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Laurent Veron

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# Nonlinear boundary value problems relative to one dimensional heat equation

Laurent Véron\*

*To Julian with high esteem and sincere friendship*

## Abstract

We consider the problem of existence of a solution  $u$  to  $\partial_t u - \partial_{xx} u = 0$  in  $(0, T) \times \mathbb{R}_+$  subject to the boundary condition  $-u_x(t, 0) + g(u(t, 0)) = \mu$  on  $(0, T)$  where  $\mu$  is a measure on  $(0, T)$  and  $g$  a continuous nondecreasing function. When  $p > 1$  we study the set of self-similar solutions of  $\partial_t u - \partial_{xx} u = 0$  in  $\mathbb{R}_+ \times \mathbb{R}_+$  such that  $-u_x(t, 0) + u^p = 0$  on  $(0, \infty)$ .

**Key Words:** Nonlinear heat flux; Singularities; Radon measures; Marcinkiewicz spaces.

**MSC2010:** 35J65, 35L71.

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# Contents

## 1 Introduction

Let  $g : \mathbb{R} \mapsto \mathbb{R}$  be a continuous nondecreasing function. Set  $Q_{\mathbb{R}_+}^T = (0, T) \times \mathbb{R}_+$  for  $0 < T \leq \infty$  and  $\partial_t Q_{\mathbb{R}_+}^T = \overline{\mathbb{R}_+} \times \{0\}$ . The aim of this article is to study the following 1-dimensional heat equation with a nonlinear flux on the parabolic boundary

$$\begin{aligned} u_t - u_{xx} &= 0 && \text{in } Q_{\mathbb{R}_+}^T \\ -u_x(\cdot, 0) + g(u(\cdot, 0)) &= \mu && \text{in } [0, T) \\ u(0, \cdot) &= \nu && \text{in } \mathbb{R}_+, \end{aligned} \quad (1.1)$$

where  $\nu, \mu$  are Radon measures in  $\mathbb{R}_+$  and  $[0, T)$  respectively. A related problem in  $Q_{\mathbb{R}_+}^\infty$  for which there exist explicit solutions is the following,

$$\begin{aligned} u_t - u_{xx} &= 0 && \text{in } Q_{\mathbb{R}_+}^\infty \\ -u_x(t, 0) + |u|^{p-1}u(t, 0) &= 0 && \text{for all } t > 0 \\ \lim_{t \rightarrow 0} u(t, x) &= 0 && \text{for all } x > 0 \end{aligned} \quad (1.2)$$

where  $p > 1$ . Problem (1.2) is invariant under the transformation  $T_k$  defined for all  $k > 0$  by

$$T_k[u](t, x) = k^{\frac{1}{p-1}} u(k^2 t, kx). \quad (1.3)$$

This leads naturally to look for existence of self-similar solutions under the form

$$u_s(t, x) = t^{-\frac{1}{2(p-1)}} \omega\left(\frac{x}{\sqrt{t}}\right). \quad (1.4)$$

Putting  $\eta = \frac{x}{\sqrt{t}}$ ,  $\omega$  satisfies

$$\begin{aligned} -\omega'' - \frac{1}{2}\eta\omega' - \frac{1}{2(p-1)}\omega &= 0 && \text{in } \mathbb{R}_+ \\ -\omega'(0) + |\omega|^{p-1}\omega(0) &= 0 \\ \lim_{\eta \rightarrow \infty} \eta^{\frac{1}{p-1}}\omega(\eta) &= 0. \end{aligned} \quad (1.5)$$

Brezis, Terman and Peletier proved in [4] the existence of a positive strongly singular function satisfying

$$u_t - \Delta u + |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^n \quad (1.6)$$

and vanishing on  $\mathbb{R}^n \setminus \{0\}$ . They called it the *very singular solution*. Their method is based upon the study of an ordinary differential equation with a phase space analysis. A new and more flexible method based upon variational analysis has been provided by [6]. Other singular solutions of (1.6) in different configurations such as boundary singularities have been studied in [11]. We set  $K(\eta) = e^{\eta^2/4}$  and

$$L_K^2(\mathbb{R}_+) = \left\{ \phi \in L_{loc}^1(\mathbb{R}_+) : \int_{\mathbb{R}_+} \phi^2 K dx := \|\phi\|_{L_K^2}^2 < \infty \right\}, \quad (1.7)$$

and, for  $k \geq 1$ ,

$$H_K^k(\mathbb{R}_+) = \left\{ \phi \in L_K^2(\mathbb{R}_+) : \sum_{\alpha=0}^k \left\| \phi^{(\alpha)} \right\|_{L_K^2}^2 := \|\phi\|_{H_K^k}^2 < \infty \right\}. \quad (1.8)$$

We denote by  $\mathcal{E}$  the subset of  $H_K^1(\mathbb{R}_+)$  of weak solutions of (1.5) that is the set of functions satisfying

$$\int_0^\infty \left( \omega' \zeta' - \frac{1}{2(p-1)} \omega \zeta \right) K(\eta) d\eta + (|\omega|^{p-1} \omega \zeta)(0) = 0, \quad (1.9)$$

and by  $\mathcal{E}_+$  the subset of nonnegative solutions. The next result gives the structure of  $\mathcal{E}$ .

