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## INFINITE CLASS FIELD TOWERS OF NUMBER FIELDS OF PRIME POWER DISCRIMINANT

by

Farshid Hajir, Christian Maire, Ravi Ramakrishna

**Abstract.** — For every prime number p, we show the existence of a solvable number field L ramified only at  $\{p, \infty\}$  whose p-Hilbert Class field tower is infinite.

#### Keywords: Hilbert class field tower, root discriminant

For a number field L of degree n over  $\mathbb{Q}$ , the root discriminant is defined to be  $D_{\rm L}^{1/n}$ where  $D_{\rm L}$  is the absolute value of the discriminant of L. Given a finite set S of places of  $\mathbb{Q}$ , it is an old question as to whether there is an infinite sequence of number fields unramified outside S with bounded root discriminant. This question is related to the constants of Martinet [10] and Odlyzko's bounds [12]. Since the root discriminant is constant in unramified extensions, an approach to answering the previous question in the positive is to find a number field L (of finite degree) unramified outside S having an infinite class field tower. In the case of K/ $\mathbb{Q}$  quadratic, it is a classical result of Golod and Shafarevich that if K/ $\mathbb{Q}$  is ramified at at least 8 places, then K has an infinite 2class field tower. On the other hand, if p is a prime, and  $S = \{p, \infty\}$ , this question becomes whether there exist number fields with p-power discriminant having an infinite unramified extension. Schmitals [13] and Schoof [14] produced a few isolated examples of this type. See also [3], [9], etc. For  $p \in \{2, 3, 5\}$ , Hoelscher [4] announced the existence of number fields unramified outside  $\{p, \infty\}$  and having an infinite Hilbert class field tower; see remark 2.2. Here we prove:

**Theorem.** — For every prime number p, there exists a solvable extension  $L/\mathbb{Q}$ , ramified only at  $\{p, \infty\}$ , having an infinite Hilbert p-class field tower. Consequently, there exists an infinite nested sequence of number fields of p-power discriminant with bounded root discriminant.

Our proof is based on the idea of "cutting" of wild towers introduced in [2]; in particular it does not involve the usual technique of genus theory. The strategy begins by choosing s such that  $\mathbb{Q}(\zeta_{p^s})$  has large class group (always possible). We then let K be the Hilbert class field of  $\mathbb{Q}(\zeta_{p^s})$ . Clearly  $K_S$ , the maximal pro-p extension of K ramified only at S, the

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places of K above p, is infinite. For each positive integer k, we define a Galois extension  $K_S^{[k]}/K$  contained inside  $K_S/K$  such that all decomposition groups of  $Gal(K_S^{[k]}/K)$  are finite and abelian. We then show that for all large enough k,  $K_S^{[k]}/K$  is of infinite degree, and that there exists a finite Galois extension L/K contained in  $K_S^{[k]}$ , with the property that the primes above p split completely from L to  $K_S^{[k]}$ , and in particular are unramified, leading to the desired result.

We do not know whether for every prime p, there is a *totally real* number field of ppower discriminant having an infinite Hilbert class field tower. In [14, Corollary 4.4] it is shown that  $\mathbb{Q}(\sqrt{39345017})$  (which is ramified only at the prime 39345017) has infinite Hilbert class field tower. In [15], Shanks studied primes of the form  $p = a^2 + 3a + 9$ and the corresponding totally real cubic subfields  $K \subset \mathbb{Q}(\mu_p)$  and showed the minimal polynomials of K are  $x^3 - ax^2 - (a + 3)x - 1$ . Taking a = 17279 so p = 298615687, one can compute that the 2-part of the class group of K has rank 6. It is not hard to see, using the Golod-Shafarevich criterion, that K has infinite 2-Hilbert class field tower. Thus some examples exist in the totally real case.

