_t - x_{it}) + \frac{(1 - g)\underline{x}_t + g - 1 - r_t}{1 - \lambda} + \frac{\eta_{it}}{1 - \lambda + \sigma_v^2/\sigma_\omega^2} + \frac{\xi\tau_{t+1}}{1 - \lambda}.$$

*The resulting firms' technology mixes at  $t + 1$  are normally distributed with mean*

$$\mu_{t+1} = g\mu_t + \frac{(1 - g)\underline{x}_t + g - 1 - r_t}{1 - \lambda} + \frac{\omega_t}{1 - \lambda + \sigma_v^2/\sigma_\omega^2} + \frac{\xi\tau_{t+1}}{1 - \lambda}, \quad (14)$$

*and standard deviation  $\sigma_x$  for all  $t$ .*

The effect of the environmental tax on the firms' productive capacity is the same under the two policy options, given by (11). The tax also produces the same positive effect on the GTI dynamics (14) and (12). Both (12) and (14) show that whatever the implementation chosen by the government, the shock  $\omega_{t+1}$  affects the diffusion of technology, rendering the path of GTI stochastic. Also, under both policy regimes, this uncertainty affects financial markets through the total demand of capital, hence the interest rate  $r_t$ . The main difference between these dynamics is that the term  $(1 - g)\mu_t + (1 + r_t)z_t - gz_{t+1}$  in (12) is replaced by  $(1 - g)\underline{x}_t$  in (14).  $\underline{x}_t$  anchors the path of GTI under the standard policy to the one desired by the social planner, whereas  $\mu_t$  reinforces the path dependency of GTI under the subsidy policy. Moreover, the incentives

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in force for light-duty vehicles).

given by this latter policy depend on  $r_t$ , and are thus subject to its uncertainty (and its path dependency). Therefore, although they have the same first term  $g\mu_t$ , we can expect the path dependence of  $\mu_t$  to be stronger under the tax-subsidy policy than under the tax-standard policy, i.e. the latter policy should be more effective than the former in stimulating the greening of the economy.

### 3 Optimal path of the economy

To better compare these two policy options, I consider in this section and the next their effectiveness in implementing the socially optimal path.<sup>19</sup>

#### 3.1 Consumers and capital markets

Capital supply comes from consumers who maximize their intertemporal utility by arbitraging between consumption and savings each period. Consumers derive well-being from consumption  $c_t$  and environmental quality (EQ)  $e_t$ . Their behavior is modeled by considering a representative consumer whose saving and consumption plans solve

$$\max \mathbb{E}_t \left\{ \sum_{h=0}^{+\infty} \beta^h u(\tilde{c}_{t+h}, \tilde{e}_{t+h}) : \tilde{c}_{t+h} = \tilde{R}_{t+h} + \tilde{r}_{t+h-1} \tilde{S}_{t+h-1} - \tilde{s}_{t+h}, \tilde{s}_{t+h} = \tilde{S}_{t+h} - \tilde{S}_{t+h-1} \right\} \quad (15)$$

each period, where  $u$  is a concave utility function,  $R_t$  is her date- $t$  revenue,  $S_{t-1}$  her savings from the previous period,  $r_{t-1}S_{t-1}$  the corresponding date- $t$  capital earnings,  $s_t$  the savings adjustment of period  $t$ , and  $\beta = \exp(-\psi)$  the intrinsic discount factor corresponding to a psychological discount rate  $\psi > 0$ . Solving (15), we obtain

**Lemma 1** *The consumption plan that solves (15) satisfies*

$$h_t^a \equiv \frac{\partial u(c_t, e_t) / \partial c}{\beta \mathbb{E}_t [\partial u(\tilde{c}_{t+1}, \tilde{e}_{t+1}) / \partial c]} = 1 + r_t \quad (16)$$

at each date  $t$ .

Equation (16) corresponds to the Ramsey-Euler rule which states that  $h_t^a$ , the expected intertemporal rate of substitution in consumption (IRS), is equal to the return

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<sup>19</sup>By considering policies that induce economic paths with the same expectation, we can compare their fluctuations. However, as these paths are stochastic, the policy specifications given in the following are not optimal. Optimal policies take into account the cost of this risk, which leads to paths whose expectations are likely to be different and to differ from the socially optimal path.

on capital each period. It also defines the supply function of capital, while (7) and (10) are the demand side coming from firms.

At equilibrium, aggregate production net of investment must be equal to total consumption of manufactured goods, i.e.

$$c_t = q_t - I_t - \mathcal{O}_t = q(k_t) - (k_{t+1} - gk_t) - (\mu_{t+1} - g\mu_t). \quad (17)$$

Hence, at equilibrium on financial markets, i.e. when the total demand for capital by firms meets the supply provided by consumers, the interest rate is affected by the network shock. Its distribution over time depends on accumulated capital, on the consumer' expectations whose preferences are affected by the quality of the environment, and on her desire to smooth consumption over time. Since productive investment depends on the interest rate, GDP is also affected by these fluctuations and thus follows a random path.

### 3.2 Environmental dynamic

Production generates pollution that deteriorates the quality and the availability of environmental goods and services provided by Nature. These effects are summarized in the dynamic of  $e_t$ , which is given by

$$e_{t+1} = \theta e_t + \hat{e} - \iota_t q_t \quad (18)$$

where  $\theta \in (0, 1)$  is the environmental inertia rate,  $\hat{e}$  the per-period maximum regeneration capacity of the environment, and

$$\iota_t = \int_0^1 q_{it} \iota_{it} di / q_t \quad (19)$$

the emission intensity of the economy at date  $t$ , which measures the total environmental damage per unit of GDP. Without human interference ( $\iota_t = 0$ ),  $e_t$  is at its pristine level  $e_N = \hat{e}/(1 - \theta)$ . More generally, it comes using  $\iota_t = \varphi - \xi \mu_t / q_t$  that the EQ dynamic follows the linear first-order recursive equation

$$e_{t+1} = \theta e_t + \xi \mu_t - \varphi q_t + \hat{e}. \quad (20)$$

### 3.3 The socially optimal path

Consider a social planner whose task is to determine the optimal path of the economy. Assuming that he can force firms to adopt the socially optimal set of machines each period, network effects are irrelevant. As a result, the economy follows a deterministic path to environmental neutrality that corresponds to the First-Best.<sup>20</sup> I suppose however that investment in productive capacity cannot be negative, i.e.<sup>21</sup>

$$k_{t+1} \geq gk_t, \quad (21)$$

and that green technologies can only reduce emissions, i.e.

$$\varphi q(k_t) \geq \xi \mu_t, \quad (22)$$

so that reaching an environmentally neutral path (ENP) is the best situation society can achieve.

In absence of shocks, the planner's problem is to solve

$$\max_{\{\mu_t, k_t, c_t, e_t\}_{t>0}} \left\{ \sum_{t=0}^{+\infty} \beta^t u(c_t, e_t) : (17), (20) - (22) \right\} \quad (23)$$

given  $e_0, k_0$  and  $\mu_0$ . I suppose  $\xi \mu_0 < \varphi q(k_0)$ , so that society is not already on a ENP. Denoting

$$h_t = \frac{\partial u(c_t, e_t) / \partial c}{\beta \partial u(c_{t+1}, e_{t+1}) / \partial c} \quad (24)$$

the date- $t$  IRS, it is shown in the appendix that:

**Proposition 4** *The First-Best path of the economy  $\{\mu_t^*, k_t^*, c_t^*, e_t^*, h_t^*\}_{t>0}$  satisfies (17), (20)–(22),*

$$q'(k_{t+1}^*) \leq \frac{h_t^* - g}{1 - (h_t^* - g)\varphi/\xi}, \quad (25)$$

for all  $t > 0$ , with an equality when (21) is not binding, and

$$h_{t+1}^*(h_t^* - g - \theta) + \theta g = \xi \frac{\partial u(c_{t+2}^*, e_{t+2}^*) / \partial e}{\partial u(c_{t+2}^*, e_{t+2}^*) / \partial c} \quad (26)$$

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<sup>20</sup>While the supporting infrastructure can adapt well in advance to the policy, shocks may still exist, requiring the social planner to revise the policy plan each period. I neglect these shocks in the determination of the optimal policy because they do not cause a coordination problem.

<sup>21</sup>Constraint (21) implies that the social planner cannot forbid the use of machines already bought that are too environmentally damaging.



when (22) is not binding.

Condition (25) indicates that when (21) is not binding, the IRS is larger than the rental rate of capital  $g + q'(k_{t+1})$  due to the impact of production on the environment (i.e. unless  $\varphi = 0$ ). To interpret (26), observe that the impact on EQ of new green investment occurs after two periods. The IRS over these two periods is given by  $h_{t+1}h_t$ . Absent stock effects, i.e.  $g = \theta = 0$ , (26) states that the optimal date- $t$  two-period IRS must be equal to the marginal rate of substitution (MRS) of consumption for EQ at date  $t + 2$ . The regenerative capacity of the environment ( $\theta > 0$ ) and the durability of capital ( $g > 0$ ) allow for larger IRSs, i.e. larger increases in consumption over time.

**Proposition 5** *The tax-subsidy and the tax-standard schemes that implement the First-Best path in expectation satisfy*

(i)  $\tau_{t+1} \leq (h_t^* - g)/\xi$  given by

$$\tau_{t+1} = \begin{cases} (1 - (h_t^* - g)/q'(gk_t^*)) / \varphi & t < t_0 \\ (h_t^* - g)/\xi & t \geq t_0 \end{cases}, \quad (27)$$

(ii)

$$z_t = \frac{(1 - \lambda)\mu_{t+1}^* - (1 - \lambda g)\mu_t^* + gz_{t+1}}{h_t^*} + \frac{h_t^* - g - \xi\tau_{t+1}}{h_t^*} \quad (28)$$

for all  $t > 0$ , with  $\lim_{t \rightarrow \infty} z_t < 0$ , and

(iii)

$$\underline{x}_t = \frac{(1 - \lambda)(\mu_{t+1}^* - g\mu_t^*)}{1 - g} + \frac{h_t^* - g - \xi\tau_{t+1}}{1 - g} \quad (29)$$

for all  $t > 0$ .

Not surprisingly, the emissions tax (27) is the same for both policy implementations. When  $k_{t+1} > gk_t$ , it equates the rental price of capital,  $h_t^* - g = 1 + r_t - g$ , with the marginal savings from investing in green technology,  $\xi\tau_{t+1}$ . When  $k_{t+1} = gk_t$ , the tax is reduced so that firms use all the productive capital that is not depreciated. This reduction is compensated by a large subsidy level or a stringent standard as shown by the last terms in (28) and (29) that are positive when  $t < t_0$  and null when  $t \geq t_0$ .

Proposition 5 also shows that the subsidy schedule entails negative values passed a certain date, i.e. it becomes a tax scheme. This is due to the herd behavior of firms generated by the network effects: firms over-invest in green technology compared to

the First-Best once GTI is high enough. As noted earlier, the expected GTI dynamic under Laissez-Faire is positive if  $\mu_t > [1 + r_t/(1 - g)]/\lambda$ . With environmental taxes (27) and absent a complementary subsidy or standard policy, this condition becomes  $\mu_t > [h_t^* - (1 + r_t)]/[(1 - g)\lambda]$ , i.e.  $\mu_t > 0$  since  $\mathbb{E}[\tilde{r}_t] = h_t^* - 1$ . Hence, green investment could be greater than  $g\mu_t$ , the level necessary to renew the capital depreciated, and firms are very likely to over-invest when  $\mu_t$  is large.

## 4 Illustrative example

To illustrate, assume that  $c_t$  and  $e_t$  can be subsumed in a ‘global wealth index’  $y_t \equiv c_t + pe_t$ , so that the consumer’s MRS is the same whatever the GDP, equal to  $p$ .

**Lemma 2** *With constant MRS,  $\{\mu_t^*, k_t^*, c_t^*, e_t^*, h_t^*\}_{t>0}$  is characterized by:*

(i)  $h_t^* = \bar{h} \equiv (g + \theta + \sqrt{(g - \theta)^2 + 4p\xi})/2 > 1/\beta$  and  $k_t^* \geq \bar{k}$  as long as  $\xi\mu_t^* < q(\bar{k})\varphi$  where

$$\bar{k} \equiv q'^{-1} \left( \frac{\bar{h} - g}{1 - (\bar{h} - g)\varphi/\xi} \right) > 0. \quad (30)$$

(ii) If  $gk_0 > \bar{k}$ ,  $k_{t+1}^* = gk_t^*$  for all  $t < t_0 \equiv \ln(\bar{k}/k_0)/\ln g$ .

(iii)  $k_t^* = \bar{k}$  and  $q_t^* = \bar{q} \equiv q(\bar{k})$  for all  $t_0 \leq t \leq T+1$  where  $T$  is given by  $\mu_T^* = q(\bar{k})\varphi/\xi$ .

