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1 A stable fixed point method for the numerical simulation  
2 of a kinetic collisional sheath

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6 **Abstract**

7 This work introduces a numerical fixed point method to approximate the so-  
8 lutions of a Vlasov-Poisson-Boltzmann boundary value problem which arises  
9 when modeling a bi-species collisional sheath. Our method relies on the ex-  
10 act integration of the transport equations by means of the characteristic  
11 curves. A special care is given about the choice of a suitable phase space  
12 discretization together with the use of adequate quadrature formulas so as  
13 to ensure that the numerical fixed point method is stable. Numerical exper-  
14 iments are carried out in order to illustrate the effects of the various physical  
15 parameters that are in the scope of the analysis. Some results going beyond  
16 the scope of the analysis are also given.

17 *Keywords:* stationary transport problems, Vlasov-Poisson-Boltzmann  
18 boundary value problem, collisional sheaths, fixed point method,  
19 characteristic curves

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20 **1. Introduction**

Plasma interacting with material boundaries are ubiquitous in applica-  
tions. A well-known physical feature of a plasma interacting with an isolated  
partially absorbing surface, is the development near the surface of a thin posi-  
tively charged boundary layer called the Debye sheath. The Debye sheath  
can be mathematically described by a steady state regime where the flows of  
ions and electrons reaching the wall are equal [21, 6, 25]. The mathematical  
and physical foundations of plasma sheaths in the case where particles do  
not collide are now well-established [3, 1, 13, 12]. When particles suffer from  
collisions several models have been proposed in the literature [24, 22, 18]. In  
this work we are interested in the numerical approximation of the bi-species

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Vlasov-Poisson-Boltzmann model studied in [2]. The model reads as follows:

$$\begin{cases} v\partial_x f_i - \partial_x \phi \partial_v f_i = -\nu Q(f_i), & (x, v) \in (0, 1) \times \mathbb{R}, & (1) \\ v\partial_x f_e + \frac{1}{\mu} \partial_x \phi \partial_v f_e = 0, & (x, v) \in (0, 1) \times \mathbb{R}, & (2) \\ -\varepsilon^2 \partial_{xx} \phi = n_i(x) - n_e(x), & x \in (0, 1), & (3) \end{cases}$$

1 where the unknowns are the ions and electrons densities in the phase space  
 2  $f_i : (x, v) \in [0, 1] \times \mathbb{R} \mapsto f_i(x, v) \in \mathbb{R}^+$ ,  $f_e : (x, v) \in [0, 1] \times \mathbb{R} \mapsto f_e(x, v) \in \mathbb{R}^+$ ,  
 3 and the electrostatic potential  $\phi : x \in [0, 1] \mapsto \phi(x) \in \mathbb{R}$ . In this model,  
 4  $\nu > 0$  is a normalized collision frequency between ions and a cold neutral  
 5 gas,  $\mu > 0$  denotes the mass ratio between electrons and ions,  $\varepsilon > 0$  is a  
 6 normalized Debye length,  $Q(f_i)$  is a collision operator which takes the form  
 7 of a linear relaxation operator towards a mono-kinetic distribution:

$$\forall (x, v) \in (0, 1) \times \mathbb{R}, \quad Q(f_i)(x, v) := f_i(x, v) - \left( \int_{\mathbb{R}} f_i(x, v) dv \right) \delta_{v=0}, \quad (4)$$

8 where  $\delta_{v=0}$  is the Dirac measure supported at the point  $v = 0$ . The macro-  
 9 scopic densities are defined by :

$$\forall x \in [0, 1] \quad n_i(x) := \int_{\mathbb{R}} f_i(x, v) dv, \quad n_e(x) := \int_{\mathbb{R}} f_e(x, v) dv. \quad (5)$$

The system (1)-(3) is supplemented with the following boundary conditions:

$$\begin{cases} f_i(0, v > 0) = f_i^{\text{inc}}(v), & f_i(1, v < 0) = 0, & (6) \\ f_e(0, v > 0) = n_0 f_e^{\text{inc}}(v), & f_e(1, v < 0) = \alpha f_e(1, -v), & (7) \\ \phi(0) = 0, & \phi(1) = \phi_{\text{wall}}, & (8) \end{cases}$$

10 where  $f_i^{\text{inc}} : (0, +\infty) \rightarrow \mathbb{R}^+$ ,  $f_e^{\text{inc}} : (0, +\infty) \rightarrow \mathbb{R}^+$  stand for incoming parti-  
 11 cles densities that model the flows of particles that come from the plasma  
 12 ( $x = 0$ ). Since electrons are usually well described by Maxwellian distribu-  
 13 tions in the core plasma (see [25] for a physical justification), the incoming  
 14 electrons density to be considered here is a normalized semi-Maxwellian

$$f_e^{\text{inc}}(v) = \sqrt{\frac{2\mu}{\pi}} e^{-\frac{\mu v^2}{2}}, \quad \forall v > 0. \quad (9)$$

At the wall ( $x = 1$ ), ions particles are absorbed while for the electrons a  
 fraction  $\alpha \in [0, 1]$  of the particles is re-emitted from the wall specularly. The  
 pair  $(n_0, \phi_{\text{wall}}) \in (0, +\infty) \times (-\infty, 0)$  plays the role of an unknown which has

to be determined in such a way that the solutions  $f_i, f_e, \phi$  to (1)-(8) satisfy the additional equations

$$\begin{cases} n_i(0) = n_e(0), & (10) \\ J_i = J_e & (11) \end{cases}$$

1 where the current densities are defined by

$$\forall x \in [0, 1] \quad J_i := \int_{\mathbb{R}} f_i(x, v) v dv, \quad J_e := \int_{\mathbb{R}} f_e(x, v) v dv. \quad (12)$$

2 Note that an integration in velocity of the equations (1)-(2) yields that  
 3 the current densities are constant in space. The existence of weak solutions  
 4 for the system (1)-(12) has been proven in [2]. We refer to Theorem 3.1 of the  
 5 aforementioned reference for a precise statement of the existence result. To  
 6 briefly summarize the result, it is shown that there is a critical re-emission  
 7 coefficient  $\alpha_c \approx 1$  such that for any  $0 \leq \alpha \leq \alpha_c < 1$  and for any incoming ions  
 8 density  $f_i^{\text{inc}}$  that belongs to a standard class of regularity and additionally  
 9 satisfies

10 - the admissibility condition

$$\frac{\int_0^{+\infty} f_i^{\text{inc}}(v) v dv}{\int_0^{+\infty} f_i^{\text{inc}}(v) dv} < \frac{(1 - \alpha) \sqrt{2}}{(1 + \alpha) \sqrt{\mu\pi}}, \quad (13)$$

11 - the Bohm criterion

$$B_\alpha(f_i^{\text{inc}}) := \frac{m_\alpha(0) + m'_\alpha(0)}{m_\alpha(0)} - \frac{\int_0^{+\infty} \frac{f_i^{\text{inc}}(v)}{v^2} dv}{\int_0^{+\infty} f_i^{\text{inc}}(v) dv} > 0, \quad (14)$$

12 where

$$\forall u \in [\phi_{\text{wall}}, 0], \quad m_\alpha(u) := 2 - (1 - \alpha) \operatorname{erfc}(\sqrt{u - \phi_{\text{wall}}}), \quad (15)$$

13 then for all  $0 < \nu < \nu^c$  where

$$\nu^c := -\phi_{\text{wall}} B_\alpha(f_i^{\text{inc}}) \left( \frac{\int_0^{+\infty} f_i^{\text{inc}}(v) dv}{\int_0^{+\infty} \frac{f_i^{\text{inc}}(v)}{v} dv} \right) > 0 \quad (16)$$

14 there exists  $\varepsilon^* > 0$  such that for all  $\varepsilon \geq \varepsilon^*$  the weak solutions to (1)-(11)  
 15 have the following properties:

1 - a non negative charge density:

$$\forall x \in [0, 1], \quad n_i(x) \geq n_e(x) > 0, \quad (17)$$

2 - a sufficiently decreasing electric potential:

$$\forall x \in [0, 1], \quad \partial_x \phi(x) \leq -\frac{\nu}{B_\alpha(f_i^{\text{inc}})} \left( \frac{\int_0^{+\infty} \frac{f_i^{\text{inc}}(v)}{v} dv}{\int_0^{+\infty} f_i^{\text{inc}}(v) dv} \right). \quad (18)$$

3 A very exhaustive literature about numerical methods to approximate  
4 kinetic plasma models is available. We refer for example to [4, 20, 9, 5, 14,  
5 11, 8, 7] for Particle-In-Cell methods, Semi-Lagrangian methods and Galerkin  
6 type methods. Convergence and stability analysis of these methods can be  
7 found in the mentioned references. In the specific context of plasma sheaths,  
8 some of these numerical methods have been used [26, 17, 19, 23] with their  
9 own specificity according to the model under consideration. We mention  
10 that non stationary based numerical methods are often more generic in a  
11 sense, and enable to avoid dealing with the delicate problem of selecting  
12 the boundary conditions that reproduce the physics of sheaths. They are  
13 however more time consuming and less robust since they may not reach  
14 the desired stationary solution. As far as the numerical difficulties are con-  
15 cerned, they are all related to the multi-scale nature of the plasma sheath  
16 formation. The main observed numerical difficulties are threefold. The first  
17 one stems from the need of a high spatial resolution due to the boundary  
18 layer that forms in the different regimes  $\varepsilon \ll 1$  and  $\nu \gg 1$ . The second one  
19 comes from the relative difference in velocities between electrons and ions  
20 due to the small mass ratio  $\mu \ll 1$ . The third one is related to the numerical  
21 treatment of the boundary conditions (6)-(7) which need a specific inter-  
22 polation procedure with ghost points outside the computational domain.  
23 These three numerical difficulties bring stringent stability conditions and  
24 prohibitive computational effort. We refer to [19, 16] for numerical studies  
25 with physical parameters taken from the literature. Some cures are likely  
26 possible by following the so called Asymptotic-Preserving approach [15, 10]  
27 except for the numerical difficulties related to the presence of boundary  
28 layer which are not well-suited for this approach. Specific numerical meth-  
29 ods must therefore be implemented in the context of plasma sheaths. The  
30 strategy followed in this work relies on the a priori knowledge of the math-  
31 ematical structure of the solution. It is based on the analysis of the phase  
32 space by means of the characteristic curves. It is somehow very specific but

1 it yields an exact integration of the transport equations (1)-(2) and thus no  
 2 numerical error related to the transport equations is introduced. It provides  
 3 a numerical method which has a strong analytical background.

4 The present work thus proposes a simple fixed point method to ap-  
 5 proximate the weak solutions of the boundary value problem (1)-(11). Our  
 6 method takes fully advantage of the one dimensional structure of the model  
 7 and follows closely the analysis developed in [2]. An exact integration of the  
 8 transport equations (1)-(2) with the method of characteristics enables to  
 9 have an exact representation of the densities  $f_i, f_e$  up to an error of approxi-  
 10 mation on  $\phi$ . These exact formula are then used to compute the macroscopic  
 11 densities and currents. It enables to reduce the two algebraic equations (10)-  
 12 (11) to one single non linear equation to be solved for  $\phi_{\text{wall}}$ . Once the couple  
 13  $(n_0, \phi_{\text{wall}})$  is computed, the core of the method then consists in solving the  
 14 non linear Poisson problem (3) with the boundary conditions (8) by a fixed  
 15 point algorithm using a finite difference scheme. A suitable choice of the  
 16 ions phase space discretization based on the geometry of the characteristics  
 17 and convenient quadrature formulas for velocity integrals are proposed in  
 18 order to ensure that the inequalities (17) and (18) are easily preserved at  
 19 the discrete level which yields the stability of the method.

