Tautological classes on moduli spaces of hyper-Kähler manifolds

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Abstract. We study algebraic cycles on moduli spaces $\mathcal{F}_h$ of h-polarized hyperkähler manifolds. Following previous work of Marian, Oprea and Pandharipande on the tautological conjecture on moduli spaces of K3 surfaces, we first define the tautological ring on $\mathcal{F}_h$. We then study the images of these tautological classes in the cohomology groups of $\mathcal{F}_h$ and prove that most of them are linear combinations of Noether-Lefschetz cycle classes. In particular, we prove the cohomological version of the tautological conjecture on moduli space of $\mathrm{K3}^{[n]}$-type hyperkähler manifolds with $n \leq 2$. Secondly, we prove the cohomological generalized Franchetta conjecture on universal family of these hyperkähler manifolds.

1. Introduction

The tautological ring of the moduli space $\mathcal{M}_g$ of genus $g \geq 2$ curves, originally studied by Mumford, is the subring of the Chow ring generated by κ-classes, which are the ones that appear most naturally in geometry. There has been substantial progress in understanding the tautological ring of moduli spaces of curves in the past decade. In higher dimensional moduli theory, little is known even regarding definitions. Recently, there have been some developments towards the cycle theory on moduli spaces of K3 surfaces. In this paper, we investigate the tautological class problem on moduli spaces of K3 type varieties. In this introduction we specialize our results to the case of moduli spaces of K3 surfaces and refer to the general, more technical, theorems that deal with more general hyperkähler manifolds.

1.1. **MOP conjecture and its generalization.** Let $\mathcal{K}_g$ be the moduli space of primitively polarized K3 surfaces of genus g. The cycle theory on $\mathcal{K}_g$ appears to be much more complicated than on $\mathcal{M}_g$ because there are many more classes in the Chow groups. First of all, there are so called *NL-cycles* on $\mathcal{K}_g$ arising from the Noether-Lefschetz theory. For $1 \leq r \leq 19$, the *r-th higher Noether-Lefschetz locus*

$$\mathcal{N}^r(\mathcal{K}_g) \subseteq \mathcal{K}_g$$

parametrizing the K3 surfaces in $\mathcal{K}_g$ with Picard number greater than r, is a countable union of subvarieties of codimension r. Each irreducible component of $\mathcal{N}^r(\mathcal{K}_g)$ occurs as the moduli space of certain lattice-polarized K3 surfaces in $\mathcal{K}_g$ and a NL-cycle of codimension r is defined as some linear

combination of such irreducible subvarieties. They play a very important role in the cycle theory of $\mathcal{K}_g$. It has been shown that the Picard group $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{K}_g)$ with rational coefficients is generated by NL-divisors, see [6]. This motivates us to study the subring generated by all NL-cycles, denoted by $\mathrm{NL}^*(\mathcal{K}_g)$.

Secondly, in a similar way as in the case of $\mathcal{M}_g$, one can associate natural cycle classes to tautological bundles on the universal family. Let $\mathcal{K}_\Sigma \subset \mathcal{K}_g$ be the r-th higher Noether-Lefschetz locus corresponding to some Picard lattice Σ of rank $r+1$. In [42], Marian, Oprea and Pandharipande (MOP) have introduced κ-classes $\kappa^{\mathcal{B}}_{a_0,\dots,a_r;b}$ on $\mathcal{K}_\Sigma$, see Definition 4.2.1. The *tautological ring* $\mathrm{R}^*(\mathcal{K}_g) \subseteq \mathrm{CH}^*_{\mathbb{Q}}(\mathcal{K}_g)$ is the $\mathbb{Q}$-subalgebra generated by the images of the κ-classes on $\mathcal{K}_\Sigma$ via pushforward maps. The tautological conjecture on $\mathcal{K}_g$ states:

Conjecture 1 (MOP)**.** *Let* $\mathrm{R}^*(\mathcal{K}_g)$ *be the subring of* $\mathrm{CH}^*_{\mathbb{Q}}(\mathcal{K}_g)$ *generated by all the images of the* κ*-classes on* $\mathcal{K}_\Sigma$*. Then* $\mathrm{NL}^*(\mathcal{K}_g) = \mathrm{R}^*(\mathcal{K}_g)$*.*

More generally, as polarized hyperkähler manifolds are higher-dimensional generalizations of algebraic K3 surfaces and behave like K3 surfaces in many ways (cf. §2), it is natural to study tautological rings on moduli spaces of all polarized hyperkähler manifolds. Guided by the K3 case, we define κ-classes on smooth families of hyperkähler manifolds as pushforwards of "Beauville-Voisin classes" on the generic fiber. These classes are named in reference to the work of Beauville and Voisin [4] on weak splitting properties of algebraic hyperkähler manifolds. In this way, we define a tautological ring on each moduli space of polarized hyperkähler manifolds. We then formulate a generalization of Conjecture 1 for polarized hyperkähler manifolds. We will refer to it as the *hyperkähler tautological conjecture*, see Conjecture 4 for the details.

In this paper, we mainly consider the cohomological version of the hyperkähler tautological conjecture. We prove that the cohomological images of most tautological classes are in the span of the Noether-Lefschetz classes. Our main results are Theorems 4.3.1 and 8.3.1. When specialized to the case of K3 surfaces, we obtain:

Theorem 1. *Let* $\mathrm{R}^*_{\mathrm{hom}}(\mathcal{K}_g) \subseteq H^*(\mathcal{K}_g, \mathbb{Q})$ *be the image of* $\mathrm{R}^*(\mathcal{K}_g)$ *in the cohomology ring. Then, we have* $\mathrm{NL}^*_{\mathrm{hom}}(\mathcal{K}_g) = \mathrm{R}^*_{\mathrm{hom}}(\mathcal{K}_g)$*.*

Our approach is to show that all κ-classes on $\mathcal{K}_\Sigma$ are in the span of Noether-Lefschetz cycles on $\mathcal{K}_\Sigma$. However, this actually is not true when $r > 17$ (see Remark 8.3.2). Fortunately, this is not in contradiction with Theorem 1: we will prove that $\mathrm{R}^r_{\mathrm{hom}}(\mathcal{K}_g) = 0$ for $r > 17$.

While we were writing this paper, the article [56] appeared on arXiv. There Conjecture 1 is essentially proved. They use Gromov-Witten theory. Beware however that they work with a slightly different (more canonical) definition of κ-classes.

1.2. **Generalized Franchetta conjecture.** Let $\mathcal{K}_g^\circ$ denote the open and dense subset of $\mathcal{K}_g$ that parametrizes the K3 surfaces with trivial automorphism groups. It carries a universal family $\pi : \mathcal{U}_g^\circ \to \mathcal{K}_g^\circ$. Motivated by Franchetta's conjecture on the moduli spaces of curves, O'Grady raised the following conjecture [52], referred to as the generalized Franchetta conjecture.

Conjecture 2 (O'Grady)**.** *Given a class $\alpha \in \mathrm{CH}^2(\mathcal{U}_g^\circ)$, the restriction $\alpha|_S$ to any closed fiber S is a multiple of the Beauville-Voisin class c_S.*

Here c_S is a canonical class in $\mathrm{CH}_0(S)$ represented by a point on a rational curve of S, which satisfies the following properties:

(1) The intersection of two divisor classes on S lies in $\mathbb{Z}c_S \subset \mathrm{CH}_0(S)$.
(2) The second Chern class $c_2(T_S)$ equals $24c_S \in \mathrm{CH}_0(S)$.

We should point out that, as the K3 surface does not have odd degree cohomology, the famous Bloch-Beilinson conjecture predicts that Conjecture 2, with rational coefficients, holds if α is defined over $\bar{\mathbb{Q}}$. So Conjecture 2 can be viewed as a strengthening of the Bloch-Beilinson conjecture on the universal K3 surface. It has been confirmed when α is spanned by the intersection of two divisors of $\mathcal{U}_g^\circ$ (see [72]) or when $g \leq 12$ (see [57]) but remains open in general. Let $\mathcal{T}_\pi$ be the relative tangent bundle of π. According to [70, Theorem 10.19], over the rationals Conjecture 2 is equivalent to the following

Conjecture 3. *For any $\alpha \in \mathrm{CH}^2(\mathcal{U}_g^\circ)$, there exists $m \in \mathbb{Q}$ such that $\alpha - mc_2(\mathcal{T}_\pi)$ is supported on a proper subvariety of $\mathcal{K}_g^\circ$.*

Our second main result is a cohomological version of the generalized Franchetta conjecture (cf. [72]).

Theorem 1.2.1. *For any $\alpha \in \mathrm{CH}^2(\mathcal{U}_g^\circ)$, there exists a rational number m such that the class $[\alpha - mc_2(\mathcal{T}_\pi)] \in H^4(\mathcal{U}_g^\circ, \mathbb{Q})$ is supported on Noether-Lefschetz divisors. In particular, $\alpha - mc_2(\mathcal{T}_\pi)$ is cohomologically equivalent to zero on $\pi^{-1}(W)$ for some open subset $W \subseteq \mathcal{K}_g^\circ$.*

Again, Theorem 1.2.1 follows from a more general theorem concerning the cohomological generalized Franchetta conjecture on hyperkähler manifolds, see Theorem 8.1.1.

1.3. **Organization of the paper.** In Sections 2 and 3, we review the hyperkähler geometry, especially the cycle theory and the Torelli theorem. The tautological ring on moduli spaces of polarized hyperkählers is defined in Section 4. In Section 5 and Section 6, we recap the work of [7] and [6] about surjectivity results on special cycles of Shimura varieties of orthogonal type. Following [26], in Section 7 we construct a so-called Funke-Kudla-Millson ring in the space of differential forms with coefficients. Section 8 is devoted to the proof of the cohomological tautological conjecture and the generalized Franchetta conjecture. In the last section, we discuss the properties of the ring generated by special cycles on Shimura varieties.

1.4. **Notation and conventions.** Throughout this paper, we write $\widehat{\mathbb{Z}}$ for the profinite completion of $\mathbb{Z}$. We denote by $\mathbb{A}$ the adèle ring of $\mathbb{Q}$ and $\mathbb{A}_f$ the ring of finite adèles. If G is a semisimple classical group over $\mathbb{Q}$, we let $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ be the space of square integrable functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$ and denote by $\mathcal{A}(G)$ (resp. $\mathcal{A}_{\mathrm{cusp}}(G)$) the set of irreducible square integral (resp. cuspidal) representations occuring discretely in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. For a complex variety X, from now on we shall use $\mathrm{CH}^*(X)$ to denote the Chow ring of X with *rational* coefficients and denote by $\mathrm{DCH}^*(X) \subseteq \mathrm{CH}^*(X)$ the subring generated by divisor classes.

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2. Hyperkähler manifolds and moduli

2.1. **Basic theory of hyperkähler manifolds.** A smooth complex compact $2n$-dimensional manifold X is an *irreducible holomorphic symplectic* or *hyperkähler* manifold if it is simply connected and $H^0(X, \Omega_X^2)$ is spanned by an everywhere nondegenerate holomorphic 2-form ω_X. It carries an integral, primitive quadratic form q_X on $H^2(X, \mathbb{Z})$, called the Beauville-Bogomolov (BB) form, which satisfies

(1) q_X is non-degenerate and of signature $(3, b_2(X) - 3)$
(2) There exists a positive rational number c, *the Fujiki invariant*, such that $q_X^n(\alpha) = c \int_X \alpha^{2n}$ for all classes $\alpha \in H^2(X, \mathbb{Z})$.
(3) The Hodge decomposition $H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$ is orthogonal with respect to $q_X \otimes \mathbb{C}$.

From (2), the BB-form q_X is a deformation invariant and we will thus restrict to compact hyperkähler manifolds of a fixed deformation class. In such a class, the isomorphism type, say Λ of the lattice realized by the BB-form is unique. We say an isometry $\varphi : H^2(X, \mathbb{Z}) \to \Lambda$ is a *marking* of X. If we let R denote an abstract ring such that the cohomology ring $H^*(X, \mathbb{Z})$ is isomorphic to R, then we refer to a ring isomorphism

$$\Phi : H^*(X, \mathbb{Z}) \xrightarrow{\sim} R,$$

as a *full marking* of X. If X comes with an ample line bundle H, we say that (X, H) is a *polarized hyperkähler manifold*.

There are few known constructions of such manifolds. The two well-known series of examples, found by Beauville [3], are the Hilbert scheme of points on K3 surfaces and the generalized Kummer varieties. The only other

known examples were constructed by O'Grady [50, 51]. See the details in Example 2.1.1 below.

Example 2.1.1. (i) For $n > 0$, the length n Hilbert scheme $S^{[n]}$ of a K3 surface S is a hyperkähler manifold of dimension $2n$. The second cohomology $H^2(S^{[n]}, \mathbb{Z})$ under BB-form is an even lattice of signature $(3, 20)$ and it is isomorphic to

$$L_n = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2(n-1) \rangle \tag{2.1}$$

where U is the hyperbolic lattice of rank two and E_8 is the positive definite lattice associated to the Lie group of the same name. We say that a hyperkähler manifold X is of $\mathrm{K3}^{[n]}$-*type* if it is deformation equivalent to $S^{[n]}$. In that case the Fujiki invariant is $c = (2n)!/(n!2n)$.

(ii) If A is an abelian surface and $s : A^{[n+1]} \to A$ is the morphism induced by the additive structure of A, then the generalized Kummer variety $K_n(A) := s^{-1}(0)$ is a projective hyperkähler manifold of dimension $2n$. We say that X is of *generalized Kummer type* if it is deformation equivalent to $K_n(A)$. In this case, $H^2(X, \mathbb{Z})$ is isomorphic to

$$L_{K,n} = U^{\oplus 3} \oplus \langle -2(n+1) \rangle \,. \tag{2.2}$$

while the Fujiki invariant is $c = (n+1)(2n)!/(n!2n)$.

(iii) **(OG6)** Let S be a K3 surface, and M the moduli space of stable rank 2 vector bundles on S, with Chern classes $c_1 = 0$, $c_2 = 4$. It admits a natural compactification $\overline{M}$, obtained by adding classes of semi-stable torsion free sheaves. There is a desingularization X of $\overline{M}$ which is a hyperkähler manifold of dimension 10. Its second cohomology is a lattice of signature $(3, 21)$. We say that a hyperkähler manifold is of type **OG6** if it is deformation equivalent to X.

(iv) **(OG10)** A similar construction can be done starting from rank 2 bundles with $c_1 = 0$, $c_2 = 2$ on an abelian surface, giving hyperkähler manifolds of dimension 6 as in (iii). Its second cohomology is a lattice of signature $(3, 5)$.

2.2. **Automorphisms and monodromy group.** Let X be a hyperkähler manifold in the list above and dimension $2n$. Let $\mathrm{Aut}(X)$ be the group of automorphisms of X. Its action on the cohomology gives a morphism

$$\mathrm{Aut}(X) \to \mathrm{GL}(H^*(X, \mathbb{Z})) \tag{2.3}$$

which has finite kernel. We say that X is *cohomologically rigidified* if $\mathrm{Aut}(X)$ acts faithfully on $H^*(X, \mathbb{Z})$, i.e. (2.3) is injective. All the hyperkähler manifolds of Example 2.1.1 are known to be cohomologically rigidified except for the examples of **OG6** type for which it is still an open question (cf. [3, 55]). In general, it remains open whether (2.3) is injective for all hyperkähler manifolds.

The group of automorphisms $\mathrm{Aut}(X)$ moreover acts on $H^2(X, \mathbb{Z})$ preserving the BB-form. The projection map to the second cohomology therefore

yields a morphism

$$\mathrm{Aut}(X) \to \mathrm{O}(H^2(X,\mathbb{Z})). \tag{2.4}$$

with finite kernel. According to [29], this kernel is deformation invariant. It is trivial if X is of $\mathrm{K3}^{[n]}$ or of **OG10** type and it is nontrivial if X is of generalized Kummer type or of **OG6** type.

2.3. The *monodromy group* $\mathrm{Mon}(X) \subseteq \mathrm{GL}(H^*(X,\mathbb{Z}))$ is defined as the subgroup generated by the images of the monodromy representations of all the connected families containing X. Let $\mathrm{Mon}^2(X)$ be the image of the monodromy group into $\mathrm{GL}(H^2(X,\mathbb{Z}))$. Verbitsky [67] has shown that the group $\mathrm{Mon}^2(X)$ is an arithmetic subgroup of $\mathrm{O}(H^2(X,\mathbb{Z}))$ and that there is an exact sequence

$$1 \to T \to \mathrm{Mon}(X) \to \mathrm{Mon}^2(X) \to 1 \tag{2.5}$$

with T a finite group. Moreover, T is trivial if (2.4) is injective (cf. [67, Corollary 7.3]).

Furthermore, given a polarized hyperkähler manifold (X,H), the *polarized monodromy group*

$$\mathrm{Mon}^2(X,H) \subseteq \mathrm{Mon}^2(X)$$

is defined to be the stabilizer of $c_1(H)$. Then $\mathrm{Mon}^2(X,H)$ is isomorphic to an arithmetic subgroup of $\mathrm{O}(\Lambda)$ (cf. [67, Theorem 3.4]) via any given marking. And it has been shown by Markman [44, Proposition 7.1] that the image is independent of the marking.

2.4. **Group actions on cohomology.** Let G_X be the $\mathbb{Q}$-algebraic group associated to $\mathrm{SO}(\Lambda)$. It acts naturally on $H^2(X,\mathbb{Q})$ through the standard representation. As G_X is a connected component of the Zariski closure of $\mathrm{Mon}^2(X)$ and $\mathrm{Mon}(X) \to \mathrm{Mon}^2(X)$ has finite kernel, the monodromy action of $\mathrm{Mon}(X)$ gives rise to an action of G_X on the whole cohomology ring $H^*(X,\mathbb{Q})$ of X via automorphisms (cf. [28, Proposition 4.1]) that extends the natural action of G_X on $H^2(X,\mathbb{Q})$.

This G_X-action on $H^*(X,\mathbb{Q})$ respects the Hodge structure. For instance, the trivial summand in $H^*(X,\mathbb{Q})$ consists of the Hodge classes. In particular, the Chern classes $c_i(T_X)$ of the tangent bundle lie in a trivial summand of $H^{2i}(X,\mathbb{Q})$ and $c_i(T_X) = 0$ is zero when i is odd.

Remark 2.4.1. The theory of Lefschetz modules, developed by Verbitsky [66] and Looijenga-Lunts [40], provides another action of the group

$$\widetilde{G}_X := \mathrm{Spin}(H^2(X,\mathbb{Q}))$$

on $H^*(X,\mathbb{Q})$. The action also preserves the ring structure and on the even cohomology it factors through a representation

$$\rho : G_X \to \mathrm{Aut}(H^{\mathrm{even}}(X,\mathbb{Q})) \tag{2.6}$$

that coincides with the standard representation on $H^2(X,\mathbb{Q})$. Therefore, the two representation of $\mathrm{Mon}^2(X)$ in $\mathrm{GL}(H^*(X,\mathbb{Q}))$ agree on a finite index subgroup (cf. [43, §4.6]).