**Theorem 1.1** 1- If  $p \geq 2$ , then  $\mathcal{E} = \{0\}$ .

2- If  $1 < p \leq \frac{3}{2}$ , then  $\mathcal{E}_+ = \{0\}$

3- If  $\frac{3}{2} < p < 2$  then  $\mathcal{E} = \{\omega_s, -\omega_s, 0\}$  where  $\omega_s$  is the unique positive solution of (1.5). Furthermore for any  $\epsilon > 0$  there exists  $c_\epsilon > 0$  such that

$$c_\epsilon \eta^{\frac{1}{p-1}-1-\epsilon} \leq e^{\frac{\eta^2}{4}} \omega_s(\eta) \leq c \eta^{\frac{1}{p-1}-1} \quad \text{for all } \eta > 0. \quad (1.10)$$

Whenever it exists the function  $u_s$  defined in (1.4) is the limit, when  $\ell \rightarrow \infty$  of the positive solutions  $u_{\ell\delta_0}$  of

$$\begin{aligned} u_t - u_{xx} &= 0 && \text{in } Q_{\mathbb{R}_+}^\infty \\ -u_x(t, \cdot) + |u|^{p-1}u(t, \cdot) &= \ell\delta_0 && \text{in } [0, T] \\ \lim_{t \rightarrow 0} u(t, x) &= 0 && \text{for all } x \in \mathbb{R}_+. \end{aligned} \quad (1.11)$$

When such function  $u_s$  does not exist the sequence  $\{u_{\ell\delta_0}\}$  tends to infinity. This is a characteristic phenomenon of an underlying fractional diffusion associated to the linear equation

$$\begin{aligned} u_t - u_{xx} &= 0 && \text{in } Q_{\mathbb{R}_+}^\infty \\ -u_x(\cdot, 0) &= \mu && \text{in } [0, \infty) \\ u(0, \cdot) &= 0 && \text{in } \mathbb{R}_+. \end{aligned} \quad (1.12)$$

More generally we consider problem (1.1). We define the set  $\mathbb{X}(Q_{\mathbb{R}_+}^T)$  of test functions by

$$\mathbb{X}(Q_{\mathbb{R}_+}^T) = \left\{ \zeta \in C_c^{1,2}([0, T] \times [0, \infty)) : \zeta_x(t, 0) = 0 \text{ for } t \in [0, T] \right\}. \quad (1.13)$$

**Definition 1.2** Let  $\nu, \mu$  be Radon measures in  $\mathbb{R}_+$  and  $[0, T]$  respectively. A function  $u$  defined in  $\overline{Q_{\mathbb{R}_+}^T}$  and belonging to  $L_{loc}^1(\overline{Q_{\mathbb{R}_+}^T}) \cap L^1(\partial_\ell Q_{\mathbb{R}_+}^T; dt)$  such that  $g(u) \in L^1(\partial_\ell Q_{\mathbb{R}_+}^T; dt)$  is a weak solution of (1.1) if for every  $\zeta \in \mathbb{X}(Q_{\mathbb{R}_+}^T)$  there holds

$$-\int_0^T \int_0^\infty (\zeta_t + \zeta_{xx}) u dx dt + \int_0^T (g(u)\zeta)(t, 0) dt = \int_0^\infty \zeta d\nu(x) + \int_0^T \zeta(t, 0) d\mu(t). \quad (1.14)$$

We denote by  $E(t, x)$  the Gaussian kernel in  $\mathbb{R}_+ \times \mathbb{R}$ . The solution of

$$\begin{aligned} v_t - v_{xx} &= 0 && \text{in } Q_{\mathbb{R}_+}^\infty \\ -v_x &= \delta_0 && \text{in } \overline{\mathbb{R}_+} \\ v(0, \cdot) &= 0 && \text{in } \mathbb{R}_+, \end{aligned} \quad (1.15)$$

has explicit value

$$v(t, x) = E(t, x) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}. \quad (1.16)$$

If  $x, y > 0$  and  $s < t$  we set  $\tilde{E}(t-s, x, y) = E(t-s, x-y) + E(t-s, x+y)$ . When  $\nu \in \mathfrak{M}^b(\mathbb{R}_+)$  and  $\mu \in \mathfrak{M}^b(\overline{\mathbb{R}_+})$  the solution of

$$\begin{aligned} v_t - v_{xx} &= 0 && \text{in } Q_{\mathbb{R}_+}^\infty \\ -v_x(\cdot, 0) &= \mu && \text{in } \overline{\mathbb{R}_+} \\ u(0, \cdot) &= \nu && \text{in } \mathbb{R}_+, \end{aligned} \quad (1.17)$$

is given by

$$\begin{aligned} v_{\nu, \mu}(t, x) &= \int_0^\infty \tilde{E}(t, x, y) d\nu(y) + 2 \int_0^t E(t-s, x) d\mu(s) \\ &= \mathcal{E}_{\mathbb{R}_+}[\nu](t, x) + \mathcal{E}_{\mathbb{R}_+ \times \{0\}}[\mu](t, x) = \mathcal{E}_{Q_{\mathbb{R}_+}^\infty}[(\nu, \mu)](t, x). \end{aligned} \quad (1.18)$$

We prove the following existence and uniqueness result

**Theorem 1.3** *Let  $g : \mathbb{R} \mapsto \mathbb{R}$  be a continuous nondecreasing function such that  $g(0) = 0$ . If  $g$  satisfies*

$$\int_1^\infty (g(s) - g(-s)) s^{-3} ds < \infty, \quad (1.19)$$

*then for any bounded Borel measures  $\nu$  in  $\mathbb{R}_+$  and  $\mu$  in  $(0, T)$ , there exists a unique weak solution  $u := u_{\nu, \mu} \in L^1(Q_{\mathbb{R}_+}^T)$  of (1.1). Furthermore the mapping  $(\nu, \mu) \mapsto u_{\nu, \mu}$  is nondecreasing.*