### 20 1.1. Summary of the numerical method

21 We now briefly summarize our method, it consists in the three following  
 22 steps.

#### 23 *Step I - The method of characteristics for the Vlasov-Boltzmann equations.*

24 Following [2], provided  $\phi \in W^{2,\infty}(0,1)$  is decreasing with  $\phi(0) = 0$ , the  
 25 method of characteristics yields an explicit representation of the particles  
 26 densities  $f_i$  and  $f_e$  as functions of the potential  $\phi$ . Namely, they are given  
 27 by

$$\begin{aligned}
 f_i(x, v) = & \mathbf{1}_{\{v > \sqrt{-2\phi(x)}\}}(x, v) f_i^{\text{inc}} \left( \sqrt{v^2 + 2\phi(x)} \right) e^{\nu t_{\text{inc}}(x, v)} \\
 & + \mathbf{1}_{\{0 < v < \sqrt{-2\phi(x)}\}}(x, v) \frac{-\nu e^{\nu s_0(x, v)} n_i \left( \phi^{-1}(\phi(x) + v^2/2) \right)}{\partial_x \phi \left( \phi^{-1}(\phi(x) + v^2/2) \right)}. \tag{19}
 \end{aligned}$$

28

$$\begin{aligned}
 f_e(x, v) = & n_0 \mathbf{1}_{\{v > \sqrt{\frac{2}{\mu}(\phi(x) - \phi(1))}\}}(x, v) f_e^{\text{inc}} \left( \sqrt{v^2 - \frac{2}{\mu} \phi(x)} \right) \\
 & + n_0 \alpha \mathbf{1}_{\{v < \sqrt{\frac{2}{\mu}(\phi(x) - \phi(1))}\}}(x, v) f_e^{\text{inc}} \left( \sqrt{v^2 - \frac{2}{\mu} \phi(x)} \right). \tag{20}
 \end{aligned}$$

In (19),  $t_{\text{inc}}$  and  $s_0$  are negative times given respectively by

$$t_{\text{inc}}(x, v) = - \int_0^x \frac{du}{\sqrt{v^2 + 2\phi(u)}}, \quad (21)$$

$$s_0(x, v) = - \int_{\phi^{-1}(\phi(x)+v^2/2)}^x \frac{du}{\sqrt{2(\phi(x) + v^2/2 - \phi(u))}}. \quad (22)$$

Integrating with respect to the velocity  $v \in \mathbb{R}$  both (19) and (20), one then obtains the following formulas for the macroscopic densities  $n_i \equiv n_i[\phi]$  and  $n_e \equiv n_e[\phi]$ :

$$\begin{aligned} n_i[\phi](x) &= \int_{\sqrt{-2\phi(x)}}^{+\infty} f_i^{\text{inc}}\left(\sqrt{v^2 + 2\phi(x)}\right) e^{\nu t_{\text{inc}}(x,v)} dv \\ &\quad - \int_0^{\sqrt{-2\phi(x)}} \frac{\nu e^{\nu s_0(x,v)} n_i[\phi](\phi^{-1}(\phi(x) + v^2/2))}{\partial_x \phi(\phi^{-1}(\phi(x) + v^2/2))} dv, \end{aligned} \quad (23)$$

$$n_e[\phi](x) = n_0 e^{\phi(x)} m_\alpha(\phi(x)), \quad (24)$$

where  $m_\alpha$  is given by (15). The ions density  $n_i$  solves an integral equation while the electrons density  $n_e$  is given explicitly. As for the current densities, they are given by

$$J_i = \int_0^{+\infty} f_i^{\text{inc}}(v) v dv, \quad (25)$$

$$J_e = n_0(1 - \alpha) \sqrt{\frac{2}{\pi\mu}} e^{\phi_{\text{wall}}}. \quad (26)$$

*Step II- Determination of the wall potential.* Assuming  $\phi(1) = \phi_{\text{wall}}$ , the two algebraic equations (10) and (11) yields the equivalent system of unknown  $(n_0, \phi_{\text{wall}})$

$$\int_0^{+\infty} f_i^{\text{inc}}(v) dv = n_0 m_\alpha(0), \quad (27)$$

$$\int_0^{+\infty} f_i^{\text{inc}}(v) v dv = n_0(1 - \alpha) \sqrt{\frac{2}{\pi\mu}} e^{\phi_{\text{wall}}}, \quad (28)$$

1 where we remember that  $m_\alpha(0)$  is given by

$$m_\alpha(0) = 2 - (1 - \alpha) \text{erfc}(\sqrt{-\phi_{\text{wall}}})$$

and thus also depends on  $\phi_{\text{wall}}$ . Eliminating  $n_0$  yields that  $\phi_{\text{wall}}$  is the unique negative solution of the non linear equation

$$m_\alpha(0) \int_0^{+\infty} f_i^{\text{inc}}(v) v dv - (1 - \alpha) \sqrt{\frac{2}{\pi\mu}} \int_0^{+\infty} f_i^{\text{inc}}(v) dv e^{\phi_{\text{wall}}} = 0. \quad (29)$$

1 This equation has a unique negative solution if and only if the inequality  
 2 (13) holds. It is solved using a standard Newton method.

*Step III-Solving the non linear Poisson problem.* The non linear Poisson problem consists in finding a decreasing potential  $\phi$  satisfying (18) such that

$$-\varepsilon^2 \partial_{xx} \phi(x) = n_i[\phi](x) - n_e[\phi](x), \quad x \in [0, 1], \quad (30)$$

$$\phi(0) = 0 \quad \text{and} \quad \phi(1) = \phi_{\text{wall}} < 0, \quad (31)$$

3 where  $n_i[\phi] > 0$  solves the integral equation (23),  $n_e[\phi] > 0$  is given explic-  
 4 itly on  $\phi$  by (24) and the solution  $\phi$  must ensure the inequality (17). We  
 5 note that in the case  $\nu = 0$ , the Poisson problem (30)-(31) reformulates as  
 6 a minimization problem [3]. Since  $\nu > 0$ ,  $n_i[\phi]$  solves a non trivial inte-  
 7 gral equation (23). The Poisson problem (30)-(31) is a strongly non linear  
 8 integro-differential equation. Our numerical method consists in solving (30)  
 9 with a fixed point method. The stability of the method is ensured by the  
 10 inequalities (17) and (18). Our main concern in the rest of the paper is the  
 11 preservation of these inequalities at the discrete level.

## 12 1.2. Organization

13 The outline of the paper is as follows. In the next section, we define  
 14 the numerical scheme to solve the non linear Poisson problem (23)-(24)-  
 15 (30)-(31). We detail the discretization, paying a particular attention to the  
 16 ions phase space discretization in order to avoid the computation of  $\phi^{-1}$   
 17 involved in the definition of  $n_i[\phi]$ , given by (23) and in  $t_{\text{inc}}$  defined by (21).  
 18 Then in Section 3 we establish the stability properties to be satisfied by  
 19 the numerical scheme, namely the discrete analogue of inequalities (17) and  
 20 (18). In Section 4, several numerical experiments that are performed in  
 21 the scope of the analysis and an interpretation of the results is proposed.  
 22 Some results going beyond our analysis are also given. Eventually, a short  
 23 conclusion is given in Section 5.

## 24 2. The numerical scheme

25 In all the sequel, we shall assume that the incoming ions density  $f_i^{\text{inc}} :$   
 26  $(0, +\infty) \rightarrow \mathbb{R}^+$  is at least piecewise continuous so that the upcoming quadra-  
 27 ture formulas make sense. Moreover, we assume  $\phi_{\text{wall}} < 0$  to solve exactly or  
 28 approximately the equation (29). We begin with introducing a uniform dis-  
 29 cretization of the interval  $[0, 1]$  of size  $\Delta x = 1/(N + 1)$  where  $N + 1$  denotes

- 1 the number of intervals of discretization so that  $x_j = j\Delta x$  for  $0 \leq j \leq N+1$ .  
2 We denote  $\phi_j$  the approximation of  $\phi(x_j)$  while  $n_i[\phi]_j$  and  $n_e[\phi]_j$  respectively  
3 denote the approximations of  $n_i[\phi](x_j)$  and  $n_e[\phi](x_j)$ .

To approximate the sequence  $(\phi_j)_{0 \leq j \leq N+1}$  with the boundary conditions given by

$$\phi_0 = 0 \quad \text{and} \quad \phi_{N+1} = \phi_{\text{wall}},$$

- 4 we use the following fixed point numerical procedure:

$$-\frac{\varepsilon^2}{\Delta x^2} \left( \phi_{j+1}^{n+1} - 2\phi_j^{n+1} + \phi_{j-1}^{n+1} \right) = n_i[\phi^n]_j - n_e[\phi^n]_j, \quad (32)$$

where the initial sequence  $(\phi_j^0)_{0 \leq j \leq N+1}$  must satisfy the boundary conditions

$$\phi_0^0 = 0 \quad \text{and} \quad \phi_{N+1}^0 = \phi_{\text{wall}}.$$

- 5 In view of the inequality (18), we also impose this initial sequence to verify  
6 for all  $0 \leq j \leq N$

$$\frac{1}{\Delta x} (\phi_{j+1}^0 - \phi_j^0) \leq M_\phi < 0, \quad (33)$$

- 7 where  $M_\phi$  will be defined as an approximation of the right hand side in the  
8 inequality (18).

9 Regarding the definition of  $n_i[\phi]_j$ , to approximate  $n_i[\phi](x_j)$  defined by  
10 (23), we shall introduce a suitable discretization of half the phase space  
11  $(0, 1) \times [0, +\infty)$  that avoid the computation of  $\phi^{-1}$  involved in (23). As  
12 in [2], one uses the characteristic curves. These are the curves of algebraic  
13 equation  $\frac{v^2}{2} + \phi(x) = \text{const}$ . They span the domain  $(0, 1) \times [0, +\infty)$  so that  
14 one has the natural decomposition

$$(0, 1) \times [0, +\infty) = D_1 \cup D_2,$$

with

$$D_1 = \left\{ (x, v) \in (0, 1) \times [0, +\infty) : v \geq \sqrt{-2\phi(x)} \right\}, \quad (34)$$

$$D_2 = \left\{ (x, v) \in (0, 1) \times [0, +\infty) : 0 \leq v < \sqrt{-2\phi(x)} \right\}. \quad (35)$$

- 15 The domain  $D_1$  corresponds to characteristic curves that originate from  
16  $x = 0$  with positive velocities, namely

$$\forall (x, v) \in D_1, \exists v_0 > 0, \frac{v_0^2}{2} = \frac{v^2}{2} + \phi(x). \quad (36)$$

1 The domain  $D_2$  corresponds to characteristic curves that crosses  $v = 0$ ,  
 2 namely

$$\forall(x, v) \in D_2, \exists 0 < x_0 < 1, \phi(x_0) = \frac{v^2}{2} + \phi(x). \quad (37)$$

3 In particular for a pair  $(x_j, v)$  where  $x_j \in (0, 1)$  is a grid point and  $v \in$   
 4  $[0, \sqrt{-2\phi(x_j)})$ , the corresponding  $x_0$  is solution of the equation

$$\phi(x_0) = \frac{v^2}{2} + \phi(x_j). \quad (38)$$

5 From a numerical perspective, the equation (38) is not convenient since  $\phi$   
 6 may be replaced by an approximation which is only known on a discrete set.  
 7 We need somehow an ad hoc reconstruction of  $\phi$ . A way to circumvent this  
 8 issue is to choose  $v$  in such a way that  $x_0$  belongs to the grid  $(x_k)_{0 \leq k \leq N+1}$   
 9 so that the equation (38) does not need to be solved. Namely, we define  
 10 a discretization of the velocity interval which depends both on the grid  
 11  $(x_k)_{0 \leq k \leq N+1}$  using the equation of the characteristics (38) (see Fig 1). In  
 12 this regard, for  $0 \leq v \leq \sqrt{-2\phi_j}$  with a given  $j$ , we set

$$v_{j-k}^j = \sqrt{2(\phi_k - \phi_j)}, \quad 0 \leq k \leq j, \quad (39)$$

13 where  $\phi_k - \phi_j \geq 0$  for all  $0 \leq k \leq j$  provided  $(\phi_j)_{0 \leq j \leq N+1}$  satisfies

$$\frac{1}{\Delta x} (\phi_{j+1} - \phi_j) \leq M_\phi < 0. \quad (40)$$

For  $v \geq \sqrt{-2\phi_j}$ , we consider a uniform discretization of size  $\Delta v > 0$  such  
 that we set

$$\bar{v}_k^j = \sqrt{-2\phi_j} + k\Delta v, \quad k \geq 0.$$

14 To avoid some possible confusion in the velocity discretization, we have  
 15 denoted  $v_\ell^j$  the discretization of  $v$  in  $[0, \sqrt{-2\phi_j}]$  while  $\bar{v}_\ell^j$  denotes the dis-  
 16 cretization of  $v$  in  $[\sqrt{-2\phi_j}, +\infty)$ .

We need now to define an approximation of  $n_i[\phi](x_j)$  for all  $0 \leq j \leq N + 1$ . To do so, the integrals involved in (23) are approximated using the trapezoidal rules. More precisely, the first integral in (23) is approximated by  $\mathcal{I}^\infty[\phi]_j$  for all  $0 \leq j \leq N + 1$  defined by

$$\begin{aligned} \mathcal{I}^\infty[\phi]_j = \sum_{k \geq 0} \frac{\Delta v}{2} & \left( f_i^{\text{inc}} \left( \sqrt{(\bar{v}_k^j)^2 + 2\phi_j} \right) e^{\nu \bar{t}_{\text{inc}}[\phi](x_j, \bar{v}_k^j)} \right. \\ & \left. + f_i^{\text{inc}} \left( \sqrt{(\bar{v}_{k+1}^j)^2 + 2\phi_j} \right) e^{\nu \bar{t}_{\text{inc}}[\phi](x_j, \bar{v}_{k+1}^j)} \right) \end{aligned} \quad (41)$$

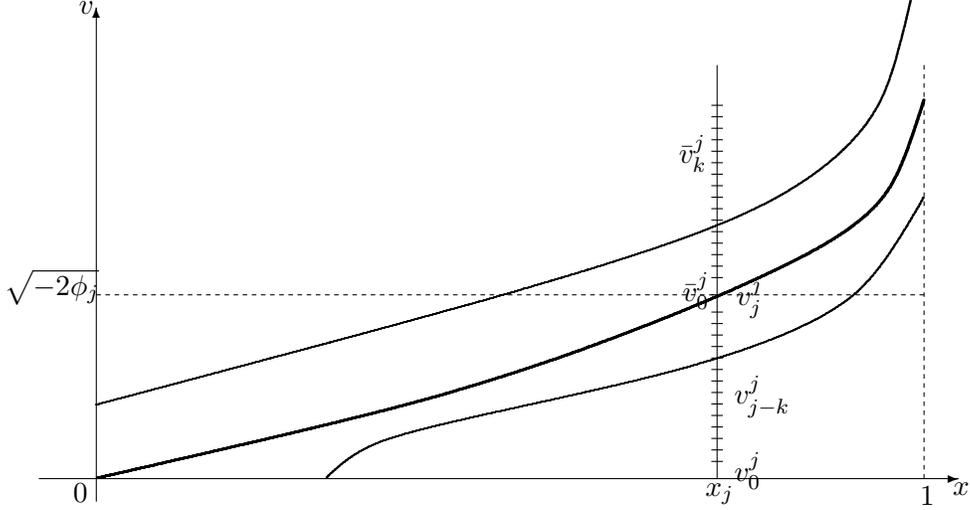


Figure 1: Presentation of the velocity mesh.