Also in [5] Joshi and McLeman, using ideas and data of [15], showed that  $\mathbb{Q}(\zeta_p)$  has infinite Hilbert class field tower for sufficiently large primes p of the form  $a^2 + 3a + 9$ . Inasmuch as it specifies an explicit and easily constructed number field as base of the tower, this result is stronger than ours for the primes for which it applies.

#### 1. The results we need

Let p be a prime number. Let  $K/\mathbb{Q}$  be a finite Galois extension. Assume  $\mu_p \subset K$  and moreover that K is totally imaginary when p = 2. For a prime  $\mathfrak{p}$  of K dividing p denote by e (resp. f) the ramification index (resp. the residue degree) of  $\mathfrak{p}$  in  $K/\mathbb{Q}$ .

**1.1. On the group**  $G_S$ . — Denote by S the set of places of K above p, and consider  $K_S$  the maximal pro-p extension of K unramified outside S; put  $G_S = \text{Gal}(K_S/K)$ .

Theorem 1.1 below is well-known, see for example [11, Corollary 8.7.5 and Theorem 10.7.3].

**Theorem 1.1.** — Let  $K/\mathbb{Q}$  be a totally imaginary Galois extension containing  $\mu_p$ . Let  $S = \{p, \infty\}$ . If  $p \nmid h_K$ , the class number of K, then

dim 
$$H^1(\mathbf{G}_S, \mathbb{F}_p) = g\left(\frac{ef}{2} + 1\right)$$
 and dim  $H^2(\mathbf{G}_S, \mathbb{F}_p) = g - 1$ 

where (p) has the usual efg decomposition in  $K/\mathbb{Q}$ .

#### 1.2. The cutting towers strategy. —

1.2.1. The Golod-Shafarevich Theorem. — Let G be a finitely presented pro-p group. Consider a minimal presentation  $1 \to \mathbb{R} \to \mathbb{F} \xrightarrow{\varphi} \mathbb{G}$  of G, where F is a free pro-p group on  $d = d(\mathbb{G})$  generators  $\sigma_1, \ldots, \sigma_d$  and r relations  $\rho_1, \ldots, \rho_r$  with normal closure  $\mathbb{R} = \langle \rho_1, \cdots, \rho_r \rangle^{\text{Norm}}$ . We note that  $d = \dim H^1(\mathbb{G}, \mathbb{F}_p)$  and  $r = \dim H^2(\mathbb{G}, \mathbb{F}_p)$ . We recall the depth function  $\omega$  on F. See [8, Appendice A.3] or [7] for more details. The augmentation ideal I of  $\mathbb{F}_p[\![\mathbb{F}]\!]$  is, by definition, generated by the set of elements  $\{g - 1\}_{g \in \mathbb{F}}$ . Then for  $1 \neq g \in \mathbb{F}$ , define  $\omega(g) = \max\{k \ge 1 \mid g - 1 \in I^k\}$ ; we put  $\omega(1) = \infty$ . It is not difficult to see that  $\omega([g,g']) \ge 2$  and that  $\omega(g^{p^k}) \ge p^k$  for every  $g, g' \in F$  and  $k \in \mathbb{Z}_{>0}$ . Observe also that as the presentation  $\varphi$  is minimal,  $\omega(\rho_i) \ge 2$  for all the relations  $\rho_i$ .

The Golod-Shafarevich polynomial associated to the presentation  $\varphi$  of G is  $P_{\rm G}(t) = 1 - dt + \sum_i t^{\omega(\rho_i)}$ .

**Theorem 1.2** (Golod-Shafarevich, Vinberg [16]). — If G is finite then  $P_{\rm G}(t) > 0$  for all  $t \in ]0, 1[$ .

Of course in generic situations, we have no information about the  $\rho_i$ 's other than their being elements of F of depth at least 2. With that in mind, let us note that if  $P_G(t) \leq P(t)$ for all  $t \in ]0, 1[$ , for some polynomial P(t) which takes on a non-negative value somewhere on the open unit interval, then G must be infinite. For example, for a pro-p group of generator rank d and relation rank r, since all r relations have depth at least 2, we have  $P_G(t) \leq 1 - dt + rt^2$  for all  $t \in ]0, 1[$ . Theorem 1.2 then yields the usual Golod-Shafarevich criterion: if G is a non-trivial finite p-group of generator rank d and relation rank r, then  $r > d^2/4$ .