Assuming moreover an exponential consumer’s utility, i.e.  $u(c_t, e_t) = -e^{-\gamma(c_t + pe_t)}$  and a Cobb-Douglas production function  $q_t = Ak_t^\alpha$ , we obtain

**Proposition 6** *With constant MRS, exponential utility function and a Cobb-Douglas production function,*

(i) *the First-Best sequence of the EQ index  $\{e_t^*\}_{t>1}$  is given by*

$$e_t^* = e_1 + \left( \frac{g^\alpha - g^{\alpha t}}{1 - g^\alpha} - \left( \frac{g^\alpha}{\bar{h}} \right)^{t_0-1} \frac{\bar{h}^t - \bar{h}}{\bar{h} - 1} \right) \nu_0 + (t-1)\nu_1 - \frac{(1-\theta)(1-g)\nu_1}{\xi p \bar{h}^T} \frac{\bar{h}^t - \bar{h}}{\bar{h} - 1}$$

for all  $1 < t < t_0$ ,

$$e_t^* = e_{t_0-1}^* + (t - t_0 + 1)\nu_1 - \frac{(1-\theta)(1-g)\nu_1}{\xi p \bar{h}^T} \frac{\bar{h}^t - \bar{h}^{t_0-1}}{\bar{h} - 1} \quad (31)$$

for all  $t_0 \leq t \leq T$ , and

$$e_t^* = e_N - \theta^{t-T}(e_N - e_T^*)$$

Table 1: Calibration parameters and equilibrium values

$q_0$	Pop	$r_0$	$g$	$\alpha$	$E_0$	$\iota_0$	$\xi$	$\varphi$	$\mu_0$	$A$
101.33	8	0.06	$4/5$	$1/3$	36.8	0.363	0.84	0.84	57.52	20.01
CO <sub>20</sub>	CO <sub>2N</sub>	TP	$\theta$	$\hat{e}$	$e_0$	$e_N$	$p$	$\gamma$	$\psi$	
417.9	280	450	0.985	19.81	249.42	1,320.9	$15/1000$	0.19	0.015	
$k_0$	$\bar{h}$	$\bar{k}$	$\bar{q}$	$\bar{\tau}$	$t_0$	$T$	$k_\infty$	$q_\infty$	$\lambda$	$\sigma_\omega$
129.91	1.038	98.71	92.46	0.283	2	64	120.04	98.69	0.005	$1/2$

Date  $t = 0$  is 2022. GDP and capital stocks ( $q_t$ ,  $k_t$ ,  $\mu_t$ ) are expressed in US\$ trillions, world population in billions, CO<sub>2</sub> concentrations (Tipping Point TP, CO<sub>20</sub>, CO<sub>2N</sub>) in ppm, 2022 emissions  $E_0$  and carbon budgets ( $e_0$ ,  $e_N$ ,  $\hat{e}$ ) in Gt of CO<sub>2</sub>. Emission intensities  $\iota_0$  and  $\varphi$  in kg of CO<sub>2</sub> for US\$1.  $\varphi$  corresponds to the emission intensity for 1960 (as estimated by the World Bank).

for all  $t > T$ , where  $\nu_1 > 0$ ,  $\nu_0 > 0$  if  $\xi/\varphi \geq 1 - g$ .

(ii) EQ increases and consumption decreases at decreasing rates for all  $t \in \{1, \dots, T\}$ .

The dynamic of the economy entails three sequences, the first being characterized by  $k_{t+1}^* = gk_t^*$ , the second by  $k_t^* = \bar{k}$ , and the third by  $\varphi q(k_t^*) = \xi \mu_t^*$  (which corresponds to the ENP). Because the IRS is constant before reaching the ENP, the increase in global wealth is constant: we have  $\bar{h} = e^{\gamma(y_{t+1} - y_t)}/\beta$  and thus  $c_{t+1}^* - c_t^* + p(e_{t+1}^* - e_t^*)$  is constant. However, while EQ is increasing, consumption decreases. This is due to investment in green technologies, as investment in productive capital is either zero or constant (equal to  $g\bar{k}$ ) over this period. Hence, during the first two sequences, when the GDP decreases ( $1 \leq t < t_0$ ) and when it is stabilized at level  $q(\bar{k})$  ( $t_0 \leq t \leq T$ ), consumption decreases.

These results are illustrated in Fig. 1 using the calibration parameters and equilibrium values presented Table 1.<sup>22</sup> Panel 1a shows that  $e_t^*$  increases rapidly until  $t = T$  and then increases at a lower rate toward  $e_N$ . Panel 1b shows the sharp decrease in  $k_t^*$  and the sharp increase in  $\mu_t^*$  from period 0 to period 1 (from  $k_0 = 129.91$  to  $\bar{k} = 98.71$  and  $\mu_0 = 57.52$  to  $\mu_1^* = 86.33$ , respectively). Then,  $k_t^* = \bar{k}$  until  $t = T$ , while  $\mu_t^*$  increases. Both capital stocks increase afterward, at the slow rate permitted

<sup>22</sup>The reference year  $t = 0$  is 2022. The corresponding world GDP and population are  $q_0 = \text{US\$}101.33$  trillions and  $\text{Pop} = 8$  billions.  $p$  is set at  $\text{US\$}15/\text{tCO}_2$ .  $k_0$  is derived from  $k_0 = \alpha q_0 / (1 + r_0 - g)$  where  $r_0 = 6\%$ .  $e_t$  is defined as a global ‘‘carbon budget’’ at date  $t$  (expressed in Gt of CO<sub>2</sub> in the atmosphere), i.e. the difference between a tipping point (TP) and the level of GHG at date  $t$  expressed in CO<sub>2</sub> equivalent.  $e_0 = 249.42$  Gt CO<sub>2</sub>. Given a pre-industrial level  $\text{CO}_{2N} = 280$  ppm, the pre-industrial budget is  $e_N = 1321$  Gt CO<sub>2</sub>.  $\theta$  is set at .985, leading to  $\hat{e} = (1 - \theta)e_N = 19.81$  Gt CO<sub>2</sub>. The initial emission intensity  $\iota_0$  corresponds to the ratio  $\text{CO}_{20}/q_0 = 363$  g CO<sub>2</sub>/US\$, leading to  $\mu_0 = q_0(\phi - \iota_0)/\xi = \text{US\$}57.52$  trillions.

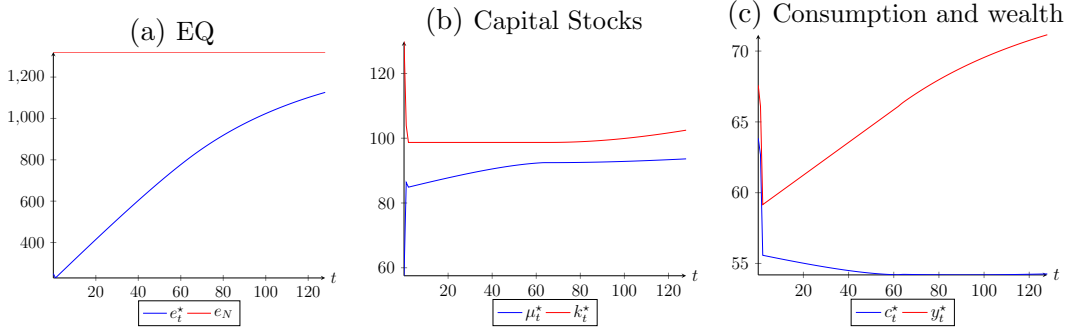


Figure 1: Optimal dynamic.

by the ENP constraint (the long term level of productive capital, which determines  $\mu_\infty = q_\infty = 98.69$ , is  $k_\infty = 120.04$ ). Panel 1c shows the decrease in  $c_t^*$ , very sharp at first (from  $c_0 = 63.84$  to  $c_2^* = 55.58$ ), then slower until period  $T$ . It increases afterward, at a very slow pace. Total wealth also decreases the first two periods, but increases sharply afterward.<sup>23</sup>

The policy schemes (27)–(29) are illustrated in Fig. 2 assuming  $\lambda = .005$ .  $\bar{\tau} = .283$  (i.e. 283 US\$/t CO<sub>2</sub>) over the period  $t_0 < t < T$ , and then slowly decreases. The subsidy scheme (panel 2a) is around 34% at  $t_0 = 2$  and decreases to become negative at  $t = 45$ . As shown panel 2b, the net impact of the tax-subsidy policy on GTI, i.e. the term  $\xi\tau_{t+1} + h_t z_t - g z_{t+1}$  in (12), is decreasing but positive over this period (it stays positive and increases slightly thereafter). Simulations show that the subsidy scheme is extremely sensitive to  $\lambda$  (the schedule is positive for only a decade when  $\lambda = .01$ , and entire negative when  $\lambda \geq .02$ ). The standard policy depicted panel 2c increases rapidly until environmental neutrality is reached, then slowly along the ENP.

Assuming rational expectations, it is possible to be more specific about the dynamic of the economy for  $t \in \{t_0, \dots, T\}$  under these policy implementations. Indeed, over this period the interest rate satisfies  $\tilde{r}_t = \bar{r}(1 + \tilde{\varepsilon}_t)$  where  $\bar{r} = \bar{h} - 1$  is the expected interest rate and  $\tilde{\varepsilon}_t$  is a random shock whose distribution depends on the present and past realizations of  $\tilde{\omega}_t$ .  $k_t$  and  $q_t$  are also randomly distributed around their stationary values  $\bar{k}$  and  $q(\bar{k})$ . Using linear approximations, the path of  $y_t$  can be approximated by a Gaussian random walk at the rational expectations equilibrium (REE), and the

<sup>23</sup>These simulations show the limit of the assumption of a constant MRS: the decreases in GDP and consumption are very sharp due to the perfect substitutability between  $c_t$  and  $e_t$ . It is however possible to relax this assumption (e.g. by allowing  $p$  to take several discrete values depending on  $c_t/e_t$ ) so that the productive capital increases during the transition period to an ENP.

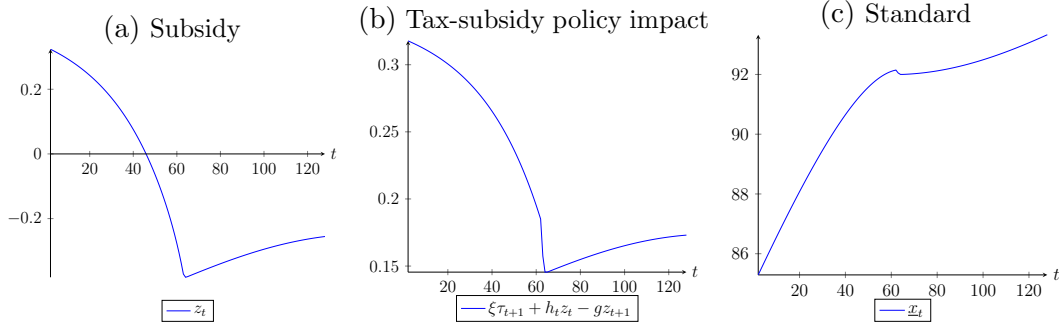


Figure 2: Tax and subsidy schemes.

supply function of capital (16) by<sup>24</sup>

$$r_t = \psi + \gamma(\mathbb{E}_t[\tilde{y}_{t+1}] - y_t) - \gamma^2 \mathbb{V}_t[\tilde{y}_{t+1}]/2. \quad (32)$$

Expression (32) exhibits the familiar effects that determine the rental price of capital: the intrinsic preference for an immediate consumption  $\psi$ , the economic trend of the global wealth index that also encourages immediate consumption if it is positive, and a precautionary effect that operates in the opposite direction and corresponds to a risk premium due to the uncertainty affecting the economy.

**Proposition 7** *Assuming a REE with constant MRS, a CARA utility function and a Cobb-Douglas production function, the interest rate under either the tax-subsidy ( $\ell = sub$ ) or the tax-standard ( $\ell = std$ ) policy is approximated by  $\tilde{r}_t^\ell = \bar{r}(1 + \tilde{\varepsilon}_t^\ell)$  for  $t \in \{t_0, \dots, T\}$ , where*

$$\begin{aligned} \tilde{\varepsilon}_t^{sub} &= \chi^{sub}(z_t)\tilde{\omega}_{t+1} + \rho^{sub}(z_t)\varepsilon_{t-1}^{sub}, \\ \tilde{\varepsilon}_t^{std} &= \chi^{std}\tilde{\omega}_{t+1} + \rho^{std}\varepsilon_{t-1}^{std}, \end{aligned} \quad (33)$$

are normally distributed. The paths of  $\tilde{\varepsilon}_t^\ell$ ,  $\tilde{\mu}_t^\ell$  and  $\tilde{y}_t^\ell$  under policy  $\ell \in \{sub, std\}$ , can be approximated by Gaussian random walks deduced from (17), (20), and

$$\mu_{t+1}^{sub} = a_1^{sub}\mu_t^{sub} + a_2^{sub}e_t^{sub} + a_3^{sub}(z_t) + Z_t^{sub} + b_1^{sub}(z_t)\varepsilon_t^{sub} + b_2^{sub}\varepsilon_{t-1}^{sub} \quad (34)$$

<sup>24</sup>This expression is derived using  $1 + r_t \approx e^{r_t}$ , and  $\mathbb{E}[e^{-\gamma\tilde{y}}] = e^{-\gamma(\mathbb{E}[\tilde{y}] - \gamma\mathbb{V}[\tilde{y}]/2)}$  when  $\tilde{y}$  is normally distributed.  $\gamma$  corresponds to both the coefficient of absolute risk aversion and the inverse of the intertemporal elasticity of substitution.