- 1 where  $\bar{t}_{\text{inc}}[\phi](x_j, \bar{v}_k^j)$  is an approximation of the time  $t_{\text{inc}}(x_j, \bar{v}_k^j)$  given in
- 2 (21). It is defined by  $\bar{t}_{\text{inc}}[\phi](x_0, \bar{v}_k^0) = 0$  and for  $1 \leq j \leq N + 1$  a midpoint
- 3 rule is used:

$$\bar{t}_{\text{inc}}[\phi](x_j, \bar{v}_k^j) = - \sum_{\ell=0}^{j-1} \frac{\Delta x}{\sqrt{(\bar{v}_k^j)^2 + \phi_\ell + \phi_{\ell+1}}}. \quad (42)$$

The second integral is approximated by  $\mathcal{I}^0[\phi]_j$  with  $\mathcal{I}^0[\phi]_0 = 0$  and for  $1 \leq j \leq N + 1$

$$\begin{aligned} \mathcal{I}^0[\phi]_j = \sum_{k=0}^{j-1} \frac{v_{j-(k+1)}^j - v_{j-k}^j}{2} & \left( \nu e^{\nu \bar{s}_0(x_j, v_{j-k}^j)} \frac{\overline{n_i[\phi]}(\phi^{-1}(\phi_j + (v_{j-k}^j)^2/2))}{\overline{\partial_x \phi}(\phi^{-1}(\phi_j + (v_{j-k}^j)^2/2))} \right. \\ & \left. + \nu e^{\nu \bar{s}_0(x_j, v_{j-(k+1)}^j)} \frac{\overline{n_i[\phi]}(\phi^{-1}(\phi_j + (v_{j-(k+1)}^j)^2/2))}{\overline{\partial_x \phi}(\phi^{-1}(\phi_j + (v_{j-(k+1)}^j)^2/2))} \right) \end{aligned}$$

where  $\overline{n_i[\phi]}(\phi^{-1}(\phi_j + (v_{j-k}^j)^2/2))$ ,  $\overline{\partial_x \phi}(\phi^{-1}(\phi_j + (v_{j-k}^j)^2/2))$  and  $\bar{s}_0(x_j, v_{j-k}^j)$  are approximations of  $n_i[\phi](\phi^{-1}(\phi_j + (v_{j-k}^j)^2/2))$ ,  $\partial_x \phi(\phi^{-1}(\phi_j + (v_{j-k}^j)^2/2))$

and  $s_0(x_j, v_{j-k}^j)$  given by (22). They are constructed using the velocity discretization defined by (39). Indeed, we have

$$\phi_j + \frac{(v_{j-k}^j)^2}{2} = \phi_k.$$

One therefore naturally defines,

$$\overline{n_i[\phi]}(\phi^{-1}(\phi_j + (v_{j-k}^j)^2/2)) = n_i[\phi]_k, \quad (43)$$

$$\overline{\partial_x \phi}(\phi^{-1}(\phi_j + (v_{j-k}^j)^2/2)) = \overline{\partial_x \phi}_k, \quad (44)$$

$$\overline{s_0}(x_j, v_{j-k}^j) = - \sum_{\ell=k}^{j-1} \frac{\Delta x}{\sqrt{2(\phi_k - \phi_{\ell+1})}}, \quad (45)$$

- 1 where  $\overline{\partial_x \phi}_k$  denotes an approximation of  $\partial_x \phi(x_k)$  such that the control of
- 2 the variations of  $(\phi_j)_{0 \leq j \leq N+1}$  through the inequality (40) also implies

$$\overline{\partial_x \phi}_k \leq M_\phi < 0. \quad (46)$$

As a consequence,  $\mathcal{I}^0[\phi]_j$  rewrites  $\mathcal{I}^0[\phi]_0 = 0$  and for  $1 \leq j \leq N+1$

$$\begin{aligned} \mathcal{I}^0[\phi]_j = & \frac{\nu}{2} \sum_{k=0}^{j-1} \left( \sqrt{\phi_{k+1} - \phi_j} - \sqrt{\phi_k - \phi_j} \right) \times \\ & \left( e^{\nu \overline{s_0}(x_j, v_{j-k}^j)} \frac{n_i[\phi]_k}{\overline{\partial_x \phi}_k} + e^{\nu \overline{s_0}(x_j, v_{j-(k+1)}^j)} \frac{n_i[\phi]_{k+1}}{\overline{\partial_x \phi}_{k+1}} \right). \end{aligned} \quad (47)$$

- 3 The approximation  $n_i[\phi]_j$  is then defined by

$$n_i[\phi]_j = \mathcal{I}^\infty[\phi]_j + \mathcal{I}^0[\phi]_j. \quad (48)$$

- 4 For the proof of the stability properties (17) and (18), it is convenient to
- 5 define several consistent approximations of the constant  $n_0$  which is given
- 6 according to equation (27) by

$$n_0 = \frac{1}{m_\alpha(0)} \int_0^{+\infty} f_i^{\text{inc}}(v) dv.$$

The trick is to remark that for all  $\varphi < 0$  we have

$$\int_0^{+\infty} f_i^{\text{inc}}(v) dv = \int_{\sqrt{-2\varphi}}^{+\infty} f_i^{\text{inc}}(\sqrt{v^2 + 2\varphi}) \frac{v}{\sqrt{v^2 + 2\varphi}} dv,$$

1 so that  $n_0$  can also be expressed in terms of the sequence  $(\phi_j)_{0 \leq j \leq N+1}$  as

$$n_0 = \frac{1}{m_\alpha(0)} \int_{\sqrt{-2\phi_j}}^{+\infty} f_i^{\text{inc}}(\sqrt{v^2 + 2\phi_j}) \frac{v}{\sqrt{v^2 + 2\phi_j}} dv.$$

One can thus define an approximation of  $n_0$  for each  $\phi_j$  by

$$n_0^j m_\alpha(0) = \sum_{k \geq 0} \frac{\Delta v}{2} \left( f_i^{\text{inc}}(\sqrt{(\bar{v}_k^j)^2 + 2\phi_j}) \frac{\bar{v}_k^j}{\sqrt{(\bar{v}_k^j)^2 + 2\phi_j}} + f_i^{\text{inc}}(\sqrt{(\bar{v}_{k+1}^j)^2 + 2\phi_j}) \frac{\bar{v}_{k+1}^j}{\sqrt{(\bar{v}_{k+1}^j)^2 + 2\phi_j}} \right). \quad (49)$$

2 The full sequence  $(n_0^j)_{0 \leq j \leq N+1}$  is of course consistent with the constant  $n_0$   
 3 defined by (27).

4 As for  $n_e[\phi]_j$ , we simply define

$$n_e[\phi]_j = n_0^j e^{\phi_j} m_\alpha(\phi_j). \quad (50)$$

5 The definition of the scheme is now achieved. Before we turn to the  
 6 study of the stability properties satisfied by this scheme, it is worth noticing  
 7 that  $n_i[\phi]_j$  is solution of a  $(N+2) \times (N+2)$  triangular linear system in the  
 8 form

$$n_i[\phi] = \mathcal{I}^\infty + M \cdot n_i[\phi], \quad (51)$$

9 where  $n_i[\phi] \in \mathbb{R}^{N+2}$  is the unknown vector made of  $(n_i[\phi]_j)_{0 \leq j \leq N+1}$ ,  $\mathcal{I}^\infty \in$   
 10  $\mathbb{R}^{N+2}$  is the vector with components  $(\mathcal{I}^\infty[\phi]_j)_{0 \leq j \leq N+1}$  and  $M$  is a triangular  
 11 matrix of size  $N+2$  such that for  $0 \leq j \leq N+1$

$$(M \cdot n_i[\phi])_i = \mathcal{I}^0[\phi]_j. \quad (52)$$

### 12 3. Stability properties

We consider the scheme (32)-(48)-(50) to approximate the solutions of  
 (1)-(12). One has to prove the discrete analogue of the inequalities (17) and  
 (18) that reads

$$\frac{1}{\Delta x} (\phi_{j+1}^n - \phi_j^n) \leq M_\phi < 0, \\ n_i[\phi^n]_j - n_e[\phi^n]_j \geq 0,$$

13 during all the fixed point iterations  $n \in \mathbb{N}$ . Our main result is the following.

1 **Theorem 3.1.** Let  $(\phi_\ell^0)_{0 \leq \ell \leq N+1}$  be a given sequence such that  $\phi_0^0 = 0$ ,  
2  $\phi_{N+1}^0 = \phi_{\text{wall}} < 0$  and  $(\phi_{\ell+1}^0 - \phi_\ell^0)/\Delta x \leq M_\phi < 0$  for all  $0 \leq \ell \leq N$  where  
3  $M_\phi$  verifies (61). Let the updated sequence  $(\phi_\ell^{n+1})_{0 \leq \ell \leq N+1}$  be defined by the  
4 scheme (32)-(48)-(50) with the boundary conditions

$$\phi_0^{n+1} = 0 \quad \text{and} \quad \phi_{N+1}^{n+1} = \phi_{\text{wall}} < 0. \quad (53)$$

Assume the parameters of the model are selected such that

$$-\phi_{\text{wall}} > \max_{0 \leq j \leq N+1} \left( \frac{\nu I_1^j}{\bar{B}_\alpha^j(f_i^{\text{inc}}) I_0^j} \right),$$

where  $\bar{B}_\alpha^j(f_i^{\text{inc}})$  is given by (60) and  $I_{0,1}^j$  by (59). Then, there exists  $\varepsilon^* > 0$ ,  
such that for all  $\varepsilon \geq \varepsilon^*$ , during all the iterations  $n \in \mathbb{N}$ , we have

$$n_i[\phi^n]_j - n_e[\phi^n]_j \geq 0, \quad 0 \leq j \leq N+1, \quad (54)$$

$$\frac{1}{\Delta x}(\phi_{j+1}^n - \phi_j^n) \leq M_\phi, \quad 0 \leq j \leq N. \quad (55)$$

5 To prove this result, we shall need several discrete a priori estimates that  
6 are the focus of the next section.

### 7 3.1. A priori estimates

8 The first one is an estimate of  $\bar{t}_{\text{inc}}[\phi]$ .

9 **Lemma 3.2.** Let  $(\phi_\ell)_{0 \leq \ell \leq N+1}$  be a given sequence such that

$$\phi_0 = 0 \quad \text{and} \quad \frac{1}{\Delta x}(\phi_{\ell+1} - \phi_\ell) \leq M_\phi, \quad (56)$$

10 where  $M_\phi < 0$  is a given constant which may eventually depend on the  
11 parameters of the model and  $\phi_{\text{wall}}$ . Then for  $0 \leq j \leq N+1$ , we have

$$\bar{t}_{\text{inc}}[\phi](x_j, \bar{v}_k^j) \geq \frac{1}{M_\phi} \left( \bar{v}_k^j - \sqrt{(\bar{v}_k^j)^2 + 2\phi_j} \right), \quad (57)$$

12 where  $\bar{t}_{\text{inc}}$  is defined by (42).