2.5. **Beauville-Voisin classes.** Recall the following theorem of Beauville and Voisin on the Chow ring of K3 surfaces.

Theorem 2.5.1 (Beauville-Voisin [4])**.** *Let X be a smooth projective K3 surface. Then there exists a canonical zero cycle $c_X \in \mathrm{CH}_0(X)$ of degree one such that*

- *c_X is represented by a point on a rational curve on X*
- *for any divisors $D_1, D_2 \in \mathrm{Pic}(X)$, the intersection $D_1 \cdot D_2$ is proportional to c_X in $\mathrm{CH}_0(X)$.*
- *$c_2(T_X) = 24c_X$.*

In particular, the cohomology class map

$$[\cdot] : \mathrm{DCH}^*(X) \to H^*(X,\mathbb{Q}) \tag{2.7}$$

is injective.

Beauville has pointed out that this can be understood in a broader framework of Bloch-Beilinson-Murre filtration, the so called *weak splitting property*, and he has conjectured that this can be generalized to hyperkähler manifolds, i.e. the map (2.7) is injective when X is hyperkähler. This was later strengthened by Voisin [72] by involving the Chern classes of the tangent bundle T_X, i.e. replacing $\mathrm{DCH}^*(X)$ in (2.7) by the subalgebra $\mathrm{BV}^*(X) \subseteq \mathrm{CH}^*(X)$ generated by all divisor classes and the Chern classes $c_i(T_X)$. We will call the elements in $\mathrm{BV}^*(X)$ the *Beauville-Voisin (BV) classes* of X.

The conjecture has been confirmed by Voisin [71] and Fu [23] when X is a K3 surface, the Fano variety of lines of a cubic fourfold, a generalized Kummer variety, or any $S^{[n]}$ if $n \leq 48 - 2\rho(S)$, where $\rho(S)$ is the Picard number of S. In all these cases, there exists a canonical zero cycle $c_X \in \mathrm{CH}_0(X)$ such that $\mathrm{BV}^n(X)$ is spanned by c_X. In general, the existence of c_X remains wide open. Note that if the canonical cycle c_X exists, then the top Chern class $c_{2n}(T_X)$ is a multiple of c_X.

3. Moduli space of hyperkähler manifolds

3.1. **Moduli of polarized hyperkähler manifolds.** In the sequel we will always assume that a $2n$-dimensional hyperkähler manifold X of type Λ has a primitive polarization H, i.e. $c_1(H) \in H^2(X,\mathbb{Z})$ is primitive. In order to discuss the moduli space of polarized hyperkähler manifolds, we shall choose a polarization type, i.e. the $\mathrm{O}(\Lambda)$-orbit of a primitive vector $h \in \Lambda$. We say that (X,H) is *h-polarized* if $\varphi(c_1(H)) = h$ for a given marking $\varphi : H^2(X,\mathbb{Z}) \cong \Lambda$.

Consider the moduli stack of h-polarized hyperkähler manifolds of dimension $2n$ and type Λ with a given Fujiki invariant, which associates to a

scheme T (over $\mathbb{C}$) the set of isomorphism classes of flat families of polarized hyperkähler manifolds over T. This stack in general is disconnected with only finitely many connected components. Let $\mathcal{F}_h$ be a connected component of this stack. Then a standard result is

Proposition 3.1.1. *$\mathcal{F}_h$ is a smooth Deligne-Mumford quotient stack and it can be coarsely represented by a quasi-projective variety F_h.*

Proof. The proof is the same as in the case of K3 surfaces. The quasi-projectivity of F_h follows from the Torelli theorem for polarized hyperkähler manifolds (cf. [44, 22]) and the standard result of Baily-Borel [2]. ♣

Remark 3.1.2. Here we work over the base field $\mathbb{C}$ only for simplicity. We also refer to [1] for the results on the moduli space of hyperkähler varieties over fields of characteristic 0.

3.2. **Period map.** Let us recall Verbitsky and Markman's Hodge theoretic Torelli theorem for hyperkähler manifolds. Given a polarization type vector $h \in \Lambda$, the orthogonal complement $\Lambda_h := h^\perp \subseteq \Lambda$ is a lattice of signature $(2, b_2(X) - 3)$. The period domain D of h-polarized hyperkähler manifolds of type Λ is a connected component of

$$D^\pm = \{x \in \mathbb{P}(\Lambda_h \otimes \mathbb{C}) \mid (x, x) = 0, (x, \bar{x}) > 0\}, \tag{3.1}$$

seen as a Hermitian symmetric domain of type IV. Consider the connected component of the coarse moduli space of marked hyperkähler manifolds

$$N_h = (X, H, \varphi)/\sim$$

parametrizing marked h-polarized hyperkähler manifolds (X, H, φ) up to equivalence. Then we have a period map

$$\widetilde{\mathcal{P}}_h : N_h \to D \tag{3.2}$$

that maps (X, H, φ) to the line $[\varphi(\omega_X)]$.

Remark 3.2.1. One can also consider the moduli (analytic) stack $\widetilde{\mathcal{N}}_h$ of fully marked h-polarized hyperkähler manifolds and denote by $\widetilde{N}_h$ its coarse moduli space. This also defines a period map $\widetilde{N}_h \to D$. See §3.3 for more details.

Write $\Gamma_h := \mathrm{Mon}^2(X, H)$: then Markman shows that the map (3.2) is a Γ_h-equivariant open immersion (cf. [44]). To forget the marking, we quotient (3.2) on both sides and get

Theorem 3.2.2. [44, Theorem 8.4] *The period map* (3.2) *induces an open immersion*

$$\mathcal{P}_h : F_h \hookrightarrow \Gamma_h \backslash D. \tag{3.3}$$

The analytic orbifold $\Gamma_h \backslash D$ has an algebraic structure and it is a quasiprojective variety. In this paper, we shall sometimes also use $\mathcal{P}_h$ to denote the map from the stack $\mathcal{F}_h$ to $\Gamma_h \backslash D$.

Remark 3.2.3. If we consider the coarse moduli space of quasipolarized hyperkähler manifolds, i.e. the line bundle H is big and nef, the open immersion (3.3) becomes an isomorphism, see [32].

3.3. There is an analytic construction of $\mathcal{F}_h$ via Teichmüller theory. As the hyperkähler manifolds only differ by their complex structures, we can write X as (M, I), where M is a hyperkähler manifold and I is a complex structure on M. Then the moduli space $\mathcal{F}_h$ can be described as the orbifold

$$\mathrm{Tei}_h(M)/\mathrm{MCG}(M, h),$$

given as the quotient of a divisor $\mathrm{Tei}_h(M)$ in a connected component of the Teichmüller space of M by a subgroup $\mathrm{MCG}(M, h)$ of the mapping class group.

The space $\mathrm{Tei}_h(M)$ actually coarsely represents the moduli stack $\widetilde{\mathcal{N}}_h$ in Remark 3.2.1 and the image of the group $\mathrm{MCG}(M, h)$ in $\mathrm{GL}(H^*(M, \mathbb{Z}))$ is nothing but the polarized monodromy group $\mathrm{Mon}(X, H)$. In this way, most of the results we discussed in this section can be found in [67] using the language of Teichmüller spaces.

3.4. **Level structures.** We now introduce the level structures on polarized hyperkähler manifolds to help rigidify our moduli problem. Just as the case for moduli of curves with level structures, this can be viewed as the quotient of $\mathrm{Tei}_h(M)$ by a finite index subgroup of $\mathrm{MCG}(M, h)$. For polarized hyperkähler manifolds, there are two natural ways to add level structures by taking the kernel of either the map

$$\mathrm{MCG}(M, h) \to \mathrm{GL}(H^*_{\mathrm{prim}}(M, \mathbb{Z}/\ell\mathbb{Z})) \tag{3.4}$$

or the map

$$\mathrm{MCG}(M, h) \to \mathrm{O}(H^2_{\mathrm{prim}}(M, \mathbb{Z}/\ell\mathbb{Z})), \tag{3.5}$$

for $\ell > 0$. In this paper, we shall be mainly concerned with the moduli space of polarized hyperkähler manifolds with a level structure on the total cohomology, i.e. the quotient of $\mathrm{Tei}_h(M)$ by the kernel of (3.4).

In the algebraic setting, we say that a *full ℓ-level structure* on a h-polarized hyperkähler (X, H) is an isomorphism

$$H^*(X, \mathbb{Z}/\ell\mathbb{Z}) \cong H^*(M, \mathbb{Z}) \otimes \mathbb{Z}/\ell\mathbb{Z},$$

mapping the class $c_1(H)$ to h. We let $\mathcal{F}_h^\ell$ be the connected component (associated to $\mathcal{F}_h$) of the moduli stack of h-polarized hyperkähler manifold with a full ℓ-level structure. The forgetful map $\mathcal{F}_h^\ell \to \mathcal{F}_h$ is finite and étale.

Denote by

$$\mathrm{Mon}_\ell(X, H) \subseteq \mathrm{Mon}(X, H)$$

the polarized monodromy group with a full ℓ-level structure, which is the image of the kernel of (3.4) in $\mathrm{Mon}(X, H)$. If ℓ is sufficiently large, the projection to the second cohomology yields an isomorphism

$$\mathrm{Mon}_\ell(X, H) \cong \mathrm{Mon}^2_\ell(X, H) \tag{3.6}$$

because of (2.5). We let $\Gamma_h^\ell \subseteq \Gamma_h$ be the image of $\mathrm{Mon}_\ell^2(X,H)$ via markings. Then the coarse moduli space F_h^ℓ of $\mathcal{F}_h^\ell$ is the quotient of the coarse moduli space of fully marked h-polarized hyperkähler manifolds by $\mathrm{Mon}_\ell^2(X,H)$ and it admits an open immersion

$$\mathcal{P}_\ell : F_h^\ell \to \Gamma_h^\ell \backslash D. \tag{3.7}$$

induced by the period map.

If the kernel of (2.3) is trivial, then $\mathcal{F}_h^\ell$ is represented by F_h^ℓ for $\ell \gg 1$ because any object in $\mathcal{F}_h^\ell$ has only trivial automorphisms (cf. [60, Lemma 1.5.12]). In this case, F_h^ℓ carries a universal family in the category of varieties. As mentioned in §2.2, this has been confirmed for all known examples except **OG6**.

3.5. Let us make a few more remarks about the orthogonal group version (3.5). Rizov [60] has dealt with the case of K3 surfaces in adelic language (see also [46, 41]), but there is no difficulty to extend his construction to hyperkähler manifolds, see e.g. [16, §3.3]. The resulting moduli space better fits with our notions of Shimura varieties defined in Section 5.

Write $G = \mathrm{SO}(\Lambda_h)$ and let $K \subseteq G(\mathbb{A}_f)$ be an open compact subgroup. Recall that there is an injective morphism

$$\{g \in \mathrm{SO}(\Lambda) \mid g(h) = h\} \to \mathrm{SO}(\Lambda_h).$$

Following [60, 2.2], we say that K is *admissible* if every element of K can be viewed as an isometry of $\Lambda(\mathbb{A}_f)$ fixing h and stabilizing $\Lambda_h(\widehat{\mathbb{Z}})$. In this case, we define a *K-level structure* on a h-polarized hyperkähler (X,H) as an element of the set

$$K\backslash\{g \in \mathrm{Isometry}(\Lambda(\hat{\mathbb{Z}}), H^2(X,\hat{\mathbb{Z}})(1)) \mid g(h) = c_1(H)\}. \tag{3.8}$$

As in [60, Definition 1.5.16], one can consider the moduli stack $\mathcal{F}_{h,K}$ of primitively h-polarized hyperkähler manifolds with a K-level structure and let $F_{h,K}$ be the corresponding coarse moduli space. The period map gives an open immersion

$$F_{h,K} \hookrightarrow \Gamma_{h,K}\backslash D, \tag{3.9}$$

where $\Gamma_{h,K} = \Gamma_h \cap K$ (cf. [60, Proposition 3.2.1]). Again, this follows from Verbitsky-Markman's Hodge theoretic Torelli theorem on hyperkähler manifolds.

3.6. **Descent of local systems.** Let $\pi : \mathcal{U}_h \to \mathcal{F}_h$ be the universal family of h-polarized hyperkähler manifolds over $\mathcal{F}_h$. We denote by

$$\mathbb{H}_\pi := R\pi_*\mathbb{Q}$$

the local system that fiberwise corresponds to the cohomology ring of the fiber. As we explained in §2.3, the special orthogonal group $\mathrm{SO}(\Lambda_h)$ (up to a finite covering) acts fiberwise on $\mathbb{H}_\pi$. As $\mathrm{SO}(\Lambda_h)$ acts naturally on D, our goal is to show that the local system $\mathbb{H}_\pi$ descends to an automorphic local

system on $\Gamma_h\backslash D$ after taking a finite cover of $\mathcal{F}_h$. We prove it by adding level structures defined in §3.4.

Lemma 3.6.1. *Consider the universal family*

$$\pi_\ell : \mathcal{U}_h^\ell \to \mathcal{F}_h^\ell \tag{3.10}$$

of h-polarized hyperkähler manifolds with a full ℓ-level structure. When ℓ is sufficiently large, the local system $\mathbb{H}_{\pi_\ell} := R(\pi_\ell)_*\mathbb{Q}$ descends to an automorphic local system $\mathbf{H}^\bullet$ on $\Gamma_h^\ell\backslash D$ via the period map (3.7).

Proof. We can consider the universal fibration $\widetilde{\pi} : U \to \widetilde{\mathcal{N}}_h$ over $\widetilde{\mathcal{N}}_h$. Let $\mathbb{H}_{\widetilde{\pi}}$ be the associated local system. One can easily see that the local system $\mathbb{H}_{\widetilde{\pi}}$ is a constant system on $\widetilde{\mathcal{N}}_h$ and it descends to the constant system

$$D \times H^*(M, \mathbb{Q})$$

on D via the period map (3.2) (cf. [45]).

For ℓ sufficiently large, the action of $\Gamma_h^\ell \cong \mathrm{Mon}_\ell(X, H)$ on $\widetilde{\mathcal{N}}_h$ naturally lifts to an action on $\mathbb{H}_{\widetilde{\pi}}$. After taking the quotient by Γ_h^ℓ, it gives the locally constant system $\mathbb{H}_\pi$ on $\mathcal{F}_h^\ell$. On the other hand, the map (3.7) is obtained by passing to the quotient on both sides of (3.2). It follows that $\mathbb{H}_\pi$ descends to the automorphic bundle $\mathbf{H}^\bullet := \Gamma_\ell\backslash(D \times H^*(M, \mathbb{Q}))$ via (3.7). ♣

3.7. **Moduli of lattice polarized hyperkähler.** We have to introduce the lattice polarized hyperkähler manifolds for later use. The moduli spaces of lattice polarized hyperkähler manifolds are briefly studied by Camere in [15, 14], which generalize the work of Dolgachev [20] for K3 surfaces. Here we review the basic notions with certain modifications. Let

$$j : \Sigma \hookrightarrow \Lambda$$

be a fixed primitive embedding of a lattice Σ with signature $(1, r)$. An ample Σ-*polarized hyperkähler manifold* (with respect to j) is a pair (X, ϕ), where X is a hyperkähler manifold and

$$\phi : \Sigma \to \mathrm{NS}(X) \subset H^2(X, \mathbb{Z})$$

is a primitive lattice embedding satisfying

- $\phi(\Sigma)$ contains an ample divisor class of X,
- there exists a marking φ such that $\varphi \circ \phi = j$.

If we fix an ample class h, we can define the so called *h-ample* Σ-*polarized hyperkähler manifold (with respect to j)* as a triple (X, H, ϕ) satisfying that (X, H) is h-polarized and $\phi(\Sigma)$ contains $c_1(H)$. Denote by $\mathcal{F}_{\Sigma,h}$ a connected component of the moduli stack of h-ample Σ-polarized hyperkähler manifolds. It comes with the natural forgetful map $\mathcal{F}_{\Sigma,h} \to \mathcal{F}_h$.

There is also a Hodge theoretic description of $\mathcal{F}_{\Sigma,h}$ via the Torelli theorem. Regard Σ as a sublattice of Λ. Let the restricted period domain $D_\Sigma \subseteq D$ be defined as a connected component of

$$\{x \in \mathbb{P}(\Sigma^\perp \otimes \mathbb{C}) \mid (x, x) = 0, (x, \bar{x}) > 0\}.$$

Let Γ_Σ be the stabilizer of $\phi(\Sigma)$ in the monodromy group $\mathrm{Mon}^2(X)$. It is an arithmetic subgroup of $\mathrm{O}(\Sigma^\perp)$. As before, let $F_{\Sigma,h}$ be the corresponding coarse moduli space. Then the restriction of the period map on the moduli space of marked polarized hyperkähler manifolds gives a map

$$\mathcal{P}_\Sigma : F_{\Sigma,h} \to \Gamma_\Sigma \backslash D_\Sigma$$

which is an open immersion, and we have a commutative diagram

$$\begin{array}{ccc} F_{\Sigma,h} & \xrightarrow{\mathcal{P}_\Sigma} & \Gamma_\Sigma \backslash D_\Sigma \\ \downarrow & & \downarrow \\ F_h & \xrightarrow{\mathcal{P}_h} & \Gamma_h \backslash D. \end{array} \tag{3.11}$$

Remark 3.7.1. In general, the definition of $\mathcal{F}_{\Sigma,h}$ depends on the primitive embedding $j : \Sigma \to \Lambda$. In some cases, e.g. when Λ is unimodular, the primitive embedding of Σ into Λ is unique up to isometry and $\mathcal{F}_{\Sigma,h}$ is thus independent of the choice of j.

Finally, one can easily see that the results in §3.4 and §3.6 naturally extend to lattice polarized hyperkähler manifolds. We shall in particular denote by $\mathcal{F}^\ell_{\Sigma,h}$ (resp. $F^\ell_{\Sigma,h}$) the moduli stack (resp. the coarse moduli space) of h-ample Σ-polarized hyperkähler manifolds with a full ℓ-level structure. Then there is an open immersion

$$\mathcal{P}^\ell_\Sigma : F^\ell_{\Sigma,h} \hookrightarrow \Gamma^\ell_\Sigma \backslash D_\Sigma, \tag{3.12}$$

for some finite index subgroup $\Gamma^\ell_\Sigma \subseteq \Gamma_\Sigma$ and the associated local systems on $F^\ell_{\Sigma,h}$ descend via $\mathcal{P}^\ell_\Sigma$.

4. Cycle classes on moduli spaces of hyperkähler manifolds

As in the case of K3 surfaces, there exist natural cycle classes on $\mathcal{F}_h$ and its universal family, which come from either Hodge theory or the geometry of the generic fiber. In this section, we discuss the relations between these cycles and define the tautological ring on the moduli space. Natural conjectures are made, motived by the MOP conjecture and generalized Franchetta conjecture.