When  $g(s) = |s|^{p-1}s$ , condition (1.19) is satisfied if

$$0 < p < 2. \quad (1.20)$$

The above result is still valid under minor modifications if  $\mathbb{R}_+$  is replaced by a bounded interval  $I := (a, b)$  and problem (1.1) by

$$\begin{aligned} u_t - u_{xx} &= 0 && \text{in } Q_I^T \\ u_x(\cdot, b) + g(u(\cdot, b)) &= \mu_1 && \text{in } [0, T) \\ -u_x(\cdot, a) + g(u(\cdot, a)) &= \mu_2 && \text{in } [0, T) \\ u(0, \cdot) &= \nu && \text{in } (a, b). \end{aligned} \quad (1.21)$$

where  $\nu, \mu_j$  ( $j = 1, 2$ ) are Radon measures in  $I$  and  $(0, T)$  respectively.

In the last section we present the scheme of the natural extensions of this problem to a multidimensional framework

$$\begin{aligned} u_t - \Delta u &= 0 & \text{in } Q_{\mathbb{R}_+^n}^T \\ -u_{x_n} + g(u) &= \mu & \text{in } \partial_\ell Q_{\mathbb{R}_+^n}^T \\ u(0, \cdot) &= \nu & \text{in } \mathbb{R}_+^n, \end{aligned} \quad (1.22)$$

The construction of solutions with measure data can be generalized but there are some difficulties in the obtention of self-similar solutions. The equation with a source flux

$$\begin{aligned} u_t - \Delta u &= 0 & \text{in } Q_{\mathbb{R}_+^n}^T \\ u_{x_n} + g(u) &= 0 & \text{in } \partial_\ell Q_{\mathbb{R}_+^n}^T \\ u(0, \cdot) &= \nu & \text{in } \mathbb{R}_+^n, \end{aligned} \quad (1.23)$$

has been studied by several authors, in particular Fila, Ishige, Kawakami and Sato [7], [8], [9]. Their main concern deals with global existence of solutions.

## 2 Self-similar solutions

### 2.1 The symmetrization

We define the operator  $\mathcal{L}_K$  in  $C_0^2(\mathbb{R})$  by

$$\mathcal{L}_K(\phi) = -K^{-1}(K\phi)'$$

The operator  $\mathcal{L}_K$  has been thoroughly studied in [6]. In particular

$$\inf \left\{ \int_{-\infty}^{\infty} \phi'^2 K(\eta) \eta : \int_{-\infty}^{\infty} \phi^2 K(\eta) d\eta = 1 \right\} = \frac{1}{2}. \quad (2.1)$$

Furthermore the infimum is achieved by  $\phi_1 = (4\pi)^{-\frac{1}{2}} K^{-1}$  and  $\mathcal{L}_K$  is an isomorphism from  $H_K^1(\mathbb{R})$  onto its dual  $(H_K^1(\mathbb{R}))' \sim H_K^{-1}(\mathbb{R})$ . Finally  $\mathcal{L}_K^{-1}$  is compact from  $L_K^2(\mathbb{R})$  into  $H_K^1(\mathbb{R})$ , which implies that  $\mathcal{L}_K$  is a Fredholm self-adjoint operator with

$$\sigma(\mathcal{L}_K) = \left\{ \lambda_j = \frac{1+j-1}{2} : j = 1, 2, \dots \right\},$$

and

$$\ker(\mathcal{L}_K - \lambda_j I_d) = \text{span} \left\{ \phi_1^{(j)} \right\}.$$

If  $\phi$  is defined in  $\mathbb{R}_+$ ,  $\tilde{\phi}(x) = \phi(-x)$  is the symmetric with respect to 0 while  $\phi^*(x) = -\phi(-x)$  is the antisymmetric with respect to 0. The operator  $\mathcal{L}_K$  restricted to  $\mathbb{R}_+$  is denoted by  $\mathcal{L}_K^+$ . The operator  $\mathcal{L}_K^{+,N}$  with Neumann condition at  $x = 0$  is a Fredholm operator. This is also valid for the operator  $\mathcal{L}_K^{+,D}$  with Dirichlet condition at  $x = 0$ . Hence, if  $\phi$  is an eigenfunction of  $\mathcal{L}_K^{+,N}$ , then  $\tilde{\phi}$  is an eigenfunction of  $\mathcal{L}_K$  in  $L_K^2(\mathbb{R})$ . Similarly, if  $\phi$  is an eigenfunction of  $\mathcal{L}_K^{+,D}$ , then  $\phi^*$  is an eigenfunction of  $\mathcal{L}_K$  in  $L_K^2(\mathbb{R})$ . Conversely, any even (resp. odd) eigenfunction of  $\mathcal{L}_K$  in  $L_K^2(\mathbb{R})$  satisfies Neumann (resp. Dirichlet) boundary condition at  $x = 0$ . Hence its restriction to  $L_K^2(\mathbb{R}_+)$  is an eigenfunction of  $\mathcal{L}_K^{+,N}$

(resp.  $\mathcal{L}_K^{+,D}$ ). Since  $\phi_1^{(j)}$  is even (resp. odd) if and only if  $j$  is even (resp. odd), we derive

$$H_K^{1,0}(\mathbb{R}_+) = \bigoplus_{\ell=1}^{\infty} \text{span} \left\{ \phi_1^{(2\ell+1)} \right\}, \quad (2.2)$$

and

$$H_K^1(\mathbb{R}_+) = \bigoplus_{\ell=0}^{\infty} \text{span} \left\{ \phi_1^{(2\ell)} \right\}. \quad (2.3)$$