*Proof.* With  $\bar{t}_{\text{inc}}[\phi](x_j, \bar{v}_k^j)$  given by (42), we write

$$\bar{t}_{\text{inc}}[\phi](x_j, \bar{v}_k^j) = \sum_{\ell=0}^{j-1} \frac{1}{(\phi_{\ell+1} - \phi_\ell)/\Delta x} \frac{1}{\sqrt{(\bar{v}_k^j)^2 + \phi_\ell + \phi_{\ell+1}}} (\phi_\ell - \phi_{\ell+1}).$$

Since  $(\phi_\ell)_{0 \leq \ell \leq N+1}$  is imposed to satisfy (56), we get

$$\bar{t}_{\text{inc}}[\phi](x_j, \bar{v}_k^j) \geq \frac{1}{M_\phi} \sum_{\ell=0}^{j-1} \frac{1}{\sqrt{(\bar{v}_k^j)^2 + \phi_\ell + \phi_{\ell+1}}} (\phi_\ell - \phi_{\ell+1}).$$

Let us emphasize that the function  $g(\varphi) = 1/\sqrt{(\bar{v}_k^j)^2 + 2\varphi}$  is decreasing and convex. As a consequence, for all  $\phi_{\ell+1} < \varphi < \phi_\ell$  with  $(\bar{v}_k^j)^2 + \phi_\ell + \phi_{\ell+1} \geq 0$ , we have

$$g\left(\frac{\phi_\ell + \phi_{\ell+1}}{2}\right) + \left(\varphi - \frac{\phi_\ell + \phi_{\ell+1}}{2}\right) g'\left(\frac{\phi_\ell + \phi_{\ell+1}}{2}\right) \leq g(\varphi).$$

By integrating the above relation with respect of  $\varphi$  over  $[\phi_{\ell+1}, \phi_\ell]$ , we obtain

$$(\phi_\ell - \phi_{\ell+1}) g\left(\frac{\phi_\ell + \phi_{\ell+1}}{2}\right) \leq \int_{\phi_{\ell+1}}^{\phi_\ell} g(\varphi) d\varphi,$$

that re-writes

$$\frac{\phi_\ell - \phi_{\ell+1}}{\sqrt{(\bar{v}_k^j)^2 + \phi_\ell + \phi_{\ell+1}}} \leq \int_{\phi_{\ell+1}}^{\phi_\ell} \frac{d\varphi}{\sqrt{(\bar{v}_k^j)^2 + 2\varphi}}.$$

As a consequence, we have the following sequence of inequalities ( $M_\phi$  being negative):

$$\begin{aligned} \bar{t}_{\text{inc}}[\phi](x_j, \bar{v}_k^j) &\geq \frac{1}{M_\phi} \sum_{\ell=0}^{j-1} \int_{\phi_{\ell+1}}^{\phi_\ell} \frac{d\varphi}{\sqrt{(\bar{v}_k^j)^2 + 2\varphi}} \\ &\geq \frac{1}{M_\phi} \int_{\phi_j}^0 \frac{d\varphi}{\sqrt{(\bar{v}_k^j)^2 + 2\varphi}} \\ &\geq \frac{1}{M_\phi} \left( \bar{v}_k^j - \sqrt{(\bar{v}_k^j)^2 + 2\phi_j} \right). \end{aligned}$$

1

□

2 Using the estimate of  $\bar{t}_{\text{inc}}$ , given by (57), we now establish that  $n_i[\phi]_j -$   
3  $n_e[\phi]_j$  stays non-negative.

4 **Lemma 3.3.** *Let us introduce*

$$\bar{w}_k^j = \sqrt{(\bar{v}_k^j)^2 + 2\phi_j}, \quad (58)$$

1 so that  $\bar{v}_k^j = \sqrt{(\bar{w}_k^j)^2 - 2\phi_j}$ . Let us set for  $0 \leq j \leq N + 1$

$$I_\delta^j = \sum_{k \geq 0} \frac{\Delta v}{2} \left( f_i^{\text{inc}}(\bar{w}_k^j) \frac{\bar{v}_k^j}{(\bar{w}_k^j)^{\delta+1}} + f_i^{\text{inc}}(\bar{w}_{k+1}^j) \frac{\bar{v}_{k+1}^j}{(\bar{w}_{k+1}^j)^{\delta+1}} \right). \quad (59)$$

2 Let the parameters of the model, namely  $\alpha$  and  $f_i^{\text{inc}}$ , be defined such that  
 3  $\bar{B}_\alpha^j(f_i^{\text{inc}}) > 0$  for all  $0 \leq j \leq N + 1$  where we have set

$$\bar{B}_\alpha^j(f_i^{\text{inc}}) = \frac{m_\alpha(0) + m'_\alpha(0)}{m_\alpha(0)} - \frac{I_2^j}{I_0^j}. \quad (60)$$

4 Let  $(\phi_\ell)_{0 \leq \ell \leq N+1}$  be a given sequence such that (56) holds for a constant  $M_\phi$   
 5 defined as follows:

$$M_\phi = - \max_{0 \leq j \leq N+1} \left( \frac{\nu}{\bar{B}_\alpha^j(f_i^{\text{inc}})} \frac{I_1^j}{I_0^j} \right). \quad (61)$$

6 With  $\Delta x$  small enough, we have for all  $0 \leq j \leq N + 1$

$$n_i[\phi]_j - n_e[\phi]_j \geq 0, \quad (62)$$

7 where  $n_i[\phi]_j$  and  $n_e[\phi]_j$  are defined by (48) and (50).

We underline that  $\bar{B}_\alpha^j(f_i^{\text{inc}})$ , defined by (60), is nothing but the discrete version of the Bohm number given by (14). Indeed, for all  $0 \leq j \leq N + 1$ , the quantities  $I_\delta^j$  are easily shown to be consistent with

$$\int_{\sqrt{-2\phi_j}}^{+\infty} f_i^{\text{inc}}(\sqrt{v^2 + 2\phi_j}) \frac{v}{(\sqrt{v^2 + 2\phi_j})^{\delta+1}} dv = \int_0^{+\infty} f_i^{\text{inc}}(w) \frac{dw}{w^\delta},$$

8 so that the consistency with  $B_\alpha(f_i^{\text{inc}})$  holds. We now prove the Lemma 3.3.

*Proof.* First, we notice that  $n_i[\phi]_j$  for all  $0 \leq j \leq N + 1$  is non-negative provided  $\Delta x$  is small enough. Indeed,  $n_i[\phi]_j$  is defined by (48) which reformulates as a triangular linear system (51) in the form

$$(\text{I}_d - M) \cdot n_i[\phi] = \mathcal{I}^\infty,$$

9 where  $\text{I}_d$  stands for the identity matrix in  $\mathbb{R}^{N+2} \times \mathbb{R}^{N+2}$ . We easily remark  
 10 that all the non-diagonal components of the triangular matrix  $\text{I}_d - M$  are  
 11 non-positive. Moreover, the diagonal components of  $M$  are negative defined  
 12 by

$$M_{jj} = -\frac{\sqrt{\phi_{j-1} - \phi_j}}{\partial_x \phi_j} e^{\nu \bar{s}_0(x_j, v_0^j)},$$

- 1 so that  $M_{jj} = \mathcal{O}(\sqrt{\Delta x})$ . Then,  $1 - M_{jj} > 0$  provided  $\Delta x$  is small enough.  
 2 As a consequence, we deduce that  $n_i[\phi]_j \geq 0$  for all  $0 \leq j \leq N + 1$ .

Now, since  $n_i[\phi]_j \geq 0$ , we immediately obtain  $\mathcal{I}^0[\phi]_j \geq 0$  for all  $0 \leq j \leq N + 1$ . Then, it is sufficient to establish  $\mathcal{I}^\infty[\phi]_j - n_e[\phi]_j \geq 0$  for all  $0 \leq j \leq N + 1$  to get the estimate (62). One remarks that with  $\mathcal{I}^\infty[\phi]_j$  defined by (41) and  $n_e[\phi]_j$  by (50), we have for all  $0 \leq j \leq N + 1$

$$\mathcal{I}^\infty[\phi]_j - n_e[\phi]_j = e^{\phi_j} \left( \sum_{k \geq 0} \frac{\Delta v}{2} \left( f_i^{\text{inc}}(\bar{w}_k^j) e^{-\phi_j + \nu \bar{t}_{\text{inc}}(x_j, \bar{v}_k^j)} + \right. \right. \\ \left. \left. f_i^{\text{inc}}(\bar{w}_{k+1}^j) e^{-\phi_j + \nu \bar{t}_{\text{inc}}(x_j, \bar{v}_{k+1}^j)} \right) - n_0^j m_\alpha(\phi_j) \right),$$

where  $\bar{w}_k^j$  are defined by (58). Because of the estimate (57), we get

$$\mathcal{I}^\infty[\phi]_j - n_e[\phi]_j \geq e^{\phi_j} \left( \sum_{k \geq 0} \frac{\Delta v}{2} \left( f_i^{\text{inc}}(\bar{w}_k^j) e^{-\phi_j + \frac{\nu}{M_\phi}(\bar{v}_k^j - \bar{w}_k^j)} + \right. \right. \\ \left. \left. f_i^{\text{inc}}(\bar{w}_{k+1}^j) e^{-\phi_j + \frac{\nu}{M_\phi}(\bar{v}_{k+1}^j - \bar{w}_{k+1}^j)} \right) - n_0^j m_\alpha(\phi_j) \right). \quad (63)$$

- 3 We now use the following convexity inequality which is a consequence of the  
 4 inequality (61) of Lemma 6.2 of [2] :

$$e^{-\phi_j + \frac{\nu}{M_\phi}(\bar{v}_k^j - \bar{w}_k^j)} \geq \frac{\bar{v}_k^j}{\bar{w}_k^j} + \phi_j \bar{v}_k^j \left( \frac{1}{(\bar{w}_k^j)^3} - \frac{1}{\bar{w}_k^j} - \frac{\nu}{M_\phi} \frac{1}{(\bar{w}_k^j)^2} \right). \quad (64)$$

Since the function  $\varphi \in (-\infty, 0] \mapsto m_\alpha(\varphi)$  is a concave function, we have

$$m_\alpha(\phi_j) \leq m_\alpha(0) + \phi_j m'_\alpha(0).$$

Gathering the above inequality and (64), we glean from (63) the following

estimate:

$$\begin{aligned} \mathcal{I}^\infty[\phi]_j - n_e[\phi]_j &\geq e^{\phi_j} \left( \sum_{k \geq 0} \frac{\Delta v}{2} \left( f_i^{\text{inc}}(\bar{w}_k^j) \frac{\bar{v}_k^j}{\bar{w}_k^j} \alpha_k^j + f_i^{\text{inc}}(\bar{w}_{k+1}^j) \frac{\bar{v}_{k+1}^j}{\bar{w}_{k+1}^j} \alpha_{k+1}^j \right) \right. \\ &\quad \left. - n_0^j m_\alpha(0) - n_0^j \phi_j m'_\alpha(0) \right), \end{aligned}$$

where we have set

$$\alpha_k^j = 1 + \phi_j \left( \frac{1}{(\bar{w}_k^j)^2} - 1 - \frac{\nu}{M_\phi} \frac{1}{\bar{w}_k^j} \right).$$

Using the definition of  $n_0^j$  given by (49), we obtain

$$\begin{aligned} \mathcal{I}^\infty[\phi]_j - n_e[\phi]_j &\geq \phi_j e^{\phi_j} \left( \sum_{k \geq 0} \frac{\Delta v}{2} \left( f_i^{\text{inc}}(\bar{w}_k^j) \frac{\bar{v}_k^j}{\bar{w}_k^j} \beta_k^j + f_i^{\text{inc}}(\bar{w}_{k+1}^j) \frac{\bar{v}_{k+1}^j}{\bar{w}_{k+1}^j} \beta_{k+1}^j \right) \right. \\ &\quad \left. - n_0^j (m_\alpha(0) + m'_\alpha(0)) \right), \end{aligned}$$

where we have set

$$\beta_k^j = \frac{1}{(\bar{w}_k^j)^2} - \frac{\nu}{M_\phi} \frac{1}{\bar{w}_k^j}.$$

With  $I_0^j$ ,  $I_1^j$  and  $I_2^j$  defined by (59), we have

$$\mathcal{I}^\infty[\phi]_j - n_e[\phi]_j \geq \phi_j e^{\phi_j} \left( I_2^j - \frac{\nu}{M_\phi} I_1^j - n_0^j (m_\alpha(0) + m'_\alpha(0)) \right).$$

- 1 According to (49), we have  $n_0^j = I_0^j/m_\alpha(0)$  so that  $\mathcal{I}^\infty[\phi]_j - n_e[\phi]_j \geq 0$  if
- 2  $M_\phi$  satisfies (61). The proof is thus completed.  $\square$

- 3 In order to establish the control of the discrete variation of the sequence
- 4  $(\phi_j)_{0 \leq j \leq N+1}$  (18), we have to establish a uniform (with respect to  $\phi_j$ ) upper-
- 5 bound for  $n_i[\phi]_j$ .

- 6 **Lemma 3.4.** *Let  $(\phi_\ell)_{0 \leq \ell \leq N+1}$  be a given sequence such that  $\phi_0 = 0$ ,  $\phi_{N+1} =$*
- 7  *$\phi_{\text{wall}} < 0$  and  $(\phi_{\ell+1} - \phi_\ell)/\Delta x \leq M_\phi < 0$  for all  $0 \leq \ell \leq N$  where  $M_\phi$  verifies*
- 8 *(61). Then, there exists a constant  $\tau > 0$  independent of  $M_\phi$  such that the*
- 9 *following estimate holds for all  $0 \leq j \leq N$ :*

$$\frac{1}{2} (n_i[\phi]_j + n_i[\phi]_{j+1}) \leq 2e^{-\tau \frac{\phi_j + \phi_{j+1}}{2}} \min_{0 \leq j \leq N+1} I_0^j, \quad (65)$$

- 10 where  $n_i[\phi]_j$  is given by (48) and  $I_0^j$  is defined by (59).

*Proof.* We use the triangular linear system (51), given by

$$(\text{Id} - M) \cdot n_i[\phi] = \mathcal{I}^\infty,$$

to determine the unknown vector  $n_i[\phi] \in \mathbb{R}^{N+2}$ . We consider a vectorial norm in  $\mathbb{R}^{N+2}$  as follows:

$$\|V\|_\tau := \max_{0 \leq j \leq N} \left( \frac{1}{2} (|V_j| + |V_{j+1}|) e^{\tau \frac{\phi_j + \phi_{j+1}}{2}} \right),$$

1 where  $\tau > 0$  is a constant to be defined.