4.1. Noether-Lefschetz cycles.

We first recall the Noether-Lefschetz theory for hyperkähler manifolds. Let $\pi : \mathcal{U} \to \mathcal{F}$ be a family of projective hyperkähler manifolds over $\mathcal{F}$. When $\mathcal{F}$ is a quasiprojective variety, we define the r-th *Noether-Lefschetz locus on* $\mathcal{F}$ (with respect to π) as

$$\mathcal{N}^r(\mathcal{F}) := \{b \in \mathcal{F} \mid \mathrm{rank}(\mathrm{Pic}(\mathcal{U}_b)) \geq r + \rho\}. \tag{4.1}$$

where ρ is the Picard number of the generic fiber of π. We can also define the Noether-Lefschetz loci with respect to the universal family $\pi : \mathcal{U}_h \to \mathcal{F}_h$ over

the moduli stack $\mathcal{F}_h$. The irreducible components of the Noether-Lefschetz loci on $\mathcal{F}_h$ correspond to the proper maps

$$\mathcal{F}_{\Sigma,h} \to \mathcal{F}_h, \tag{4.2}$$

associated to Lorentzian lattices $\Sigma \hookrightarrow \Lambda$ containing h. In all cases, we define

$$\mathrm{NL}^*_\pi(\mathcal{F}) \subseteq \mathrm{CH}^*(\mathcal{F}) \tag{4.3}$$

as the subalgebra generated by the Noether-Lefschetz locus on $\mathcal{F}$ (cf. [56]). We may omit the subscript π when $\mathcal{F}$ is the moduli stack.

Proposition 4.1.1 (see [6])**.** *The first Chern class of the Hodge line bundle is in the span of NL-divisors. Moreover, let $m_\Lambda = \mathrm{rank}\,\Lambda$ be the second Betti number of hyperkähler manifolds in $\mathcal{F}_{\Sigma,h}$ and assume that $m_\Lambda - \mathrm{rank}(\Sigma) \geq 5$: then*

$$\mathrm{NL}^1(\mathcal{F}_{\Sigma,h}) = \mathrm{Pic}(\mathcal{F}_{\Sigma,h}).$$

In particular, we have an inclusion $\mathrm{DCH}^*(\mathcal{F}_{\Sigma,h}) \subseteq \mathrm{NL}^*(\mathcal{F}_{\Sigma,h})$.

Let us roughly explain the proof. It actually relies on the study of Picard groups on the coarse moduli space $F_{\Sigma,h}$ as we have $\mathrm{Pic}(\mathcal{F}_{\Sigma,h}) \cong \mathrm{Pic}(F_{\Sigma,h})$. In [6, Theorem 1.5], we deal with orthogonal Shimura varieties whereas $F_{\Sigma,h}$ is only an open subset of $\Gamma_\Sigma \backslash D_\Sigma$. However the map

$$\mathcal{P}^*_\Sigma : \mathrm{Pic}(\Gamma_\Sigma \backslash D_\Sigma) \to \mathrm{Pic}(F_{\Sigma,h})$$

is onto and we are easily reduced to $\Gamma_\Sigma \backslash D_\Sigma$ – see Proposition 5.3.4 below for details. Then we can conclude the assertion by applying [6, Theorem 1.5] and the fact $\mathrm{Pic}(\Gamma_\Sigma \backslash D_\Sigma) \cong H^{1,1}(\Gamma \backslash D_\Sigma)$ (cf. [30]). This is similar to the proof of [6, Corollary 3.8]. Note that the assumption $m_\Lambda - \mathrm{rank}(\Sigma) \geq 5$ is equivalent to $\dim D_\Sigma \geq 3$; this is used to apply [6, Theorem 1.5].

4.2. **Kappa classes and tautological ring.** In this subsection, we construct the κ-classes on $\mathcal{F}_{\Sigma,h}$ by taking the pushforward of BV classes on the generic fiber of universal families of hyperkähler manifolds of (complex) dimension n. This idea was initiated in [42] in the case of K3 surfaces. To work in greater generality, we first define κ-classes with respect to an arbitrary smooth family.

Definition 4.2.1. Let $\pi : \mathcal{U} \to \mathcal{F}$ be a smooth projective family of hyperkähler manifolds over a smooth Deligne-Mumford stack. Let

$$\mathcal{B} = \{\mathcal{L}_0, \ldots, \mathcal{L}_r\} \subseteq \mathrm{Pic}(\mathcal{U})$$

be a collection of line bundles whose image in $\mathrm{Pic}(\mathcal{U}/\mathcal{F})$ form a basis. Then we define the *κ-classes*

$$\kappa^{\mathcal{B}}_{a_0,\ldots,a_r;b_1,\ldots,b_{2n}} = \pi_*\Big(\prod_{i=0}^{r} c_1(\mathcal{L}_i)^{a_i} \prod_{j=1}^{2n} c_j(T_\pi)^{b_j}\Big) \in \mathrm{CH}^m(\mathcal{F}), \tag{4.4}$$

where $m = \sum_{i=0}^{r} a_i + \sum_{j=1}^{2n} jb_j - 2n$. For simplicity, we may write

$$\kappa^{\mathcal{B}}_{a_0,\dots,a_r} = \kappa^{\mathcal{B}}_{a_0,\dots,a_r;0,\dots,0}. \tag{4.5}$$

and refer to these as the *special κ-classes*.

In particular, we can define κ-classes and also special κ-classes on $\mathcal{F}_{\Sigma,h}$ as the κ-classes with respect to the universal family $\pi_\Sigma : \mathcal{U}_\Sigma \to \mathcal{F}_{\Sigma,h}$.

Definition 4.2.2. The *tautological ring* $\mathrm{R}^*(\mathcal{F}_h) \subseteq \mathrm{CH}^*(\mathcal{F}_h)$ of $\mathcal{F}_h$ is defined to be the subalgebra generated by the images of all κ-classes on $\mathcal{F}_{\Sigma,h}$ via the pushforward maps

$$(i_\Sigma)_* : \mathrm{CH}_k(\mathcal{F}_{\Sigma,h}) \to \mathrm{CH}_k(\mathcal{F}_h) \tag{4.6}$$

for all Σ. Instead, if we only use the images of the *special* κ-classes on $\mathcal{F}_{\Sigma,h}$ via (4.6), we obtain a subring $\mathrm{DR}^*(\mathcal{F}_h) \subseteq \mathrm{R}^*(\mathcal{F}_h)$.

Clearly, we have natural inclusions

$$\mathrm{NL}^*(\mathcal{F}_h) \subseteq \mathrm{DR}^*(\mathcal{F}_h) \subseteq \mathrm{R}^*(\mathcal{F}_h). \tag{4.7}$$

Note that we can vary the choices $\mathcal{B}$ of universal line bundles on $\mathcal{U}_\Sigma$. (These choices differ by the pullback of line bundles on $\mathcal{F}_{\Sigma,h}$.) It follows that $\mathrm{DR}^*(\mathcal{F}_h)$ and hence $\mathrm{R}^*(\mathcal{F}_h)$ contains all classes in $\mathrm{DCH}^*(\mathcal{F}_{\Sigma,h})$ via (4.6). Then we propose the following generalization of the MOP conjecture:

Conjecture 4 (Hyperkähler Tautological Conjecture)**.**

$$\mathrm{NL}^*(\mathcal{F}_h) = \mathrm{R}^*(\mathcal{F}_h).$$

We can obviously generalize Conjecture 4 to moduli spaces with level structures. One advantage of adding level structures is that usually we will deal with a smooth variety.

4.3. **Cohomological tautological conjecture.** We shall mainly consider a weak, or cohomological, version of the hyperkähler tautological conjecture that we now describe. Let $H^*(\mathcal{F}_h, \mathbb{Q})$ denote the singular cohomology of $\mathcal{F}_h$; it is isomorphic to the singular cohomology of the coarse moduli space F_h (cf. [5]). There is a cycle class map

$$cl : \mathrm{CH}^*(\mathcal{F}_h) \to H^*(\mathcal{F}_h, \mathbb{Q}) \tag{4.8}$$

and we add subscript hom to denote the image of the corresponding ring in $H^*(\mathcal{F}_h, \mathbb{Q})$ via the cycle class map. Then we have the cohomological tautological conjecture (CTC).

Conjecture 5. $\mathrm{NL}^*_{\mathrm{hom}}(\mathcal{F}_h) = \mathrm{R}^*_{\mathrm{hom}}(\mathcal{F}_h)$.

One of our main results is

Theorem 4.3.1. *Assume that the lattice Λ is of rank $m_\Lambda \geq 6$, or equivalently that $\dim \mathcal{F}_h \geq 3$. Then we have*

$$\mathrm{NL}^*_{\mathrm{hom}}(\mathcal{F}_h) = \mathrm{DR}^*_{\mathrm{hom}}(\mathcal{F}_h). \tag{4.9}$$

Moreover, if $n < \frac{m_\Lambda - 3}{8}$, then we have

$$\mathrm{NL}^*_{\mathrm{hom}}(\mathcal{F}_h) = \mathrm{R}^*_{\mathrm{hom}}(\mathcal{F}_h). \tag{4.10}$$

In particular, Conjecture 5 holds when Λ is of $\mathrm{K3}^{[n]}$-type with $n \leq 2$.

We prove Theorem 4.3.1 in Section 8. In fact, we prove a more general result concerning κ-classes with respect to general families of hyperkähler manifolds. Unfortunately, our results can not imply the full Conjecture 5 for hyperkähler manifolds of generalized Kummer type, **OG6**, **OG10**, or $\mathrm{K3}^{[n]}$-type with large n, because their second Betti number is too small. Instead, we show that a large part of the tautological classes is lying in $\mathrm{R}^*_{\mathrm{hom}}(\mathcal{F}_h)$.

4.4. **Examples.** As a complement to Theorem 4.3.1, we follow [65] to give some examples of κ-classes on $\mathcal{F}_h$ with $a_i = 0$ and $b_j \neq 0$ lying in $\mathrm{NL}^*(\mathcal{F}_h)$. For simplicity, we may write

$$\tilde{\kappa}_{b_1,\ldots,b_{2n}} = \kappa_{0,\ldots,0;b_1,\ldots;b_{2n}}.$$

Let $\Omega^i_{\mathcal{U}_h/\mathcal{F}_h}$ be the relative sheaf of differential i-forms. We denote by $\lambda = c_1(R^0\pi_*\Omega^2_{\mathcal{U}_h/\mathcal{F}_h})$ the first Chern class of the line bundle on $\mathcal{F}_h$ that fiberwise corresponds to the vector space spanned by the non-degenerate holomorphic 2-form of the fiber. By [6, Corollary 8.5], we know that $\lambda \in \mathrm{NL}^*(\mathcal{F}_h)$. One can easily get that

$$c_1(R^0\pi_*\Omega^{2i}_{\mathcal{U}_h/\mathcal{F}_h}) = i\lambda \in \mathrm{NL}^*(\mathcal{F}_h).$$

Now, note that there is an isomorphism $\wedge^i(R^2\pi_*\mathcal{O}_{\mathcal{U}_h}) \cong R^{2i}\pi_*\mathcal{O}_{\mathcal{U}_h}$ via cup product and an isomorphism $R^{2n}\pi_*\mathcal{O}_{\mathcal{U}_h} \cong (R^0\pi_*\Omega_\pi)^\vee$. This yields

$$c_1(R^{2i}\pi_*\mathcal{O}_{\mathcal{U}_h}) = -i\lambda.$$

Using the Grothendieck-Riemann-Roch Theorem, we have

$$\begin{aligned} \mathrm{ch}(\pi_!\mathcal{O}_{\mathcal{U}_h}) &= e^{\sum_{i=0}^{n}(-i\lambda)} \\ &= \pi_*(\mathrm{Td}(\Omega^\vee_{\mathcal{U}_h/\mathcal{F}_h})). \end{aligned} \tag{4.11}$$

Let td_i be the i-th Todd class of $(\Omega^1_{\mathcal{U}_h/\mathcal{F}_h})^\vee = T_{\mathcal{U}_h/\mathcal{F}_h}$. By definition, the class

$$\pi_*(\mathrm{td}_i) \in \mathrm{CH}^{i-2n}(\mathcal{F}_h)$$

is a linear combination of $\tilde{\kappa}_{b_1\ldots,b_n}$ and we find

$$\pi_*(\mathrm{td}_{i+2n}) = \sum_{j=0}^{n} \frac{(-j\lambda)^i}{i!} \tag{4.12}$$

by comparing terms in (4.11) of degree i. The first Chern class $c_1(\Omega^1_{\mathcal{U}_h/\mathcal{F}_h})$ is the pull back of a divisor class on $\mathcal{F}_h$. By the projection formula, this gives

$$\tilde{\kappa}_{b_1,\dots,b_{2n}} = 0 \tag{4.13}$$

if $\sum_{j\geq 2} jb_j < 2n$ or $\sum_{j\geq 2} jb_j = 2n$ with some $b_{2k+1} \neq 0$. When $n = 1$, the relations in (4.12) and (4.13) suffice to show that all classes $\tilde{\kappa}_{b_1\dots,b_{2n}}$ lie in the subalgebra $\mathbb{Q}\langle\lambda\rangle$ of $\mathrm{CH}^*(\mathcal{F}_h)$ generated by λ (cf. [65]). However, it seems more complicated as n gets larger. For instance, let us compute the first few terms of $\tilde{\kappa}_{b_1,\dots,b_4}$ (i.e. $n = 2$). Using (4.12), we have

- $i = 0$, $\frac{1}{720}(3\kappa_{0,2,0,0} - \kappa_{0,0,0,1}) = 3$;
- $i = 1$, $c_1(\Omega^1_{\mathcal{U}_h/\mathcal{F}_h}) = 2\pi^*(\lambda)$;
- $i = 2$, $\frac{1}{60480}(\tilde{\kappa}_{0,3,0,0} - 9\tilde{\kappa}_{0,1,0,1} - 5\tilde{\kappa}_{2,0,0,1} + 11\tilde{\kappa}_{2,2,0,0}) = \frac{5\lambda^2}{2}$

$\vdots$

Let $\chi = \chi(X)$ be the Euler characteristic of $X \in \mathcal{F}_h$, e.g. $\chi(S^{[2]}) = 324$. It follows that

$$\tilde{\kappa}_{0,0,0,1} = \chi,\ \tilde{\kappa}_{0,2,0,0} = 720 + \frac{\chi}{3},\ \tilde{\kappa}_{2,0,0,1} = 4\chi\lambda^2,\ \tilde{\kappa}_{2,2,0,0} = (2880 + \frac{4\chi}{3})\lambda^2;$$

and

$$\tilde{\kappa}_{0,3,0,0} - 9\tilde{\kappa}_{0,1,0,1} = (\frac{42}{84} + \frac{11\chi}{45360})\lambda^2.$$

It would be interesting to know if all such classes lie in $\mathbb{Q}\langle\lambda\rangle$.

4.5. **Generalized Franchetta conjecture for hyperkählers.** Assume that general members in $\mathcal{F}_h$ have trivial group of automorphisms. Then let $\mathcal{F}_h^\circ \subseteq \mathcal{F}_h$ denote the open subset that parametrizes the hyperkähler manifolds with trivial automorphism groups. It carries a universal family

$$\pi^\circ : \mathcal{U}_h^\circ \to \mathcal{F}_h^\circ.$$

Motivated by O'Grady's generalized Franchetta conjecture we formulate the following conjecture for $\mathcal{U}_h^\circ$. We will referred to it also as the generalized Franchetta conjecture.

Conjecture 6. *Let $\mathcal{T}_{\pi^\circ}$ be the relative tangent bundle of $\pi^\circ : \mathcal{U}_h^\circ \to \mathcal{F}_h^\circ$. Then given any class $\alpha \in \mathrm{CH}^{2n}(\mathcal{U}_h^\circ)$, there exists $m \in \mathbb{Q}$ such that $\alpha - mc_{2n}(T_{\pi^\circ})$ is supported on a proper subvariety of $\mathcal{F}_h^\circ$.*

After our work, Fu, Laterveer, Vial and Shen have strengthened Conjecture 2 to deal with relative Chow groups of $\mathcal{U}_h^\circ \to \mathcal{F}_h^\circ$ in all codimensions.

Conjecture 7. [24, Conjecture 1.3] *For any $\alpha \in \mathrm{CH}^i(\mathcal{U}_h^\circ)$, then the restriction of α to the very general fiber of $\mathcal{U}_h^\circ \to \mathcal{F}_h^\circ$ is rationally equivalent to zero if and only it is homologus equivalent to zero.*

In [24], Conjecture 7 has been proved for some polarized $K3^{[n]}$-type hyperähler manifolds. Their idea is similar as [57], which needs the concrete projective models of hyperkähler manifolds. In general, the conjectures are all widely open.

The conjecture can be stated for the moduli stack. Then the assumption on the triviality of the automorphism group of the generic fiber of the universal family can be removed. In §8.1, we prove the cohomological version of this conjecture for hyperkähler manifolds whose second Betti number is sufficiently large (with respect to the dimension).

4.6. **Tautological classes in Deligne-Beilinson Cohomology.** It is actually more natural to consider these questions in the Deligne-Beilinson cohomology of our moduli spaces. This is motivated by the theory of Bloch-Beilinson. Let X be a smooth quasi-projective variety over $\bar{\mathbb{Q}}$ and let $X_{\mathbb{C}}$ be the corresponding complex variety. Let $H^{\bullet}_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{Z}(p))$ be the Deligne-Beilinson cohomology of a smooth variety $X_{\mathbb{C}}$; it is defined as the hypercohomology of the Deligne-Beilinson complex (cf. [21]). There is a composite cycle class map

$$cl_{\mathcal{D}} : \mathrm{CH}^k(X) \to \mathrm{CH}^k(X_{\mathbb{C}}) \to H^{2k}_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{Z}(k)) \otimes \mathbb{Q}. \tag{4.14}$$

Then the Bloch-Beilinson Conjecture and the Hodge conjecture predict that this map is injective (cf. [17, Proposition 3.2.6], see also [36, 37]). As both our moduli spaces and our tautological classes are defined over number fields, both Conjecture 4 and Conjecture 6 would follow from a proof of their Deligne-Beilinson cohomological versions and the injectivity of (4.14). All these conjectures remain wide open in general. In a sequel to this paper, we plan to study the Deligne-Beilinson cohomological versions of Conjectures 4 and 6 for moduli spaces of K3 surfaces.

5. Shimura varieties of orthogonal type

In this section, we discuss Kudla-Millson's construction of special cycles on Shimura varieties of orthogonal type as well as its generalization by Funke and Millson to special cycles with coefficients. We also discuss the connection with Noether-Lefschetz cycles on moduli spaces of polarized hyperkähler manifolds.