Table 2: REE coefficients

$a_0^\ell$	$a_1^\ell$	$a_2^\ell$	$\bar{a}_3^\ell$	$\bar{b}_1^\ell$	$b_2^\ell$	$\bar{\rho}^\ell$	$\bar{\chi}^\ell$	$\bar{\sigma}_{y_{t+1 t}}^\ell$
0.870	0.999	4.513 <sub>-5</sub>	-5.019 <sub>-2</sub>	33.824	-32.886	0.971	0.030	1.515 <sub>-1</sub>
0.842	0.798	4.338 <sub>-5</sub>	-4.857 <sub>-2</sub>	27.065	-26.296	0.970	0.037	6.398 <sub>-2</sub>

Values and average values of the coefficients of (33)–(37) under the tax-subsidy ( $\ell = sub$ , first row) and the tax-standard ( $\ell = std$ , second row) policies, with  $\bar{a}_3^{sub} \equiv \sum_{t=t_0}^T a_3^{sub}(z_t)/(T-t_0+1)$ ,  $\bar{a}_3^{std} \equiv a_3^{std}$  and similarly for  $\bar{b}_1^\ell$ ,  $\bar{\rho}^\ell$ ,  $\bar{\chi}^\ell$  and  $\bar{\sigma}_{y_{t+1|t}}^\ell$ . The last column corresponds to the average value of the one-period-ahead standard deviation of the aggregate wealth. Subscripts correspond to exponents (i.e.  $2.3_{-2} = 2.3 \cdot 10^{-2}$ ).

with

$$Z_t^{sub} = a_0^{sub} \sum_{i=0}^{+\infty} (a_0^{sub} \gamma)^i (\xi \tau_{t+1+i} + h_{t+i} z_{t+i} - g z_{t+1+i}), \quad (35)$$

and

$$\mu_{t+1}^{std} = a_1^{std} \mu_t^{std} + a_2^{std} e_t^{std} + a_3^{std} + Z_t^{std} + b_1^{std} \varepsilon_t^{std} + b_2^{std} \varepsilon_{t-1}^{std} \quad (36)$$

with

$$Z_t^{std} = a_0^{std} \sum_{i=0}^{+\infty} (a_0^{std} \gamma)^i (\xi \tau_{t+1+i} + (1-g) \underline{x}_{t+i}). \quad (37)$$

Eq. (33) shows that  $\tilde{r}_t^\ell$  follows a first-order autoregressive process where  $\tilde{\omega}_t$  is the innovation, with constant parameters under the tax-standard policy, and parameters function of  $z_t$  under the tax-subsidy policy. Similarly, (34) and (36) are linear first-order recursive equations with constant parameter in the latter, and some parameters ( $a_3^{sub}$  and  $b_1^{sub}$ ) function of  $z_t$  in the former. Both include forward looking term  $Z_t^\ell$  given by either (35) or (37) that is an exponential smoothing of future tax and subsidy levels or future tax and standard levels respectively.

Coefficient values are reported Table 2.<sup>25</sup> As  $\bar{\rho}^{sub} > \rho^{std}$  and  $\chi^{std} > \bar{\chi}^{sub}$ , the autocorrelation of  $\tilde{\varepsilon}_t^\ell$  is larger, and the impact of innovations lower, under the tax-subsidy policy than under the tax-standard policy. Also, as  $a_1^{sub} > a_1^{std}$ , the path dependency of  $\tilde{\mu}_t^{sub}$  is higher than the one of  $\tilde{\mu}_t^{std}$ . The innovation coefficients  $\bar{b}_1^\ell$  and  $b_2^\ell$  have large absolute values of opposite signs. As a result,  $\sigma_{y_{t+1|t}}^\ell$ , the volatility of  $\tilde{y}_t^\ell$  from period to period, is small, but larger under the tax-subsidy policy than under the tax-standard policy.

The distributions of  $\tilde{\varepsilon}_t^\ell$  and  $\tilde{r}_t^\ell$  are illustrated in Fig. 3. In panel 3a, the realizations of  $\tilde{\omega}_t$  are randomly distributed around 0, while  $\tilde{\varepsilon}_t^\ell$  fluctuates somehow smoothly and stay

<sup>25</sup>  $\rho_1^{sub}(z_t)$  and  $\chi^{sub}(z_t)$  are slightly increasing while  $a_3^{sub}(z_t)$ ,  $b_1^{sub}(z_t)$  and  $\sigma_{y_{t+1|t}}^{sub}$  are slightly decreasing. All stay very close to their average values reported Table 2.

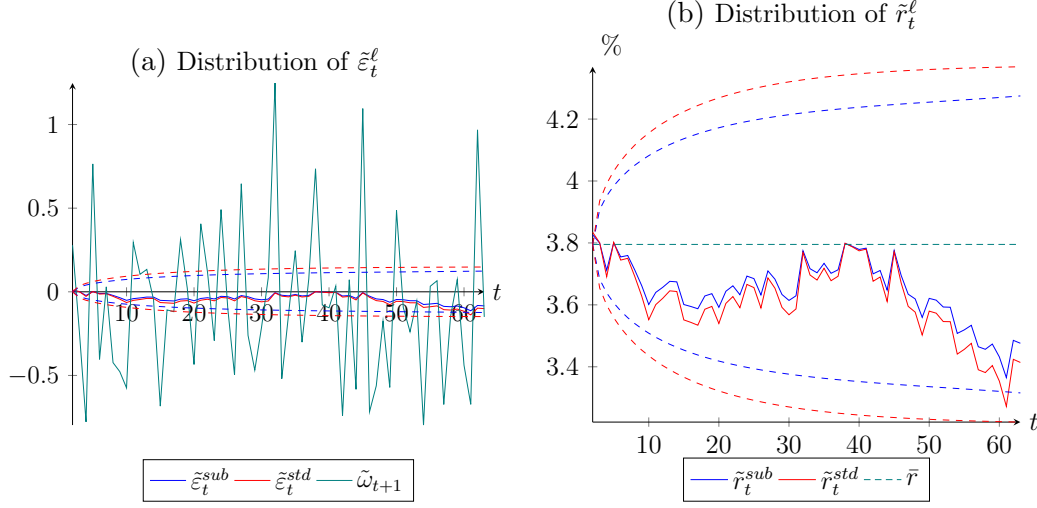


Figure 3: Shocks and the interest rate (95% CIs are delineated by the dashed black lines).

below  $\varepsilon_{t_0}^\ell = 0$  under both policies. The large autocorrelation coefficient  $\rho^\ell$  compared to the low impact of the innovation  $\chi^\ell$  explain this path dependency of  $\tilde{\varepsilon}_t^\ell$  under both policies. As a result,  $\tilde{r}_t^\ell$  departs from  $\bar{r} = \bar{h} - 1 = 3.8\%$  following the same variations as  $\tilde{\varepsilon}_t^\ell$  (panel 3b), the amplitude of its path under the tax-standard policy being often larger than under the tax-subsidy policy due to  $\chi^{std} > \bar{\chi}^{sub}$ . The dashed curves above and below the horizontal lines located at  $\varepsilon_{t_0}^\ell = 0$  and  $\bar{r}$  delineate the 95% confidence interval (CI) deduced from the Gaussian distributions  $\mathcal{N}(0, \mathbb{V}[\tilde{\varepsilon}_t^\ell])$  (panel 3a) and  $\mathcal{N}(\bar{r}, \bar{r}^2 \mathbb{V}[\tilde{\varepsilon}_t^\ell])$  (panel 3b). Because  $\chi^{std} > \bar{\chi}^{sub}$ , the CI is larger at first under the tax-standard policy than under the tax-subsidy policy. This effect is progressively attenuated due to the difference in the autocorrelation coefficients ( $\rho^{sub}$  is larger than  $\rho^{std}$  on average and increasing). CIs for the paths of  $\tilde{\mu}_t^\ell, \tilde{\varepsilon}_t^\ell$  and  $\tilde{q}_t^\ell$  over  $\{t_0, \dots, T\}$  are derived from (20) and (34)-(37) using

$$\tilde{Y}_t^\ell = B_t^\ell Y_{t-1}^\ell + H_t^\ell \tilde{\nu}_t \quad (38)$$

where  $\tilde{Y}_t^\ell = (\tilde{\mu}_t^\ell, \tilde{\varepsilon}_t^\ell, \tilde{q}_t^\ell, 1, \tilde{\varepsilon}_t^\ell, \tilde{\varepsilon}_{t-1}^\ell)'$ ,

$$B_t^\ell = \begin{bmatrix} a_1^\ell & a_2^\ell & 0 & a_{3t}^\ell + Z_t^\ell & b_{1t}^\ell & b_2^\ell \\ \xi & \theta & -\varphi & \hat{e} & 0 & 0 \\ 0 & 0 & 0 & \bar{q} & q'(\bar{k})/q''(\bar{k}) & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \rho_t^\ell & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad H_t^\ell = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \chi_t^\ell \sigma_\omega \\ 0 \end{bmatrix},$$

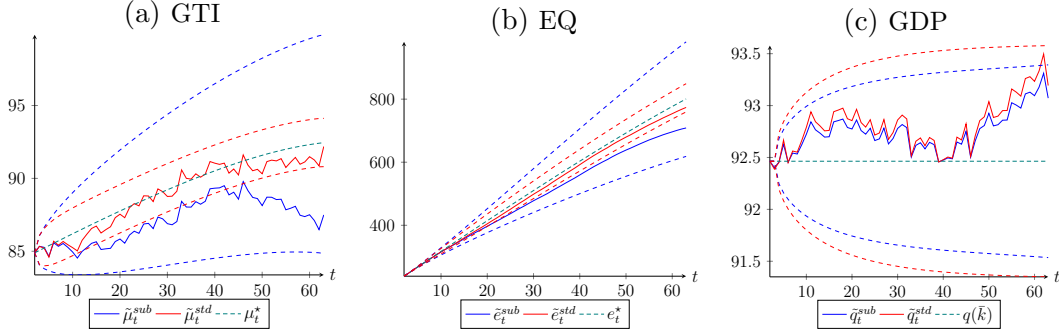


Figure 4: GTI, EQ and GDP under the two policies (95% CIs are delineated by the dashed blue (red) lines under the tax-subsidy (tax-standard) implementation).

and where  $\tilde{\nu}_t$  is an independent standardized Gaussian variable.  $\tilde{Y}_t^\ell$  follows a Gaussian random walk with  $\mathbb{E}[\tilde{Y}_t^\ell] = (\prod_{i=0}^{t-t_0} B_i^\ell) Y_{t_0}$  and  $\mathbb{V}[\tilde{Y}_t^\ell] = \sum_{i=0}^{t-t_0} (\prod_{j=0}^i B_j^\ell) H_\ell H_\ell' (\prod_{j=0}^i B_j^\ell)'$ .

Fig. 4 depicts the paths and the CIs of GTI, EQ and GDP. While  $\tilde{\mu}_t^{std}$  fluctuates around  $\mu_t^*$ ,  $\tilde{\mu}_t^{sub}$  do not show this positive trend (panel 4a). The consequence on EQ is apparent (panel 4b): the stochastic path of  $\tilde{e}_t^{std}$  is much closer to  $e_t^*$  than  $\tilde{e}_t^{sub}$ . Also, there is a striking difference in the magnitude of the CIs for these variables: The CI for the tax-subsidy policy delineated by the dashed blue curves are much larger than those for the tax-standard policy (dashed red curves). Panel 4c,  $\tilde{q}_t^{sub}$  and  $\tilde{q}_t^{std}$  stay (relatively) close. This is because  $\tilde{k}_t^\ell$  is not directly affected by  $\tilde{\omega}_t$ , but only by  $\tilde{r}_t^\ell$ .

## 5 Conclusion

This paper analyzes the effect of coordination problems on the adoption of green technologies. Governments design policies to drive their economies toward environmental neutrality, but investment choices are ultimately made by private agents who respond to policies according to their own expectations about the future of the economy. The effectiveness of green technologies is partly the result of their choices, through complex network and spillover effects. As a result, regulatory instruments can only imperfectly guide economies on their paths to environmental neutrality. By modeling the coordination problem of firms as a succession of global games, the paths of the economy that are derived in this paper are not subject to arbitrary assumptions about firms' self-fulfilling beliefs. Instead, it is assumed that agents are aware that the development and diffusion of technologies are subject to shocks that they imperfectly anticipate. While the resulting equilibrium path does not depend on firms' beliefs, it is indirectly affected by these shocks and the equilibrium conditions in real and financial markets.



Although highly stylized, the model presented in this article shows that these shocks cause economic fluctuations that can move the economy away from the First-Best path, or any path envisioned by the authorities. Technology standards permit a transition to green technologies more in line with the regulator's intended path than subsidies do. This is because producers have less flexibility to adapt their green investments to economic conditions when standards rather than subsidies are implemented. The latter instrument allows firms to better react to changes on financial markets, for better or for worse in terms of the resulting environmental quality.

The First-Best path of the economy is derived assuming that governments know the technologies that make such a path feasible, which is unrealistic. Environmental policies are based on scenarios that depend heavily on assumptions about how technologies will evolve –both in terms of performance and cost– in the distant future. As noted by the IEA (2021), its model's forecasts based on already existing technologies have a time horizon of about a decade. Forecasts beyond 2030 (and up to 2070) rely on assumptions about the evolution of green technologies. These assumptions are based on the opinion of experts who cannot have complete knowledge of all the technologies that will be deployed, but only indications of those that are under development, at the research stage or in demonstration projects. The likely evolution of these technologies, in terms of cost and performance, is subject to many uncertainties. Determining the appropriate policy is therefore a very difficult task. Technology standards, by imposing minimum environmental quality characteristics that machines must meet, seem to be the most appropriate instrument for achieving binding objectives such as those imposed by international environmental agreements. And indeed, emission standards policies such as those implemented by the European Union for vehicles, which oblige car manufacturers to produce low-emission models, seem to be more effective in guiding society towards environmental neutrality than have been the economic incentives given to car buyers to date.