2 First, we establish that there exists  $\tau > 0$ , large enough and independent  
3 of  $M_\phi$ , such that we have for all  $V \in \mathbb{R}^{N+2}$

$$\frac{1}{2} (|(M \cdot V)_j| + |(M \cdot V)_{j+1}|) e^{\tau \frac{\phi_j + \phi_{j+1}}{2}} \leq \frac{1}{2} \|V\|_\tau, \quad 0 \leq j \leq N. \quad (66)$$

To address such an issue, we need a suitable estimate of  $|(M \cdot V)_j|$ . Since we have (52), with  $(M \cdot V)_0 = 0$  we easily get for  $1 \leq j \leq N+1$

$$\begin{aligned} |(M \cdot V)_j| &\leq \frac{\nu}{\sqrt{2}} \sum_{k=0}^{j-1} \left| \sqrt{\phi_{k+1} - \phi_j} - \sqrt{\phi_k - \phi_j} \right| \times \\ &\quad \left( e^{\nu \bar{s}_0(x_j, v_{j-k}^j)} \frac{|V_k|}{|\partial_x \phi_k|} + e^{\nu \bar{s}_0(x_j, v_{j-(k+1)}^j)} \frac{|V_{k+1}|}{|\partial_x \phi_{k+1}|} \right), \end{aligned}$$

where  $\bar{s}_0(x_j, v_{j-k}^j)$ , given by (45) is negative. As a consequence, the above inequality rewrites

$$|(M \cdot V)_j| \leq \frac{\nu}{\sqrt{2}} \sum_{k=0}^{j-1} \left| \sqrt{\phi_{k+1} - \phi_j} - \sqrt{\phi_k - \phi_j} \right| \times \left( \frac{|V_k|}{|\partial_x \phi_k|} + \frac{|V_{k+1}|}{|\partial_x \phi_{k+1}|} \right).$$

According to (46), with  $(\phi_\ell)_{0 \leq \ell \leq N+1}$  a decreasing sequence, we obtain

$$\begin{aligned} |(M \cdot V)_j| &\leq \frac{\nu}{2|M_\phi|} \sum_{k=0}^{j-1} \left( \sqrt{2(\phi_k - \phi_j)} - \sqrt{2(\phi_{k+1} - \phi_j)} \right) (|V_k| + |V_{k+1}|) \\ &\leq \frac{\nu}{2|M_\phi|} \sum_{k=0}^{j-1} \left( \sqrt{2(\phi_k - \phi_j)} - \sqrt{2(\phi_{k+1} - \phi_j)} \right) (|V_k| + |V_{k+1}|) e^{\tau \frac{\phi_k + \phi_{k+1}}{2}} e^{-\tau \frac{\phi_k + \phi_{k+1}}{2}} \\ &\leq \frac{\nu}{|M_\phi|} \|V\|_\tau \sum_{k=0}^{j-1} \left( \sqrt{2(\phi_k - \phi_j)} - \sqrt{2(\phi_{k+1} - \phi_j)} \right) e^{-\tau \frac{\phi_k + \phi_{k+1}}{2}}, \end{aligned}$$

1 so that we get for  $1 \leq j \leq N$

$$|(M \cdot V)_j| \leq \frac{\nu}{|M_\phi|} \|V\|_\tau \mathcal{S}_j, \quad (67)$$

where we have set for  $1 \leq j \leq N + 1$

$$\mathcal{S}_j = \sum_{k=0}^{j-1} \left( \sqrt{2(\phi_k - \phi_j)} - \sqrt{2(\phi_{k+1} - \phi_j)} \right) e^{-\tau \frac{\phi_k + \phi_{k+1}}{2}}.$$

In order to obtain the expected inequality (66), we give an estimate satisfied by  $\mathcal{S}_j$ . Indeed, we have for  $1 \leq j \leq N + 1$

$$\mathcal{S}_j = \sum_{k=0}^{j-1} \frac{\sqrt{2(\phi_k - \phi_j)} - \sqrt{2(\phi_{k+1} - \phi_j)}}{\phi_k - \phi_{k+1}} e^{-\tau \frac{\phi_k + \phi_{k+1}}{2}} (\phi_k - \phi_{k+1}).$$

Let consider  $p > 0$  and  $q > 0$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ . Then using the Holder inequality, we get

$$\begin{aligned} \mathcal{S}_j &\leq \left( \sum_{k=0}^{j-1} e^{-q\tau \frac{\phi_k + \phi_{k+1}}{2}} (\phi_k - \phi_{k+1}) \right)^{\frac{1}{q}} \times \\ &\quad \left( \sum_{k=0}^{j-1} \left( \frac{\sqrt{2(\phi_k - \phi_j)} - \sqrt{2(\phi_{k+1} - \phi_j)}}{\phi_k - \phi_{k+1}} \right)^p (\phi_k - \phi_{k+1}) \right)^{\frac{1}{p}}. \end{aligned}$$

Notice that

$$e^{-q\tau \frac{\phi_k + \phi_{k+1}}{2}} (\phi_k - \phi_{k+1}) \leq \int_{\phi_{k+1}}^{\phi_k} e^{-q\tau\varphi} d\varphi,$$

so that we have, since  $\phi_0 = 0$ ,

$$\begin{aligned} \sum_{k=0}^{j-1} e^{-q\tau \frac{\phi_k + \phi_{k+1}}{2}} (\phi_k - \phi_{k+1}) &\leq \int_{\phi_j}^0 e^{-q\tau\varphi} d\varphi \\ &\leq \frac{1}{q\tau} \left( e^{-q\tau\phi_j} - 1 \right). \end{aligned} \quad (68)$$

Moreover, we have

$$\frac{\sqrt{2(\phi_k - \phi_j)} - \sqrt{2(\phi_{k+1} - \phi_j)}}{\phi_k - \phi_{k+1}} = \frac{1}{\phi_k - \phi_{k+1}} \int_{\phi_{k+1}}^{\phi_k} \frac{d\varphi}{\sqrt{2(\varphi - \phi_j)}}.$$

Using Jensen's inequality, we then obtain

$$\left( \frac{\sqrt{2(\phi_k - \phi_j)} - \sqrt{2(\phi_{k+1} - \phi_j)}}{\phi_k - \phi_{k+1}} \right)^p \leq \frac{1}{\phi_k - \phi_{k+1}} \int_{\phi_{k+1}}^{\phi_k} (2(\varphi - \phi_j))^{-\frac{p}{2}} d\varphi,$$

so that for  $p < 2$ ,

$$\begin{aligned} \sum_{k=0}^{j-1} \left( \frac{\sqrt{2(\phi_k - \phi_j)} - \sqrt{2(\phi_{k+1} - \phi_j)}}{\phi_k - \phi_{k+1}} \right)^p (\phi_k - \phi_{k+1}) &\leq \int_{\phi_j}^{\phi_0} (2(\varphi - \phi_j))^{-\frac{p}{2}} d\varphi \\ &\leq \frac{1}{2-p} (-2\phi_j)^{1-\frac{p}{2}}. \end{aligned} \quad (69)$$

Now, from (68) and (69), it results the following estimate of  $\mathcal{S}_j$  for  $1 \leq j \leq N+1$ :

$$\mathcal{S}_j \leq \left( \frac{e^{-q\tau\phi_j} - 1}{q\tau} \right)^{\frac{1}{q}} \frac{(-2\phi_j)^{\frac{1}{p}-\frac{1}{2}}}{(2-p)^{\frac{1}{p}}}.$$

Since we have  $\phi_{\text{wall}} \leq \phi_{j+1} \leq \phi_j \leq 0$ , then we get

$$\frac{1}{2} (\mathcal{S}_j + \mathcal{S}_{j+1}) e^{\tau \frac{\phi_j + \phi_{j+1}}{2}} \leq \frac{1}{(q\tau)^{\frac{1}{q}}} \left( \frac{(-2\phi_{\text{wall}})^{1-\frac{p}{2}}}{2-p} \right)^{\frac{1}{p}}.$$

As a consequence, it suffices to consider

$$\tau > \frac{1}{q} \left( 2 \left( \frac{(-2\phi_{\text{wall}})^{1-\frac{p}{2}}}{2-p} \right)^{\frac{1}{p}} \max_{0 \leq j \leq N+1} \left( \bar{B}_\alpha^j(f_i^{\text{inc}}) \frac{I_0^j}{I_1^j} \right) \right)^q,$$

where  $\bar{B}_\alpha^j(f_i^{\text{inc}})$  is given by (60) and  $I_{0,1}^j$  are defined by (59), to obtain for  $0 \leq j \leq N$

$$\frac{1}{2} (\mathcal{S}_j + \mathcal{S}_{j+1}) e^{\tau \frac{\phi_j + \phi_{j+1}}{2}} \leq \frac{1}{2} \min_{0 \leq j \leq N+1} \left( \frac{1}{\bar{B}_\alpha^j(f_i^{\text{inc}})} \frac{I_1^j}{I_0^j} \right).$$

- 1 Involving (67) and the condition (61), we deduce the required estimate (66)
- 2 where  $\tau$  does not depend on  $M_\phi$ .

The proof is easily completed since, from (66), we immediately obtain  $\|M\|_\tau \leq \frac{1}{2}$  so that

$$n_i[\phi] = \sum_{\ell \geq 0} M^\ell \mathcal{I}^\infty,$$

to write

$$\|n_i[\phi]\|_\tau \leq \|\mathcal{I}^\infty\|_\tau \sum_{\ell \geq 0} \|M\|_\tau^\ell \leq 2\|\mathcal{I}^\infty\|_\tau.$$

Next, with  $\mathcal{I}^\infty[\phi]_j$  given by (41) and  $\bar{t}_{\text{inc}}[\phi](x_j, \bar{v}_k^j) < 0$ , we directly obtain  $\mathcal{I}^\infty[\phi]_j \leq I_0^j$  for all  $0 \leq j \leq N+1$  where  $I_0^j$  is given by (59). Moreover, since  $\phi_{j+1} \leq \phi_j \leq 0$  then we have

$$\frac{1}{2} (I_j^\infty + I_{j+1}^\infty) e^{\tau \frac{\phi_j + \phi_{j+1}}{2}} \leq \min_{0 \leq j \leq N+1} I_0^j,$$

so that

$$\|\mathcal{I}^\infty\|_\tau \leq \min_{0 \leq j \leq N+1} I_0^j.$$

As a consequence,  $\|n_i[\phi]\|_\tau \leq 2 \min_{0 \leq j \leq N+1} I_0^j$  and we have

$$\begin{aligned} \frac{1}{2} (n_i[\phi]_j + n_i[\phi]_{j+1}) &\leq \frac{1}{2} (n_i[\phi]_j + n_i[\phi]_{j+1}) e^{\tau \frac{\phi_j + \phi_{j+1}}{2}} e^{-\tau \frac{\phi_j + \phi_{j+1}}{2}} \\ &\leq e^{-\tau \frac{\phi_j + \phi_{j+1}}{2}} \|n_i[\phi]\|_\tau. \end{aligned}$$

1 The proof is thus achieved. □

2 *3.2. The proof of stability*

3 We are now able to prove the Theorem 3.1.

4 *Proof.* By induction, both inequalities (54) and (55) are established provided  
5 that for a given sequence  $(\phi_\ell)_{0 \leq \ell \leq N+1}$  such that  $\phi_0^n = 0$ ,  $\phi_{N+1}^n = \phi_{\text{wall}}$  and  
6  $(\phi_{\ell+1}^0 - \phi_\ell^0)/\Delta x \leq M_\phi$  for all  $0 \leq \ell \leq N$ , we prove (54) for  $\phi^n$  and (55) for  
7  $\phi^{n+1}$ . The first estimate (54) for  $\phi^n$  immediately comes from Lemma 3.3.  
8 Now, we have to show

$$\frac{1}{\Delta x} (\phi_{j+1}^{n+1} - \phi_j^{n+1}) \leq M_\phi. \quad (70)$$

9 To address such an issue, let us consider the summation of (32) from  $j$  equals  
10 one to  $\ell$ , to write

$$-\frac{\varepsilon^2}{\Delta x} (\phi_{\ell+1}^{n+1} - \phi_\ell^{n+1} - \phi_1^{n+1} + \phi_0^{n+1}) = \sum_{j=1}^{\ell} (n_i[\phi^n]_j - n_e[\phi^n]_j) \Delta x. \quad (71)$$

11 By Lemma 3.3, we have  $n_i[\phi^n]_j - n_e[\phi^n]_j \geq 0$  so that  $\phi_{\ell+1}^{n+1} - \phi_\ell^{n+1} \leq \phi_1^{n+1} -$   
12  $\phi_0^{n+1}$ . As a consequence, the required estimate (70) is established provided  
13 we prove

$$\frac{1}{\Delta x} (\phi_1^{n+1} - \phi_0^{n+1}) \leq M_\phi, \quad (72)$$

where we remember that

$$M_\phi = - \max_{0 \leq j \leq N+1} \left( \frac{\nu I_1^j}{\bar{B}_\alpha^j(f_i^{\text{inc}}) I_0^j} \right)$$

1 with  $\bar{B}_\alpha^j(f_i^{\text{inc}})$  given by (60) and  $I_{0,1}^j$  by (59).