If  $\phi \in H_K^1(\mathbb{R}_+)$  implies  $\tilde{\phi} \in H_K^1(\mathbb{R}^n)$ . Furthermore, if  $\phi_1$  is an eigenfunction of  $\mathcal{L}_K^+$  in  $H_K^1(\mathbb{R}_+^n)$  with Neumann boundary condition on  $\partial\mathbb{R}_+^n$  while  $\partial_{x_n}\phi_1$  is an eigenfunction of  $\mathcal{L}_K^+$  in  $H_K^1(\mathbb{R}_+^n)$  with Dirichlet boundary condition on  $\partial\mathbb{R}_+^n$ . We recall below some properties of  $H_K^1(\mathbb{R}_+^n)$  (actually they proved them in [6, Prop. 1.12] with  $H_{K^\beta}^1(\mathbb{R}^n)$  but the proof is valid with  $H_{K^\beta}^1(\mathbb{R}_+^n)$ ) for any  $\beta > 0$ .

$$\begin{aligned} (i) \quad & \phi \in H_{K^\beta}^1(\mathbb{R}_+) \implies K^{\frac{\beta}{2}}\phi \in C^{0,\frac{1}{2}}(\mathbb{R}_+) \\ (ii) \quad & H_{K^\beta}^1(\mathbb{R}_+) \hookrightarrow L_{K^\beta}^2(\mathbb{R}_+) \quad \text{is compact for all } n \geq 1. \end{aligned} \quad (2.4)$$

## 2.2 Proof of Theorem 1.1-(i)-(ii)

Assume  $p \geq 2$ , then  $\frac{1}{2(p-1)} \leq \frac{1}{2}$ . If  $\omega$  is a weak solution, then

$$\int_0^\infty \left( \omega'^2 - \frac{1}{2(p-1)}\omega^2 \right) K d\eta + |\omega|^{p+1}(0) = 0$$

If  $\frac{1}{2} > \frac{1}{2(p-1)}$  we deduce that  $\omega = 0$ . Furthermore, when  $\frac{1}{2} = \frac{1}{2(p-1)}$  then

$$|\omega|^{p+1}(0) = 0.$$

If  $\omega$  is nonzero, it is an eigenfunction of  $\mathcal{L}_K^{+,D}$ . Since the first eigenvalue is 1 it would imply  $1 = \frac{1}{2(p-1)} \leq \frac{1}{2}$ , contradiction.

Assume  $1 < p \leq \frac{3}{2}$  and  $\omega$  is a nonnegative weak solution. We take  $\zeta(\eta) = \eta e^{-\frac{\eta^2}{4}} = -2\phi_1'(\eta)$ , then

$$\int_0^\infty \left( -\zeta'' - \frac{1}{2(p-1)}\zeta \right) \omega K(\eta) d\eta + \zeta'(0)\omega^p(0) = 0$$

Since  $-\zeta'' = \zeta|_{\mathbb{R}_+} > 0$  and  $\zeta'(0) = \phi_1(0) = 1$ , we derive  $\omega\zeta = 0$  if  $1 > \frac{1}{2(p-1)}$  and  $\omega(0) = 0$  if  $1 = \frac{1}{2(p-1)}$ . Hence  $\omega'(0) = 0$  by the equation and  $\omega \equiv 0$  by the Cauchy-Lipschitz theorem.  $\square$

## 2.3 Proof of Theorem 1.1-(iii)

We define the following functional on  $H_K^1(\mathbb{R}_+^n)$

$$J(\phi) = \frac{1}{2} \int_0^\infty \left( \phi'^2 - \frac{1}{2(p-1)}\phi^2 \right) K d\eta + \frac{1}{p+1} |\phi(0)|^{p+1}. \quad (2.5)$$

**Lemma 2.1** *The functional  $J$  is lower semicontinuous in  $H_K^1(\mathbb{R}_+)$ . It tends to infinity at infinity and achieves negative values.*

*Proof.* We write

$$J(\psi) = J_1(\psi) - J_2(\psi) = J_1(\psi) - \frac{1}{2(p-1)} \|\psi\|_{L_K^2}^2.$$

Clearly  $J_1$  is convex and  $J_2$  is continuous in the weak topology of  $H_K^1(\mathbb{R}_+)$  since the imbedding of  $H_K^1(\mathbb{R}_+)$  into  $L_K^2(\mathbb{R}_+)$  is compact. Hence  $J$  is weakly semicontinuous in  $H_K^1(\mathbb{R}_+)$ .

Let  $\epsilon > 0$ , then

$$J(\epsilon\phi_1) = \left( \frac{1}{4} - \frac{1}{4(p-1)} \right) \frac{\epsilon^2 \sqrt{\pi}}{2} + \frac{\epsilon^{p+1}}{p+1}.$$

Since  $1 < p < 2$ ,  $\frac{1}{4} - \frac{1}{4(p-1)} < 0$ . Hence  $J(\epsilon\phi_1) < 0$  for  $\epsilon$  small enough, thus  $J$  achieves negative values on  $H_K^1(\mathbb{R}_+)$ .