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# Appendix

## A Proof of Proposition 1

As firm  $i$ 's profit is separable in revenues from productive capacity and network effects from its green index, the firm's investment problem can be split into two independent programs. Applying the principle of optimality, the investment strategy in productive capacity can be derived by using the Bellman equation

$$\mathcal{W}(k_{it}) = \max_{I_{it}, \ell_{it}} Q(k_{it}, \ell_{it}) - w\ell_{it} - I_{it} + \delta_t \mathbb{E}_t[\mathcal{W}(gk_{it} + I_{it})]. \quad (39)$$

Maximizing (39) with respect to  $\ell_{it}$  gives

$$\partial Q(k_{it}, \ell_{it})/\partial \ell = w_t,$$

while the first-order condition with respect to  $I_{it}$  leads to

$$-1 + \delta_t \mathbb{E}_t[\mathcal{W}'(gk_{it} + I_{it})] = 0. \quad (40)$$

The envelop condition yields

$$\mathcal{W}'(k_{it}) = \partial Q(k_{it}, \ell_{it})/\partial k + g\delta_t \mathbb{E}_t[\mathcal{W}'(gk_{it} + I_{it})]$$

implying using (40)  $\mathcal{W}'(k_{it}) = \partial Q(k_{it}, \ell_{it})/\partial k + g$ . Plugging this expression evaluated for period  $t + 1$  in (40) yields

$$\partial Q(gk_{it} + I_{it}, \ell_{it+1})/\partial k = 1 + r_t - g.$$

As  $Q$  is homogeneous of degree 1, we thus get  $q'(k_{it+1}/\ell_{it+1}) = 1 + r_t - g$  where  $q(k) \equiv Q(k, 1)$  is an increasing and concave function. Inverting, it comes that  $k_{it+1}/\ell_{it+1} = q'^{-1}(1 + r_t - g)$ , and using  $\int_0^1 \ell_{it} di = \ell_t = 1$ ,  $k_{it}/\ell_{it} = k_t/\ell_t = k_t$  for all  $i$  and  $t$ ,  $k_{t+1} = q'^{-1}(1 + r_t - g)$  which gives (7). Using  $w_t = \partial Q(k_{it}, \ell_{it})/\partial \ell = d[\ell_{it}q(k_{it}/\ell_{it})]/d\ell_{it} = q(k_{it}/\ell_{it}) - q'(k_{it}/\ell_{it})k_{it}/\ell_{it}$  yields  $w_t = q_t - (1 + r_{t-1} - g)k_t$ .

Neglecting the constants in (6), the Bellman equation corresponding to the technology mix is given by

$$\mathcal{V}(x_{it}; x_t^*) = \max_{o_{it}} -(x_{it} - x_t^*)^2/2 - o_{it} + \delta_t \mathbb{E}[\mathcal{V}(gx_{it} + o_{it}; \tilde{x}_{t+1}^*) | \eta_{it}, \mu_t, r_t]. \quad (41)$$

Maximizing (41) with respect to  $o_{it}$  leads to

$$-1 + \delta_t \mathbb{E} [\partial \mathcal{V}(gx_{it} + o_{it}; \tilde{x}_{t+1}^*) / \partial x | \eta_{it}, \mu_t, r_t] = 0. \quad (42)$$

The envelop condition yields

$$\partial \mathcal{V}(x_{it}; x_t^*) / \partial x = x_t^* - x_{it} + g \delta_t \mathbb{E} [\partial \mathcal{V}(gx_{it} + o_{it}; \tilde{x}_{t+1}^*) / \partial x | \eta_{it}, \mu_t, r_t] \quad (43)$$

implying  $\partial \mathcal{V}(x_{it}; x_t^*) / \partial x = x_t^* - x_{it} + g$ . Using this expression for  $t + 1$  in (42) yields

$$1 + r_t = \mathbb{E} [\tilde{x}_{t+1}^* - (gx_{it} + o_{it}) + g | \eta_{it}, \mu_t, r_t] = \mathbb{E} [\tilde{x}_{t+1}^* | \eta_{it}, \mu_t, r_t] - (gx_{it} + o_{it}) + g,$$

which gives (8). Following Angeletos & Pavan (2004), the resulting investment strategy is linear in the variables observed by the firm at date  $t$ , i.e. it is given by

$$o(\mu_t, \eta_{jt}, x_{jt}, r_t) = \beta_1 \mu_t + \beta_2 \eta_{jt} + \beta_3 x_{jt} + \beta_4 r_t + \beta_5 \quad (44)$$

where the coefficients  $\beta_k, k = 1, \dots, 5$ , are derived as follows. On average, as  $\int_0^1 \eta_{jt} dj = \omega_{t+1}$  and  $\int_0^1 x_{jt} dj = \mu_t$ , we have

$$\int_0^1 o(\mu_t, \eta_t, \varepsilon_{jt}, x_{jt}, r_t) dj = (\beta_1 + \beta_3) \mu_t + \beta_2 \omega_{t+1} + \beta_4 r_t + \beta_5$$

and thus, from (5),

$$x_{t+1}^* = \mu_t + \lambda \int_0^1 o_{it} di + \omega_{t+1} = [1 + \lambda(\beta_1 + \beta_3)] \mu_t + (1 + \lambda\beta_2) \omega_{t+1} + \lambda\beta_4 r_t + \lambda\beta_5.$$

Using (8) and  $\mathbb{E}[\tilde{\omega}_{t+1} | \eta_{it}] = \eta_{it} \sigma_\omega^2 / (\sigma_\omega^2 + \sigma_v^2)$ , we get

$$\begin{aligned} o_{it} &= \mathbb{E}[\tilde{x}_{t+1}^* | \eta_{it}, \mu_t, r_t] - 1 + g - gx_{it} - r_t \\ &= [1 + \lambda(\beta_1 + \beta_3)] \mu_t + (1 + \lambda\beta_2) \eta_{it} \sigma_\omega^2 / (\sigma_\omega^2 + \sigma_v^2) - (1 - \lambda\beta_4) r_t + \lambda\beta_5 - 1 + g - gx_{it}. \end{aligned}$$

Identifying with (44) yields  $\beta_3 = -g$ ,  $\beta_1 = 1 + \lambda(\beta_1 + \beta_3) = (1 - \lambda g) / (1 - \lambda)$ ,  $\beta_2 = (1 + \lambda\beta_2) \sigma_\omega^2 / (\sigma_\omega^2 + \sigma_v^2) = 1 / (1 - \lambda + \sigma_v^2 / \sigma_\omega^2)$ ,  $\beta_4 = -1 + \lambda\beta_4 = -1 / (1 - \lambda)$ ,  $\beta_5 = -(1 - g) / (1 - \lambda)$ , hence (9). Consequently,

$$x_{it+1} = gx_{it} + o_{it} = \frac{(1 - \lambda g) \mu_t + \varpi \eta_{it} + g - 1 - r_t}{1 - \lambda},$$

where  $\varpi \equiv (1 - \lambda)/(1 - \lambda + \sigma_v^2/\sigma_\omega^2)$ . Integrating, it comes

$$\mu_{t+1} = \frac{(1 - \lambda g)\mu_t + \varpi\omega_{t+1} + g - (1 + r_t)}{1 - \lambda}$$

using  $\int_0^1 \eta_{it} di = \omega_{t+1}$ . Re-organizing terms gives (10). As idiosyncratic investments depend on signals that are normally distributed,  $x_{it+1}$  is normally distributed around  $\mu_{t+1}$  with variance  $\mathbb{V}[x_{it+1}] = \sigma_v^2/(1 - \lambda + \sigma_v^2/\sigma_\omega^2)^2 \equiv \sigma_x^2$ .

## B Proof of Proposition 2

The problem of firm  $i$  is to solve

$$\mathcal{V}(k_{it}, x_{it}; x_t^*) = \max_{I_{it}, \ell_{it}, o_{it}} \pi_{it} + \delta_t \mathbb{E}[\mathcal{V}(gk_{it} + I_{it}, gx_{it} + o_{it}; \tilde{x}_{t+1}^*) | \eta_{it}, \mu_t, r_t]$$

where

$$\pi_{it} \equiv Q(k_{it}, \ell_{it}) - w_t \ell_{it} - I_{it} - (x_{it} - x_t^*)^2/2 - o_{it}(1 - z_t) - \tau_t (\varphi Q(k_{it}, \ell_{it}) - \xi x_{it}).$$

Using the same steps as those described in the proof of Proposition 1, it comes (11) and

$$o_{it} = \mathbb{E}[\tilde{x}_{t+1}^* | \eta_{it}, \mu_t, r_t] - gx_{it} + \xi \tau_{t+1} + g(1 - z_{t+1}) - (1 + r_t)(1 - z_t).$$

At equilibrium, we get

$$o_{it} = \frac{1}{1 - \lambda} [(1 - \lambda g)\mu_t + \varpi\eta_{it} + \xi\tau_{t+1} + g(1 - z_{t+1}) - (1 + r_t)(1 - z_t)] - gx_{it},$$

where  $\varpi \equiv (1 - \lambda)/(1 - \lambda + \sigma_v^2/\sigma_\omega^2)$ , hence

$$x_{it+1} = gx_{it} + o_{it} = \frac{1}{1 - \lambda} [(1 - \lambda g)\mu_t + \varpi\eta_{it} + \xi\tau_{t+1} + g(1 - z_{t+1}) - (1 + r_t)(1 - z_t)].$$

Summing over all firms and using  $\int_0^1 \eta_{it} di = \omega_{t+1}$  gives (12).

## C Proof of Proposition 3

Using

$$\pi_{it} = Q(k_{it}, \ell_{it}) - w_t \ell_{it} - I_{it} - (x_{it} - x_t^*)^2/2 - o_{it} - \tau_t (\varphi Q(k_{it}, \ell_{it}) - \xi x_{it})$$

where  $\tilde{x}_{t+1}^*$  is given by (13), it comes

$$o_{it} = o_t(\mu_t, \underline{x}_t, \eta_{it}, x_{it}, r_t) \equiv \mathbb{E} [\tilde{x}_{t+1}^* | \eta_{it}, \mu_t, \underline{x}_t, r_t] - gx_{it} + \xi\tau_{t+1} + g - 1 - r_t. \quad (45)$$

Assuming linearity, i.e.

$$o_t(\mu_t, \eta_{it}, x_{it}, r_t) = \beta_0 \underline{x}_t + \beta_1 \mu_t + \beta_2 \eta_{it} + \beta_3 x_{it} + \beta_4 r_t + \beta_5 + \beta_6 \tau_{t+1}, \quad (46)$$

which leads to

$$\int_0^1 o_{jt} dj = \beta_0 \underline{x}_t + (\beta_1 + \beta_3) \mu_t + \beta_2 \omega_t + \beta_4 r_t + \beta_5 + \beta_6 \tau_{t+1},$$

and substituting in (13), we arrive at

$$\begin{aligned} x_{t+1}^* &= g\mu_t + (1-g)\underline{x}_t + \lambda \int_0^1 o_{jt} dj + \omega_{t+1} \\ &= (1-g + \lambda\beta_0)\underline{x}_t + [g + \lambda(\beta_1 + \beta_3)]\mu_t + \lambda\beta_4 r_t + \lambda\beta_5 + \beta_6 \tau_{t+1} + (1 + \lambda\beta_2)\omega_{t+1}. \end{aligned}$$

Using (45) and  $\mathbb{E}[\tilde{\omega}_{t+1} | \eta_{it}] = \eta_{it}/(1 + \sigma_v^2/\sigma_\omega^2)$  we get

$$\begin{aligned} o_t(\mu_t, \eta_{it}, x_{it}, r_t) &= (1-g + \lambda\beta_0)\underline{x}_t + [g + \lambda(\beta_1 + \beta_3)]\mu_t + (\lambda\beta_4 - 1)r_t + \lambda\beta_5 + (\lambda\beta_6 + \xi)\tau_{t+1} \\ &\quad + \eta_{it}(1 + \lambda\beta_2)/(1 + \sigma_v^2/\sigma_\omega^2) - gx_{it} + g - 1. \end{aligned}$$

Identifying with (46) yields  $\beta_2 = 1/(1 - \lambda + \sigma_v^2/\sigma_\omega^2)$ ,  $\beta_0 = (1-g)/(1-\lambda)$ ,  $\beta_1 = g$ ,  $\beta_3 = -g$ ,  $\beta_4 = -1/(1-\lambda)$ ,  $\beta_5 = (g-1)/(1-\lambda)$ ,  $\beta_6 = \xi/(1-\lambda)$ . It comes

$$o_{it} = g(\mu_t - x_{it}) + \frac{(1-g)\underline{x}_t + \varpi\eta_{it} - 1 - r_t + g + \xi\tau_{t+1}}{1-\lambda},$$

where  $\varpi \equiv (1-\lambda)/(1-\lambda + \sigma_v^2/\sigma_\omega^2)$ , hence

$$x_{it+1} = gx_{it} + o_{it} = g\mu_t + \frac{(1-g)\underline{x}_t + \varpi\eta_{it} - 1 - r_t + g + \xi\tau_{t+1}}{1-\lambda}.$$

Summing over  $[0, 1]$  and using  $\int_0^1 \eta_{it} di = \omega_{t+1}$  gives (14). We thus have  $x_{it+1} = \mu_{t+1} + v_{it}/(1-\lambda + \sigma_v^2/\sigma_\omega^2)$  which is normally distributed with standard deviation  $\sigma_v/(1-\lambda + \sigma_v^2/\sigma_\omega^2) = \sigma_x$ .

## D Proof of Lemma 1

At each date  $t$ , the Bellman equation corresponding to (15) can be written as

$$v(S_{t-1}; e_t) = \max_{s_t} u(R_t + r_{t-1}S_{t-1} - s_t, e_t) + \beta \mathbb{E}_t[v(S_{t-1} + s_t; \tilde{e}_{t+1})]$$

where  $S_t$  and  $s_t$  are the state and the control variables respectively. The first-order equation gives

$$\partial u(c_t, e_t)/\partial c = \beta \mathbb{E}_t [\partial v(S_t; \tilde{e}_{t+1})/\partial S], \quad (47)$$

and the envelope theorem gives

$$\partial v(S_{t-1}; e_t)/\partial S = r_{t-1} \partial u(c_t, e_t)/\partial c + \beta \mathbb{E}_t [\partial v(S_t; \tilde{e}_{t+1})/\partial S].$$

Replacing the last term using (47), we get

$$\partial v(S_{t-1}; e_t)/\partial S = (1 + r_{t-1}) \partial u(c_t, e_t)/\partial c.$$

Taking the expectation and replacing in (47) yields (16) where  $1 + r_t$  on the RHS is factorized out of the expected value since the date- $t$  interest rate is a known parameter.