We can also define a depth function  $\omega_{\rm G}$  on G associated to the augmentation ideal  $I_G$  of  $\mathbb{F}_p[\![{\rm G}]\!]$  by  $\omega_{\rm G}(g) = \max\{k \ge 1 \mid g - 1 \in I_{\rm G}^k\}$ , for  $1 \ne g \in {\rm G}$ ; put  $\omega_{\rm G}(1) = \infty$ . Then:

**Proposition 1.3**. — For every  $g \in G$ , we have

$$\omega_{\rm G}(g) = \max\{\omega(y) \mid \varphi(y) = g\}.$$

*Proof.* — See [8, Appendice 3, Theorem 3.5].

We now study quotients  $\Gamma$  of G such that  $d(G) = d(\Gamma)$ . In this case, the initial minimal presentation of G induces a minimal presentation of  $\Gamma$ 

$$1 \longrightarrow R \longrightarrow F \xrightarrow{\varphi} G \longrightarrow 1$$

Suppose that  $\Gamma = G/\langle x_1, \cdots, x_m \rangle^{\text{Norm}}$ . Lift the  $x_i$ 's to  $y_i \in F$  such that  $\omega_G(x_i) = \omega(y_i)$  for each *i*. Hence,  $\Gamma = F/R'$ , where  $R' = R\langle y_1, \cdots, y_m \rangle^{\text{Norm}}$ . In particular, if  $R = \langle \rho_1, \cdots, \rho_r \rangle^{\text{Norm}}$ , then  $R' = \langle \rho_1, \cdots, \rho_r, y_1, \cdots, y_m \rangle^{\text{Norm}}$ . In this situation, we say that we have 'cut' the group *G* by the elements  $y_1, \ldots, y_m$ . Even if we have no additional information about the  $\rho_i$ 's, the estimate  $P_{\Gamma}(t) \leq 1 - dt + rt^2 + \sum_i t^{\omega(y_i)}$  is valid on the open unit interval.

1.2.2. Cutting of  $G_S$ . — Fixing a prime p and a number field K, we let S be the set of primes of K above p. Recall that  $G_S$  is the Galois group over K of the maximal pextension of K unramified outside S. We want to consider some special quotients  $\Gamma$  of  $G_S$ of the type that were introduced in [2]. In [2] tame ramification was allowed, and then a quotient was taken. Here  $G_S$  is wildly ramified and the quotient we take will have abelian decomposition groups with wild but finite image, and hence finite image of inertia. This quotient of course corresponds to a sub-extension so we will use the term 'cut' to apply both to Galois groups and the corresponding tower of fields.

Each place  $v \in S$  corresponds to (a conjugacy class of) a decomposition group and hence to some extension  $K_v/\mathbb{Q}_p$  of degree ef (in fact these fields are isomorphic as  $K/\mathbb{Q}$  is Galois). Then, as  $\mu_p \subset K_v$ , the  $\mathbb{F}_p$ -vector space  $K_v^{\times}/(K_v^{\times})^p$  has dimension ef + 2, and

local class field theory implies the Galois group of the maximal pro-p extension of  $K_v$  is generated by ef + 2 elements. Thus the decomposition subgroup  $G_v$  of v in  $K_S/K$  is generated by at most ef + 2 elements  $z_{i,v}$ . Consider now the commutators  $[z_{i,v}, z_{j,v}]$  of all these elements; they clearly yield at most  $\binom{ef+2}{2}$  distinct elements of  $G_S$ . Now we cut  $G_S$  by the closed normal subgroup

$$\mathbf{R}_0 = \langle [z_{i,v}, z_{j,v}] \mid 1 \leq i, j \leq ef + 2; v \in S \rangle^{\text{Norm}}$$

these elements generate, and denote by  $\Gamma_0$  the corresponding quotient. As  $\omega_{G_S}([z_{i,v}, z_{j,v}]) \ge 2$ , we have the following estimate for the Golod-Shafarevich polynomial of  $\Gamma_0$ :

$$P_{\Gamma_0}(t) \leq 1 - dt + rt^2 + g\binom{ef+2}{2}t^2, \qquad \forall t \in ]0,1[.$$