5.1. **Shimura varieties of orthogonal type.** Let V be a non-degenerate quadratic space over $\mathbb{Q}$ of signature $(2, b)$ and let $G = \mathrm{SO}(V)$ be the corresponding special orthogonal group. The group $G(\mathbb{R})$ of real points of G is isomorphic to $\mathrm{SO}(2, b)$. We denote by $G(\mathbb{R})_+ \cong \mathrm{SO}_0(2, b)$ the component of the identity; recall that it is precisely the kernel of the spinor norm. Let $K_{\mathbb{R}} \cong \mathrm{SO}(2) \times \mathrm{SO}(b)$ be a maximal compact subgroup of $G(\mathbb{R})_+$ and let $\widehat{D} = G(\mathbb{R})/K_{\mathbb{R}}$. We denote by D the connected component of $\widehat{D}$ associated to $G(\mathbb{R})_+$. We have $D \cong \mathrm{SO}_0(2, b)/\mathrm{SO}(2) \times \mathrm{SO}(b)$, it is a Hermitian symmetric domain of dimension b.

Let $\widetilde{G}$ be the general spin group $\mathrm{GSpin}(V)$ associated to V. For any compact open subgroup $K \subseteq G(\mathbb{A}_f)$, we set $\widetilde{K}$ to be its preimage in $\widetilde{G}(\mathbb{A}_f)$ and denote by X_K the double coset space

$$\widetilde{G}(\mathbb{Q})\backslash \widehat{D} \times \widetilde{G}(\mathbb{A}_f)/\widetilde{K}.$$

Let $\widetilde{G}(\mathbb{Q})_+ \subseteq \widetilde{G}(\mathbb{Q})$ be the subgroup consisting of elements with totally positive spinor norm, which can be viewed as the subgroup of $\widetilde{G}(\mathbb{Q})$ lying in the identity component of the adjoint group of $\widetilde{G}(\mathbb{R})$. Write

$$\widetilde{G}(\mathbb{A}_f) = \coprod_j \widetilde{G}(\mathbb{Q})_+ g_j \widetilde{K}.$$

The decomposition of X_K into connected components is

$$X_K = \coprod_{g_j} \Gamma_{g_j}\backslash D,$$

where Γ_{g_j} is the image of $\widetilde{G}(\mathbb{Q})_+ \cap g_j \tilde{K} g_j^{-1}$ in $\mathrm{SO}_0(2,b)$. When $g_j = 1$, we denote by Γ_K the arithmetic group $\Gamma_1 = K \cap G(\mathbb{Q})$ and $Y_K = \Gamma_K\backslash D$ the connected component of X_K.

5.2. **Connected cycles with coefficients.** Let U be a $\mathbb{Q}$-subspace of V with $\dim U = r$ and such that the quadratic form is negative definite when restricted to U. Let $\widehat{D}_U \subset \widehat{D}$ be the subset consisting of the 2-planes that are perpendicular to U and let $D_U = \widehat{D}_U \cap D$.

For a fixed compact open subgroup $K \subseteq G(\mathbb{A}_f)$ and an element $g \in \widetilde{G}(\mathbb{A}_f)$ we let Γ_g be the image of $\widetilde{G}(\mathbb{Q})_+ \cap g\tilde{K}g^{-1}$ in $\mathrm{SO}_0(2,b)$. Denote by $\Gamma_{g,U}$ the image in $\mathrm{SO}_0(2,b)$ of the pointwise stabilizer of U in $\widetilde{G}(\mathbb{Q})_+ \cap g\tilde{K}g^{-1}$. We then have a map

$$\Gamma_{g,U}\backslash D_U \longrightarrow \Gamma_g\backslash D. \tag{5.1}$$

Following Kudla [34] we will denote this connected cycle (with trivial coefficients) by $c(U,g,K)$; it is of codimension r. We let $\mathrm{SC}^r(Y_K) \subseteq \mathrm{CH}^r(Y_K)$ be the subgroup spanned by connected cycles $c(U,1,K)$ of codimension r and set

$$\mathrm{SC}^*(Y_K) \subseteq \mathrm{CH}^*(Y_K)$$

to be the subalgebra generated by connected cycles of all codimensions. We list some simple properties of connected cycles.

- Let $K' \subseteq K$ be an open compact subgroup and let $f : Y_{K'} \to Y_K$ be the covering map. Then

$$f(c(U,1,K')) = c(U,1,K) \text{ and } f^*(\mathrm{SC}^r(Y_K)) \subseteq \mathrm{SC}^r(Y_{K'}). \tag{5.2}$$

- For the map $\iota : \Gamma_{1,U}\backslash D_U \to Y_K$ in (5.1), we have

$$\iota_*(\mathrm{SC}_k(\Gamma_{1,U}\backslash D_U)) \subseteq \mathrm{SC}_k(Y_K). \tag{5.3}$$

  Here we write $\mathrm{SC}_k(X) := \mathrm{SC}^{\dim X - k}(X)$.

5.3. Now we give some description of the ring $\mathrm{SC}^*(Y_K)$. Firstly, besides the connected cycles, there is the Hodge line bundle on Y_K. We denote by λ its first Chern class in $\mathrm{CH}^1(Y_K)$. It is preserved under the pullback via (5.1).

Lemma 5.3.1. [6, Corollary 8.7] *The class* $\lambda \in \mathrm{SC}^1(Y_K)$.

This yields the following

Theorem 5.3.2. $\mathrm{SC}^*(Y_K) = \bigoplus_r \mathrm{SC}^r(Y_K)$.

Proof. We need to show that the image of the intersection product

$$\mathrm{SC}^{r_1}(Y_K) \times \mathrm{SC}^{r_2}(Y_K) \xrightarrow{\cap} \mathrm{SC}^*(Y_K)$$

lies in $\mathrm{SC}^{r_1+r_2}(Y_K)$. Let us first consider the case when $r_1 = 1$. For any two connected cycles $\alpha = c(U_1, 1, K) \in \mathrm{SC}^{r_1}(Y_K)$ and $\beta = c(U_2, 1, K) \in \mathrm{SC}^{r_2}(Y_K)$, there are two possibilities:

(1) If α and β intersect properly, then $\alpha \cdot \beta$ is a linear combination of connected cycles $c(U, 1, K)$ where U is spanned by U_2 and an element in the Γ_K-orbit of U_1.

(2) If β is contained in α, then we have

$$\alpha \cdot \beta \in \langle -\lambda \cdot \beta \rangle$$

via a standard computation of the normal bundle of a connected cycle of codimension one (cf. [54, Lemma 1.2] [53]). By Lemma 5.3.1 and (5.3), we know that

$$-\lambda \cdot \beta \in \mathrm{SC}^{1+r_2}(Y_K).$$

It follows that $\alpha \cdot \beta \in \mathrm{SC}^{1+r_2}(Y_K)$.

This further gives

$$\alpha_1 \cdot \alpha_2 \cdots \alpha_r \cdot \gamma \in \mathrm{SC}^{r+r_2}(Y_K) \tag{5.4}$$

for any $\alpha_1, \ldots, \alpha_r \in \mathrm{SC}^1(Y_K)$ and $\gamma \in \mathrm{SC}^{r_2}(Y_K)$.

When $r_1 > 1$, given a connected cycle $c(U_1, 1, K)$ of codimension r_1 on Y_K, an easy inductive argument shows that there exists an open compact subgroup $K' \subseteq K$ such that the connected cycle

$$c(U_1, 1, K') \subseteq Y_{K'}$$

can be written as an intersection of r_1 connected cycles of codimension one, i.e.

$$c(U_1, 1, K') = c(W_1, 1, K') \cdot c(W_2, 1, K') \cdots c(W_{r_1}, 1, K'), \tag{5.5}$$

for some subspaces W_j with $\dim W_j = 1$. If $f : Y_{K'} \to Y_K$ is the covering map, then

$$c(U_1, 1, K') \cdot f^*(\beta) \in \mathrm{SC}^{r_1+r_2}(Y_{K'}), \quad \forall \beta \in \mathrm{SC}^{r_2}(Y_K) \tag{5.6}$$

by (5.2) and (5.4). As $f(c(U_1, 1, K')) = c(U_1, 1, K)$, the projection formula gives $c(U_1, 1, K) \cdot \beta \in \mathrm{SC}^{r_1+r_2}(Y_k)$.

♣

Remark 5.3.3. Similar questions have been asked by Kudla in [34], where one consider the ring generated by special cycles instead of connected cycles. We believe our argument might still work.

To conclude this paragraph we relate connected cycles to Noether-Lefschetz cycles via the period map $\mathcal{P}_h : \mathcal{F}_h \to \Gamma_h\backslash D$. The following proposition is then straightforward.

Proposition 5.3.4. *The irreducible Noether-Lefschetz cycles are the restriction of connected cycles on $\Gamma_h\backslash D$ to $\mathcal{F}_h$. In particular,*

$$\mathrm{NL}^*(\mathcal{F}_h) = \mathcal{P}_h^*(\mathrm{SC}^*(\Gamma_h\backslash D)). \tag{5.7}$$

More generally, one can easily see that (5.7) holds for the Noether-Lefschetz ring on the moduli space of lattice polarized hyperkähler manifolds with level structures.

5.4. Now following Funke and Millson [26] we promote the connected cycles $c(U, g, K)$ to cycles with non-trivial coefficients. First recall that given any partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, with $k = \left[\frac{b+2}{2}\right]$, the harmonic Schur functor associates to the quadratic space V a finite dimensional $\mathrm{O}(V)$-module $\mathbb{S}_{[\lambda]}(V)$ which is irreducible of highest weight λ. As such it defines a fiber bundle

$$\Gamma_g\backslash(D \times \mathbb{S}_{[\lambda]}(V)) \to \Gamma_g\backslash D. \tag{5.8}$$

We will use $\mathbf{E}$ to denote the associated local system.

Now let $\mathbf{x} = (x_1, \dots, x_r)$ be a rational basis of U. The vectors $x_1, \dots, x_n$ are all fixed by $\Gamma_{g,U}$. Hence any tensor word in the x_j's is also fixed by $\Gamma_{g,U}$. Given a tableau T on λ we denote by $\mathbf{x}_T$ the corresponding harmonic tensor in $\mathbb{S}_{[\lambda]}(V)$, see [25].[1] It is fixed by $\Gamma_{g,U}$ and therefore gives rise to a parallel section of the restriction of $\mathbf{E}$ over the connected cycle $c(U, g, K)$. We define connected cycles with coefficients in $\mathbf{E}$ by setting

$$c(\mathbf{x}, g, K)_T = c(U, g, K) \otimes \mathbf{x}_T \text{ where } U = \mathrm{span}(x_1, \dots, x_n).$$

The scalar product on V induces an inner product on $\mathbb{S}_{[\lambda]}(V)$. Composing the wedge product with this inner product and integrating the result over Y_K we define a pairing

$$\langle\ ,\ \rangle : H^{2r}(Y_K, \mathbf{E}) \times H_c^{2(b-r)}(Y_K, \mathbf{E}) \to \mathbb{C}, \tag{5.9}$$

where $H_c^*(-)$ denotes the de Rham cohomology with compact support. It follows from Poincaré duality that the pairing (5.9) is perfect. Now if η is a compactly supported $\mathbb{S}_{[\lambda]}(V)$-valued closed $2(b-r)$-form on Y_K we can form the period

$$\int_{c(\mathbf{x},g,K)_T} \eta = \int_{c(U,g,K)} (\eta, \mathbf{x}_T).$$

[1] Beware that a tableau is called a semi standard filling in [26].

Thus, corresponding to $c(\mathbf{x}, g, K)_T$ (a connected cycle with coefficients) we have a linear form on $H_c^{2(b-r)}(Y_K, \mathbf{E})$. Since the pairing (5.9) is perfect this linear form in turn defines a class $[c(\mathbf{x}, g, K)_T]$ in $H^{2r}(Y_K, \mathbf{E})$. We shall refer to the corresponding map as the cycle class map (with coefficients). We denote by $\mathrm{SC}^*_{\mathrm{hom}}(Y_K, \mathbf{E})$ the subgroup of $H^*(Y_K, \mathbf{E})$ generated by the cycle classes $[c(\mathbf{x}, g, K)_T]$ (as $\mathbf{x}$ and T vary).

Remark 5.4.1. When $\lambda = 0$ the representation $\mathbb{S}_{[\lambda]}(V)$ is trivial and the cycle class map is obtained by composing the inclusion map $\mathrm{SC}^r(Y_K) \subseteq \mathrm{CH}^r(Y_K)$ with the usual cycle class map $\mathrm{CH}^r(Y_K) \to H^{2r}(Y_K)$.

If $K' \subseteq K$ is a compact open normal subgroup, the finite covering group $\Gamma_K/\Gamma_{K'}$ that acts on $H^{2r}(Y_{K'}, \mathbf{E})$ preserves the image $\mathrm{SC}^r_{\mathrm{hom}}(Y_{K'}, \mathbf{E})$ of the cycle class map and, since the covering projection map $Y_{K'} \to Y_K$ maps connected cycles to connected cycles, we have:

$$\mathrm{SC}^r_{\mathrm{hom}}(Y_{K'}, \mathbf{E})^{\Gamma_K/\Gamma_{K'}} = \mathrm{SC}^r_{\mathrm{hom}}(Y_K, \mathbf{E}). \tag{5.10}$$

5.5. **Zucker's conjecture and Hodge theory.** As a key ingredient, we shall explain the connection between the ordinary cohomology and L^2-cohomology of Y_K. As we will see later, the latter is well understood as relative Lie algebra cohomology, which naturally links to representation theory.

We assume that Y_K is smooth and fix a local system $\mathbf{E}$ as in the preceding paragraph. Let $H^i_{(2)}(Y_K, \mathbf{E})$ be the i-th L^2-cohomology of Y_K with coefficients in $\mathbf{E}$. By Hodge theory, the group $H^i_{(2)}(Y_K, \mathbf{E})$ is isomorphic to the space of L^2-harmonic i-forms, which is a finite dimensional vector space with a natural Hodge structure (see [9]). Let $\overline{Y}^{bb}_K$ be the Baily-Borel-Satake compactification of Y_K. Zucker's conjecture, proved by Looijenga [39] and by Saper and Stern [62] states that

Theorem 5.5.1. *There is an isomorphism*

$$H^k_{(2)}(Y_K, \mathbf{E}) \cong IH^k(\overline{Y}^{bb}_K, \mathbf{E}), \tag{5.11}$$

where $IH^\bullet(\overline{Y}^{bb}_K, \mathbf{E})$ is the intersection cohomology with middle perversity on $\overline{Y}^{bb}_K$.

For our purpose, we shall also consider the Hodge, or mixed Hodge, structures on these cohomology groups. Here, as the local system $\mathbf{E}$ underlies a natural variation of Hodge structure over Y_K (cf. [74]), $IH^k(\overline{Y}^{bb}_K, \mathbf{E})$ carries a mixed Hodge structure by Saito's [61] mixed Hodge module theory. A priori the two Hodge structures need not correspond under the isomorphism but Harris and Zucker [27, Theorem 5.4] nevertheless prove:

Theorem 5.5.2. *Let*

$$\xi_k : H^k_{(2)}(Y_K, \mathbf{E}) \to H^k(Y_K, \mathbf{E}) \tag{5.12}$$

be the composition of the isomorphism $H^k_{(2)}(Y_K, \mathbf{E}) \cong IH^k(\overline{Y}^{bb}_K, \mathbf{E})$ *of Theorem 5.5.1 with the natural map* $IH^k(\overline{Y}^{bb}_K, \mathbf{E}) \to H^k(Y_K, \mathbf{E})$. *Then* ξ_k *is a morphism of mixed Hodge structures.*

The image of ξ_k is the lowest non-zero weight in the mixed Hodge structure of $H^k(Y_K, \mathbf{E})$ (see Remark 5.5 of [27]). In particular, the subspace $\mathrm{SC}^r_{\mathrm{hom}}(Y_K, \mathbf{E})$ lies in the image of ξ_{2r}.

Corollary 5.5.3. *We have*

$$H^k_{(2)}(Y_K, \mathbf{E}) \cong H^k(Y_K, \mathbf{E}) \tag{5.13}$$

as a morphism of Hodge structure for all $k < b - 1$.

Proof. The isomorphism (5.13) holds because the boundary of $\overline{Y}^{bb}_K$ has dimension at most one. See [27, Remark 5.5] for the second statement. ♣

6. A surjectivity result for theta lifting

In this section, we recap the work in [6] on the surjectivity of global theta lifting for cohomological automorphic representations with trivial coefficients of orthogonal groups. We also explain how to extend to the case of nontrivial coefficients.

6.1. **Global theta correspondence.** Here we briefly review Howe's global theta correspondence — see e.g. [31, 47] for more details. Let us first fix some notations. Let V be a non-degenerate quadratic space over $\mathbb{Q}$ of signature $(2, b)$ and let W be a symplectic space over $\mathbb{Q}$ of dimension $2r$. We denote by $G' = \mathrm{Mp}_{2r}(W)$ the symplectic group $\mathrm{Sp}(W)$ if b is even and the metaplectic double cover of $\mathrm{Sp}(W)$ if b is odd.

Fix a nontrivial additive character ψ of $\mathbb{A}/\mathbb{Q}$. We denote by ω_ψ the (automorphic) Weil representation of $\mathrm{O}(V)(\mathbb{A}) \times G'(\mathbb{A})$ realized in the space $\mathcal{S}(V(\mathbb{A})^r)$ of Schwartz-Bruhat functions on $V(\mathbb{A})^r$ — the so-called Schrödinger model of the Weil representation. For each $\phi \in \mathcal{S}(V(\mathbb{A})^r)$ we form the theta function on $\mathrm{O}(V)(\mathbb{A}) \times G'(\mathbb{A})$:

$$\theta_{\psi,\phi}(g, g') = \sum_{\xi \in V(\mathbb{Q})^r} \omega_\psi(g, g')(\phi)(\xi). \tag{6.1}$$

Given an irreducible cuspidal automorphic representation (τ, H_τ) of $G'(\mathbb{A})$ which occurs as an irreducible subspace of $L^2(G'\mathbb{Q})\backslash G'(\mathbb{A}))$ and given an element f in that subspace, we can form the theta integral

$$\theta^f_{\psi,\phi}(g) = \int\limits_{G'(\mathbb{Q})\backslash G'(\mathbb{A})} \theta_{\psi,\phi}(g, g') f(g') dg'. \tag{6.2}$$

It is absolutely convergent and defines an automorphic function on $\mathrm{O}(V)(\mathbb{A})$, called the *global theta lift* of f. We shall denote by $\theta_{\psi,V}(\tau)$ the space of the automorphic representation generated by all the global theta lifts $\theta^f_{\psi,\phi}$ as f

and ϕ vary. We shall refer to the corresponding automorphic representation of $\mathrm{O}(V)(\mathbb{A})$ as *the global ψ-theta lift* of τ to $\mathrm{O}(V)$.