## E Proof of Proposition 4

The planner's program is equivalently stated as

$$\max_{\{\mu_t, k_t, e_t\}_{t>0}} \left\{ \sum_{t=0}^{+\infty} \beta^t u(q(k_t) - k_{t+1} + gk_t - \mu_{t+1} + g\mu_t, e_t) : (20), (21), (22) \right\},$$

given  $e_0, k_0$  and  $\mu_0$ . Neglecting the constraints (21) and (22), and denoting by  $\hat{\lambda}_t$  the multiplier associated to (20), the Lagrangian of this program is given by

$$\mathcal{L} = \sum_{t=0}^{+\infty} \beta^t u(q(k_t) - k_{t+1} + gk_t - \mu_{t+1} + g\mu_t, e_t) - \hat{\lambda}_t (e_{t+1} - \theta e_t - \xi \mu_t + \varphi q(k_t) - \hat{e}).$$



When (22) is not binding, the FOCs are

$$\frac{\partial \mathcal{L}}{\partial k_t} = \beta^t \frac{\partial u_t}{\partial c} (q'(k_t) + g) - \beta^{t-1} \frac{\partial u_{t-1}}{\partial c} - \hat{\lambda}_t \varphi q'(k_t) \leq 0 \perp k_t \geq gk_{t-1}, \quad (48)$$

$$\frac{\partial \mathcal{L}}{\partial \mu_t} = \beta^t \frac{\partial u_t}{\partial c} g - \beta^{t-1} \frac{\partial u_{t-1}}{\partial c} + \hat{\lambda}_t \xi = 0, \quad (49)$$

for all  $t > 0$ , and

$$\frac{\partial \mathcal{L}}{\partial e_t} = \beta^t \frac{\partial u_t}{\partial e} - \hat{\lambda}_{t-1} + \theta \hat{\lambda}_t = 0, \quad (50)$$

for all  $t > 1$ . Eq. (49) gives

$$\hat{\lambda}_t = \left( \beta^{t-1} \frac{\partial u_{t-1}}{\partial c} - \beta^t \frac{\partial u_t}{\partial c} g \right) / \xi$$

and, using (24),

$$\hat{\lambda}_t = \beta^t \frac{\partial u_t}{\partial c} (h_{t-1} - g) / \xi. \quad (51)$$

Substituting in (50) evaluated at  $t + 2$ , yields

$$\begin{aligned} 0 &= \beta^{t+2} \frac{\partial u_{t+2}}{\partial e} - \beta^{t+1} \frac{\partial u_{t+1}}{\partial c} (h_t - g) / \xi + \theta \beta^{t+2} \frac{\partial u_{t+2}}{\partial c} (h_{t+1} - g) / \xi \\ &= \beta^{t+2} \frac{\partial u_{t+2}}{\partial e} - \beta^{t+2} \frac{\partial u_{t+2}}{\partial c} [h_{t+1} (h_t - g) - \theta (h_{t+1} - g)] / \xi, \end{aligned}$$

using (24), hence

$$\xi \frac{\partial u_{t+2} / \partial e}{\partial u_{t+2} / \partial c} = h_{t+1} (h_t - g - \theta) + \theta g$$

for all  $t > 1$  such that (22) is not binding. Using (51) to substitute for  $\hat{\lambda}_{t+1}$  in (48) evaluated at  $t + 1$  gives

$$\begin{aligned} 0 &\geq \beta^{t+1} \frac{\partial u_{t+1}}{\partial c} (q'(k_{t+1}) + g) - \beta^t \frac{\partial u_t}{\partial c} - \beta^{t+1} \frac{\partial u_{t+1}}{\partial c} (h_t - g) q'(k_{t+1}) \varphi / \xi \\ &= \beta^{t+1} \frac{\partial u_{t+1}}{\partial c} [q'(k_{t+1}) - (h_t - g)(1 + q'(k_{t+1}) \varphi / \xi)], \end{aligned}$$

using (24), hence

$$h_t - g \geq \frac{q'(k_{t+1})}{1 + q'(k_{t+1}) \varphi / \xi} \perp k_{t+1} \geq gk_t.$$

Assuming (22) is binding for all  $t \geq T$ , the planer's program becomes

$$\max_{\{k_t\}_{t>T}} \left\{ \sum_{t=T}^{+\infty} \beta^t u(q(k_t) - k_{t+1} + gk_t - [q(k_{t+1}) - gq(k_t)]\varphi/\xi, e_t) : e_{t+1} = \theta e_t + \hat{e} \right\},$$

given  $e_T, k_T$ . The FOCs are

$$\frac{\partial \mathcal{L}}{\partial k_t} = \beta^t \frac{\partial u_t}{\partial c} [q'(k_t)(1 + g\varphi/\xi) + g] - \beta^{t-1} \frac{\partial u_{t-1}}{\partial c} [1 + q'(k_t)\varphi/\xi] = 0$$

for all  $t > T$ , which gives

$$h_{t-1} = \frac{g + q'(k_t)(1 + g\varphi/\xi)}{1 + q'(k_t)\varphi/\xi},$$

or, equivalently,

$$h_t - g = \frac{q'(k_{t+1})}{1 + q'(k_{t+1})\varphi/\xi}$$

for all  $t \geq T$ .

## F Proof of Proposition 5

To implement the First-Best, the realized IRS given by (16), that satisfies  $h_t^a = 1 + r_t$ , must be equal to  $h_t^*$ . For the tax policy, we have from (11),  $1 + r_t - g = q'(k_{t+1})(1 - \tau_{t+1}\varphi)$ , while (25) gives  $h_t^* - g \geq q'(k_{t+1}^*)[1 - (h_t^* - g)\varphi/\xi]$ , with an equality when (21) is not binding. Identifying the two equations when  $k_{t+1}^* > gk_t^*$ , i.e. when  $t \geq t_0$ , gives  $\tau_{t+1} = (h_t^* - g)/\xi$ . When  $k_{t+1}^* = gk_t^*$ , substituting  $h_t^*$  for  $1 + r_t$  in (11) defines  $\tau_{t+1}$  for  $t < t_0$ , and we have  $h_t^* - g = q'(gk_t^*)[1 - \tau_{t+1}\varphi] > q'(gk_t^*)[1 - (h_t^* - g)\varphi/\xi]$  implying  $\tau_{t+1} < (h_t^* - g)/\xi$ . Substituting  $h_t^*$  for  $1 + r_t$  in (12) taken in expectation gives

$$\mathbb{E}[\mu_{t+1}] = [(1 - \lambda g)\mu_t + h_t^* z_t - g z_{t+1} + \xi \tau_{t+1} - (h_t^* - g)] / (1 - \lambda).$$

Re-arranging terms gives (28) using  $\mu_t = \mu_t^*$  and  $\mathbb{E}[\mu_{t+1}] = \mu_{t+1}^*$ . (29) is obtained similarly from (14). At the stationary state, denoting with subscript  $\infty$  the values of the variables, we have  $e_\infty = e_N, c_\infty = q(k_\infty) - (1 - g)(k_\infty + \mu_\infty)$ ,  $h_\infty = 1/\beta$ ,  $\tau_\infty = (1/\beta - g)\varphi/\xi$  and  $k_\infty = q^{-1}((1/\beta - g)/(1 - (1/\beta - g)\varphi/\xi))$ . (22) and (12) imply  $\mu_\infty = (\varphi/\xi)q(k_\infty)$  and  $(1 - \lambda)\mu_\infty = (1 - \lambda g)\mu_\infty + \xi \tau_\infty - (1 - z_\infty)(h_\infty - g)$ . Replacing and reorganizing terms yields  $z_\infty = -\lambda(1 - g)(\xi/\varphi)q(k_\infty)/(1/\beta - g) < 0$ .

## G Proof of Lemma 2

With a constant MRS, (26) simplifies to

$$h_{t+1}^* h_t^* - h_{t+1}^* (g + \theta) + \theta g = p\xi, \quad (52)$$

which must hold for all  $t$  as long as (22) is not binding.

(i). The discriminant of  $P(h) = 0$  where

$$P(h) \equiv p\xi - (h - g)(h - \theta). \quad (53)$$

is  $\Delta \equiv (g - \theta)^2 + 4p\xi > 0$ , and the equation admits two roots,  $\bar{h} = (g + \theta + \sqrt{\Delta})/2 > \max\{g, \theta\}$  and  $\underline{h} = (g + \theta - \sqrt{\Delta})/2 < \min\{g, \theta\}$ . We have  $\underline{h} > 0$  iff  $\xi p < \theta g$ . The First-Best policy corresponds to a sequence of IRS with a subsequence defined by (52) as long as the ENP constraint (22) is not binding, i.e.  $t < T$  where  $T$  is the first period (22) binds. The sequence  $\{h_t^*\}_{t=0}^{T-1}$  is either degenerate, i.e.  $h_t^* = \underline{h}$  or  $h_t^* = \bar{h}$  for all  $t \in \{0, \dots, T-1\}$ , or  $h_0^* \notin \{\underline{h}, \bar{h}\}$ , and, reorganizing (52),

$$h_t^* = (\xi p - \theta g) / (h_{t-1}^* - g - \theta) \quad (54)$$

for all  $t = 1, \dots, T-1$ . If  $T$  is large, this sequence eventually converges to a root of  $P(h) = 0$  that we denote by  $h_\infty$ . We can derive this non-degenerate sequence as follows. Defining  $v_t = (h_t^* - h_\#)^{-1}$ ,  $h_\# \in \{\underline{h}, \bar{h}\}$ , we have  $h_t^* = 1/v_t + h_\#$  and (54) becomes

$$\frac{1}{v_{t+1}} + h_\# = \frac{p\xi - \theta g}{1/v_t + h_\# - g - \theta},$$

which gives

$$\frac{1}{v_{t+1}} = \frac{-h_\#}{1 + v_t(h_\# - g - \theta)},$$

using (53). We thus have

$$v_{t+1} = v_t(g + \theta - h_\#) / h_\# - 1/h_\# \equiv v_t b_1 - b_0$$

with  $b_1 = \underline{h}/\bar{h}$  if  $h_\# = \bar{h}$  and  $b_1 = \bar{h}/\underline{h}$  if  $h_\# = \underline{h}$ . With an initial value  $v_0$  at  $t = t_0$ , the

solution of this recurrence equation is given by

$$\begin{aligned} v_t &= v_0 b_1^t - b_0(1 - b_1^t)/(1 - b_1) \\ &= [v_0 + b_0/(1 - b_1)]b_1^t - b_0/(1 - b_1). \end{aligned} \quad (55)$$

where

$$\frac{b_0}{1 - b_1} = \frac{1/h_{\#}}{1 + (h_{\#} - g - \theta)/h_{\#}} = \frac{1}{2h_{\#} - g - \theta}.$$

If  $|b_1| < 1$ , i.e. if  $h_{\#} = \bar{h}$ ,  $v_t$  converges toward  $v_{\infty} = -b_0/(1 - b_1) = 1/(g + \theta - 2\bar{h}) = (h_{\infty} - \bar{h})^{-1}$ , hence  $h_t^*$  converges toward  $h_{\infty} = g + \theta - \bar{h} = \underline{h}$ . If  $|b_1| > 1$ , i.e. if  $h_{\#} = \underline{h}$ ,  $v_t$  diverges and thus  $h_{\infty} = \underline{h}$ . Hence, the recursion does not converge to  $\bar{h}$  (unless in the degenerate case  $h_0 = \bar{h}$ ), while it converges to  $\underline{h}$  from any initial value  $h_0 \neq \bar{h}$ . This cannot be optimal if  $\xi p \geq \theta g$  since  $\underline{h} \leq 0$ . If  $\xi p < \theta g$ , as  $\underline{h} < \min\{g, \theta\} \leq g$ , there is no stock of productive capital that can satisfy (25) when  $h_t^* \rightarrow \underline{h}$ . The optimal solution is thus  $h_t^* = \bar{h}$  for all  $t \in \{0, \dots, T - 1\}$ .

(ii). If  $gk_0 > \bar{k}$ , (21) is binding as long as  $g^t k_0 > \bar{k}$ , i.e.  $t \leq t_0$  given by  $g^{t_0} k_0 = \bar{k}$ , hence  $t_0 = \ln(\bar{k}/k_0)/\ln g$ .

(iii). For  $t > t_0$ ,  $k_{t+1} = \bar{k}$  until (22) is binding, i.e. until  $T$  given by  $\mu_T^* = q(k_{T+1})\varphi/\xi$ . Using (52) for  $t = T - 1$  yields  $h_T = \bar{h}$ , hence  $k_{T+1} = \bar{k}$ .

Finally, the path of the economy for  $t > T$  is defined recursively by (20), (17), (22), (24) and (25) that holds for all  $t > t_0$ . Given the initial values  $e_T$ ,  $k_{T+1} = k_T = \bar{k}$ , the IRS can be written as  $h_t = h(k_t, k_{t+1}, k_{t+2})$  and (25) gives the implicit equation  $q'(k_{t+1})[1 - (h(k_t, k_{t+1}, k_{t+2}) - g)\varphi/\xi] - h(k_t, k_{t+1}, k_{t+2}) + g = 0$  defining  $k_{t+2}$  for all  $t \geq T$ .