If  $\psi \in H_K^1(\mathbb{R}_+)$  it can be written under the form  $\psi = a\phi_1 + \psi_1$  where  $a \in \mathbb{R}$  and  $\psi_1 \in H_K^{1,0}(\mathbb{R}_+)$ . Hence, for any  $\epsilon > 0$ ,

$$\begin{aligned} J(\psi) &= \frac{1}{2} \int_0^\infty \left( \psi_1'^2 - \frac{1}{2(p-1)} \psi_1^2 \right) K d\eta + \frac{a^2}{2} \int_0^\infty \left( \phi_1'^2 - \frac{1}{2(p-1)} \phi_1^2 \right) K d\eta \\ &\quad + a \int_0^\infty \left( \psi_1' \phi_1' - \frac{1}{2(p-1)} \psi_1 \phi_1 \right) K d\eta + \frac{1}{p+1} |a|^{p+1} \\ &\geq \frac{2p-3}{4(p-1)} \int_0^\infty \psi_1'^2 K d\eta - \frac{a\epsilon}{2} \int_0^\infty \left( \psi_1'^2 + \frac{1}{2(p-1)} \psi_1^2 \right) K d\eta \\ &\quad + \frac{a^2(p-2)\sqrt{\pi}}{4(p-1)} - \frac{ap\sqrt{\pi}}{4(p-1)\epsilon} + \frac{1}{p+1} |a|^{p+1}. \end{aligned}$$

Note that  $\|\psi\|_{H_K^1}^2 \leq 4 \left( \|\psi_1'\|_{L_K^2}^2 + a^2 \right)$ . Since  $2p-3 > 0$ , we can take  $\epsilon > 0$  small enough in order that

$$\lim_{\|\psi\|_{H_K^1} \rightarrow \infty} J(\psi) = \infty. \quad (2.6)$$

□

*End of the proof of Theorem 1.1-(iii).* By Lemma 2.1 the functional  $J$  achieves its minimum in  $H_K^1(\mathbb{R}_+)$  at some  $\omega_s \neq 0$ , and  $\omega_s$  can be assumed to be non-negative since  $J$  is even. By the strong maximum principle  $\omega_s > 0$ , and by the method used in the proof of [13, Proposition 1] is easy to prove that positive solutions belong to  $H_K^2(\mathbb{R}_+)$ . Assume that  $\tilde{\omega}$  is another positive solution, then

$$\int_0^\infty \left( \frac{(K\omega_s)'}{\omega_s} - \frac{(K\tilde{\omega}_s)'}{\tilde{\omega}_s} \right) (\omega_s^2 - \tilde{\omega}_s^2) d\eta = 0.$$



Integration by parts, easily justified by regularity, yields

$$\begin{aligned}
\int_0^\infty \left( \frac{(K\omega_s)'}{\omega_s} - \frac{(K\tilde{\omega}_s)'}{\tilde{\omega}_s} \right) (\omega_s^2 - \tilde{\omega}_s^2) d\eta &= \left[ K\omega_s' \left( \omega_s - \frac{\tilde{\omega}_s^2}{\omega_s} \right) - K\tilde{\omega}_s' \left( \frac{\omega_s^2}{\tilde{\omega}_s} - \tilde{\omega}_s \right) \right]_0^\infty \\
&\quad - \int_0^\infty \left( \omega_s - \frac{\tilde{\omega}_s^2}{\omega_s} \right)' K\omega_s' d\eta + \int_0^\infty \left( \frac{\omega_s^2}{\tilde{\omega}_s} - \tilde{\omega}_s \right)' K\omega_s' d\eta \\
&= -(\omega_s^{p-1} - \tilde{\omega}_s^{p-1}) (\omega_s^2 - \tilde{\omega}_s^2) (0) \\
&\quad - \int_0^\infty \left( \left( \frac{\omega_s' \tilde{\omega}_s - \omega_s \tilde{\omega}_s'}{\tilde{\omega}_s} \right)^2 + \left( \frac{\omega_s \tilde{\omega}_s' - \tilde{\omega}_s \omega_s'}{\omega_s} \right)^2 \right) d\eta.
\end{aligned}$$

This implies that  $\omega_s = \tilde{\omega}_s$ . The proof of (1.10) is similar as the proof of estimate (2.5) in [11, Theorem 4.1].  $\square$

### 3 Problem with measure data

#### 3.1 The regular problem

Set  $G(r) = \int_0^r g(s) ds$ . We consider the functional  $J$  in  $L^2(\mathbb{R}_+)$  with domain  $D(J) = H^1(\mathbb{R}_+)$  defined by

$$J(u) = \frac{1}{2} \int_0^\infty u_x^2 dx + G(v(0)).$$

It is convex and lower semicontinuous in  $L^2(\mathbb{R}_+)$  and its subdifferential  $\partial J$  satisfies

$$\int_0^\infty \partial J(u) \zeta dx = \int_0^\infty u_x \zeta_x dx + g(u(0)) \zeta(0)$$

for all  $\zeta \in H^1(\mathbb{R}_+)$ . Therefore

$$\int_0^\infty \partial J(u) \zeta dx = - \int_0^\infty u_{xx} \zeta dx + (g(u(0)) - u_x(0)) \zeta(0).$$

Hence

$$\partial J(u) = -u_{xx} \quad \text{for all } u \in D(\partial J) = \{v \in H^1(\mathbb{R}_+) : v_x(0) = g(v(0))\}. \quad (3.1)$$