Definition 6.1.1. We say that an irreducible automorphic representation π occurring discretely in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$, i.e. $\pi \in \mathcal{A}(G)$, is in the image of the cuspidal ψ-theta correspondence from a smaller symplectic group if there exists a symplectic space W with $\dim W \leq 2\left[\frac{b+2}{2}\right]$ and an extension $\widetilde{\pi}$ of π to $\mathrm{O}(V)$ such that $\widetilde{\pi}$ is the global ψ-theta lift of an irreducible cuspidal automorphic representation of $G' = \mathrm{Mp}_{2r}(W)$, i.e. there exists $\tau \in \mathcal{A}_{\mathrm{cusp}}(G')$ such that $\tilde{\pi} \hookrightarrow \theta_{\psi,V}(\tau)$.

6.2. **Cohomological representations.** Throughout this section, we let E be a finite dimensional irreducible representation of $G(\mathbb{R})$. Let $K_{\mathbb{R}} \cong \mathrm{SO}(2) \times \mathrm{SO}(b)$ be as above. Denote by θ the corresponding Cartan involution and let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the associated Cartan decomposition of the Lie algebra $\mathfrak{g}_0$ of $G(\mathbb{R})$. We fix a Cartan subalgebra $\mathfrak{t}_0 \subset \mathfrak{k}_0$. We shall denote by $\mathfrak{a}$ the complexification of a real Lie algebra $\mathfrak{a}_0$.

A unitary representation $\pi_{\mathbb{R}}$ of $G(\mathbb{R})$ is *cohomological* (with respect to E) if its associated $(\mathfrak{g}, K_{\mathbb{R}})$-module $\pi_{\mathbb{R}}^{\infty}$ has nonzero relative Lie algebra cohomology, i.e.

$$H^{\bullet}(\mathfrak{g}, K_{\mathbb{R}}; \pi_{\mathbb{R}}^{\infty} \otimes E) \neq 0. \tag{6.3}$$

The unitary $(\mathfrak{g}, K_{\mathbb{R}})$-modules with nonzero cohomology have been classified by Vogan and Zuckerman [69]. They are determined by θ-stable parabolic subalgebras $\mathfrak{q} \subset \mathfrak{g}$: then $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$, where $\mathfrak{l}$ is the centralizer of an element $X \in i\mathfrak{t}_{\mathfrak{o}}$ and $\mathfrak{u}$ is the span of the positive roots of X in $\mathfrak{g}$. Then the Lie algebra $\mathfrak{l}$ is the complexification of $\mathfrak{l}_0 = \mathfrak{l} \cap \mathfrak{g}_0$ and we let L be the connected subgroup of $G(\mathbb{R})$ with Lie algebra $\mathfrak{l}_0$. Associated to $\mathfrak{q}$, there is a well-defined, irreducible representation $V(\mathfrak{q})$ of $K_{\mathbb{R}}$ that occurs with multiplicity one in $\wedge^R \mathfrak{p}$ where $R = \dim(\mathfrak{u} \cap \mathfrak{p})$. Now if (6.3) holds there exists some $\mathfrak{q}$ such that the Cartan product of $V(\mathfrak{q})$ and E^* occurs with multiplicity one in $\wedge^R \mathfrak{p} \otimes E^*$ and the $(\mathfrak{g}, K_{\mathbb{R}})$-module $\pi_{\mathbb{R}}^{\infty}$ is isomorphic to some irreducible $(\mathfrak{g}, K_{\mathbb{R}})$-module $A_{\mathfrak{q}}(E)$ characterized by the following two properties:

(1) $A_{\mathfrak{q}}(E)$ is unitary with the same infinitesimal character as E.
(2) The Cartan product of $V(\mathfrak{q})$ and E^* occurs (with multiplicity one) in $A_{\mathfrak{q}}(E)$.

In our case $K_{\mathbb{R}} = \mathrm{SO}(2) \times \mathrm{SO}(b)$ acts on $\mathfrak{p} = (\mathbb{C}^2)^* \otimes \mathbb{C}^b$ through the standard representation of $\mathrm{SO}(2)$ on $\mathbb{C}^2$ and the standard representation of $\mathrm{SO}(b)$ on $\mathbb{C}^b$. We denote by $\mathbb{C}^+$ and $\mathbb{C}^-$ the $\mathbb{C}$-span of the vectors $e_1 + ie_2$ and $e_1 - ie_2$ in $\mathbb{C}^2$. The two lines $\mathbb{C}^+$ and $\mathbb{C}^-$ are left stable by $\mathrm{SO}(2)$. This yields a decomposition $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ which corresponds to the decomposition given by the natural complex structure on $\mathfrak{p}_0$. For each non-negative integer p the $K_{\mathbb{R}}$-representation $\wedge^p \mathfrak{p} = \wedge^p(\mathfrak{p}^+ \oplus \mathfrak{p}^-)$ decomposes as the sum:

$$\wedge^p \mathfrak{p} = \bigoplus_{r+s=p} \wedge^r \mathfrak{p}^+ \otimes \wedge^r \mathfrak{p}^-.$$

The $K_{\mathbb{R}}$-representations $\wedge^r\mathfrak{p}^+ \otimes \wedge^s\mathfrak{p}^-$ are not irreducible in general: there is at least a further splitting given by the Lefschetz decomposition:

$$\wedge^r\mathfrak{p}^+ \otimes \wedge^s\mathfrak{p}^- = \bigoplus_{k=0}^{\min(r,s)} \tau_{r-k,s-k}.$$

One can check that for $2(r+s) < b$ each $K_{\mathbb{R}}$-representation $\tau_{r,s}$ is irreducible. Moreover in the range $2(r+s) < b$ only those with $r = s$ can occur as a $K_{\mathbb{R}}$-type $V(\mathfrak{q})$ associated to a cohomological module. In the special case $r = s$ one can moreover check that each $\tau_{r,r}$ is irreducible as long as $r < b$; it is isomorphic to some $V(\mathfrak{q})$ where the Levi subgroup L associated to $\mathfrak{q}$ is isomorphic to $C \times \mathrm{SO}_0(2, b-2r)$ with $C \subset K_{\mathbb{R}}$.

It follows in particular that if $\pi_{\mathbb{R}}$ is an irreducible unitary representation of $G(\mathbb{R})$ that satisfies

$$H^{r,r}(\mathfrak{g}, K_{\mathbb{R}}; \pi_{\mathbb{R}}^\infty \otimes E) \neq 0$$

for some $r < b$ then $\pi_{\mathbb{R}}^\infty$ is isomorphic to the unique unitary $(\mathfrak{g}, K_{\mathbb{R}})$-module that has the same infinitesimal character as E and contains the Cartan product of $\tau_{r,r}$ and E^*. We shall denote by $A_{r,r}(E)$ this $(\mathfrak{g}, K_{\mathbb{R}})$-module. We have:

$$H^{i,j}(\mathfrak{g}, K_{\mathbb{R}}; A_{r,r}(E) \otimes E) = \begin{cases} \mathbb{C} & \text{if } r \leq i = j \leq b-r,\ 2i \neq b \\ \mathbb{C} + \mathbb{C} & \text{if } 2i = 2j = b \\ 0 & \text{otherwise.} \end{cases}$$

We refer to [7, §5.2 and 5.4] for more details.

Note that $A_{r,r}(E)$ can only contribute to even degree cohomology. The following vanishing result therefore follows from the above classification (and Matsushima's formula (6.4) below).

Proposition 6.2.1. *For any* odd *degree $i < b/2$ and any local system $\mathbf{E}$ we have*

$$H^i(Y_K, \mathbf{E}) = 0.$$

Proof. Since $i < b/2$ implies $i < b-1$ we have $H^i(Y_K, \mathbf{E}) \cong H^i_{(2)}(Y_K, \mathbf{E})$. The proposition then follows from the fact that in the range $r+s < b/2$ the only $K_{\mathbb{R}}$-types $\tau_{r,s}$ that can occur in a cohomological representation are the ones with $r = s$. ♣

6.3. **Surjectivity of the theta lift.** The following theorem is proved in [6, Theorem 7.7] when E is the trivial representation. It was proved in [7] for general E but under the hypothesis that π is cuspidal. To be able to deal with residual representations as well was the main input of [6]. Though the results of [6] only address the case where E is the trivial representation, the proofs (see especially the key Proposition 6.2) only make use of the fact that cohomological representations have integral regular infinitesimal characters, which is still true for representations that are cohomological with respect to E. We shall therefore not repeat the proof and simply refer to [6] for the proof of:

Theorem 6.3.1. *Let $\pi = \otimes \pi_v \in \mathcal{A}(G)$ be a square integrable automorphic representation of G. Suppose that the $(\mathfrak{g}, K_{\mathbb{R}})$-module $\pi_{\mathbb{R}}^{\infty}$ of the local Archimedean component of π is isomorphic to some cohomological module $A_{r,r}(E)$ with $3r < b+1$. Then there exists a cuspidal representation τ of $\mathrm{Mp}_{2r}(\mathbb{A})$ such that π (up to a twist by a quadratic character) is in the image of the theta lift of τ.*

6.4. **Theta classes in cohomology groups.** We keep all notations as in §5.1. We construct some special cohomology classes (with respect to the local system E) on Y_K from the relative Lie algebra cohomology of $(\mathfrak{g}, K_{\mathbb{R}})$-modules. We write

$$\pi = \pi_{\mathbb{R}} \otimes \pi_f \in \mathcal{A}_2(G),$$

and let π_f^K be the finite dimensional subspaces of K-invariant vectors in π_f. According to Matsushima's formula and Langlands spectral decomposition (see [8], [6, §7.8]), we have

$$H^i_{(2)}(X_K, \mathbf{E}) \cong \bigoplus_{\pi \in \mathcal{A}(G)} m(\pi) H^i(\mathfrak{g}, K_{\mathbb{R}}; \pi_{\mathbb{R}}^{\infty} \otimes E) \otimes \pi_f^K \tag{6.4}$$

where π occurs discretely in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ with multiplicity $m(\pi)$. Following [6], we define the space of theta classes

$$H^i_{\theta}(X_K, \mathbf{E}) \subseteq H^i(X_K, \mathbf{E})$$

as the subspace generated by the image of $H^i(\mathfrak{g}, K_{\mathbb{R}}; \pi_{\mathbb{R}}^{\infty} \otimes E)$ via (6.4) and (5.12), where π varies among the irreducible representations in $\mathcal{A}(G)$ which are in the image of the ψ-cuspidal theta correspondence from a smaller symplectic group. The space $H^i_{\theta}(Y_K, \mathbf{E})$ of theta classes on Y_K is naturally defined by restriction. The main result in [6] can be reformulated as:

Theorem 6.4.1. *There is an inclusion*

$$H^{2r}_{\theta}(Y_K, \mathbf{E}) \subseteq \mathrm{SC}^r_{\mathrm{hom}}(Y_K, \mathbf{E}). \tag{6.5}$$

Moreover, $H^{2r}_{\theta}(Y_K, \mathbf{E}) = H^{r,r}(Y_K, \mathbf{E})$ when $r < \frac{b+1}{3}$ or $r > \frac{2b-1}{3}$.

Proof. The inclusion (6.5) follows from (the proof of) [7, Proposition 11.3].[2] The last assertion follows from Theorem 6.5, the decomposition (6.4) and the fact that if $\pi_{\mathbb{R}}$ is an irreducible unitary representation of $G(\mathbb{R})$ that satisfies

$$H^{r,r}(\mathfrak{g}, K_{\mathbb{R}}; \pi_{\mathbb{R}}^{\infty} \otimes E) \neq 0$$

for some $r < b$ then $\pi_{\mathbb{R}}^{\infty}$ is isomorphic to $A_{r,r}(E)$. Note that the fact that $H^{2r}_{\theta}(Y_K, \mathbf{E})$ is contained in $H^{r,r}(Y_K, \mathbf{E})$ obviously follows from the inclusion (6.5). It also follows from the fact that the only cohomological representations that are in the image of the local Archimedean theta correspondence are the $A_{r,r}(E)$, see [6, §7.2] and [38]. ♣

[2] Beware that our space SC^* is a priori larger than the one defined in [7].

Let $H^{2k}_{\mathrm{alg}}(Y_K,\mathbb{C})$ be the image of $\mathrm{CH}^k(Y_K)\otimes\mathbb{C}$ in $H^{2k}(Y_K,\mathbb{C})$ via the cycle class map. Following [59], one can use Zucker's conjecture and the hard Lefschetz theorem on intersection cohomology to get the following result

Corollary 6.4.2. *When $b>3$, we have*

$$H^{2b-2}_{alg}(Y_K,\mathbb{C})=0. \tag{6.6}$$

Proof. Let us quickly sketch the proof. Let λ be the Hodge line bundle on Y_K. By [6] and Zucker's conjecture (Theorem 5.5.1), we have that

$$IH^2(\overline{Y}^{bb}_K,\mathbb{C})\cong H^2(Y_K,\mathbb{C}) \tag{6.7}$$

is generated by connected cycles of codimension one. Next, as λ is ample, the hard Lefschetz theorem for intersection cohomology implies that the image of the map

$$\xi_{2b-2}: IH^{2b-2}(\overline{Y}^{bb}_K,\mathbb{C})\to H^{2b-2}(Y_K,\mathbb{C}) \tag{6.8}$$

is spanned by the class of $\lambda^{b-2}\cdot c(U,g,K)$. By [65], the class $\lambda^{b-2}\cdot c(U,g,K)$ is zero in $H^{2b-2}(Y_K,\mathbb{C})$ when $\dim c(U,g,K)\geq 3$. Since $H^{2b-2}_{alg}(Y_K,\mathbb{C})$ lies in the image of ξ_{2b-2}, this proves the assertion. ♣

Let us finally define theta classes in the cohomology groups of the moduli spaces discussed in Section 3. Assume that E is a finite dimensional representation of $\mathrm{SO}(\Lambda_{\mathbb{R}})$. Recall that there is a period map $\mathcal{P}_\Sigma:\mathcal{F}_{\Sigma,h}\to\Gamma_\Sigma\backslash D_\Sigma$. Let $\mathbb{E}=\mathcal{P}_\Sigma^*(\mathbf{E})$ be the pullback of $\mathbf{E}$ to $\mathcal{F}_{\Sigma,h}$. Then we define

$$H^\bullet_\theta(\mathcal{F}_{\Sigma,h},\mathbb{E}):=\mathcal{P}_\Sigma^*(H^\bullet_\theta(\Gamma_\Sigma\backslash D_\Sigma,\mathbf{E})). \tag{6.9}$$

as the subspace of theta classes on $\mathcal{F}_{\Sigma,h}$. It follows from Theorem 6.4.1 that:

Corollary 6.4.3. *There is an inclusion*

$$H^{2r}_\theta(\mathcal{F}_{\Sigma,h},\mathbb{E})\subseteq\mathcal{P}_\Sigma^*(\mathrm{SC}^r_{\mathrm{hom}}(\Gamma_\Sigma\backslash D_\Sigma,\mathbf{E}))$$

for all r.

7. The Funke-Kudla-Millson ring

In this section, we introduce special Schwartz forms at the Archimedean place and Kudla-Millson's special theta lift. Following [7] and [6], this establishes a connection between special cycle classes and theta classes as defined in §5.3. More importantly for us, we show that this yields a so called *Funke-Kudla-Millson ring*, or just FKM ring, in the space of differential forms with coefficients, which plays the key role in our study of the cohomological tautological conjecture.

7.1. **Special Schwartz forms.** In this subsection we let V be a real quadratic space of signature $(b,2)$ of dimension $m=b+2$. We use signature $(b,2)$ rather than signature $(2,b)$ in order to follow the notations of the works of Kudla-Millson and Funke-Millson; when applied to our geometric situation we will reverse the variables.

Pick an oriented orthogonal basis $\{v_i\}$ of V such that $(v_\alpha,v_\alpha)=1$ for $\alpha=1,\dots,b$ and $(v_\mu,v_\mu)=-1$ for $\mu=b+1,b+2$. We shall use the notations of [7].

Kudla and Millson [35] and then Funke and Millson [26] have constructed special Schwartz forms φ in

$$\Big[\mathcal{S}(V^r)\otimes A^{2r}(D)\otimes T^\ell(V)\Big]^{\mathrm{O}(V)}\cong \mathrm{Hom}_{K_\mathbb{R}}(\wedge^{2r}\mathfrak{p},\mathcal{S}(V^r)\otimes T^\ell(V)).$$

By abuse of notations we also denote by $T^\ell(V)$ the local system on D associated to $T^\ell(V)$. If $\mathbf{x}\in V^r$ we then have $\varphi(\mathbf{x})\in A^{2r}(D,T^\ell(V))$.

We shall rather work with the polynomial Fock model for the dual pair $\mathrm{O}(b,2)\times\mathrm{Sp}_{2r}(\mathbb{R})$, see [7, Section 7]. We denote by ι the intertwining operator from the Schrödinger model to the Fock model, see [35, Section 6]. The map ι maps the vectors of $\mathcal{S}(V^r)$ that are finite under the action of a maximal compact subgroup $\mathrm{U}(2rm)$ of the symplectic group Sp_{2rm} containing the dual pair onto the space of polynomials

$$\mathcal{P}(\mathbb{C}^{mr})\cong\mathbb{C}[z_{1,j},\dots,z_{m,j}\ :\ j=1,\dots r].$$

Here if $\mathbf{x}=(x_1,\dots,x_r)\in V^r$ we have

$$x_j=\sum_{\alpha=1}^{b} z_{\alpha,j}v_\alpha+\sum_{\mu=b+1}^{b+2} z_{\mu,j}v_\mu.$$

We will use the notation $\mathcal{P}(\mathbb{C}^{mr})_+$ to denote the polynomials in the "positive" variables $z_{\alpha,j}$, $1\le\alpha\le b$, $1\le j\le r$. As suggested by the notation, we shall think of $(z_{\alpha,j})$ as a complex $b\times r$-matrix. In this way $\mathcal{P}(\mathbb{C}^{mr})_+$ is identified with the space $\mathrm{Pol}(M_{b,r})$ of polynomials in the entries of complex $b\times r$-matrices.

As a subgroup of $\mathrm{U}(2rm)$, the intersection

$$K_\infty\times K'_\infty=(\mathrm{O}(b,2)\times\mathrm{Sp}_{2r}(\mathbb{R}))\cap\mathrm{U}(2rm)$$

acts on $\mathcal{P}(\mathbb{C}^{mr})$ (Fock model of the Weil representation) and preserves the subspace $\mathcal{P}(\mathbb{C}^{mr})_+$. Here $K_\infty\cong\mathrm{O}(b)\times\mathrm{O}(2)$ is a maximal compact subgroup of $\mathrm{O}(b,2)$ and K'_∞ is a maximal compact subgroup of $\mathrm{Sp}_{2r}(\mathbb{R})$. Through the isomorphism

$$\mathcal{P}(\mathbb{C}^{mr})_+\cong\mathrm{Pol}(M_{b,r}(\mathbb{C}))$$

the action of $\mathrm{O}(b)\times\{1\}\subset K_\infty$ on $\mathcal{P}(\mathbb{C}^{mr})_+$ corresponds to the natural action of $\mathrm{O}(b)$ on $\mathrm{Pol}(M_{b,r}(\mathbb{C}))$ induced by the linear left action on the columns of the matrices.