Finally, the solution of (52) exists if

**H1:**  $\xi/\varphi > 1/\beta - (3g - \theta)/2$ .

**H2:**  $(1/\beta - \theta)(1/\beta - g) < p\xi < (\xi/\varphi)^2 + (g - \theta)\xi/\varphi$ .

The path of  $y_t$  is increasing if  $\bar{h} > 1/\beta$ . From the concavity of  $P$ , as  $1/\beta > \max\{g, \theta\}$ , we have  $\bar{h} > 1/\beta$  iff  $P(1/\beta) > 0$ , i.e. iff  $p\xi > (1/\beta - \theta)(1/\beta - g)$ . Also, we have to verify that (21) is not binding indefinitely. From (25), using  $\lim_{k \rightarrow 0} q'(k) = +\infty$ , this is the case if  $\bar{h}$  is lower than  $g + \xi/\varphi$ , i.e.  $p < \xi/\varphi^2 + (g - \theta)/\varphi$ . The conditions  $P(1/\beta) > 0$  and  $p < \xi/\varphi^2 + (g - \theta)/\varphi$  imply that  $(\xi/\varphi)^2 + (g - \theta)\xi/\varphi > p\xi > (1/\beta - \theta)(1/\beta - g)$  which is assumption H2. H2 is possible only if  $F(\xi/\varphi) > 0$ , where  $F(x) = x^2 + (g - \theta)x - (1/\beta - \theta)(1/\beta - g)$  is a second degree polynomial. The discriminant

of  $F(x) = 0$  is given by  $[2/\beta - (g + \theta)]^2$ , implying that  $F(x) = (x - \underline{x})(x - \bar{x})$  where  $\underline{x} < 0 < \bar{x}$  are the two real roots of  $F(x) = 0$ .  $F(x)$  is positive if  $x < \underline{x}$  or  $x > \bar{x}$ , and since  $\xi/\varphi > 0$ , we must have  $\xi/\varphi > \bar{x} = 1/\beta - (3g - \theta)/2$  (assumption H1).

## H Proof of Proposition 6

With exponential preferences  $h_t = e^{\gamma[c_{t+1} - c_t + p(e_{t+1} - e_t)]}/\beta$ .  $h_t = \bar{h}$  gives

$$c_{t+1} - c_t + p(e_{t+1} - e_t) = \ln(\beta\bar{h})^{1/\gamma} \quad (56)$$

for all  $t \leq T$ . Multiplying both sides by  $\xi$  and using (20) and (17) leads to

$$\begin{aligned} \ln(\beta\bar{h})^{\xi/\gamma} &= \xi[q_{t+1} - q_t - k_{t+2} + (1+g)k_{t+1} - gk_t] - \varphi[q_{t+2} - (1+g)q_{t+1} + gq_t] \\ &\quad - (e_{t+3} - e_{t+2}) + (\theta + g)(e_{t+2} - e_{t+1}) + (\xi p - g\theta)(e_{t+1} - e_t) \end{aligned} \quad (57)$$

for all  $t \leq T$ . For  $t < t_0$ , we have  $k_t = g^t k_0$ , and using  $q(k) = Ak^\alpha$ ,  $q_t = g^{\alpha t} q_0$ . For  $t_0 \leq t \leq T+1$ , we have  $k_t = \bar{k}$  and  $q_t = q(\bar{k}) \equiv \bar{q}$ . (57) can thus be expressed as

$$m_{t+2} = (\theta + g)m_{t+1} + (\xi p - g\theta)m_t - \kappa g^{\alpha t} - \ln(\beta\bar{h})^{\xi/\gamma} \quad (58)$$

for all  $t \leq T-1$ , where  $m_t \equiv e_{t+1} - e_t$  and

$$\kappa = \begin{cases} [\xi + \varphi(g - g^\alpha)](1 - g^\alpha)q_0 & t < t_0 \\ 0 & t_0 \leq t \leq T-1 \end{cases} .$$

For  $t \geq T$ , as  $\varphi q(k_t) = \xi \mu_t$ , we have  $e_{t+1} = \theta e_t + \hat{e}$  implying  $m_{t+1} = \theta m_t$ .

The solution of (58) is  $m_t = n_t + v_t$  where  $n_t$  and  $v_t$  are the solutions of the corresponding homogeneous and particular equations. The characteristic equation of the homogeneous equation is  $P(x) = 0$ . As  $\bar{h}$  is irrelevant,  $n_t = \phi \bar{h}^t$  where  $\phi$  is a constant. The particular solution of (58) is given by  $v_t = g^{\alpha t} \nu_0 + \nu_1$  where  $\nu_0$  and  $\nu_1$  solve

$$g^{\alpha(t+2)} \nu_0 + \nu_1 = (\theta + g)(\nu_0 g^{\alpha(t+1)} + \nu_1) + (\xi p - g\theta)(\nu_0 g^{\alpha t} + \nu_1) - \kappa g^{\alpha t} - \ln(\beta\bar{h})^{\xi/\gamma}$$

for all  $t \leq T-1$ , which gives  $\nu_0 = \kappa/P(g^\alpha)$  and  $\nu_1 = \ln(\beta\bar{h})^{\xi/\gamma}/P(1)$ . As  $P$  is concave with  $P(g) = p\xi$  and  $P(1/\beta) > 0$ , we have  $P(g^\alpha) > 0$  and  $P(1) > 0$ , hence  $\nu_1 > 0$  since  $P(1/\beta) > 0$  implies  $\bar{h} > 1/\beta$ . We also have  $\nu_0 \geq 0$  if  $\xi/\varphi \geq 1 - g > g^\alpha - g$ . The

solution of (58) is thus

$$m_t = \phi \bar{h}^t + g^{\alpha t} \nu_0 + \nu_1, \quad (59)$$

where  $\phi$  is specific to each sub-sequence and is derived from their initial conditions, i.e. from (58) evaluated at  $t = t_0 - 1$ , (57) evaluated at  $t = T$ , and from (20) at  $t = 1$ .

Using (57) at  $t = T$ , yields

$$\begin{aligned} \ln(\beta \bar{h})^{\xi/\gamma} &= \xi(\bar{k} - k_{T+2}) - \varphi(q(k_{T+2}) - \bar{q}) - m_{T+2} + (\theta + g)m_{T+1} + (\xi p - g\theta)m_T \\ &= \xi\bar{k} + \varphi\bar{q} - [\xi k_{T+2} + \varphi q(k_{T+2})] - \theta^2 m_T + \theta(\theta + g)m_T + (\xi p - g\theta)m_T \\ &= \xi\bar{k} + \varphi\bar{q} - [\xi k_{T+2} + \varphi q(k_{T+2})] + \xi p m_T, \end{aligned}$$

where the first line comes from  $k_t = \bar{k}$  and  $q_t = \bar{q}$  for  $t_0 \leq t \leq T + 1$ , and the second from  $m_{t+1} = \theta m_t$  for  $t \geq T$ . Identifying each term with (56) multiplied by  $\xi$  on both sides, it comes

$$\xi(c_{T+1} - c_T) = \xi\bar{k} + \varphi\bar{q} - [\xi k_{T+2} + \varphi q(k_{T+2})] \equiv \kappa_1. \quad (60)$$

We thus get, using (59) where  $\nu_0 = 0$  and  $\ln(\beta \bar{h})^{\xi/\gamma} = \nu_1 P(1)$ ,

$$\phi \bar{h}^T + \nu_1 = [\nu_1 P(1) - \kappa_1]/\xi p.$$

It comes

$$\phi = \nu_1(P(1)/\xi p - 1)/\bar{h}^T - \kappa_1/(\xi p \bar{h}^T) = -[(1 - \theta)(1 - g)\nu_1 + \kappa_1]/(\xi p \bar{h}^T)$$

for all  $t_0 \leq t \leq T$ . Substituting into (59) and using  $\nu_0 = 0$  for all  $t_0 \leq t \leq T$  gives

$$m_t = \nu_1 - [(1 - \theta)(1 - g)\nu_1 + \kappa_1]/(\xi p \bar{h}^{T-t})$$

for all  $t_0 \leq t \leq T$ . Using (56) evaluated at  $T$  and (60) we get

$$c_{T+1} - c_T + p m_T = \kappa_1/\xi + p \nu_1 - [(1 - \theta)(1 - g)\nu_1 + \kappa_1]/\xi = P(1)\nu_1/\xi = \ln(\beta \bar{h})^{1/\gamma},$$

and it comes

$$h_T = e^{\gamma[c_{T+1} - c_T + p(e_{T+1} - e_T)]}/\beta = \bar{h} = g + q'(k_{T+2})/[1 + q'(k_{T+2})\varphi/\xi].$$

We thus have  $k_{T+2} = \bar{k}$ , and from and (60)  $c_{T+1} - c_T = 0 = \kappa_1$ , implying

$$m_t = \nu_1 - [(1 - \theta)(1 - g)\nu_1]/(\xi p \bar{h}^{T-t})$$

for all  $t_0 \leq t \leq T$ . As  $m_t \geq P(1)\nu_1/\xi p = \ln(\beta \bar{h})^{1/p\gamma} > 0$ ,  $e_{t+1} > e_t$  for all  $t_0 \leq t \leq T$ .

Using (58) for  $t = t_0 - 1$ , it comes

$$\begin{aligned} m_{t_0+1} &= \nu_1 - [(1 - \theta)(1 - g)\nu_1]/(\xi p \bar{h}^{(T-t_0-1)}) \\ &= (\theta + g)\{\nu_1 - [(1 - \theta)(1 - g)\nu_1]/(\xi p \bar{h}^{(T-t_0)})\} + (\xi p - \theta g)(\phi \bar{h}^{t_0-1} + g^{\alpha(t_0-1)}\nu_0 + \nu_1) - \nu_1 P(1). \end{aligned}$$

As  $P(1) = \xi p - \theta g - 1 + \theta + g$ , this equation simplifies to

$$\begin{aligned} [(1 - \theta)(1 - g)\nu_1]/(\xi p \bar{h}^{(T-t_0-1)}) &= (\theta + g)[(1 - \theta)(1 - g)\nu_1]/(\xi p \bar{h}^{(T-t_0)}) \\ &\quad - (\xi p - \theta g)(\phi \bar{h}^{t_0-1} + g^{\alpha(t_0-1)}\nu_0). \end{aligned}$$

Multiplying by  $\xi p \bar{h}^{T-t_0+1}$  yields

$$[(1 - \theta)(1 - g)\nu_1]\bar{h}^2 = (\theta + g)[(1 - \theta)(1 - g)\nu_1]\bar{h} - (\xi p - \theta g)(\phi \bar{h}^{t_0-1} + g^{\alpha(t_0-1)}\nu_0)\xi p \bar{h}^{T-t_0+1},$$

and re-organizing terms,

$$[(1 - \theta)(1 - g)\nu_1][\bar{h}^2 - (\theta + g)\bar{h}] = -(\xi p - \theta g)(\phi \bar{h}^{t_0-1} + g^{\alpha(t_0-1)}\nu_0)\xi p \bar{h}^{T-t_0+1},$$

where, as  $P(\bar{h}) = 0$ ,  $p\xi - \theta g = \bar{h}^2 - (\theta + g)\bar{h}$ . Simplifying, we get

$$(1 - \theta)(1 - g)\nu_1 = -[\phi + (g^\alpha/\bar{h})^{t_0-1}\nu_0]\xi p \bar{h}^T,$$

which gives

$$\phi = -\left(\frac{g^\alpha}{\bar{h}}\right)^{t_0-1}\nu_0 - \frac{(1 - \theta)(1 - g)\nu_1}{\xi p \bar{h}^T}.$$

Substituting into (59) yields

$$m_t = \nu_0 g^{\alpha t} [1 - (g^\alpha/\bar{h})^{t_0-1-t}] + \nu_1 - [(1 - \theta)(1 - g)\nu_1]/(\xi p \bar{h}^{T-t})$$

for all  $0 < t < t_0$ . As  $m_t > 0$ ,  $e_{t+1} > e_t$  for all  $0 < t < t_0$ . (31) is deduced from

$$\begin{aligned} \sum_{\tau=1}^{t-1} m_\tau &= e_t - e_1 = \sum_{\tau=1}^{t-1} (\phi \bar{h}^\tau + g^{\alpha\tau} \nu_0 + \nu_1) = \phi \frac{\bar{h}^t - \bar{h}}{\bar{h} - 1} + \nu_0 \frac{g^\alpha - g^{\alpha t}}{1 - g^\alpha} + (t-1)\nu_1 \\ &= \nu_0 \left( \frac{g^\alpha - g^{\alpha t}}{1 - g^\alpha} - \left( \frac{g^\alpha}{\bar{h}} \right)^{t_0-1} \frac{\bar{h}^t - \bar{h}}{\bar{h} - 1} \right) + \nu_1(t-1) - \frac{(1-\theta)(1-g)\nu_1}{\xi p \bar{h}^T} \frac{\bar{h}^t - \bar{h}}{\bar{h} - 1} \end{aligned}$$

for all  $1 < t < t_0$ , where  $e_1 = \theta e_0 + \xi \mu_0 - \varphi q_0 + \hat{e}$ , and from

$$\begin{aligned} \sum_{\tau=t_0-1}^{t-1} m_\tau &= e_t - e_{t_0-1} = \phi \sum_{\tau=t_0-1}^{t-1} \bar{h}^\tau + (t - t_0)\nu_1 = \phi \frac{\bar{h}^t - \bar{h}^{t_0-1}}{\bar{h} - 1} + (t - t_0 + 1)\nu_1 \\ &= \nu_1(t - t_0 + 1) - \frac{(1-\theta)(1-g)\nu_1}{\xi p \bar{h}^T} \frac{\bar{h}^t - \bar{h}^{t_0-1}}{\bar{h} - 1}, \end{aligned}$$