The operator  $\partial J$  is maximal monotone, hence it generates a semi-group of contractions. Furthermore, for any  $u_0 \in L^2(\mathbb{R}_+)$  and  $F \in L^2(0, T; L^2(L^2(\mathbb{R}_+)))$  there exists a unique strong solution to

$$\begin{aligned}
U_t + \partial J(U) &= F \quad \text{a.e. on } (0, T) \\
U(0) &= u_0
\end{aligned} \quad (3.2)$$

**Proposition 3.1** *Let  $\mu \in H^1(0, T)$  and  $\nu \in L^2(\mathbb{R}_+)$ . Then there exists a unique function  $u \in C([0, T]; L^2(\mathbb{R}_+))$  such that  $\sqrt{t}u_{xx} \in L^2((0, T) \times \mathbb{R}_+)$  which satisfies (3.3). The mapping  $(\mu, \nu) \mapsto u := u_{\mu, \nu}$  is non-decreasing and  $u$  is a weak solution in the sense that it satisfies (1.14).*

*Proof.* Let  $\eta \in C_0^2([0, \infty))$  such that  $\eta(0) = 0$ ,  $\eta'(0) = 1$ . If  $f \in H^1(0, T)$ ,  $\nu \in L^2(\mathbb{R}_+)$ , and  $u$  is a solution of

$$\begin{aligned} u_t - u_{xx} &= 0 && \text{in } Q_{\mathbb{R}_+}^T \\ -u_x(\cdot, 0) + g(u(\cdot, 0)) &= \mu(t) && \text{in } [0, T] \\ u(0, \cdot) &= \nu && \text{in } \mathbb{R}_+, \end{aligned} \quad (3.3)$$

where  $\nu \in L^2(\mathbb{R}_+)$ , the function  $v(t, x) = u(t, x) - \mu(t)\eta(x)$  satisfies

$$\begin{aligned} v_t - v_{xx} &= F && \text{in } Q_{\mathbb{R}_+}^T \\ -v_x(\cdot, 0) + g(v(\cdot, 0)) &= 0 && \text{in } [0, T] \\ v(0, \cdot) &= \nu - \mu(0)\eta && \text{in } \mathbb{R}_+, \end{aligned} \quad (3.4)$$

with  $F(t, x) = -(\mu'(t)\eta(x) + \mu(t)\eta''(x))$ . The proof of the existence follows by using [2, Theorem 3.6].

Next, let  $(\tilde{\mu}, \tilde{\nu}) \in H^1(0, T) \times L^2(\mathbb{R}_+)$  such that  $\tilde{\mu} \leq \mu$  and  $\tilde{\nu} \leq \nu$  and let  $\tilde{u} = u_{\tilde{\mu}, \tilde{\nu}}$ , then

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^\infty (\tilde{u} - u)_+^2 dx + \int_0^\infty (\partial_x(\tilde{u} - u)_+)^2 dx - (\tilde{\mu}(t) - \mu(t)) (\tilde{u}(t, 0) - u(t, 0))_+ \\ + (g(\tilde{u}(t, 0)) - g(u(t, 0))) (\tilde{u}(t, 0) - u(t, 0)) = 0 \end{aligned}$$

Then

$$\int_0^\infty (\tilde{u} - u)_+^2 dx|_{t=0} \implies \int_0^\infty (\tilde{u} - u)_+^2 dx = 0 \quad \text{on } [0, T].$$

We can also use (1.18) to express the solution of (3.3):

$$u(t, x) = \int_0^\infty \tilde{E}(t, x, y)\nu(y)dy + 2 \int_0^t E(t-s, x)(\mu(s) - g(u(s, 0)))ds.$$

In particular, if  $g(0) = 0$ , then

$$|u(t, x)| \leq \int_0^\infty \tilde{E}(t, x, y)|\nu(y)|dy + 2 \int_0^t E(t-s, x)|\mu(s)|ds.$$

The proof of (1.14) follows since  $u$  is a strong solution.  $\square$

Next, we prove that the problem is well-posed if  $\mu \in L^1(0, T)$ .

**Proposition 3.2** *Assume  $\{\nu_n\} \subset C_c(\mathbb{R}_+)$  and  $\{\mu_n\} \subset C^1([0, T])$  are Cauchy sequences in  $L^1(\mathbb{R}_+)$  and  $L^1(0, T)$  respectively. Then the sequence  $\{u_n\}$  of solutions of*

$$\begin{aligned} u_{n,t} - u_{n,xx} &= 0 && \text{in } Q_{\mathbb{R}_+}^T \\ -u_{n,x}(\cdot, 0) + g(u_n(\cdot, 0)) &= \mu_n(t) && \text{in } [0, T] \\ u_n(0, \cdot) &= \nu_n && \text{in } \mathbb{R}_+ \end{aligned} \quad (3.5)$$

*converges in  $C([0, T]; L^1(\mathbb{R}_+))$  to a function  $u$  which satisfies (1.14).*

*Proof.* For  $\epsilon > 0$  let  $p_\epsilon$  be an odd  $C^1$  function defined on  $\mathbb{R}$  such that  $p'_\epsilon \geq 0$  and  $p_\epsilon(r) = 1$  on  $[\epsilon, \infty)$ , and put  $j_\epsilon(r) = \int_0^r p_\epsilon(s) ds$ . Then

$$\begin{aligned} & \frac{d}{dt} \int_0^\infty j_\epsilon(u_n - u_m) dx + \int_0^\infty (u_{n,x} - u_{m,x})^2 p'_\epsilon(u_n - u_m) dx \\ & \quad + (g(u_n(t, 0)) - g(u_m(t, 0))) p_\epsilon(u_n(t, 0) - u_m(t, 0)) \\ & = (\mu_n(t) - \mu_m(t)) p_\epsilon(u_n(t, 0) - u_m(t, 0)) \end{aligned}$$