First consider the case $r=1$. Then we simply set $z_\alpha=z_{\alpha,1}$ and the space $\mathcal{P}(\mathbb{C}^{mr})_+=\mathcal{P}(\mathbb{C}^m)_+$ is the space of polynomials in the variables z_α. Note

that $\mathfrak{p} \cong \mathbb{C}^b \otimes \mathbb{C}^2$; we let $\omega_{\alpha,\mu}$ be the linear form which maps an element of $\mathfrak{p}$ to its (α,μ)-coordinate. For any multi-index $\underline{\alpha} = (\alpha_1, \alpha_2)$ we write $\omega_{\underline{\alpha}} = \omega_{\alpha_1,b+1} \wedge \omega_{\alpha_2,b+2}$ and $z_{\underline{\alpha}} = z_{\alpha_1} z_{\alpha_2}$. The form

$$\sum_{\underline{\alpha}} z_{\underline{\alpha}} \otimes \omega_{\underline{\alpha}} \in \mathrm{Hom}_{K_\infty}(\wedge^2 \mathfrak{p}, \mathcal{P}(\mathbb{C}^m)_+)$$

is precisely the image $\iota(\varphi_{1,0})$ of the Schwartz form $\varphi_{1,0} \in \mathrm{Hom}_{K_\infty}(\wedge^2 \mathfrak{p}, \mathcal{S}(V))$, constructed by Kudla and Millson, under the intertwining operator ι. The natural product

$$\bigotimes_{j=1}^{r} \mathbb{C}[z_{1,j}, \dots, z_{b,j}] \to \mathbb{C}[z_{1,j}, \dots, z_{b,j} \ : \ j = 1, \dots r]$$

induces a natural map

$$\mathrm{Hom}_{K_\infty}(\wedge^2 \mathfrak{p}, \mathcal{P}(\mathbb{C}^m)_+)^{\otimes r} \to \mathrm{Hom}_{K_\infty}(\wedge^{2r} \mathfrak{p}, \mathcal{P}(\mathbb{C}^{mr})_+)$$

which maps

$$\bigotimes_{j=1}^{r} \left(\sum_{\underline{\alpha}} z_{\underline{\alpha},j} \otimes \omega_{\underline{\alpha}} \right)$$

onto $\iota(\varphi_{r,0})$, the image of the Kudla-Millson form $\varphi_{r,0} \in \mathrm{Hom}_{K_\infty}(\wedge^{2r} \mathfrak{p}, \mathcal{S}(V^r))$. We shall abusively denote it $\varphi_{r,0}$ as well.

We now describe Funke-Millson forms. For any multi-index $\underline{\alpha} = (\alpha_1, \dots, \alpha_\ell)$ we let

$$v_{\underline{\alpha}} = v_{\alpha_1} \otimes \cdots \otimes v_{\alpha_\ell} \in T^\ell(V).$$

Given any ℓ-tuple $I = (i_1, \dots, i_\ell)$ of integers, with $1 \leq i_j \leq b$ we define

$$\varphi^I_{0,\ell} \in \mathrm{Hom}(\mathbb{C}, [\mathcal{P}(\mathbb{C}^{mr})_+ \otimes T^\ell(V)]^{K_\infty})$$

by

$$\varphi^I_{0,\ell} = \sum_{\underline{\alpha}} (z_{\alpha_1,i_1} \cdots z_{\alpha_\ell,i_\ell}) \otimes v_{\underline{\alpha}}.$$

Using the natural product on $\mathcal{P}(\mathbb{C}^{mr})_+$, we have

$$\varphi^I_{r,\ell} = \varphi_{r,0} \cdot \varphi^I_{0,\ell} \in \mathrm{Hom}_{K_\infty}(\wedge^{2r} \mathfrak{p}, \mathcal{P}(\mathbb{C}^{mr})_+ \otimes T^\ell(V)).$$

We shall denote $\Phi_{T^\ell(V)}$ the subspace of $\mathrm{Hom}_{K_\infty}(\wedge^{2r} \mathfrak{p}, \mathcal{P}(\mathbb{C}^{mr}) \otimes T^\ell(V))$ spanned by the Schwarz forms $\varphi^I_{r,\ell}$.

Now fix a partition $\lambda = (\lambda_1, \dots, \lambda_k)$, with $k = \left[\frac{m}{2}\right]$, and let $\ell = \lambda_1 + \dots + \lambda_k$. The finite dimensional $\mathrm{O}(V)$-module $\mathbb{S}_{[\lambda]}(V)$ can be obtained as the image of the classical Schur functor $\mathbb{S}_\lambda(V) \subset T^\ell(V)$ under the $\mathrm{O}(V)$-equivariant projection of $T^\ell(V)$ onto the harmonic tensors. We denote by $\pi_{[\lambda]}$ the corresponding $\mathrm{O}(V)$-equivariant projection of $T^\ell(V)$ onto $\mathbb{S}_{[\lambda]}(V)$. We set

$$\varphi^I_{r,[\lambda]} = (1 \otimes \pi_{[\lambda]}) \circ \varphi^I_{r,\ell}.$$

We shall denote $\Phi^{2r}_{[\lambda]}$ the subspace of $\mathrm{Hom}_{K_\infty}(\wedge^{2r} \mathfrak{p}, \mathcal{P}(\mathbb{C}^{mr}) \otimes T^\ell(V))$ spanned by the Schwarz forms $\varphi^I_{r,[\lambda]}$.

More generally if E is *any* finite dimensional representation, it decomposes into a finite direct sum (with multiplicities) of $\mathrm{O}(V)$-modules $\mathbb{S}_{[\lambda]}(V)$ and we shall denote Φ_E^{2r} the finite direct sum of the corresponding $\Phi_{[\lambda]}^{2r}$'s identified as a subspace of $\mathrm{Hom}_{K_\infty}(\wedge^{2r}\mathfrak{p}, \mathcal{P}(\mathbb{C}^{mr})\otimes E)$. We will in fact abusively identify Φ_E^{2r} with a subspace

$$\Phi_E^{2r} \subset \left[\mathcal{S}(V^r)\otimes A^{2r}(D)\otimes E\right]^{\mathrm{O}(V)}$$

using the intertwining operator ι.

7.2. **The Funke-Kudla-Millson ring.** Consider a finite graded ring of finite dimensional representations $\mathbf{H}^*$ of $\mathrm{O}(V)$. We use the above special Schwartz forms to construct a subring of the bi-graded complex

$$\bigoplus_{i,j}\left[\mathcal{S}(V^i)\otimes A^{2i}(D)\otimes \mathbf{H}^j\right]^{\mathrm{O}(V)}. \tag{7.1}$$

Indeed, the wedge product

$$\wedge : A^{2i}(D,\mathbf{H}^j)\times A^{2i'}(D,\mathbf{H}^{j'}) \to A^{2(i+i')}(D,\mathbf{H}^j\otimes\mathbf{H}^{j'})$$

composed with the product $\mathbf{H}^j\otimes\mathbf{H}^{j'}\to\mathbf{H}^{j+j'}$ yields a ring structure on (7.1) and since the latter obviously maps $\Phi^{2i}_{\mathbf{H}^j}\times\Phi^{2i'}_{\mathbf{H}^{j'}}$ into $\Phi^{2(i+i')}_{\mathbf{H}^{j+j'}}$ we conclude:

Proposition 7.2.1. *The subspace*

$$\bigoplus_{i,j}\Phi^{2i}_{\mathbf{H}^j}\subset\bigoplus_{i,j}\left[\mathcal{S}(V^i)\otimes A^{2i}(D)\otimes\mathbf{H}^j\right]^{\mathrm{O}(V)}$$

is a subring.

Consider now a (global) arithmetic quotient Y_K with $K\subset\mathrm{O}(V)(\mathbb{A}_f)$ an open compact subgroup. Let $\varphi\in\mathcal{S}(V(\mathbb{A}_f)^i)$ be a K-invariant Schwartz function for some i. Then for any $\varphi_\mathbb{R}\in\Phi^{2i}_{\mathbf{H}^j}$, we define a global Schwartz form

$$\phi=\varphi_\mathbb{R}\otimes\varphi\in\left[\mathcal{S}(V(\mathbb{A})^i)\otimes A^{2i}(D)\otimes\mathbf{H}^j\right]^{\mathrm{O}(V)}.$$

Applying the theta distribution (6.1) to ϕ yields a theta function $\theta_{\psi,\phi}(g,g')$ which, as a function of $g\in\mathrm{O}(V)$, defines a differential form in $A^{2i}(X_K,\mathbf{H}^j)$. Varying φ and g' we obtain a subspace

$$A^{2i}_{\mathrm{FKM}}(X_K,\mathbf{H}^j)\subset A^{2i}(X_K,\mathbf{H}^j).$$

It follows from Proposition 7.2.1 that

$$\Phi_{\mathbf{H}^*}:=\bigoplus_{i,j}A^{2i}_{\mathrm{FKM}}(X_K,\mathbf{H}^j) \tag{7.2}$$

is a subring of the graded ring $\bigoplus_{i,j}A^{2i}(X_K,\mathbf{H}^j)$; we shall refer to it as the *Funke-Kudla-Millson ring* of X_K. Restricting to the connected component $Y_K\subset X_K$ yields the Funke-Kudla-Millson ring of Y_K.

7.3. **Relation with θ-classes.** By construction, the cohomology class of a differential form in $A^{2i}_{\mathrm{FKM}}(Y_K, \mathbf{H}^j)$ is a θ-class in

$$H^{i,i}_\theta(Y_K, \mathbf{H}^j) := H^{2i}_\theta(Y_K, \mathbf{H}^j) \cap H^{i,i}(Y_K, \mathbf{H}^j).$$

It turns out that the converse is also true:

Proposition 7.3.1 (see Theorem 11.2 of [7])**.** *Any class in $H^{i,i}_\theta(Y_K, \mathbf{H}^j)$ can be represented by a differential form in $A^{2i}_{\mathrm{FKM}}(Y_K, \mathbf{H}^j)$.*

8. Universal families of polarized hyperkähler manifolds

In this section, we study the cohomology groups of universal families of polarized hyperkähler manifolds via Deligne's decomposition theorem and prove the cohomological Franchetta conjecture. Furthermore, we prove the cohomological tautological conjecture using the FKM ring. Throughout this section, we shall be concerned with hyperkähler manifolds of type Λ and rank $\Lambda = 3 + b$.

8.1. **Cohomology of universal families of hyperkähler manifolds.** Let $\pi : \mathcal{U} \to \mathcal{F}$ be a smooth family of polarized hyperkähler manifolds over a variety $\mathcal{F}$. Let $\mathbb{H}^i_\pi$ denote the local system on $\mathcal{F}$ whose fiber at a point $p \in \mathcal{F}$ is $H^i(\mathcal{U}_p, \mathbb{Q})$, with $\mathcal{U}_p = \pi^{-1}(p)$. According to the decomposition theorem of Deligne [19], we have

$$H^k(\mathcal{U}, \mathbb{Q}) \cong \bigoplus_{i+j=k} H^i(\mathcal{F}; \mathbb{H}^j_\pi) \tag{8.1}$$

compatible with the mixed Hodge structure on both sides (cf. [18, §4.3]).

For our purpose, we shall consider the universal family (as stacks)

$$\pi_\ell : \mathcal{U}^\ell_h \to \mathcal{F}^\ell_h$$

of polarized hyperkähler manifolds of type Λ and dimension $2n$ with a full ℓ-level structure defined in §3.4. The decomposition theorem still applies in this case via equivariant cohomology. Our first result is a cohomological version of O'Grady's generalized Franchetta conjecture for hyperkähler manifolds. Here we work with stack cohomology.

Theorem 8.1.1. *Let m_Λ be the second Betti number of hyperkähler manifolds in $\mathcal{F}^\ell_h$. Assume that $r < \frac{m_\Lambda - 3}{4}$. For any class $\alpha \in \mathrm{CH}^r(\mathcal{U}^\ell_h, \mathbb{Q})$, if the restriction of α to the very general fiber of π_ℓ is homologous to zero, then its fudamental class $[\alpha]$ is supported on proper subvarieties, i.e. there exists an open subset $U \subsetneq \mathcal{F}^\ell_h$ such that $[\alpha]$ is homologous to zero in $\pi_\ell^{-1}(U)$. In particular, Theorem 1.2.1 holds.*

Proof. Obviously, we can assume ℓ is sufficiently large. The local system $\mathbb{H}^i_{\pi_\ell} = R^i(\pi_\ell)_*\mathbb{Q}$ on $\mathcal{F}^\ell_h$ descends to its coarse moduli space F^ℓ_h. There is a natural isomorphism

$$H^k(\mathcal{F}^\ell_h, \mathbb{H}^i_{\pi_\ell}) \cong H^k(F^\ell_h, \mathbb{H}^i_{\pi_\ell}) \tag{8.2}$$

and we may regard F_h^ℓ as an open subset of $\Gamma_h^\ell \backslash D$.

Next, under our hypothesis, Proposition 6.2.1 implies that, in the degrees $i \leq 2r$ we are concerned with, only *even* degrees contribute non-trivially to the algebraic part of the decomposition (8.1). By assumption, we have

$$[\alpha] \in \bigoplus_{i=1}^{r} H^{2i}(\mathcal{F}_h^\ell, \mathbb{H}_{\pi_\ell}^{2(r-i)}). \tag{8.3}$$

Denote by $\gamma_i \in H^{2i}(\mathcal{F}_h^\ell, \mathbb{H}_{\pi_\ell}^{2(r-i)})$ the i-th component of (8.3). Since $[\alpha]$ is algebraic, it lies in the lowest weight subspace of $H^{2r}(\mathcal{U}_h^\ell, \mathbb{Q})$. As (8.1) is compatible with the mixed Hodge structure on both sides, each γ_i lies in the lowest weight subspace of $H^{2i}(\mathcal{F}_h^\ell, \mathbb{H}_{\pi_\ell}^{2(r-i)})$.

Recall that Lemma 3.6.1 implies that there exists an automorphic local system

$$\mathbf{H}^\bullet = \bigoplus \mathbf{H}^j$$

on $\Gamma_h^\ell \backslash D$ such that $\mathbb{H}_{\pi_\ell}^j = R^j(\pi_\ell)_*\mathbb{Q}$ on $\mathcal{F}_h^\ell$ is the pullback $(\mathcal{P}_h^\ell)^*\mathbf{H}^j$. So we have a natural map

$$(\mathcal{P}_h^\ell)^* : H^{2i}(\Gamma_h^\ell \backslash D, \mathbf{H}^{2r-2i}) \to H^{2i}(F_h^\ell, \mathbb{H}_{\pi_\ell}^{2r-2i}) \cong H^{2i}(\mathcal{F}_h^\ell, \mathbb{H}_{\pi_\ell}^{2r-2i})$$

induced by $\mathcal{P}_h^\ell$, which is a mixed Hodge structure morphism, and it is surjective onto the non-zero lowest weight part. We can therefore lift γ_i to an element

$$\tilde{\gamma}_i \in H^{2i}(\Gamma_h^\ell \backslash D, \mathbf{H}^{2r-2i}),$$

with lowest weight (cf. [61, 58]). Now it suffices to show

$$\tilde{\gamma}_i \in \mathrm{SC}^i_{\mathrm{hom}}(\Gamma_h^\ell \backslash D, \mathbf{H}^{2r-2i}) \tag{8.4}$$

since elements in $\mathrm{SC}^i_{\mathrm{hom}}(\Gamma_h^\ell \backslash D, \mathbf{H}^{2r-2i})$ are obviously supported on proper Shimura subvarieties.

Finally, as the local system $\mathbf{H}^\bullet$ arises from a finite dimensional representation of $\mathrm{O}(V)$, Theorem 6.4.1 can be applied and this directly implies (8.4). In particular, the conditions in Theorem 8.1.1 hold for $\mathrm{K3}^{[n]}$-type hyperkähler manifolds with $n \leq 2$ for all r and hence Theorem 1.2.1 holds. ♣

8.2. **Leray spectral sequence and cup product.** There are natural cup products on both sides of (8.1). In general these coincide only on the associated graded rings, see e.g. [10]. However Voisin [72] has proved that these two cup products are compatible for K3 surfaces after shrinking to a smaller open subset. In what follows we partially extend her results to families of hyperkähler manifolds.

Let us first make some conventions. In this subsection, we will deal with the universal family of lattice polarized hyperkähler manifolds with level structures (see §3.3 for all notions). For simplicity of notation, we will use $\pi : \mathcal{U} \to \mathcal{F}$ to denote the universal family $\mathcal{U}_{\Sigma,h}^\ell \to \mathcal{F}_{\Sigma,h}^\ell$ of h-ample Σ-polarized hyperkähler manifolds with a full ℓ-level structure and let $Y =$

$\Gamma_\Sigma^\ell \backslash D_\Sigma$. We denote by

$$\mathcal{P} : \mathcal{F} \to Y$$

the period map. Moreover, we choose ℓ sufficiently large so that Y is smooth and Lemma 3.6.1 applies.

Consider on $\mathcal{U}$ the short exact sequence of vector bundles

$$0 \to \pi^*\Omega_{\mathcal{F}} \to \Omega_{\mathcal{U}} \to \Omega_{\mathcal{U}/\mathcal{F}} \to 0$$

that defines the fiber bundle of relative differential forms $\Omega_{\mathcal{U}/\mathcal{F}}$. This gives a decreasing filtration $L^*\Omega^q_{\mathcal{U}}$ of the fiber bundle $\Omega^q_{\mathcal{U}}$ defined by

$$L^p\Omega^q_{\mathcal{U}} = \pi^*\Omega^p_{\mathcal{F}} \wedge \Omega^{q-p}_{\mathcal{U}}.$$

The associated graded vector bundle is $\mathrm{Gr}^p_L\Omega^q_{\mathcal{U}} = \pi^*\Omega^p_{\mathcal{F}} \otimes \Omega^{q-p}_{\mathcal{U}}$. This yields a filtration on the complex $A^q(\mathcal{U})$ which is the space of smooth sections of $\Omega^q_{\mathcal{U}}$. We therefore have a corresponding (Leray) spectral sequence that computes the cohomology of $\mathcal{U}$ from that of the base $\mathcal{F}$ and the fiber. The E_1 term is

$$E_1^{p,q} = A^p(\mathcal{F}, R^q\pi_*\mathbb{R}).$$

Note that the filtration is compatible with the cup-product meaning that if $\alpha \in L^{p_1}A^{k_1}(\mathcal{U})$ and $\beta \in L^{p_2}A^{k_2}(\mathcal{U})$ then $\alpha \wedge \beta \in L^{p_1+p_2}A^{k_1+k_2}(\mathcal{U})$. The cup-product on $A^*(\mathcal{U})$ induces the natural structure on E_1. Recall that $R\pi_*\mathbb{R}$ is the pullback of a local system $\mathbf{H}^\bullet$ on Y that comes from a finite dimensional representation of $\mathrm{O}(\Sigma^\perp(\mathbb{R}))$. The pullback $\mathcal{P}^*\Phi_{\mathbf{H}^\bullet}$ of the FKM ring on Y associated to $\Phi_{\mathbf{H}^\bullet}$ defines a subring of E_1; we will abusively refer to it as the FKM subring of the E_1 term of the Leray spectral sequence that computes the cohomology of $\mathcal{U}$ from that of the base $\mathcal{F}$ and the fiber.