In order to derive the above inequality, we sum (71) from  $\ell$  equals from 0 to  $N$ , we then obtain

$$-\varepsilon^2 ((\phi_{N+1}^{n+1} - \phi_0^{n+1}) - (N+1)(\phi_1^{n+1} - \phi_0^{n+1})) = \sum_{\ell=0}^N \sum_{j=1}^{\ell} (n_i[\phi]_j - n_e[\phi]_j) \Delta x^2.$$

Since  $\Delta x = 1/(N+1)$ ,  $\phi_0^{n+1} = 0$  and  $\phi_{N+1}^{n+1} = \phi_{\text{wall}}$ , the above relation reformulates as follows:

$$\frac{1}{\Delta x} (\phi_1^{n+1} - \phi_0^{n+1}) = \phi_{\text{wall}} + \frac{1}{\varepsilon^2} \sum_{\ell=0}^N \sum_{j=1}^{\ell} (n_i[\phi]_j - n_e[\phi]_j) \Delta x^2.$$

Now, the definition of  $n_e[\phi]_j$  and the estimate (65) can be applied to get an upper bound of  $n_i[\phi]_j - n_e[\phi]_j$ . Indeed, we have for all  $0 \leq j \leq N$

$$\begin{aligned} n_i[\phi]_j - n_e[\phi]_j &= \frac{1}{2}(n_i[\phi]_j + n_i[\phi]_{j+1}) - n_e[\phi]_j + \frac{1}{2}(n_i[\phi]_j - n_i[\phi]_{j+1}) \\ &\leq 2e^{-\tau \frac{\phi_j + \phi_{j+1}}{2}} \min_{0 \leq j \leq N+1} (I_0^j) - \frac{I_0^j}{m_\alpha(0)} m_\alpha(\phi_j) e^{\phi_j} + \frac{1}{2}(n_i[\phi]_j - n_i[\phi]_{j+1}) \\ &\leq C_{\text{wall}} + \frac{1}{2}(n_i[\phi]_j - n_i[\phi]_{j+1}), \end{aligned}$$

where we have set

$$C_{\text{wall}} = \left( 2e^{-\tau \phi_{\text{wall}}} - \frac{m_\alpha(\phi_{\text{wall}})}{m_\alpha(0)} e^{\phi_{\text{wall}}} \right) \min_{0 \leq j \leq N+1} I_0^j.$$

Since  $n_i[\phi]_j - n_e[\phi]_j > 0$  then  $C_{\text{wall}} > 0$ . As a consequence, we obtain

$$\begin{aligned} \frac{1}{\Delta x} (\phi_1^{n+1} - \phi_0^{n+1}) &\leq \phi_{\text{wall}} + \frac{1}{\varepsilon^2} \left( C_{\text{wall}} \frac{1 - \Delta x}{2} + \frac{\Delta x^2}{2} \sum_{\ell=0}^N \sum_{j=1}^{\ell} (n_i[\phi]_j - n_i[\phi]_{j+1}) \right) \\ &\leq \phi_{\text{wall}} + \frac{1}{\varepsilon^2} \left( C_{\text{wall}} \frac{1 - \Delta x}{2} + \frac{\Delta x^2}{2} \sum_{\ell=0}^N (n_i[\phi]_1 - n_i[\phi]_{\ell+1}) \right) \\ &\leq \phi_{\text{wall}} + \frac{1}{\varepsilon^2} \left( C_{\text{wall}} \frac{1 - \Delta x}{2} + \frac{\Delta x}{2} \max_{0 \leq j \leq N} |n_i[\phi]_1 - n_i[\phi]_{j+1}| \right). \end{aligned}$$

1 The required inequality (72) is obtained for  $\varepsilon \geq \varepsilon^*$  where  $\varepsilon^*$  is given by

$$\varepsilon^* = \max_{0 \leq j \leq N+1} \left( \frac{1}{-\phi_{\text{wall}} - \frac{\nu I_1^j}{\bar{B}_\alpha^j(f_i^{\text{inc}}) I_0^j}} \times \left( I_0^j e^{-\tau \phi_{\text{wall}}} - \frac{I_0^j}{2m_\alpha(0)} m_\alpha(\phi_{\text{wall}}) e^{\phi_{\text{wall}}} + \mathcal{O}(\Delta x) \right) \right)^{\frac{1}{2}}. \quad (73)$$

2 The proof is then completed.  $\square$

### 3 4. Numerical experiments

4 This section is devoted to the numerical simulation of problem (1)-(12)  
5 in different regimes, using the numerical scheme presented in Section 2. Sev-  
6 eral studies are proposed. First, we consider a non collisional problem in  
7 Subsection 4.1. We are able to compare our results with those of [3], so that  
8 this regime is seen as a validating case for our approach. Then, we are inter-  
9 ested in collisional regimes in order to observe the influence of the collision  
10 frequency on the electrostatic potential  $\phi$  and on the ionic and electronic  
11 distribution functions. In Subsection 4.2, we propose different numerical  
12 simulations compatible with hypotheses (13)-(14) and with a subcritical fre-  
13 quency  $0 < \nu < \nu_c$ . In this case, Theorem 3.1 holds (as well as its continuous  
14 version: Theorem 3.1 of [2]). Finally, we are interested in simulations that  
15 violate hypotheses of Theorem 3.1, in order to test our numerical scheme  
16 beyond the scope of this theorem. This numerical study is presented in  
17 Subsection 4.3.

18 In the simulations, we consider the following incoming functions

$$f_e^{\text{inc}}(v) = \sqrt{\frac{2\mu}{\pi}} e^{-\frac{\mu v^2}{2}}, \quad \forall v > 0, \quad (74)$$

$$f_i^{\text{inc}}(v) = \sqrt{\frac{2}{\pi}} v^2 e^{-\frac{v^2}{2}}, \quad \forall v > 0, \quad (75)$$

19 and the mass ratio of a Deuterium plasma  $\mu = \frac{1}{3672}$ , so that the critical  
20 re-emission coefficient  $\alpha_c = \frac{1 - \sqrt{(\pi\mu)/2}}{1 + \sqrt{(\pi\mu)/2}} \approx 0.95$ . We take  $N + 1 = 1000$  for  
21 the space discretization. The velocity domain for the computation of ion  
22 density is truncated to  $[-8, 8]$  and domain  $D_1$  is divided into 100 intervals.

1 We stop the fixed point method when two successive iterates  $k$  and  $k + 1$   
 2 are such that

$$\left\| \frac{\phi^{(k)} - \phi^{(k+1)}}{\phi^{(k)}} \right\|_{\infty} + \left\| \frac{(n_i - n_e)^{(k)} - (n_i - n_e)^{(k+1)}}{(n_i - n_e)^{(k)}} \right\|_{\infty} < 10^{-8}. \quad (76)$$

3 Let us emphasize that the convergence of our fixed point method is not  
 4 ensured for small values of  $\varepsilon$ . In practice, Theorem 3.1 implies the existence  
 5 of  $\varepsilon^* > 0$  such that our fixed point algorithm converges for all  $\varepsilon \geq \varepsilon^*$ , where  
 6 the bound (73) is not optimal.

#### 7 4.1. The non collisional sheath

8 We are first interested in a non collisional problem, with parameters  $\nu =$   
 9  $0$ ,  $\alpha = 0$ . Admissibility condition (13) and Bohm criterion (14) are satisfied.  
 10 A Newton iterative method solving (29) up to  $10^{-16}$  gives  $\phi_{\text{wall}} \approx -2.719$   
 11 and a density reference  $n_0 \approx 0.505$ . Here, we consider two values of  $\varepsilon$ : 1 and  
 12 0.2. We did not obtain convergence of our fixed point method for smaller  
 13 values. We plot the electrostatic potential  $\phi(x)$  in Figure 2 and the densities  
 14  $n_i(x)$ ,  $n_e(x)$  in Figure 3.

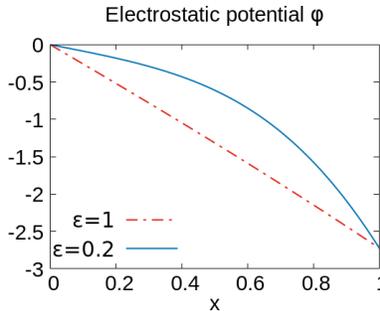


Figure 2: Electrostatic potential  $\phi(x)$  for two values of  $\varepsilon$ : 1 and 0.2, in the non collisional regime  $\nu = 0$ ,  $\alpha = 0$ .

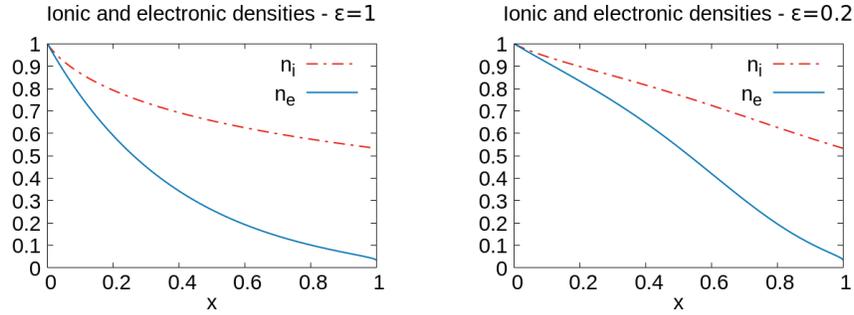


Figure 3: Ionic  $n_i(x)$  and electronic  $n_e(x)$  densities for two values of  $\varepsilon$ : 1 (left) and 0.2 (right), in the non collisional regime  $\nu = 0$ ,  $\alpha = 0$ .

1 Then, we represent distribution functions:  $f_i$  in Figure 4 and  $f_e$  in Figure  
 2 5, for both values of  $\varepsilon$  (we propose two different views for each figure).

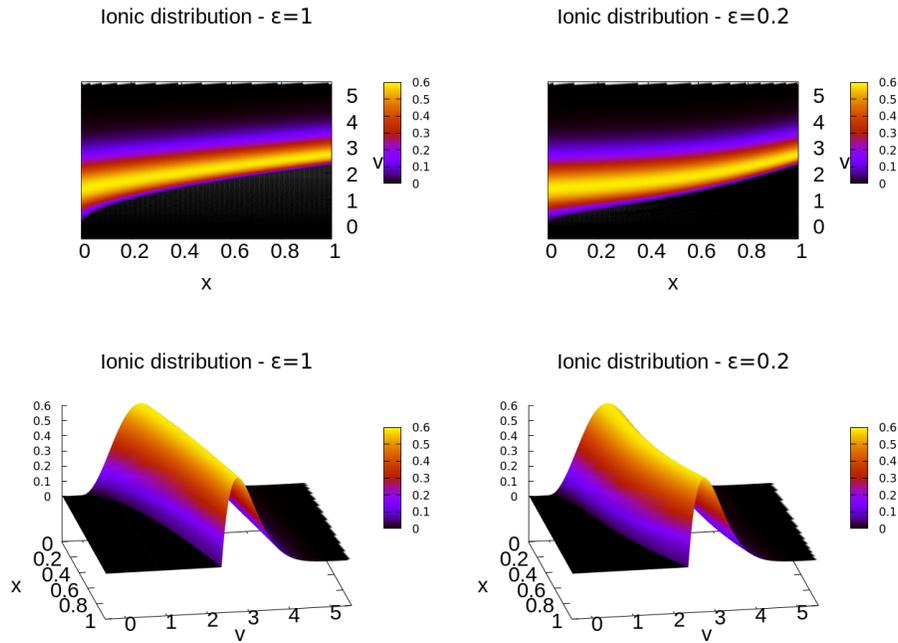


Figure 4: Ionic distribution function  $f_i(x, v)$  for two values of  $\varepsilon$ : 1 (left) and 0.2 (right), in the non collisional regime  $\nu = 0$ ,  $\alpha = 0$ . Two different views on top and bottom.

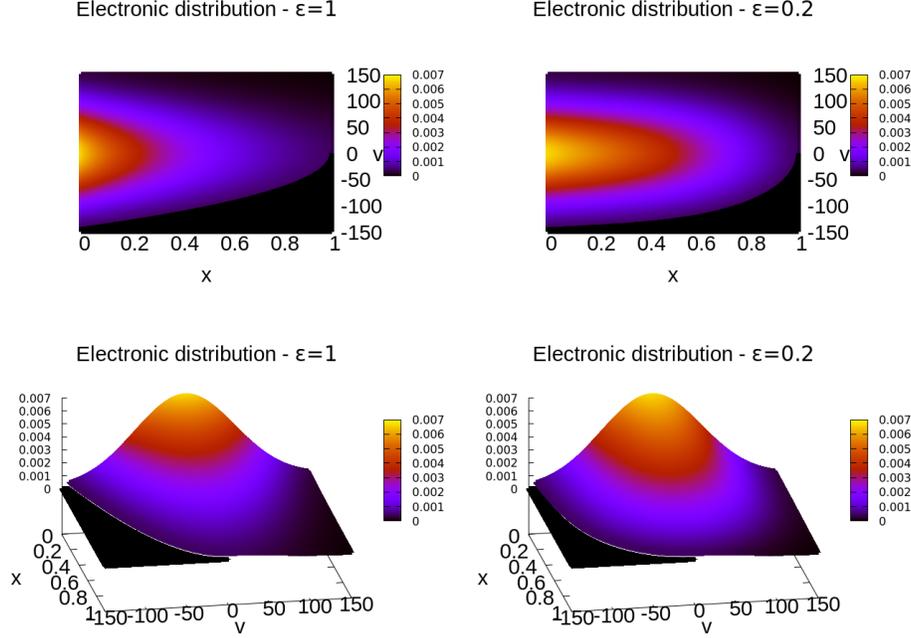


Figure 5: Electronic distribution function  $f_e(x, v)$  for two values of  $\varepsilon$ : 1 (left) and 0.2 (right), in the non collisional regime  $\nu = 0$ ,  $\alpha = 0$ . Two different views on top and bottom.