Hence

$$\begin{aligned} & \int_0^\infty j_\epsilon(u_n - u_m)(t, x) dx + (g(u_n(t, 0)) - g(u_m(t, 0))) p_\epsilon(u_n(t, 0) - u_m(t, 0)) \\ & \leq \int_0^\infty j_\epsilon(\nu_n - \nu_m) dx + (\mu_n(t) - \mu_m(t)) p_\epsilon(u_n(t, 0) - u_m(t, 0)) \end{aligned}$$

Letting  $\epsilon \rightarrow 0$ , then  $p_\epsilon \rightarrow \text{sgn}_0$ , hence for any  $t \in [0, T]$ ,

$$\begin{aligned} & \int_0^\infty |u_n - u_m|(t, x) dx + |g(u_n(t, 0)) - g(u_m(t, 0))| \\ & \leq \int_0^\infty |\nu_n - \nu_m| dx + |\mu_n(t) - \mu_m(t)|. \end{aligned} \quad (3.6)$$

Hence  $\{u_n\}$  and  $\{g(u_n(\cdot, 0))\}$  are Cauchy sequences in  $C([0, T]; L^1(\mathbb{R}_+))$  and  $C([0, T])$  respectively with limit  $u$  and  $g(u)$  and  $u = u_{\nu, \mu}$  satisfies (1.14). If we assume that  $(\nu, \tilde{\nu})$  and  $(\mu, \tilde{\mu})$  are couples of elements of  $L^1(\mathbb{R}_+)$  and  $L^1(0, T)$  respectively and if  $u = u_{\nu, \mu}$  and  $\tilde{u} = u_{\tilde{\nu}, \tilde{\mu}}$ , there holds by the above technique,

$$\begin{aligned} & \int_0^\infty |u - \tilde{u}|(t, x) dx + |g(u(t, 0)) - g(\tilde{u}(t, 0))| \\ & \leq \int_0^\infty |\tilde{\nu} - \nu| dx + |\tilde{\mu}(t) - \mu(t)| \quad \text{for all } t \in [0, T]. \end{aligned} \quad (3.7)$$

□

The following lemma is a parabolic version of an inequality due to Brezis.

**Lemma 3.3** *Let  $\nu \in L^1(\mathbb{R}_+)$  and  $\mu \in L^1(0, T)$  and  $v$  be a function defined in  $[0, T] \times \mathbb{R}_+$ , belonging to  $L^1(Q_{\mathbb{R}_+}^T) \cap L^1(\partial_t Q_{\mathbb{R}_+}^T)$  and satisfying*

$$-\int_0^T \int_0^\infty (\zeta_t + \zeta_{xx}) v dx dt = \int_0^T \zeta(\cdot, 0) \mu dt + \int_0^\infty \nu \zeta dx. \quad (3.8)$$

Then for any  $\zeta \in \mathbb{X}(Q_{\mathbb{R}_+}^T)$ ,  $\zeta \geq 0$ , there holds

$$-\int_0^T \int_0^\infty (\zeta_t + \zeta_{xx}) |v| dx dt \leq \int_0^\infty \zeta(\cdot, 0) \text{sign}(v) \mu dt + \int_0^\infty |\nu| \zeta dx. \quad (3.9)$$

Similarly

$$-\int_0^T \int_0^\infty (\zeta_t + \zeta_{xx}) v_+ dx dt \leq \int_0^\infty \zeta(\cdot, 0) \text{sign}_+(v) \mu dt + \int_0^\infty \nu_+ \zeta dx. \quad (3.10)$$

*Proof.* Let  $p_\epsilon$  be the approximation of  $\text{sign}_0$  used in Proposition 3.2 and  $\eta_\epsilon$  be the solution of

$$\begin{aligned} -\eta_{\epsilon t} - \eta_{\epsilon xx} &= p_\epsilon(v) && \text{in } Q_{\mathbb{R}_+}^T \\ \eta_{\epsilon x}(\cdot, 0) &= 0 && \text{in } [0, T] \\ \eta_\epsilon(0, \cdot) &= 0 && \text{in } \mathbb{R}_+. \end{aligned}$$

Then  $|\eta_\epsilon| \leq \eta^*$  where  $\eta^*$  satisfies

$$\begin{aligned} -\eta_t^* - \eta_{xx}^* &= 1 && \text{in } Q_{\mathbb{R}_+}^T \\ \eta_x^*(\cdot, 0) &= 0 && \text{in } [0, T] \\ \eta^*(0, \cdot) &= 0 && \text{in } \mathbb{R}_+. \end{aligned}$$

Although  $\eta_\epsilon$  does not belong to  $\mathbb{X}(Q_{\mathbb{R}_+}^T)$  (it is not in  $C^{1,2}([0, T] \times \mathbb{R}_+)$ ), it is an admissible test function and we deduce that there exists a unique solution to (3.8). Thus  $v$  is given by expression (1.18).