Theorem 8.2.1. *Let the notations be as above. Given any two cycles $\alpha_1, \alpha_2 \in \mathrm{CH}^*(\mathcal{U})$ of codimension $< \frac{m_\Lambda - 3}{4}$, the two cup-products of $[\alpha_1]$ and $[\alpha_2]$ in $H^*(\mathcal{U}, \mathbb{C})$ associated to the two sides of* (8.1) *differ by a class supported on the Noether-Lefschetz locus of $\mathcal{F}$.*

Proof. Let $2k_j$ be the degree of α_j $(j = 1, 2)$. Denote by $\alpha_j^{(i)}$ the $2i$-th component of α_j in the decomposition (8.1). Since $k_j < \frac{m_\Lambda - 3}{4}$, the argument in the proof of Theorem 8.1.1 implies that each $\alpha_j^{(i)}$ can be lifted as a θ-class in $H^{2i}_\theta(Y, \mathbf{H}^{2(k_j-i)})$. Now Proposition 7.3.1 implies that such a θ-class can be represented by a differential form in the FKM ring. It follows that both α_1 and α_2 can be represented by elements of the FKM subring of the E_1 term of the Leray spectral sequence that computes the cohomology of $\mathcal{U}$ from that of the base $\mathcal{F}$ and the fiber. Now, being a subring, the FKM subring is stable by cup-products. It follows that $\alpha_1 \wedge \alpha_2$ belongs to the FKM ring and that

$$\alpha_1 \wedge \alpha_2 - \alpha_1^{(0)} \wedge \alpha_2^{(0)} \in \bigoplus_{i=1}^{k_1+k_2} H^{i,i}_\theta(Y, \mathbf{H}^{2(k_1+k_2-i)})$$

is supported on the Noether-Lefschetz locus of $\mathcal{F}$ by Corollary 6.4.3. ♣

8.3. **Towards the Cohomological Tautological Conjecture.** Suppose that $\mathcal{U} \to \mathcal{F}$ is a smooth connected family of projective hyperkähler manifolds. We denote by $R^*_\pi(\mathcal{F}) \subseteq \mathrm{CH}^*(\mathcal{F})$ the subring generated by all κ-classes. An easy fact is that inclusions

$$\mathrm{NL}^*_\pi(\mathcal{F}) \subseteq R^*_\pi(\mathcal{F}) \tag{8.5}$$

are preserved under the pullback by morphisms preserving the relative Picard group (modulo $\mathrm{DCH}^*(\mathcal{F})$). Here, a morphism between two families $\pi : \mathcal{U} \to \mathcal{F}$ and $\pi' : \mathcal{U}' \to \mathcal{F}'$ preserving the relative Picard group is a commutative diagram

$$\begin{array}{ccc} \mathcal{U}' & \xrightarrow{\bar{f}} & \mathcal{U} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ \mathcal{F}' & \xrightarrow{f} & \mathcal{F} \end{array} \tag{8.6}$$

with $\bar{f}^*\mathrm{Pic}(\mathcal{U}/\mathcal{F}) \cong \mathrm{Pic}(\mathcal{U}'/\mathcal{F}')$. This is because $f^*(\mathrm{NL}^*_\pi(\mathcal{F})) \subseteq \mathrm{NL}^*_{\pi'}(\mathcal{F}')$ and $f^*R^*_\pi(\mathcal{F}) = R^*_{\pi'}(\mathcal{F}')$ modulo $\mathrm{DCH}^*(\mathcal{F}')$.

Using this fact, we can work with κ-classes on a general family of hyperkähler manifolds and Theorem 4.3.1 will be deduced from the following result as a direct consequence.

Theorem 8.3.1. *Let $\pi : \mathcal{U} \to \mathcal{F}$ be a smooth family of h-polarized hyperkähler manifolds of type Λ over an irreducible quasi-projective variety $\mathcal{F}$ of dimension $b \geq 3$. Let $r+1$ be the Picard number of the generic fiber of π and let*

$$\mathcal{B} = \{\mathcal{L}_0, \dots, \mathcal{L}_r\} \subset \mathrm{Pic}_{\mathbb{Q}}(\mathcal{U})$$

be a collection of line bundles whose images in $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{U}/\mathcal{F})$ form a basis. If $b_j = 0$ for $j \geq \frac{1}{4}(m_\Lambda - 3)$, the κ-class $[\kappa^{\mathcal{B}}_{a_0,\dots,a_r,b_1,\dots b_{2n}}] \in H^*(\mathcal{F}, \mathbb{Q})$ is lying in $\mathrm{NL}^*_{\mathrm{hom}}(\mathcal{F})$.

Proof. According to the discussion above, it suffices to show that the assertion holds when $\pi : \mathcal{U} \to \mathcal{F}$ is a connected component of the universal family (as stacks) of lattice-polarized hyperkähler manifolds. Because of (5.10), we can add level structures and further be reduced to the case where $\pi : \mathcal{U} \to \mathcal{F}$ is the universal family $\mathcal{U}^\ell_{\Sigma,h} \to \mathcal{F}^\ell_{\Sigma,h}$ in §8.2.

To prove the assertion, we only need to show that the cup product

$$c_1^{a_i}(\mathcal{L}_i) \dots c_1(\mathcal{L}_r)^{a_r} \dots c_1(\mathcal{T}_\pi)^{b_1} \dots c_{2n}(\mathcal{T}_\pi)^{b_{2n}} \tag{8.7}$$

lies in the FKM ring because the push-forward of classes in FKM ring lie in $H^*_\theta(\mathcal{F}, \mathbb{C})$. Then the same argument as in the proof of Theorem 8.2.1 implies that as long as $b > 2$, it suffices to show $c_1(\mathcal{L}_i)$ and $c_j(\mathcal{T}_\pi)$ lie in the FKM ring on $\mathcal{U}$.

Using the notations of §8.2, the cycle classes of $c_1(\mathcal{L}_i)$ and $c_j(\mathcal{T}_\pi)$ can be lifted to classes in $\bigoplus_{p+q=2} H^p(Y, \mathbf{H}^q)$ and $\bigoplus_{p+q=2j} H^p(Y, \mathbf{H}^q)$ respectively via the period map $\mathcal{P} : \mathcal{F} \to Y$, as in the proof of Theorem 8.1.1.

Next, by our assumption $b \geq 3$, Theorem 6.4.1 implies that the FKM ring on Y contains $\bigoplus_{p+q=2} H^p(Y, \mathbf{H}^q)$ and $\bigoplus_{p+q=2j} H^p(Y, \mathbf{H}^q)$ when $j < \frac{m_\Lambda - 3}{4}$. This proves our assertion.

Last, let us explain how we get the bound of j. Note that Theorem 6.4.1 actually only applies when $j < \frac{\dim Y}{4}$. However, recall that the moduli space $\mathcal{F}_h^\ell$ has dimension $m_\Lambda - 3$, and $c_j(\mathcal{T}_\pi)$ can be lifted to the universal family $\mathcal{U}_h^\ell \to \mathcal{F}_h^\ell$ as $\mathcal{T}_\pi$ descends. The lifted class lies in the FKM ring of $\mathcal{U}_h^\ell$ by Theorem 6.4.1 and the same argument as above. Then our claim follows from the fact that the pullback of the FKM ring on $\mathcal{U}_h^\ell$ is contained in the FKM ring of $\mathcal{U}$. ♣

Remark 8.3.2. Theorem 8.3.1 is no longer true when $\dim \mathcal{F} \leq 2$. For instance, there exist divisors on Hilbert modular surfaces which are not in the span of modular curves. (cf. [64])

Remark 8.3.3. One may have a weaker result if we replace the singular cohomology of $\mathcal{F}_h^\ell$ by Deligne-Beilinson cohomology. e.g. See Theorem 9.3.3.

8.4. As one can see from the proof, the only obstruction for proving the cohomological tautological conjecture for all hyperkähler moduli spaces is that we do not know whether the i-th components of $c_k(\mathcal{T}_\pi)$ in the decomposition (8.1) are theta classes when $\frac{m_\Lambda - 3}{2} \leq i \leq \min\{2k, \frac{3(m_\Lambda - 3)}{2}\}$.

As pointed out in [72] and [68, 63], the Beauville-Voisin conjecture actually predicts that every i-th component of $c_k(\mathcal{T}_\pi)$ lies in the last component (after possibly shrinking to an open subset of $\mathcal{F}_h$). This can be deduced from the existence of so called "Chow-Kunneth" decomposition (cf. [48]). Therefore, the Beauville-Voisin conjecture implies that Conjecture 5 is true at least after shrinking to an open subset of $\mathcal{F}_h$.

8.5. **Further remarks.** For polarized hyperkähler manifolds of generalized Kummer type, we know that the moduli space $\mathcal{F}_h$ is 4-dimensional. In this case, the κ-classes on moduli space of lattice-polarized hyperkähler manifolds automatically map to $\mathrm{NL}^*_{\mathrm{hom}}(\mathcal{F}_h)$ via the push-forward map. This is because $\mathrm{CH}^1(\mathcal{F}_h)$ is spanned by NL-divisors and $R^3_{\mathrm{hom}}(\mathcal{F}_h) = R^4_{\mathrm{hom}}(\mathcal{F}_h) = 0$. So we have

$$\mathrm{NL}^i_{\mathrm{hom}} = R^i_{\mathrm{hom}}(\mathcal{F}_h), \text{ for all } i \neq 2.$$

Then it suffices to check whether the κ-class

$$[\kappa_{a;b_1,\dots,b_{2n}}] \in H^4(\mathcal{F}_h, \mathbb{Q}), \tag{8.8}$$

(i.e. $a + \sum_{j=1}^{2n} j b_j - 2n = 2$) is lying in $\mathrm{NL}^2_{\mathrm{hom}}(\mathcal{F}_h)$. By Theorem 8.3.1, we know this is true when $b_i = 0$ for $i > 1$. When $b_i \neq 0$ for some $i > 1$, one can see some examples computed in §4.11. In the case of $n = 2$, combining these results together, we know that the conjecture holds if and only if

$$\kappa_{0;0,3,0,0}, \kappa_{1;0,1,1,0}, \kappa_{2;0,2,0,0}, \kappa_{2;0,0,0,1}, \kappa_{3;0,0,1,0}, \kappa_{4;0,1,0,0}$$

are contained in $\mathrm{NL}^2_{\mathrm{hom}}(\mathcal{F}_h)$.

9. Ring of Special cycles

In Kudla's program, it is more natural to consider the weighted cycles on Shimura varieties rather than the connected cycles. These are the so-called special cycles. In this section, we discuss the properties of the special cycles and their applications.

9.1. **Kudla's special cycles.** Keep the same notations as in §5.1. We let λ be the first Chern class of the Hodge line bundle on Y_K. For any $\beta \in \mathrm{Sym}^r(\mathbb{Q})$, we set

$$\Omega_\beta = \{\mathbf{v} \in V^n \mid \frac{1}{2}(\mathbf{v},\mathbf{v}) = \beta,\ \dim U(\mathbf{v}) = \mathrm{rank}\,\beta\},$$

and the special cycles on Y_Γ of codimension r are defined as

$$Z(\beta,\varphi,K) = \lambda^{r-\mathrm{rank}\beta} \cdot \sum_{\substack{\mathbf{v}\in\Omega_\beta \\ \mathrm{mod}\ \Gamma_1}} \varphi(\mathbf{v}) c(U(\mathbf{v}),1,K), \tag{9.1}$$

with φ being a K-invariant Schwartz function on $V^r(\mathbb{A}_f)$. Then $Z(\beta,\varphi,K)$ can be viewed as a cycle class in $\mathrm{CH}^r(Y_K)$ if $\mathrm{rank}\beta = r$. As before, we let $\widetilde{\mathrm{SC}}^r(Y_K)$ be the subspace of $\mathrm{CH}^r(Y_K)$ spanned by $Z(\beta,\varphi,K)$ and let

$$\widetilde{\mathrm{SC}}^*(Y_K) \subseteq \mathrm{CH}^*(Y_K)$$

be the subring generated by all special cycles. The special cycles have many nice properties, e.g. they behave well under pullback (cf. [34]). More importantly, although in general the Chow ring $\mathrm{CH}^*(Y_K)$ is not finitely generated, we have

Theorem 9.1.1. *The ring $\widetilde{\mathrm{SC}}^*(Y_K)$ is finitely generated and its image in $H^*(Y_K,\mathbb{Q})$ contains $H^*_{\theta}(Y_K,\mathbb{Q})$.*

The first statement follows from the work of Zhang [73] and a recent result of Bruinier and Westerholt-Raum [13], while the second assertion is proved in [6]. Clearly, there is an inclusion

$$\widetilde{\mathrm{SC}}^*(Y_K) \subseteq \mathrm{SC}^*(Y_K).$$

Moreover, we believe that the following statement should be true.

Conjecture 8. $\mathrm{SC}^*(Y_K) = \widetilde{\mathrm{SC}}^*(Y_K)$.

We shall remark below (see Proposition 9.1.2) that this conjecture can be easily checked in some simple cases such as the moduli space of polarized K3 surfaces, or the moduli space of some polarized $K3^{[n]}$-type hyperkähler manifolds (See Proposition 9.1.2). Recall that the second cohomology of a K3 surface is isomorphic to the even unimodular lattice

$$L_{K3} = U^{\oplus 2} \oplus E_8^{\oplus 2}(-1),$$

and the primitive cohomology of a polarized K3 surface (X, H) in $\mathcal{K}_g$ is isomorphic to

$$L_g := v^\perp = (2-2g) \oplus U^{\oplus 2} \oplus E_8^{\oplus 2}(-1),$$

where $v \in L_{K3}$ represents the class $c_1(H)$. The moduli space $\mathcal{K}_g$ of polarized K3 surfaces of genus g is an open subset of the arithmetic quotient $Y_{L_g} = \Gamma_{L_g}\backslash D$, where Γ_{L_g} is the associated polarized monodromy group. In this situation, we have

Proposition 9.1.2. $\mathrm{SC}^r(Y_{L_g}) = \widetilde{\mathrm{SC}}^r(Y_{L_g})$ *for all* r.

Proof. The arithmetic subgroup Γ_{L_g} is nothing but the collection of isometries of L_{K_3} preserving v. Every connected cycle $c(U)$ in Y_{L_g} of codimension r can be described as the map

$$\Gamma_\Sigma\backslash D_\Sigma \to Y_{L_g},$$

where $\Sigma = \mathrm{span}\{U, v\} \cap L_{K3}$ is a primitive sublattice of L_{K3} and D_Σ and Γ_Σ are defined as in §3.7. We write $c(\Sigma) = c(U)$ for this connected cycle.

As L_{K_3} is unimodular, every even lattice of signature $(1, r)$ $(r \leq 19)$ admits a unique (up to isometry) primitive embedding in L_{K_3}. Let Σ be an even lattice of signature $(1, r)$. The special cycle on Y_{L_g} is of the form

$$Z(\Sigma) = \sum_{\substack{\widehat{\phi(\Sigma)}=\Sigma' \\ \mathrm{mod}\ \Gamma_{L_g}}} c(\Sigma'),$$

where the sum runs over all embeddings $\phi : \Sigma \hookrightarrow L_{K3}$ modulo the isometries in Γ_{L_g} and $\widehat{\phi(\Sigma)}$ is the saturation of $\phi(\Sigma)$ in L_{K3}. We can write

$$Z(\Sigma) = c(\Sigma) + \sum_{\substack{\Sigma \not\cong \Sigma' \\ \Sigma \hookrightarrow \Sigma'}} m(\Sigma') c(\Sigma')$$

for some integers $m(\Sigma') > 0$. Then it is easy to see that $c(\Sigma)$ is in the span of $Z(\Sigma')$ and this proves the assertion.

♣

9.2. **Dimension of** $\widetilde{\mathrm{SC}}^r(Y_K)$**.** Let us give some examples to explain how to use the results of Bruinier and Westerholt-Raum in [13] and Zhang in [73] to construct relations between special cycles and estimate the dimension of $\widetilde{\mathrm{SC}}^i(Y_K)$. Let Λ be an even lattice of signature $(2, b)$. For simplicity, we restrict ourselves to the case where Y_K is the arithmetic quotient $Y_\Lambda := \Gamma_\Lambda\backslash D$, where Γ_Λ is the arithmetic subgroup

$$\{g \in \mathrm{SO}(\Lambda) |\ g \text{ acts trivially on } d(\Lambda)\}. \tag{9.2}$$

Let $\mathbb{H}_r$ be the Siegel upper half-plane of genus r. The metaplectic double cover $\mathrm{Mp}_{2r}(\mathbb{Z})$ of $\mathrm{Sp}_{2r}(\mathbb{Z})$ consists of pairs $(M, \phi(\tau))$, where

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2r}(\mathbb{Z}),\ \phi : \mathbb{H}_r \to \mathbb{C} \text{ with } \phi^2(\tau) = \det(c\tau + d).$$

Let $\rho_\Lambda^{(r)}$ be the Weil representation of $\mathrm{Mp}_{2r}(\mathbb{Z})$ on $\mathbb{C}[d(\Lambda)^r]$. For any $k \in \frac{1}{2}\mathbb{Z}$, a vector-valued Siegel modular form $f(\tau)$ of weight k and type $\rho_\Lambda^{(r)}$ is a $\mathbb{C}[d(\Lambda)^r]$-valued holomorphic function on $\mathbb{H}_r$, such that

$$f(M\tau) = \phi(\tau)^{2k} \cdot \rho_\Lambda^{(r)}(M,\phi)(f), \text{ for all } (M,\phi) \in \mathrm{Mp}_{2r}(\mathbb{Z})$$

and it is a Siegel cusp form if $f(\tau)$ is vanishes at all cusps. We denote by $M^r_{k,\Lambda}(\rho_\Lambda)$ (resp. $S^r_{k,\Lambda}(\rho_\Lambda)$) the space of $\mathbb{C}[d(\Lambda)^r]$-valued modular (resp. cusp) forms of weight k and type $\rho_\Lambda^{(r)}$. Then

Proposition 9.2.1. *For $1 \le r \le b$, we have*

$$\dim \widetilde{\mathrm{SC}}^r(Y_\Lambda) \le \dim \widetilde{\mathrm{SC}}^{r-1}(Y_\Lambda) + \dim S^r_{\frac{b+2}{2},M}(\rho_\Lambda^\vee). \tag{9.3}$$

Proof. The proof is similar to the case $r = 1$ proved in [11, §5]. By [13] and [73] the generating series

$$\Phi(\tau) = \sum Z(\beta,\varphi) q^\beta \varphi^*, \tag{9.4}$$

is a $\mathbb{C}[d(\Lambda)^r]$-valued Siegel modular form of weight $\frac{b+2}{2}$ and type ρ_Λ, where $q^\beta = \exp(2\pi \mathrm{Tr}(\beta\tau))$ (cf. [13, Theorem 5.2]). Using the Petersson inner product

$$M^r_{\frac{b+2}{2},\Lambda}(\rho_\Lambda) \times S^r_{\frac{b+2}{2},\Lambda}(\rho_\Lambda^\vee) \to \mathbb{C},$$

we obtain a natural map

$$S^r_{\frac{b+2}{2},\Lambda}(\rho_\Lambda^\vee) \to \widetilde{\mathrm{SC}}^r(Y_\Lambda)/\lambda \cdot \widetilde{\mathrm{SC}}^{r-1}(Y_\Lambda) \tag{9.5}$$

by tensoring with $\Phi(\tau)$. The assertion then follows directly from the surjectivity of (9.5). ♣

For general Y_K, the statement is still true once we replace $S^r_{\frac{b+2}{2},M}(\rho_\Lambda^\vee)$ by the space of cuspidal vector-valued modular forms with respect to certain congruence subgroup of $\mathrm{Mp}_{2r}(\mathbb{Z})$. This also follows from the modularity result of Zhang, Bruinier and Westerholt-Raum.