Here  $d = \dim H^1(\mathcal{G}_S, \mathbb{F}_p)$  and  $r = \dim H^2(\mathcal{G}_S, \mathbb{F}_p)$ . We note that the kernel  $R_0$  of the surjection  $\mathcal{G}_S \twoheadrightarrow \Gamma_0$  fixes the maximal sub-extension  $\mathcal{K}_S^{loc-ab}/\mathcal{K}$  of  $\mathcal{K}_S/\mathcal{K}$  with abelian decomposition groups everywhere. Observe that  $\mathcal{K}_S^{loc-ab}/\mathcal{K}$  contains the compositum of all  $\mathbb{Z}_p$ -extensions of  $\mathcal{K}$ .

Next, we cut  $G_S$  a little bit further as follows. For each integer  $k \ge 1$ , define

$$\mathbf{R}_{k} = \mathbf{R}_{0} \langle z_{i,v}^{p^{k}} \mid 1 \leq i \leq ef + 2, v \in S \rangle^{\mathrm{Norm}}.$$

Let  $\Gamma_k$  be the corresponding quotient of  $G_S$  and denote the fixed field of  $R_k$  by  $K_S^{[k]}$ , so that  $\Gamma_k = \text{Gal}(K_S^{[k]}/\text{K})$ . Since  $\omega_{\Gamma}(z_{i,v}^{p^k}) \ge p^k$  for all  $z_{i,v}$ , we observe that for  $k \ge 1$ ,

$$P_{\Gamma_k}(t) \leqslant P_{\Gamma_0}(t) + g(ef+2)t^{p^k} \qquad \forall t \in ]0,1[.$$

Suppose that there exists some  $t_0 \in ]0, 1[$  such that  $P_{\Gamma_0}(t_0) < 0$ . Then, evidently for all sufficiently large  $k, P_{\Gamma_0}(t_0) < 0 \implies P_{\Gamma_k}(t_0) < 0 \implies K_S^{[k]}/K$  is infinite.

We now show there exists a finite Galois extension L/K such that the infinite extension  $K_S^{[k]}/L$  is unramified everywhere. We need a lemma.

**Lemma 1.4**. — With the notation as above, fix an integer  $k \ge 1$ , set  $K_{(0)} = K$  and for  $i \ge 1$ , define  $K_{(i+1)}$  to be the compositum of all  $\mathbb{Z}/p\mathbb{Z}$ -extensions of  $K_{(i)}$  contained in  $K_S^{[k]}$ . Set  $N_n = \text{Gal}(K_S^{[k]}/K_{(n)})$ . Then  $\bigcap_{n=0}^{\infty} N_n = \{1\}$ .

*Proof.* — It suffices to show  $\bigcup_{n=0}^{\infty} K_{(n)} = K_S^{[k]}$ . Let  $\alpha \in K_S^{[k]}$ . The Galois closure  $M(\alpha)$  over K of  $K(\alpha)$  has Galois group a finite *p*-group. The solvability of finite *p*-groups implies that  $\alpha \in K_{(r)}$  for some *r*.

**Proposition 1.5.** — With the above notation, suppose  $K/\mathbb{Q}$  is Galois and  $K_S^{[k]}/K$  is infinite. Then there exists a finite subextension L/K of  $K_S^{[k]}/K$  which is Galois over  $\mathbb{Q}$ , has an infinite Hilbert p-class field tower, and has the property that all primes above p split completely from L to  $K_S^{[k]}$ .