1        These figures show the influence of  $\varepsilon$  and can be compared to numerical  
2 results proposed in [3]. Even if our approach prevents us from taking small  
3 values of  $\varepsilon$ , it presents the good behavior. The numerical scheme presented  
4 in [3] is valid for smaller values of  $\varepsilon$ , but is specific to the non collisional case  
5  $\nu = 0$ . It is based on the minimization formulation of the non linear Poisson  
6 problem. In the collisional case  $\nu > 0$ , such an approach cannot be applied  
7 directly because the non linear Poisson problem (3) does not reformulate  
8 as a minimization problem. The difficulty stems from the integral equation  
9 satisfied by  $n_i$  (23).

#### 10 4.2. Collisional problems in the scope of Theorem 3.1

11        We are now interested in collisional problem staying in the scope of the  
12 numerical analysis done in this paper, and in particular  $0 < \nu < \nu_c$  where  
13 the critical collisional frequency is defined by (16). We study the influence  
14 of physical parameters  $\nu$ ,  $\alpha$  and  $\varepsilon$ .

1 We point out that Theorem 3.1 ensures the existence of  $\varepsilon^*$  such that  
 2 the numerical scheme preserves the physical properties (54) and (55) for  
 3 all  $\varepsilon \geq \varepsilon^*$ . The value of  $\varepsilon^*$  given by (73) is in practice not optimal. In  
 4 the numerical approach, we define the minimal value  $\varepsilon^{min} \leq \varepsilon^*$  for which  
 5 we are able to make the fixed point method converge. The convergence  
 6 depends on the initial iterate, that is why we have implemented a numerical  
 7 continuation method over  $\varepsilon$ . If the fixed point method converges for a value  
 8 of  $\varepsilon$ , we use the obtained  $\phi[\varepsilon]$  as initial guess for the algorithm with  $\varepsilon - 0.001$   
 9 and we continue until the fixed point method diverges. The smallest value  
 10 for which we have convergence is denoted as  $\varepsilon^{min}$ . The step 0.001 is chosen  
 11 as a compromise between precision on  $\varepsilon^{min}$  and computational time.

12 On the one hand, we fix  $\alpha = 0$ , so that  $\nu_c \approx 3.526$ , and present results  
 13 for varying  $\nu$  from 1 to 3.5 and two values of  $\varepsilon$ : 1 and  $\varepsilon^{min}$ . We plot the  
 14 electrostatic potential  $\phi(x)$ , resp. densities  $n_i(x)$  and  $n_e(x)$ , in Figure 6,  
 resp. in Figure 7.

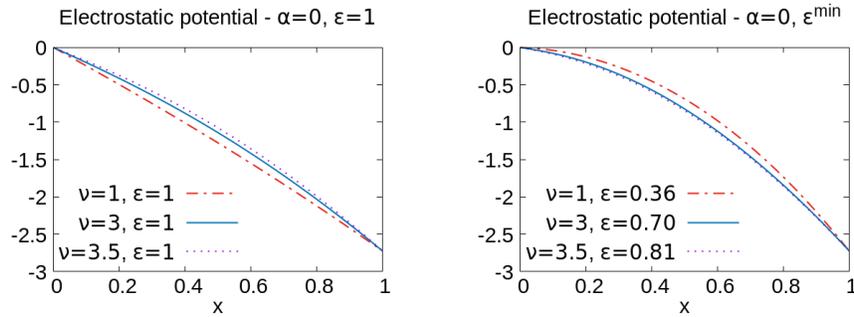


Figure 6: Electrostatic potential  $\phi(x)$  for  $\alpha = 0$ , three values of  $\nu$ : 1, 3 and 3.5. On the left:  $\varepsilon = 1$ , on the right:  $\varepsilon = \varepsilon^{min}$ .

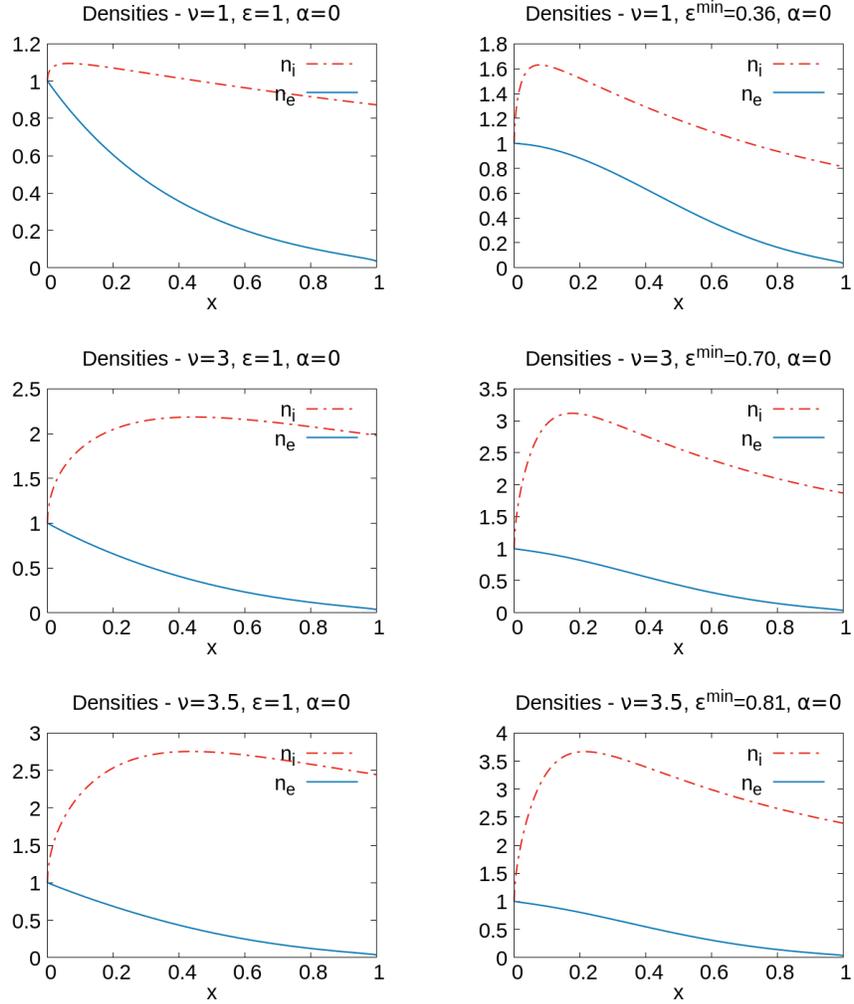
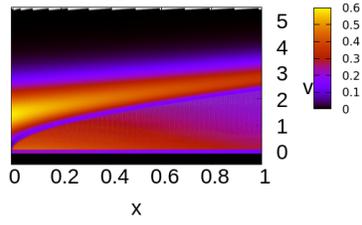


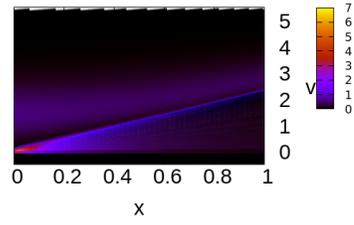
Figure 7: Ionic  $n_i(x)$  and electronic  $n_e(x)$  densities for  $\alpha = 0$ , three values of  $\nu$ : 1, 3 and 3.5 (from top to bottom). On the left:  $\varepsilon = 1$ , on the right:  $\varepsilon = \varepsilon^{min}$ .

- 1 Then, ionic distribution function is presented in Figure 8 and the elec-
- 2 tronic one in Figure 9.

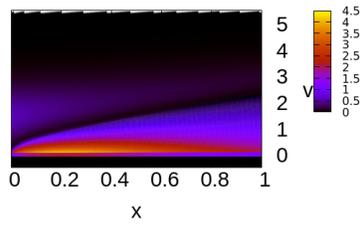
Ionic distribution -  $\nu=1, \varepsilon=1, \alpha=0$



Ionic distribution -  $\nu=1, \varepsilon^{\min}=0.36, \alpha=0$



Ionic distribution -  $\nu=3.5, \varepsilon=1, \alpha=0$



Ionic distribution -  $\nu=3.5, \varepsilon^{\min}=0.81, \alpha=0$

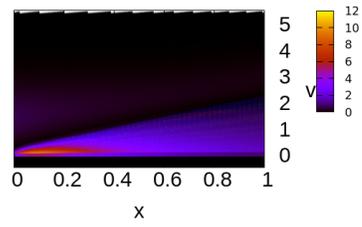
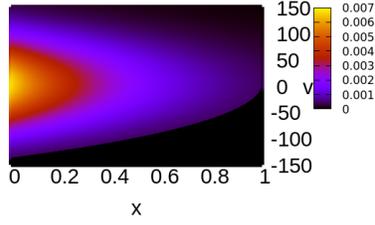
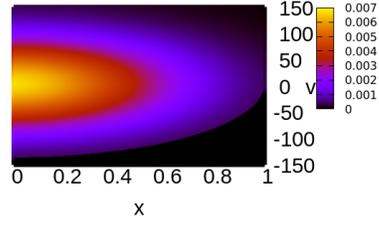


Figure 8: Ionic distribution function  $f_i(x, v)$  for  $\alpha = 0$ , two values of  $\nu$ : 1 and 3.5 (from top to bottom). On the left:  $\varepsilon = 1$ , on the right:  $\varepsilon = \varepsilon^{\min}$ .

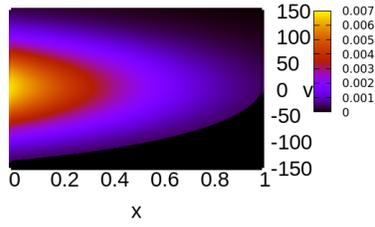
Electronic distribution -  $\nu=1, \epsilon=1, \alpha=0$



Electronic distribution -  $\nu=1, \epsilon^{min}=0.36, \alpha=0$



Electronic distribution -  $\nu=3.5, \epsilon=1, \alpha=0$



Electronic distribution -  $\nu=3.5, \epsilon^{min}=0.81, \alpha=0$

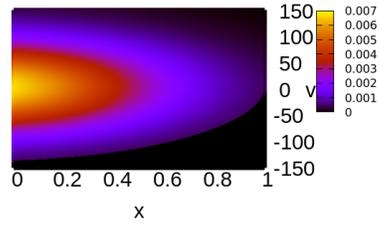


Figure 9: Electronic distribution function  $f_e(x, v)$  for  $\alpha = 0$ , two values of  $\nu$ : 1 and 3.5 (from top to bottom). On the left:  $\epsilon = 1$ , on the right:  $\epsilon = \epsilon^{min}$ .

- 1 On the other hand, we fix  $\nu = 0.2$  and present results for varying  $\alpha$
- 2 from 0 to  $0.9 < \alpha_c$  and two values of  $\epsilon$ : 1 and  $\epsilon^{min}$ . Figures 10, 11, 12 and
- 3 13 represent respectively the electrostatic potential, the densities, the ionic
- 4 distribution function and the electronic one.

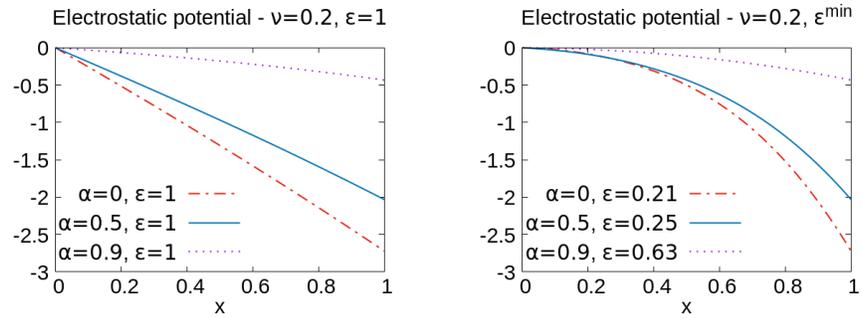


Figure 10: Electrostatic potential  $\phi(x)$  for  $\nu = 0.2$ , three values of  $\alpha$ : 0, 0.5 and 0.9. On the left:  $\varepsilon = 1$ , on the right:  $\varepsilon = \varepsilon^{\min}$ .