Remark 9.2.2. The inequality (9.3) is in general strict (see [12, §6.1]). A very interesting question to investigate is when the equality holds (see [12, Theorem 2]). This would enable us to compute the Betti numbers of Y_K and also the dimension of $\widetilde{\mathrm{SC}}^r(Y_K)$.

9.3. **Further questions.** Let us step back to $\mathrm{SC}^r(Y_K)$. Following [65], when $b \ge 3$, one can show that

$$\lambda^{b-2} \in \mathrm{SC}^{b-2}(Y_K) \neq 0$$

while $\lambda^{b-1} = \lambda^b = 0$ in $\mathrm{CH}^*(Y_K)$. Furthermore, we have

Proposition 9.3.1. *For $i > b-2$ and $j > 1$, the subgroup*

$$\lambda^j \cdot \mathrm{SC}^{i-j}(Y_K) \subseteq \mathrm{SC}^i(Y_K)$$

is zero.

Proof. The proof is the same as in [65], where one can show that the self intersection of the Hodge line bundle of codimension $\geq b-1$ on a Shimura variety of orthogonal type with dimension $b \geq 3$ is rationally equivalently to zero. ♣

Recall that $\mathrm{SC}^r_{\mathrm{hom}}(Y_K) = 0$ for $r > b-2$ by Corollary 6.4.2, we wonder if the questions below have a positive answer.

Question 9.3.2. Assume that $\dim Y_K \geq 3$. Do the following hold?

(1) $\mathrm{SC}^r(Y_K) = 0$ for $r > b-2$.
(2) The cycle class map induces an isomorphism $cl : \mathrm{SC}^r(Y_K) \to \mathrm{SC}^r_{\mathrm{hom}}(Y_K)$ for all r.

Theorem 9.3.3. *For $r \leq \frac{b}{4}$, there is an isomorphism*

$$H^{2r}_{\mathcal{D}}(Y_K, \mathbb{Z}(r)) \otimes \mathbb{Q} \cong H^{2r}(Y_K, \mathbb{Q}).$$

In particular, $\mathrm{SC}^r(Y_K) \to H^{2r}(Y_K, \mathbb{Q})$ is an isomorphism for $r \leq \frac{b}{4}$ provided both the Hodge conjecture and the Bloch-Beilinson conjecture hold.

Proof. Using the standard exact sequence of Deligne-Beilinson cohomology and singular cohomology groups (cf. [21, Corollary 2.10]), we can get an injection

$$H^{2i}_{\mathcal{D}}(Y_K, \mathbb{Z}(i)) \otimes \mathbb{Q} \hookrightarrow H^{2i}(Y_K, \mathbb{Q}),\ i \leq \frac{b}{4}$$

because of the vanishing result Proposition 6.2.1. As the connected cycles are all defined over $\bar{\mathbb{Q}}$, the assertion then follows from the generalization of the Noether-Lefschetz conjecture (cf. [6, Theorem 3.6]), which shows that $H^{2i}(Y_K, \mathbb{Q})$ is equal to $\mathrm{SC}^i_{\mathrm{hom}}(Y_K)$.

♣

Remark 9.3.4. Note that all the connected cycles are defined over $\bar{\mathbb{Q}}$. According to the proof of Theorem 9.3.3, when $r < \frac{b}{2}$, Question 9.3.2(2) has an affirmative answer provided both the Bloch-Beilinson conjecture and the Hodge conjecture hold.

Let us come back to $\mathcal{K}_g$, where the first part of Question 9.3.2 indicates that $\mathrm{NL}^r(\mathcal{K}_g) = 0$ for $r > 17$. This can be viewed as analogous to Faber's conjecture on $\mathcal{M}_g$. The second part is then equivalent to saying that there is an isomorphism

$$\mathrm{NL}^*(\mathcal{K}_g) \cong \mathrm{NL}^*_{\mathrm{hom}}(\mathcal{K}_g).$$

For small g, one can get partial results towards these questions via the geometric construction of $\mathcal{K}_g$.

Proposition 9.3.5. *When $g \leq 14$ or $g \in \{16, 18, 20\}$, $\mathrm{NL}^{19}(\mathcal{K}_g) = 0$.*

Proof. This follows from the fact that $\mathcal{K}_g$ is unirational for small g. Indeed, Mukai has proved that $\mathcal{K}_g$ is unirational when $g \leq 13$ or $g \in \{16, 18, 20\}$ and Nuer [49] recently proved that $\mathcal{K}_{14}$ is also unirational.

Note that this method of proof can not work in general as $\mathcal{K}_g$ is of general type for g sufficiently large. ♣

Remark 9.3.6. In this case, as the Betti number of $\mathcal{K}_2$ has been computed in [33], one can make explicit computations to verify Question 9.3.2 (2) via Proposition 9.2.1.

References

[1] Yves André. On the Shafarevich and Tate conjectures for hyper-Kähler varieties. *Math. Ann.*, 305(2):205–248, 1996.

[2] W. L. Baily, Jr. and A. Borel. Compactification of arithmetic quotients of bounded symmetric domains. *Ann. of Math. (2)*, 84:442–528, 1966.

[3] Arnaud Beauville. Some remarks on Kähler manifolds with $c_1 = 0$. In *Classification of algebraic and analytic manifolds (Katata, 1982)*, volume 39 of *Progr. Math.*, pages 1–26. Birkhäuser Boston, Boston, MA, 1983.

[4] Arnaud Beauville and Claire Voisin. On the Chow ring of a $K3$ surface. *J. Algebraic Geom.*, 13(3):417–426, 2004.

[5] K. Behrend. Cohomology of stacks. In *Intersection theory and moduli*, ICTP Lect. Notes, XIX, pages 249–294 (electronic). Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004.

[6] Nicolas Bergeron, Zhiyuan Li, John Millson, and Colette Moeglin. The Noether-Lefschetz conjecture and generalizations. *Invent. Math.*, 208(2):501–552, 2017.

[7] Nicolas Bergeron, John Millson, and Colette Moeglin. Hodge type theorems for arithmetic manifolds associated to orthogonal groups. *Int. Math. Res. Not. IMRN*, (15):4495–4624, 2017.

[8] A. Borel and W. Casselman. L^2-cohomology of locally symmetric manifolds of finite volume. *Duke Math. J.*, 50(3):625–647, 1983.

[9] A. Borel and H. Garland. Laplacian and the discrete spectrum of an arithmetic group. *Amer. J. Math.*, 105(2):309–335, 1983.

[10] Raoul Bott and Loring W. Tu. *Differential forms in algebraic topology*, volume 82 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.

[11] Jan H. Bruinier. *Borcherds products on* $\mathrm{O}(2, l)$ *and Chern classes of Heegner divisors*, volume 1780 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002.

[12] Jan Hendrik Bruinier. On the converse theorem for Borcherds products. *J. Algebra*, 397:315–342, 2014.

[13] Jan Hendrik Bruinier and Martin Westerholt-Raum. Kudla's modularity conjecture and formal Fourier-Jacobi series. *Forum Math. Pi*, 3:e7, 30, 2015.

[14] Chiara Camere. Some remarks on moduli spaces of lattice polarized holomorphic symplectic manifolds. *arXiv:1512.02115*, 2015.

[15] Chiara Camere. Lattice polarized irreducible holomorphic symplectic manifolds. *Ann. Inst. Fourier (Grenoble)*, 66(2):687–709, 2016.

[16] François Charles. The Tate conjecture for $K3$ surfaces over finite fields. *Invent. Math.*, 194(1):119–145, 2013.

[17] Héctor Damián Méndez Dávila. Normal Functions and the Bloch-Beilinson Filtration. 2015. Thesis (Ph.D.)–University of Alberta.

[18] Mark Andrea A. de Cataldo. The perverse filtration and the Lefschetz hyperplane theorem, (ii). *J. Algebraic Geom.*, 21(2):305–345, 2012.

[19] P. Deligne. Théorème de Lefschetz et critères de dégénérescence de suites spectrales. *Inst. Hautes Études Sci. Publ. Math.*, (35):259–278, 1968.

[20] I. V. Dolgachev. Mirror symmetry for lattice polarized $K3$ surfaces. *J. Math. Sci.*, 81(3):2599–2630, 1996. Algebraic geometry, 4.

[21] Hélène Esnault and Eckart Viehweg. Deligne-Beĭlinson cohomology. In *Beĭlinson's conjectures on special values of L-functions*, volume 4 of *Perspect. Math.*, pages 43–91. Academic Press, Boston, MA, 1988.

[22] Gavril Farkas and Ian Morrison, editors. *Handbook of moduli. Vol. I*, volume 24 of *Advanced Lectures in Mathematics (ALM)*. International Press, Somerville, MA; Higher Education Press, Beijing, 2013.
[23] Lie Fu. Beauville-Voisin conjecture for generalized Kummer varieties. *Int. Math. Res. Not. IMRN*, (12):3878–3898, 2015.
[24] Lie Fu, Robert Laterveer, Charles Vial, and Mingmin Shen. The generalized Franchetta conjecture for some hyper-Kähler varieties. *arXiv:1708.02919*, 2017.
[25] William Fulton. *Young tableaux*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry.
[26] Jens Funke and John Millson. Cycles with local coefficients for orthogonal groups and vector-valued Siegel modular forms. *Amer. J. Math.*, 128(4):899–948, 2006.
[27] Michael Harris and Steven Zucker. Boundary cohomology of Shimura varieties. III. Coherent cohomology on higher-rank boundary strata and applications to Hodge theory. *Mém. Soc. Math. Fr. (N.S.)*, (85):vi+116, 2001.
[28] David Harvey, Brendan Hassett, and Yuri Tschinkel. Characterizing projective spaces on deformations of Hilbert schemes of K3 surfaces. *Comm. Pure Appl. Math.*, 65(2):264–286, 2012.
[29] Brendan Hassett and Yuri Tschinkel. Hodge theory and Lagrangian planes on generalized Kummer fourfolds. *Mosc. Math. J.*, 13(1):33–56, 189, 2013.
[30] Hongyu He and Jerome William Hoffman. Picard groups of Siegel modular 3-folds and θ-liftings. *J. Lie Theory*, 22(3):769–801, 2012.
[31] R. Howe. θ-series and invariant theory. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 275–285. Amer. Math. Soc., Providence, R.I., 1979.
[32] Daniel Huybrechts. Compact hyper-Kähler manifolds: basic results. *Invent. Math.*, 135(1):63–113, 1999.
[33] Frances Kirwan and Ronnie Lee. The cohomology of moduli spaces of $K3$ surfaces of degree 2. II. *Proc. London Math. Soc. (3)*, 58(3):559–582, 1989.
[34] Stephen S. Kudla. Algebraic cycles on Shimura varieties of orthogonal type. *Duke Math. J.*, 86(1):39–78, 1997.
[35] Stephen S. Kudla and John J. Millson. Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables. *Inst. Hautes Études Sci. Publ. Math.*, (71):121–172, 1990.
[36] James D. Lewis. Abel-Jacobi equivalence and a variant of the Beilinson-Hodge conjecture. *J. Math. Sci. Univ. Tokyo*, 17(2):179–199, 2010.
[37] James D. Lewis. Arithmetic normal functions and filtrations on Chow groups. *Proc. Amer. Math. Soc.*, 140(8):2663–2670, 2012.
[38] Jian-Shu Li. Nonvanishing theorems for the cohomology of certain arithmetic quotients. *J. Reine Angew. Math.*, 428:177–217, 1992.
[39] Eduard Looijenga. L^2-cohomology of locally symmetric varieties. *Compositio Math.*, 67(1):3–20, 1988.
[40] Eduard Looijenga and Valery A. Lunts. A Lie algebra attached to a projective variety. *Invent. Math.*, 129(2):361–412, 1997.
[41] Keerthi Madapusi Pera. The Tate conjecture for K3 surfaces in odd characteristic. *Invent. Math.*, 201(2):625–668, 2015.
[42] Alina Marian, Dragos Oprea, and Rahul Pandharipande. Segre classes and Hilbert schemes of points. *Ann. Sci. Éc. Norm. Supér. (4)*, 50(1):239–267, 2017.
[43] Eyal Markman. On the monodromy of moduli spaces of sheaves on $K3$ surfaces. *J. Algebraic Geom.*, 17(1):29–99, 2008.

[44] Eyal Markman. A survey of Torelli and monodromy results for holomorphic-symplectic varieties. In *Complex and differential geometry*, volume 8 of *Springer Proc. Math.*, pages 257–322. Springer, Heidelberg, 2011.
[45] Eyal Markman. On the existence of universal families of marked hyperkahler varieties. *arXiv:1701.08690*, 2017.
[46] Davesh Maulik. Supersingular K3 surfaces for large primes. *Duke Math. J.*, 163(13):2357–2425, 2014. With an appendix by Andrew Snowden.
[47] Colette Mœglin, Marie-France Vignéras, and Jean-Loup Waldspurger. *Correspondances de Howe sur un corps p-adique*, volume 1291 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1987.
[48] J. P. Murre. On a conjectural filtration on the Chow groups of an algebraic variety. I. The general conjectures and some examples. *Indag. Math. (N.S.)*, 4(2):177–188, 1993.
[49] H. Nuer. Unirationality of moduli spaces of special cubic fourfolds and $K3$ surfaces. *arxiv:1503.05256*, March 2015.
[50] Kieran G. O'Grady. Desingularized moduli spaces of sheaves on a $K3$. *J. Reine Angew. Math.*, 512:49–117, 1999.
[51] Kieran G. O'Grady. A new six-dimensional irreducible symplectic variety. *J. Algebraic Geom.*, 12(3):435–505, 2003.
[52] Kieran G. O'Grady. Moduli of sheaves and the Chow group of $K3$ surfaces. *J. Math. Pures Appl. (9)*, 100(5):701–718, 2013.
[53] Kieran G. O'Grady and Radu Laza. Birational geometry of the moduli space of quartic $K3$ surfaces. *arXiv:1607.01324*, 2016.
[54] Kieran Gregory O'Grady. *MODULI OF ABELIAN AND K3 SURFACES*. ProQuest LLC, Ann Arbor, MI, 1986. Thesis (Ph.D.)–Brown University.
[55] Keiji. Oguiso. No cohomologically trivial non-trivial automorphism of generalized Kummer manifolds. *arXiv:1208.3750*, 2012.
[56] R. Pandharipande and Q. Yin. Relations in the tautological ring of the moduli space of $K3$ surfaces. *arxiv:1607.08758*, July 2016.
[57] N. Pavic, J. Shen, and Q. Yin. On O'Grady's generalized Franchetta conjecture. *arxiv:1604.02939*, April 2016.
[58] Chris Peters and Morihiko Saito. Lowest weights in cohomology of variations of Hodge structure. *Nagoya Math. J.*, 206:1–24, 2012.
[59] D. Petersen. A vanishing result for tautological classes on the moduli of $K3$ surfaces. *arxiv:1606.06716*, June 2016.
[60] Jordan Rizov. Moduli stacks of polarized $K3$ surfaces in mixed characteristic. *Serdica Math. J.*, 32(2-3):131–178, 2006.
[61] Morihiko Saito. Mixed Hodge modules. *Publ. Res. Inst. Math. Sci.*, 26(2):221–333, 1990.
[62] Leslie Saper and Mark Stern. L_2-cohomology of arithmetic varieties. *Ann. of Math. (2)*, 132(1):1–69, 1990.
[63] Mingmin Shen and Charles Vial. The motive of the Hilbert cube $X^{[3]}$. *Forum Math. Sigma*, 4:e30, 55, 2016.
[64] Gerard van der Geer. *Hilbert modular surfaces*, volume 16 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1988.
[65] Gerard van der Geer and Toshiyuki Katsura. Note on tautological classes of moduli of $K3$ surfaces. *Mosc. Math. J.*, 5(4):775–779, 972, 2005.
[66] Mikhail Sergeevic Verbitsky. *Cohomology of compact hyperkaehler manifolds*. ProQuest LLC, Ann Arbor, MI, 1995. Thesis (Ph.D.)–Harvard University.
[67] Misha Verbitsky. Mapping class group and a global Torelli theorem for hyperkähler manifolds. *Duke Math. J.*, 162(15):2929–2986, 2013. Appendix A by Eyal Markman.

[68] Charles Vial. On the motive of some hyperkähler varieties. *J.Reine Angew. Math.*, to appear.
[69] David A. Vogan, Jr. and Gregg J. Zuckerman. Unitary representations with nonzero cohomology. *Compositio Math.*, 53(1):51–90, 1984.
[70] Claire Voisin. *Hodge theory and complex algebraic geometry. II*, volume 77 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, English edition, 2007. Translated from the French by Leila Schneps.
[71] Claire Voisin. On the Chow ring of certain algebraic hyper-Kähler manifolds. *Pure Appl. Math. Q.*, 4(3, part 2):613–649, 2008.
[72] Claire Voisin. Chow rings and decomposition theorems for families of $K3$ surfaces and Calabi-Yau hypersurfaces. *Geom. Topol.*, 16(1):433–473, 2012.
[73] Wei Zhang. *Modularity of generating functions of special cycles on Shimura varieties*. ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)–Columbia University.
[74] Steven Zucker. Locally homogeneous variations of Hodge structure. *Enseign. Math. (2)*, 27(3-4):243–276 (1982), 1981.

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