for all  $t_0 \leq t \leq T$ . Using  $m_T = \ln(\beta \bar{h})^{1/(p\gamma)}$  and  $m_T = e_{T+1} - e_T = \hat{e} - (1-\theta)e_T$ , it comes that  $T$  is solution of  $e_T = e_N - \ln(\beta \bar{h})^{1/(1-\theta)p\gamma}$ . As

$$\begin{aligned} m_t - m_{t-1} &= \nu_0(g^{\alpha t} - g^{\alpha(t_0-1)})/\bar{h}^{t_0-1-t} - g^{\alpha(t-1)} + g^{\alpha(t_0-1)}/\bar{h}^{t_0-t} - \frac{\nu_1(1-\theta)(1-g)}{\xi p \bar{h}^{T-t+1}}(\bar{h} - 1) \\ &= -\nu_0[(1/g^\alpha - 1)g^{\alpha t} + (\bar{h} - 1)g^{\alpha(t_0-1)}/\bar{h}^{t_0-t}] - \frac{\nu_1(1-\theta)(1-g)}{\xi p \bar{h}^{T-t+1}}(\bar{h} - 1), \end{aligned}$$

where  $\nu_0 > 0$  if  $t < t_0$  and  $\nu_0 = 0$  if  $t_0 \leq t \leq T$ ,  $m_t - m_{t-1} < 0$  for all  $t \leq T$ : the sequence  $\{e_t\}_{2 \leq t \leq T}$  increases at a decreasing rate. From (20), the GTI index is deduced from these expressions using  $\mu_t = (e_{t+1} - \theta e_t + \varphi q_t - \hat{e})/\xi$ . From (56),

$$c_{t+1} - c_t = \ln(\beta \bar{h})^{1/\gamma} - p m_t$$

for all  $t \leq T$  where

$$\begin{aligned} m_t &= \nu_0 g^{\alpha t} [1 - (g^\alpha/\bar{h})^{t_0-1-t}] + \nu_1 [1 - (1-\theta)(1-g)/(\xi p \bar{h}^{T-t})] \\ &= \nu_0 g^{\alpha t} [1 - (g^\alpha/\bar{h})^{t_0-1-t}] + \ln(\beta \bar{h})^{1/\gamma} [\xi p - (1-\theta)(1-g)/\bar{h}^{T-t}]/[pP(1)] \end{aligned}$$

for all  $1 < t \leq T$ . We thus have

$$\begin{aligned} c_{t+1} - c_t &= -\nu_0 g^{\alpha t} [1 - (g^\alpha/\bar{h})^{t_0-1-t}] + \ln(\beta \bar{h})^{1/\gamma} [P(1) - (\xi p - (1-\theta)(1-g)/\bar{h}^{T-t})]/P(1) \\ &= -\nu_0 g^{\alpha t} [1 - (g^\alpha/\bar{h})^{t_0-1-t}] - \ln(\beta \bar{h})^{1/\gamma} (1-\theta)(1-g)(1 - 1/\bar{h}^{T-t})/P(1) \end{aligned}$$

which is negative and increasing, with a maximum equal to 0 at  $t = T$ . As  $c_{t+1} - c_t -$



$(c_t - c_{t-1}) = -p(m_t - m_{t-1}) > 0$ , the consumption sequence decreases at a decreasing rate.

## I Proof of Proposition 7

For ease of notation, superscript  $\ell$  is omitted and results are derived in the tax-subsidy policy case ( $\ell = sub$ ). Variations from these results in the tax-standard policy case are specified in the text. Also, subscript  $t$  is used as a shorthand for coefficients function of  $z_t$ , e.g.  $a_{3t} \equiv a_3^{sub}(z_t)$ . I first characterize the distribution of  $\tilde{y}_{t+1}$  given the information available in period  $t$ ,  $t_0 < t \leq T$ , a random variable denoted by  $\tilde{y}_{t+1|t}$ . Define

$$\hat{k}(r) \equiv q^{-1}((1 + r - g)/(1 - (\bar{h} - g)\varphi/\xi)),$$

and  $\hat{q}(r) \equiv q(\hat{k}(r))$ . We have  $\bar{k} = \hat{k}(\bar{r})$ , and  $\bar{q} \equiv \hat{q}(\bar{r})$  denotes the corresponding GDP.

Given the information available in period  $t$ , the prevailing interest rate  $r_t$  (and thus the realization  $\varepsilon_t$  of  $\tilde{\varepsilon}_t$ ) is known, and both  $k_{t+1} = \hat{k}(r_t)$  and  $q_{t+1} = \hat{q}(r_{t+1})$  are known. From (20),  $e_{t+1}$  is also known since it depends on variables of the previous period, that are function of  $\varepsilon_{t-1}$ . Hence, from (17), we get

$$\tilde{y}_{t+1|t} = \tilde{c}_{t+1|t} + pe_{t+1} = q_{t+1} - \tilde{k}_{t+2|t} + gk_{t+1} - (\tilde{\mu}_{t+2|t} - g\mu_{t+1}) + pe_{t+1}, \quad (61)$$

where  $\tilde{k}_{t+2}$  and  $\tilde{\mu}_{t+2}$  are unknown since they depend on the realization of  $\tilde{\varepsilon}_{t+1}$ : we have  $\tilde{k}_{t+2} = \hat{k}(\tilde{r}_{t+1})$  and, from (34),

$$\tilde{\mu}_{t+2|t} = a_1\mu_{t+1} + a_2e_{t+1} + a_{3t} + Z_{t+1} + b_{1t}\tilde{\varepsilon}_{t+1} + b_{2t}\varepsilon_t. \quad (62)$$

Replacing, we get

$$\tilde{y}_{t+1|t} = q_{t+1} - \tilde{k}_{t+2|t} + gk_{t+1} + (g - a_1)\mu_{t+1} + (p - a_2)e_{t+1} - a_{3t} - Z_{t+1} - b_{1t}\tilde{\varepsilon}_{t+1} - b_{2t}\varepsilon_t.$$

Using the linear approximation  $k_t \approx \bar{k} + \hat{k}'(\bar{r})\bar{r}\varepsilon_{t-1}$  where  $\hat{k}'(\bar{r}) = 1/[(1 - \tau\varphi)q''(\bar{k})]$ , we get using  $\mathbb{E}_t[\tilde{k}_{t+2}] \approx \bar{k} + \hat{k}'(\bar{r})\bar{r}\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t]$  and (33),

$$\begin{aligned} \tilde{y}_{t+1|t} - \mathbb{E}_t[\tilde{y}_{t+1}] &\approx -(\tilde{k}_{t+2|t} - \mathbb{E}[\tilde{k}_{t+2}|\varepsilon_t]) - b_{1t}(\tilde{\varepsilon}_{t+1} - \mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t]) \\ &= -(\hat{k}'(\bar{r})\bar{r} + b_{1t})(\tilde{\varepsilon}_{t+1} - \mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t]) \\ &= -(\hat{k}'(\bar{r})\bar{r} + b_{1t})\chi_t\tilde{\omega}_{t+2} \end{aligned}$$

when  $t_0 < t \leq T$ . Hence,  $\tilde{y}_{t+1|t}$  is approximately normally distributed with variance  $\mathbb{V}_t[\tilde{y}_{t+1}] = (\hat{k}'(\bar{r})\bar{r} + b_{1t})^2 \chi_t^2 \sigma_\omega^2 \equiv \sigma_{y_{t+1|t}}^2$  when  $t_0 < t \leq T$ . For the tax-standard policy, as  $b_1^{std}$  and  $\chi^{std}$  do not depend on  $t$  (as shown below),  $\sigma_{y_{t+1|t}}^2$  is a constant.

The coefficients  $a_1, a_2, a_3, b_{1t}, b_2$ , and  $Z_t$  in (34), and  $\chi_t$  and  $\rho_t$  in (33), are derived as follows. Using (20), (17), (61), (62) and  $\mathbb{E}_t[\tilde{k}_{t+2}] \approx \bar{k} + \hat{k}'(\bar{r})\bar{r}\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t]$ , it comes

$$\begin{aligned}
\mathbb{E}_t[\tilde{y}_{t+1}] - y_t &\approx q_{t+1} - q_t - \bar{k} - \hat{k}'(\bar{r})\bar{r}\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] + k_{t+1}(1+g) - gk_t - \mathbb{E}_t[\tilde{\mu}_{t+2}] + (1+g)\mu_{t+1} \\
&\quad - g\mu_t + p(e_{t+1} - e_t) \\
&= q_{t+1} - q_t - \bar{k} - \hat{k}'(\bar{r})\bar{r}\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] + k_{t+1}(1+g) - gk_t \\
&\quad - (a_1\mu_{t+1} + a_2e_{t+1} + a_3t + Z_{t+1} + b_{1t}\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] + b_2\varepsilon_t) + (1+g)\mu_{t+1} \\
&\quad - \mu_t g + p(e_{t+1} - e_t) \\
&= q_{t+1} - q_t - \bar{k} - \hat{k}'(\bar{r})\bar{r}\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] + k_{t+1}(1+g) - gk_t - a_3t - Z_{t+1} \\
&\quad - b_{1t}\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] - b_2\varepsilon_t + (1+g-a_1)\mu_{t+1} - \mu_t g + (p-a_2)e_{t+1} - pe_t \\
&= q_{t+1} - q_t - \bar{k} - (b_{1t} + \hat{k}'(\bar{r})\bar{r})\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] + k_{t+1}(1+g) - gk_t - a_3t - Z_{t+1} \\
&\quad - b_2\varepsilon_t + (1+g-a_1)\mu_{t+1} - \mu_t g + (p-a_2)(\theta e_t + \xi\mu_t - \varphi q_t + \hat{e}) - pe_t \\
&= q_{t+1} - q_t[1 + \varphi(p-a_2)] - \bar{k} - (b_{1t} + \hat{k}'(\bar{r})\bar{r})\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] + k_{t+1}(1+g) - gk_t \\
&\quad + (1+g-a_1)\mu_{t+1} - \mu_t[g - \xi(p-a_2)] + [\theta(p-a_2) - p]e_t + (p-a_2)\hat{e} - a_3t \\
&\quad - Z_{t+1} - b_2\varepsilon_t.
\end{aligned}$$

Using  $(1+r_t)z_t = (\bar{h} + \bar{r}\varepsilon_t)z_t$  in (12) and (32), we get

$$(1-\lambda)(g\mu_t - \mu_{t+1}) + (1-g)(\mu_t - 1) + \varpi\omega_{t+1} + \xi\tau_{t+1} + (\bar{h} + \bar{r}\varepsilon_t)z_t - gz_{t+1} \approx r_t \approx \psi - \gamma^2\sigma_{y_{t+1}}^2/2 + \gamma\{\mathbb{E}_t[\tilde{y}_{t+1}] - y_t\}$$

where  $\varpi \equiv (1-\lambda)/(1-\lambda + \sigma_v^2/\sigma_\omega^2)$ , which gives, denoting  $a_0 \equiv 1/[1-\lambda + \gamma(1+g-a_1)]$ ,

$$\begin{aligned}
\mu_{t+1}/a_0 &\approx [1 - \lambda g + \gamma g - \gamma\xi(p-a_2)]\mu_t - \gamma[\theta(p-a_2) - p]e_t + \varpi\omega_{t+1} + \xi\tau_{t+1} + (\bar{h} + \bar{r}\varepsilon_t)z_t - gz_{t+1} \\
&\quad - (\psi - \gamma^2\sigma_{y_{t+1}}^2/2) - \gamma(p-a_2)\hat{e} + \gamma a_3 + g - 1 + \gamma Z_{t+1} + \gamma(b_1 + \hat{k}'(\bar{r})\bar{r})\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] \\
&\quad + \gamma b_2\varepsilon_t - \gamma q_{t+1} + \gamma q_t[1 + \varphi(p-a_2)] + \gamma\bar{k} - \gamma k_{t+1}(1+g) + \gamma gk_t.
\end{aligned}$$

Using  $k_{t+1} \approx \bar{k} + \hat{k}'(\bar{r})\bar{r}\varepsilon_t$  and  $q_{t+1} \approx \bar{q} + \hat{q}'(\bar{r})\bar{r}\varepsilon_t$  where  $\hat{q}'(\bar{r}) = q'(\bar{k})\hat{k}'(\bar{r})$  for  $t$  and  $t-1$ , the last terms can be approximated by

$$-\gamma q_{t+1} + \gamma q_t[1 + \varphi(p-a_2)] \approx -\gamma\hat{q}'(\bar{r})\bar{r}\varepsilon_t + \gamma\bar{q}\varphi(p-a_2) + \gamma\hat{q}'(\bar{r})\bar{r}\varepsilon_{t-1}[1 + \varphi(p-a_2)]$$

and

$$\gamma\bar{k} - \gamma k_{t+1}(1+g) + \gamma g k_t \approx -\gamma\hat{k}'(\bar{r})\bar{r}\varepsilon_t(1+g) + \gamma g\hat{k}'(\bar{r})\bar{r}\varepsilon_{t-1}.$$

Substituting and, from (33), using  $\mathbb{E}[\tilde{\varepsilon}_{t+1}|\varepsilon_t] = \rho_t\varepsilon_t$ , we arrive at

$$\begin{aligned} \mu_{t+1} &\approx a_0[1 - \lambda g + \gamma g - \gamma\xi(p - a_2)]\mu_t - a_0\gamma[\theta(p - a_2) - p]e_t + a_0\varpi\omega_{t+1} \\ &\quad - a_0[\psi - \gamma^2\sigma_{y_{t+1}|t}^2/2 + \gamma(p - a_2)(\hat{e} - \bar{q}\varphi) - \gamma a_{3t} + 1 - g] + a_0[\xi\tau_{t+1} + \bar{h}z_t - gz_{t+1} + \gamma Z_{t+1}] \\ &\quad + a_0\gamma[\hat{q}'(\bar{r})[1 + \varphi(p - a_2)] + g\hat{k}'(\bar{r})\bar{r}\varepsilon_{t-1} \\ &\quad - a_0\gamma[\hat{q}'(\bar{r})\bar{r} + \hat{k}'(\bar{r})\bar{r}(1+g) - b_2 - \bar{r}z_t/\gamma - (b_{1t} + \hat{k}'(\bar{r})\bar{r})\rho_t]\varepsilon_t \end{aligned} \quad (63)$$

Identifying the non-stochastic terms with those of (34) and simplifying gives

$$\begin{aligned} a_1 &= \frac{1 - \lambda g + \gamma[g + \xi(a_2 - p)]}{1 - \lambda + \gamma(1 + g - a_1)}, \quad a_2 = \frac{\gamma p(1 - \theta)}{1 - \lambda + \gamma(1 + g - a_1 - \theta)}, \\ a_{3t} &= -\frac{\psi - \gamma^2\sigma_{y_{t+1}|t}^2/2 - \gamma(p - a_2)(\varphi\bar{q} - \hat{e}) + 1 - g}{1 - \lambda + \gamma(g - a_1)}, \end{aligned} \quad (64)$$

and  $Z_t = a_0(\xi\tau_{t+1} + \bar{h}z_t - gz_{t+1} + \gamma Z_{t+1})$ . Solving the recursion gives (35).