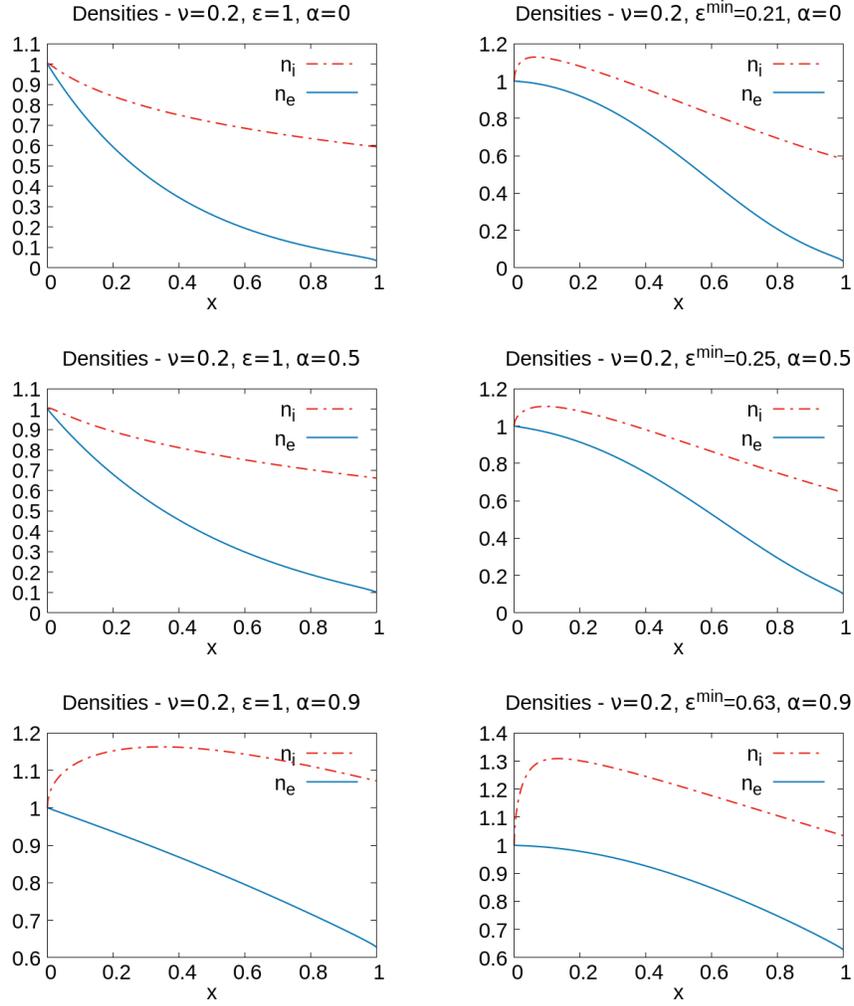
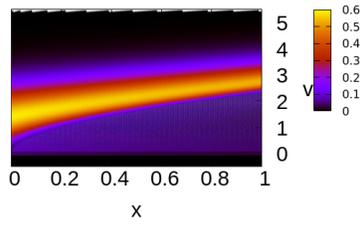
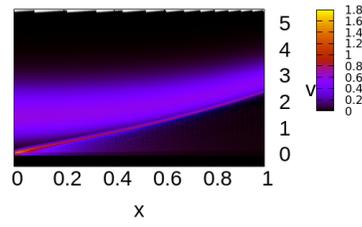


Figure 11: Ionic  $n_i(x)$  and electronic  $n_e(x)$  densities for  $\nu = 0.2$ , three values of  $\alpha$ : 0, 0.5 and 0.9 (from top to bottom). On the left:  $\epsilon = 1$ , on the right:  $\epsilon = \epsilon^{\min}$ .

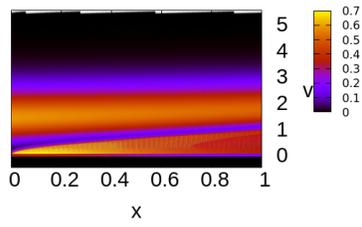
Ionic distribution -  $\nu=0.2$ ,  $\varepsilon=1$ ,  $\alpha=0$



Ionic distribution -  $\nu=0.2$ ,  $\varepsilon^{\min}=0.21$ ,  $\alpha=0$



Ionic distribution -  $\nu=0.2$ ,  $\varepsilon=1$ ,  $\alpha=0.9$



Ionic distribution -  $\nu=0.2$ ,  $\varepsilon^{\min}=0.63$ ,  $\alpha=0.9$

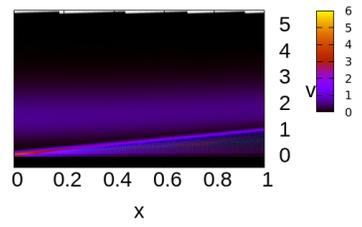


Figure 12: Ionic distribution function  $f_i(x, v)$  for  $\nu = 0.2$ , two values of  $\alpha$ : 0 and 0.9 (from top to bottom). On the left:  $\varepsilon = 1$ , on the right:  $\varepsilon = \varepsilon^{\min}$ .

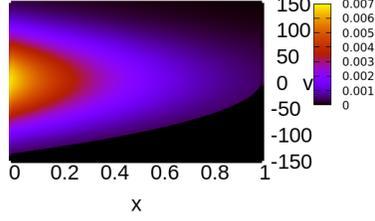
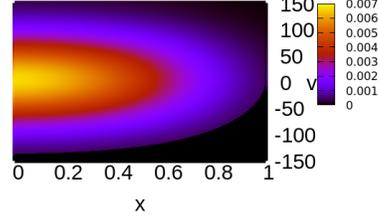
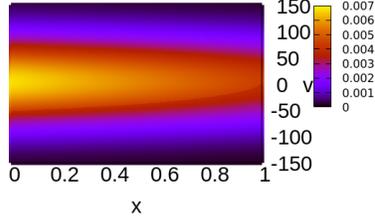
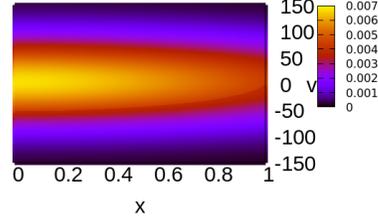
Electronic distribution -  $\nu=0.2$ ,  $\varepsilon=1$ ,  $\alpha=0$ Electronic distribution -  $\nu=0.2$ ,  $\varepsilon^{\min}=0.21$ ,  $\alpha=0$ Electronic distribution -  $\nu=0.2$ ,  $\varepsilon=1$ ,  $\alpha=0.9$ Electronic distribution -  $\nu=0.2$ ,  $\varepsilon^{\min}=0.63$ ,  $\alpha=0.9$ 

Figure 13: Electronic distribution function  $f_e(x, v)$  for  $\nu = 0.2$ , two values of  $\alpha$ : 0 and 0.9 (from top to bottom). On the left:  $\varepsilon = 1$ , on the right:  $\varepsilon = \varepsilon^{\min}$ .

1 *Interpretation.* For the ions distribution function  $f_i$ , we observe on Figures  
 2 8 and 12 that increasing the collision frequency  $\nu$  or diminishing the nor-  
 3 malized Debye length  $\varepsilon$  results in a stronger diffusion effect. Moreover,  
 4 whatever the value of  $\nu > 0$  is,  $f_i$  is non zero near the line  $v = 0$  which is the  
 5 expected effect of the collision operator (4). On Figure 8, we observe that  
 6 for a near critical collision frequency  $\nu \approx \nu_c$ , there is some concentration  
 7 of the ions distribution function in the vicinity of the line  $v = 0$  while we  
 8 can still observe low values of  $f_i$  for  $v > \sqrt{-2\phi(x)}$ . These results evidence  
 9 the competition between the collision operator (4) which tends to relax the  
 10 ions density  $f_i$  towards the mono kinetic density  $n_i(x)\delta_{v=0}$  and the trans-  
 11 port of the incoming boundary condition  $f_i^{\text{inc}}$  (75) from  $x = 0$  to  $x = 1$   
 12 with a damping effect. As for the electrons distribution function  $f_e$ , since  
 13 it does not experience collisions, there is no change as compared to the non  
 14 collisional case [3].

15 Increasing the re-emission coefficient  $\alpha$  increases the values of the po-  
 16 tential at the wall ( $x = 1$ ) as it is observed on Figure 10. It results in a

1 charge separation at  $x = 1$  which is increasing with  $\alpha$  as observed in Figure  
 2 11. As for the charge density  $n_i - n_e$ , we see on Figure 11 that it remains  
 3 non negative and that it vanishes at  $x = 0$  (as expected). We also see that  
 4 increasing  $\nu$  tends to increase the local charge  $n_i$  near  $x = 0$ . The macro-  
 5 scopic ions density  $n_i$  does not seem to be  $C^1$  at  $x = 0$ . It is not surprising  
 6 since it satisfies the weakly singular Volterra integral equation (23).

7 To conclude this study, we plot  $\varepsilon^{\min}$  as a function of  $\frac{\nu}{\nu_c} \in [0, 1[$  for the  
 8 three values of  $\alpha$  considered previously (note that  $\nu_c \approx 3.526$  for  $\alpha = 0$ ,  
 9  $\nu_c \approx 2.644$  for  $\alpha = 0.5$  and  $\nu_c \approx 0.583$  for  $\alpha = 0.9$ ). Results are presented in  
 10 Figure 14.

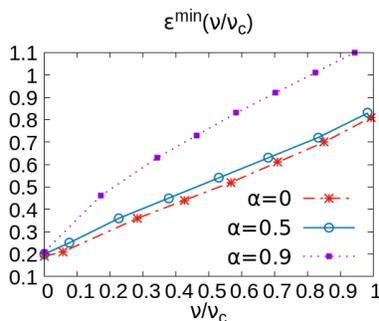


Figure 14: Minimal numerical value  $\varepsilon^{\min}$  as a function of  $\nu/\nu_c$  for different values of  $\alpha$ .

#### 11 4.3. Collisional problems beyond the scope of Theorem 3.1

12 Finally, we are interested in simulations beyond the scope of Theorem  
 13 3.1. We notice that when taking  $\nu > \nu_c$ , the fixed point method hardly  
 14 converges. In practice, we only have convergence for very large values of  
 15 the normalized Debye length  $\varepsilon$ . The obtained densities are in this case a  
 16 zoom on the space scale of the previous results. Specifically, we see the same  
 17 behavior as in a window near  $x = 0$  of Figures 7 or 11.

Our last numerical experiment, concerns an other physical scenario when  
 considering an ionic incoming function  $f_i^{\text{inc}}$  for which the Bohm criterion (14)  
 is not satisfied. We thus consider,

$$f_i^{\text{inc}}(v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}, \quad \forall v > 0.$$

18 This leads to  $B_\alpha(f_i^{\text{inc}}) = -\infty$ . We set the following physical parameters:  
 19  $\alpha = 0$ ,  $\nu = 0$  or 1, and  $\varepsilon = 1$ . We plot densities  $n_i(x)$  and  $n_e(x)$  in Figure

15. We can see that the inequality  $n_i(x) - n_e(x) \geq 0$  (17) is not verified for  $x$  near zero. It seems to show that even in the collisional case the Bohm criterion (14) is a sharp condition to ensure the non negativity of the charge density.

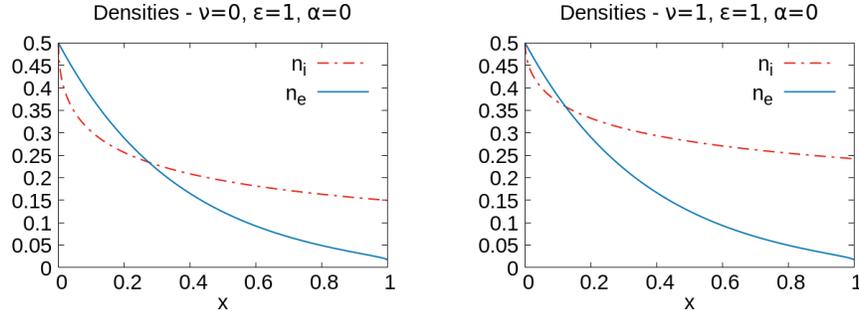


Figure 15: Ionic  $n_i(x)$  and electronic  $n_e(x)$  densities for  $\alpha = 0$ , two values of  $\nu$ : 0 (on the left) and 1 (on the right).

## 5. Conclusion

In this work we proposed a numerical method to capture kinetic collisional sheaths. The method is based on the exact integrations of the transport equations by means of the characteristics curves and the numerical resolution of a non linear Poisson problem that has the form of an integro-differential equation. According to [3], this problem has a solution provided the inequalities (14),(17) and (18) are ensured. We thus designed a suitable discretisation of the phase space together with adequate quadrature formulas that ensure easily the discrete analogue of these inequalities. We then presented some numerical experiments to assess mainly two physical scenarios. The first one was in the scope of our main result (Theorem 3.1) : when the Bohm criterion (14) holds and the different physical parameters are varied within appropriate bounds, we essentially observe that decreasing the Debye length or increasing the collision frequency results in a stronger diffusion effect on the ions distribution function. We also see some concentration of the distribution function near the line  $v = 0$ . There is a competition between the transport operator and the collision operator which yields in the extreme case some boundary layer near  $x = 0$ . Some parametric study between the Debye length and the collision frequency was carried out in order to assess a possible scaling between these two parameters. The second

1 physical scenario mainly showed that when the Bohm criterion (14) is vio-  
2 lated the charge density becomes negative near  $x = 0$ . It seems to show that  
3 the Bohm criterion (14) is a sharp condition to ensure the non negativity of  
4 the charge density. This work is up to our knowledge the very first one to  
5 propose a detailed numerical study of a fully kinetic collisional sheath model  
6 by means of a numerical method that strongly relies on a detailed analysis  
7 of the model under concern. It provided a range of physical parameters  
8 that is mathematically relevant. A possible perspective among others is to  
9 investigate asymptotic regimes  $\varepsilon \rightarrow 0$  or  $\nu \rightarrow +\infty$  though the model under  
10 concern may degenerate.

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