*Proof.* — Since  $K/\mathbb{Q}$  is Galois and S is  $Gal(K/\mathbb{Q})$ -invariant, the fields  $K_S$ ,  $K_S^{[k]}$  and  $K_{(n)}$  are all Galois over  $\mathbb{Q}$ .

Let D be a (finite!) decomposition group above v|p in  $\operatorname{Gal}(\mathrm{K}_{S}^{[k]}/\mathrm{K})$ . Suppose now that  $N_{n} \cap D$  is nontrivial for all n. As these intersections are finite and decreasing in n, if they are all nontrivial, they stabilize at a finite nontrivial group, in which case  $\bigcap_{n=0}^{\infty} N_{n}$  is nontrivial, contradicting Lemma 1.4. Thus there exists an m that  $N_{m} \cap D = \{1\}$ . Since  $\mathrm{K}_{(m)}/\mathbb{Q}$  is Galois,  $N_{m}$  intersects trivially with all  $\operatorname{Gal}(\mathrm{K}_{S}^{[k]}/\mathbb{Q})$ -conjugates of D. We

can then take  $L = K_{(m)}$  and all decomposition groups above p in  $Gal(K_S^{[k]}/L)$  are trivial so primes above p split completely from L to  $K_S^{[k]}$ .

#### 2. Proof

In Proposition 2.1 below we give a general criterion for a number field L to exist satisfying the conclusion of the Theorem. We then prove the Theorem by giving a fairly explicit conditions under which the criterion of Proposition 2.1 holds.

**Proposition 2.1.** — Let  $K/\mathbb{Q}$  be finite Galois and totally complex with  $\mu_p \subset K$ . Let S be the set of primes of K dividing p. Assume the cardinality of S, denoted  $g_K$ , is at least 8. Then there exists a finite extension L of K contained in  $K_S$  which is Galois over  $\mathbb{Q}$  and has infinite Hilbert p-class field tower.

*Proof.* — Let  $H = K_{\emptyset}$  be the "top" of the *p*-Hilbert class field tower of K, i.e. the maximal unramified *p*-extension of K. If H/K is infinite, we are done, so suppose  $[H : K] < \infty$ . Recall  $H_S = K_S$ . Note that H has class number prime to *p* so by Theorem 1.1, working over H,

$$\dim H^1(\operatorname{Gal}(\mathcal{H}_S/\mathcal{H}), \mathbb{F}_p) = g_{\mathcal{H}}\left(\frac{e_{\mathcal{H}}f_{\mathcal{H}}}{2} + 1\right) \text{ and } \dim H^2(\operatorname{Gal}(\mathcal{H}_S/\mathcal{H}), \mathbb{F}_p) = g_{\mathcal{H}} - 1.$$

As in §1.2.2, consider the quotient  $\Gamma_0$  of Gal(H<sub>S</sub>/H) by the normal subgroup generated by the local commutators at each  $v \in S$  (all commutators of generators of the decomposition group at v); one has  $\binom{e_{\rm H}f_{\rm H}+2}{2}$  such commutators. We have

$$P_{\Gamma_0}(t) \leq 1 - \dim H^1(\Gamma_0, \mathbb{F}_p)t + \dim H^2(\Gamma_0, \mathbb{F}_p)t^2 \leq 1 - dt + rt^2 \qquad \forall t \in ]0, 1[,$$

where  $d := g_{\rm H} \left(\frac{e_{\rm H}f_{\rm H}}{2} + 1\right)$ , and  $r := g_{\rm H} - 1 + g_{\rm H} \frac{(e_{\rm H}f_{\rm H}+2)(e_{\rm H}f_{\rm H}+1)}{2}$ . The first inequality we have seen simply comes from the fact that all relations have depth at least 2. For the second inequality, we note first that  $\Gamma_0$  and  ${\rm Gal}({\rm H}_S/{\rm H})$  have the same generator rank, namely d; moreover, since  $\Gamma_0$  is constructed using a presentation on d generators using at most r relations, we have dim  $H^2(\Gamma_0, \mathbb{F}_p) \leq r$ .