For the standard policy, using (14) and (32) to get

$$(1 - \lambda)(g\mu_t - \mu_{t+1}) + (1 - g)(\underline{x}_t - 1) + \varpi\omega_{t+1} + \xi\tau_{t+1} = r_t \approx \psi - \gamma^2\sigma_{y_{t+1}}^2/2 + \gamma\{\mathbb{E}_t[\tilde{y}_{t+1}] - y_t\},$$

it comes following the same steps that  $a_2^{std}$  and  $a_3^{std}$  are given also by (64) (but where  $\sigma_{y_{t+1}|t}^2$  is a constant in the latter) while we have

$$a_1^{std} = \frac{(1 - \lambda)g + \gamma[g + \xi(a_2^{std} - p)]}{1 - \lambda + \gamma(1 + g - a_1^{std})}$$

and  $Z_t^{std} = a_0^{std}(\xi\tau_{t+1} + (1 - g)\underline{x}_t + \gamma Z_{t+1}^{std})$ , with  $a_0^{std} = 1/[1 - \lambda + \gamma(1 + g - a_1^{std})]$ . Solving the recursion gives (37).

The parameters of the stochastic terms are derived as follows. When  $\varepsilon_{t-1} = \omega_{t+1} = 0$ , we have  $\varepsilon_t = 0$  implying  $r_t = \bar{r}$ . Denoting by  $\mu_{t+1|r_t=\bar{r}}$  the value of  $\mu_{t+1}$  in such a case, we get from (12),

$$\mu_{t+1} - \mu_{t+1|r_t=\bar{r}} \approx \frac{\varpi\omega_{t+1} + (1 + r_t - \bar{h})z_t + \bar{r} - r_t}{1 - \lambda} = \frac{\varpi\omega_{t+1} - (1 - z_t)\bar{r}\varepsilon_t}{1 - \lambda},$$

using  $(1 + r_t)z_t \approx (\bar{h} + \bar{r}\varepsilon_t)z_t$ , and from (63),

$$\begin{aligned} \mu_{t+1} - \mu_{t+1|r_t=\bar{r}} &\approx a_0\{\varpi\omega_{t+1} + \gamma(\hat{q}'(\bar{r})[1 + \varphi(p - a_2)] + g\hat{k}'(\bar{r}))\bar{r}\varepsilon_{t-1} \\ &\quad - \gamma[\hat{q}'(\bar{r})\bar{r} + \hat{k}'(\bar{r})\bar{r}(1 + g) - b_2 - \bar{r}z_t/\gamma - (b_{1t} + \hat{k}'(\bar{r})\bar{r})\rho_t]\varepsilon_t\}. \end{aligned} \quad (65)$$

Equalizing gives

$$\begin{aligned} \varpi\omega_{t+1} &= \frac{\bar{r}(1 - z_t) - (1 - \lambda)a_0\gamma[\hat{q}'(\bar{r})\bar{r} + \hat{k}'(\bar{r})\bar{r}(1 + g) - b_2 - \bar{r}z_t/\gamma - (b_{1t} + \hat{k}'(\bar{r})\bar{r})\rho_t]}{1 - (1 - \lambda)a_0}\varepsilon_t \\ &\quad + (1 - \lambda)\gamma a_0 \frac{\hat{q}'(\bar{r})[1 + \varphi(p - a_2)] + g\hat{k}'(\bar{r})}{1 - (1 - \lambda)a_0}\bar{r}\varepsilon_{t-1}. \end{aligned} \quad (66)$$

Substituting in (65) yields

$$\begin{aligned} \mu_{t+1} - \mu_{t+1|r_t=\bar{r}} &\approx \gamma a_0 \frac{\hat{q}'(\bar{r})[1 + \varphi(p - a_2)] + g\hat{k}'(\bar{r})}{1 - (1 - \lambda)a_0}\bar{r}\varepsilon_{t-1} \\ &\quad + a_0 \frac{\bar{r} - \gamma[\hat{q}'(\bar{r})\bar{r} + \hat{k}'(\bar{r})\bar{r}(1 + g) - b_2 - (b_{1t} + \hat{k}'(\bar{r})\bar{r})\rho_t]}{1 - (1 - \lambda)a_0}\varepsilon_t. \end{aligned}$$

Identifying with the stochastic terms of (34) gives, using  $1 - (1 - \lambda)a_0 = \gamma a_0(1 + g - a_1)$ ,

$$b_2 = \frac{\hat{q}'(\bar{r})[1 + \varphi(p - a_2)] + g\hat{k}'(\bar{r})}{1 + g - a_1}\bar{r} \quad (67)$$

$$b_{1t} = \frac{\bar{r} - \gamma[\hat{q}'(\bar{r})\bar{r} + \hat{k}'(\bar{r})\bar{r}(1 + g - \rho_t) - b_2 - \rho b_{1t}]}{\gamma(1 + g - a_1)}. \quad (68)$$

As (66) can be written as  $\varpi\omega_{t+1} = [(1 - \lambda)b_1 + \bar{r}(1 - z_t)]\varepsilon_t + b_2(1 - \lambda)\varepsilon_{t-1}$ , we get from (33) that  $\chi_t = \varpi / [(1 - \lambda)b_1 + \bar{r}(1 - z_t)]$  and

$$\rho_t = \frac{-b_2(1 - \lambda)}{(1 - \lambda)b_{1t} + \bar{r}(1 - z_t)}. \quad (69)$$

Reorganizing terms to get  $b_2 = -\rho_t [b_{1t} + \bar{r}(1 - z_t)/(1 - \lambda)]$  and substituting in (68), we arrive at

$$b_{1t} = \frac{1 - \gamma[\hat{q}'(\bar{r}) + \hat{k}'(\bar{r})(1 + g - \rho_t) + \rho_t(1 - z_t)/(1 - \lambda)]}{\gamma(1 + g - a_1)}\bar{r}.$$

From (67) and (69), we obtain that  $\rho_t$  solves

$$\rho_t = \frac{-(1-\lambda)\gamma a_0 \{\hat{q}'(\bar{r})[1 + \varphi(p - a_2)] + g\hat{k}'(\bar{r})\}}{1 - \gamma a_0 \left\{ (1-\lambda)[\hat{q}'(\bar{r}) + \hat{k}'(\bar{r})(1+g)] + \rho_t [1 - z_t - \hat{k}'(\bar{r})(1-\lambda)] \right\}}. \quad (70)$$

For the standard policy, the same steps give identical formulae but with  $z_t$  replaced by 0. As a result, parameters  $b_1^{std}$ ,  $\rho^{std}$  and  $\chi^{std}$  are constant.

The first two equations of (64) form a system involving only coefficients  $a_1$  and  $a_2$  that can be solved separately from the others. More precisely, defining  $a_1^0 \equiv g + (1 - \lambda)/\gamma$ , we can express (64) as

$$\begin{aligned} a_1 &= \frac{a_1^0 + \lambda(1-g)/\gamma + \xi(a_2 - p)}{a_1^0 - a_1 + 1}, a_2 = \frac{p(1-\theta)}{a_1^0 - a_1 + 1 - \theta}, \\ a_{3t} &= -\frac{\psi - \gamma^2 \sigma_{y_{t+1|t}}^2 / 2 - \gamma(p - a_2)(\varphi\bar{q} - \hat{e}) + 1 - g}{\gamma(a_1^0 - a_1)}, a_0 = \frac{1}{\gamma(a_1^0 - a_1 + 1)} \end{aligned} \quad (71)$$

From the expression of  $a_2$ , we get

$$a_2 - p = -p \frac{a_1^0 - a_1}{a_1^0 - a_1 + 1 - \theta}, \quad (72)$$

which, plugged into the expression of  $a_1^0 - a_1$ , gives

$$(a_1^0 - a_1) \left( a_1 - 1 + \frac{\xi p}{a_1^0 - a_1 + 1 - \theta} \right) = \frac{\lambda(1-g)}{\gamma}$$

that can be expressed as  $Q(a_1^0 - a_1) = 0$  where

$$Q(x) \equiv x^3 + (2 - \theta - a_1^0)x^2 + [(1 - \theta)(1 - a_1^0) - \xi p + \lambda(1 - g)/\gamma]x + \lambda(1 - \theta)(1 - g)/\gamma$$

is a third degree polynomial. As  $Z_t$  given by (35) converges if  $\gamma a_0 < 1$ , i.e. if  $a_1 < a_1^0$ , only positive roots are relevant. From (71) and (72),  $a_1 > 0$  if  $a_1^0 + \lambda(1 - g)/\gamma + \xi(a_2 - p) = g + (1 - \lambda g)/\gamma - p\xi(a_1^0 - a_1)/(a_1^0 - a_1 + 1 - \theta) > 0$ . Since  $a_1^0 > a_1$ , it suffices to have  $g + (1 - \lambda g)/\gamma \geq p\xi$ . Under the tax-standard policy,  $a_1^{std} > 0$  if  $a_1^0 - (1 - g)(1 - \lambda)/\gamma + \xi(a_2 - p)g = g(1 + (1 - \lambda)/\gamma) - p\xi(a_1^0 - a_1)/(a_1^0 - a_1 + 1 - \theta) > 0$ . A sufficient condition is thus  $g(1 + (1 - \lambda)/\gamma) \geq p\xi$  which is more stringent than the tax-subsidy one. Both are satisfied if  $g \geq p\xi$ .

Equation (70) can be written as  $\zeta_t(\rho) = -A_t\rho^2 + B\rho - C = 0$ , where  $A_t \equiv [1 - z_t - \hat{k}'(\bar{r})(1 - \lambda)]\gamma a_0$ ,  $B \equiv 1 - (1 - \lambda)\gamma a_0[\hat{q}'(\bar{r}) + \hat{k}'(\bar{r})(1 + g)]$ , and  $C \equiv -(1 - \lambda)\gamma a_0\{\hat{q}'(\bar{r})[1 +$

$\varphi(p - a_2)] + g\hat{k}'(\bar{r})\}$  are positive coefficients. As  $\zeta(0) < 0$  and  $\zeta'(0) > 0$ ,  $\zeta$  admits two positive roots provided that  $\Delta_t \equiv B^2 - 4A_tC > 0$ , the smallest root being given by  $(B - \sqrt{\Delta_t})/(2A_t)$ . In the tax-standard policy case, formulae are identical but with  $z_t$  replace by 0, implying that  $\rho^{std}$  is a constant.

Solving the recursion of (33), it comes

$$\tilde{\varepsilon}_t = \chi_t \tilde{\omega}_{t+1} + \rho_t \tilde{\varepsilon}_{t-1} = \sum_{k=0}^{t-t_0} \chi_{t-k} \tilde{\omega}_{t+1-k} \prod_{i=0}^{k-1} \rho_{t-i} + \varepsilon_{t_0} \prod_{k=0}^{t_0} \rho_{t-k},$$

hence  $\mathbb{E}[\tilde{\varepsilon}_t | \varepsilon_{t_0}] = \varepsilon_{t_0} \prod_{k=0}^{t_0} \rho_{t-k}$  and

$$\mathbb{V}[\tilde{\varepsilon}_t | \varepsilon_{t_0}] = \mathbb{E}[(\tilde{\varepsilon}_t - \mathbb{E}[\tilde{\varepsilon}_t | \varepsilon_{t_0}])^2 | \varepsilon_{t_0}] = \mathbb{E} \left[ \left( \sum_{k=0}^{t-t_0} \chi_{t-k} \prod_{i=0}^{k-1} \rho_{t-i} \right)^2 \right] \mathbb{E}[\tilde{\omega}]^2$$

using independence. In the tax-standard policy case, as  $\rho_t = \rho^{std}$  and  $\chi_t = \chi^{std}$ , it comes  $\mathbb{E}[\tilde{\varepsilon}_t | \varepsilon_{t_0}] = \varepsilon_{t_0} (\rho^{std})^{t-t_0}$  and  $\mathbb{V}[\tilde{\varepsilon}_t | \varepsilon_{t_0}] = (\chi^{std} \sigma_\omega)^2 (1 - (\rho^{std})^{2(t-t_0)+1}) / (1 - (\rho^{std})^2)$ .