Clearly d/2r < 1, and  $P_{\Gamma_0}(d/2r) \leq 1 - \frac{d^2}{4r}$ . If  $P_{\Gamma_0}(d/2r) < 0$ , then one has, as in §1.2.2, room to cut by some large *p*-power of the generators of the abelian decomposition group at v|p and obtain an infinite extension of K whose decomposition groups at *p* are finite. Proposition 1.5 would then give the result.

It thus suffices to check that  $4r < d^2$ , or equivalently

$$16(g_{\rm H}-1) + 8g_H(e_{\rm H}f_{\rm H}+2)(e_{\rm H}f_{\rm H}+1) \stackrel{?}{<} g_{\rm H}^2(e_{\rm H}f_{\rm H}+2)^2.$$

Replacing the  $16(g_{\rm H} - 1)$  term on the left by  $16g_{\rm H}$  and dividing by  $g_{\rm H}$ , and setting  $x = e_{\rm H}f_{\rm H}$ , it now suffices to verify

$$16 + 8(x+2)(x+1) \stackrel{?}{<} g_{\rm H}(x+2)^2$$

for all  $x \ge 1$ . It is easy to see that this inequality holds as long as  $g_{\rm H} \ge 8$ . Since  $g_{\rm H} \ge g_{\rm K}$ , we have therefore checked that  $P_{\Gamma_0}(d/2r) < 0$ . By Proposition 1.5 we conclude that  ${\rm K}_S^{[k]}/{\rm K}$  is infinite for all sufficiently large k and the field L with the desired properties exists.

Proof of Theorem : Recall that the principal prime  $\mathfrak{p} = (1 - \zeta_{p^s})$  of  $\mathbb{Q}(\zeta_{p^s})$  is the unique prime dividing p and by class field theory  $\mathfrak{p}$  splits completely in the Hilbert class field H of  $\mathbb{Q}(\zeta_{p^s})$ . Thus if the class group has order at least 8, Proposition 2.1 applied to the solvable number field H gives the result.

In the proof of [17, Corollary 11.17], the class number of  $\mathbb{Q}(\zeta_{p^s})$  is shown to be at least  $10^9$  for  $\phi(p^s) = p^{s-1}(p-1) > 220$ . Choosing  $s \ge 9$  for any p completes the proof of the Theorem.

A slightly more detailed analysis using Table §3 of [17] shows the fields below suffice:

p	Κ	$h_{\rm K}(=g_{\rm K})$
p > 23	$\mathbf{K} = \mathbb{Q}(\zeta_p)$	$\geq 8$
$7 \leqslant p \leqslant 23$	$\mathbf{K} = \mathbb{Q}(\zeta_{p^2})$	$\geq 43$
p = 5	$\mathbf{K} = \mathbb{Q}(\dot{\zeta}_{125})$	57708445601
p = 3	$\mathbf{K} = \mathbb{Q}(\zeta_{81})$	2593
p = 2	$\mathbf{K} = \mathbb{Q}(\zeta_{64})$	17
-	- ( 5 )	

**Remark 2.2.** — In [4] a proof of the Theorem for p = 2, 3 and 5 was announced. There are two cases there: Case I, where the Hilbert class field tower is infinite; and Case II, where ramification is allowed at one prime above p in the Hilbert class field H and a  $\mathbb{Z}/p$ -extension of H ramified at exactly this prime is used. Gras has given a criterion for such an extension to exist: see [1, Chapter V, Corollary 2.4.4]. Gras' criterion is not verified in [4]. Given the size of the number fields H, it seems very difficult to do so. We therefore we regard the results of [4] as incomplete. See [6] for a related description of the same error.

Our proof is partially modeled on the ideas of [4], namely considering the Hilbert class field of a cyclotomic field.

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