In order to prove (3.9), we can assume that  $\mu$  and  $\nu$  are smooth,  $\zeta \in \mathbb{X}(Q_{\mathbb{R}_+}^T)$ ,  $\zeta \geq 0$  and set  $h_\epsilon = p_\epsilon(v)\zeta$  and  $w_\epsilon = vp_\epsilon(v)$ , then

$$\begin{aligned} \int_0^\infty h_{\epsilon xx} v dx &= \int_0^\infty (2p'_\epsilon(v)v_x \zeta_x + p_\epsilon(v)\zeta_{xx} + \zeta(p_\epsilon(v))_{xx}) v dx \\ &= \int_0^\infty (2vp'_\epsilon(v)v_x \zeta_x - w_{\epsilon x} \zeta_x - (v\zeta)_x (p_\epsilon(v))_x) dx \\ &\quad - \zeta(t, 0)v(t, 0)p'_\epsilon(v(t, 0))v_x(t, 0) \\ &= - \int_0^\infty (\zeta_x (j_\epsilon(v))_x + \zeta p'(v)_\epsilon v_x^2) dx - \zeta(t, 0)v(t, 0)p'_\epsilon(v(t, 0))v_x(t, 0) \\ &= - \int_0^\infty (\zeta p'(v)_\epsilon v_x^2 - j_\epsilon(v)\zeta_{xx}) dx - \zeta(t, 0)v(t, 0)p'_\epsilon(v(t, 0))v_x(t, 0) \end{aligned} \tag{3.11}$$

and

$$\int_0^T h_{\epsilon t} v dt = \int_0^T (p_\epsilon(v)\zeta_t + p'_\epsilon(v)\zeta v_t) v dt \tag{3.12}$$

Since  $v$  is smooth

$$\begin{aligned} 0 &= \int_0^T \int_0^\infty (v_t - v_{xx}) h_\epsilon dx dt \\ &= - \int_0^T \int_0^\infty (h_{\epsilon t} + h_{\epsilon xx}) v dx dt - \int_0^\infty h_\epsilon(0, x) \nu(x) dx \\ &\quad - \int_0^T [p_\epsilon(v(t, 0)) - v(t, 0)p'_\epsilon(v(t, 0))] \zeta(t, 0) \mu(t) dt. \end{aligned}$$

Therefore, using (3.9) and (3.10),

$$\begin{aligned} - \int_0^T \int_0^\infty (j_\epsilon v)\zeta_{xx} + vp_\epsilon(v)\zeta_t dx dt + \int_0^T \int_0^\infty (\zeta p'_\epsilon(v)v_x^2 - vp'_\epsilon(v)v_t \zeta) dx dt \\ = \int_0^\infty h_\epsilon(0, x) \nu(x) dx + \int_0^T h_\epsilon(t, 0) \mu(t) dt \end{aligned} \tag{3.13}$$

Put  $\ell_\epsilon(s) = \int_0^s r p'_\epsilon(r) dr$ , then  $|\ell_\epsilon(s)| \leq c\epsilon^{-1} s^2 \chi_{[-\epsilon, \epsilon]}(s)$ . Since

$$\int_0^T \int_0^\infty \zeta v p'_\epsilon(v) v_t dx dt = - \int_0^\infty \ell_\epsilon(v(0, x)) \zeta(x) dx - \int_0^T \int_0^\infty \zeta_t \ell_\epsilon(v) dx dt,$$

and  $\zeta$  has compact support, it follows

$$\lim_{\epsilon \rightarrow 0} \int_0^T \int_0^\infty \zeta v p'_\epsilon(v) v_t dx dt = 0.$$

Letting  $\epsilon \rightarrow 0$  in (3.13), we derive (3.9) for smooth  $v$ . Using Proposition 3.2 completes the proof of (3.9). The proof of (3.10) is similar.  $\square$

*Remark.* Inequalities (3.9) and (3.10) hold if  $\zeta(t, x)$  does not vanish if  $|x| \geq R$  for some  $R$  but if it satisfies

$$\lim_{x \rightarrow \infty} \sup_{t \in [0, T]} (\zeta(t, x) + |\zeta_x(t, x)|) = 0. \quad (3.14)$$

The proof follows by replacing  $\zeta(t, x)$  by  $\zeta(t, x) \eta_n(x)$  where  $\eta_n \in C_c^\infty(\mathbb{R}_+)$  with  $0 \leq \eta_n \leq 1$ ,  $\eta_n(x) = 1$  on  $[0, n]$ ,  $\eta_n(x) = 0$  on  $[n+1, \infty)$ ,  $|\eta'_n| \leq 2$ ,  $|\eta''_n| \leq 4$ . Then  $\eta_n \zeta \in \mathbb{X}(Q_{\mathbb{R}_+}^T)$  by letting  $n \rightarrow \infty$  and the proof follows by letting  $n \rightarrow \infty$ .

### 3.2 Proof of Theorem 1.3

We give first some *heat-ball* estimates relative to our problem. For  $r > 0$ ,  $x \in \mathbb{R}_+$  and  $t \in \mathbb{R}$  we set

$$e(t, x; r) = \left\{ (s, y) \in (0, T) \times \mathbb{R}_+ : s \leq t, \tilde{E}(t-s, x, y) \geq r \right\}. \quad (3.15)$$

Since

$$e(t, x; r) \subset \left[ t - \frac{1}{4\pi e r^2}, t \right] \times \left[ x - \frac{1}{r\sqrt{\pi e}}, x + \frac{1}{r\sqrt{\pi e}} \right],$$

there holds

$$|e(t, x; r)| \leq \frac{1}{2r^3(\pi e)^{\frac{3}{2}}} \quad (3.16)$$

and if

$$e^*(t; r) = \{s \in (0, T) : s \leq t, E(t-s, 0, 0) \geq r\} \quad (3.17)$$

then we have

$$e^*(t; r) \subset \left[ t - \frac{1}{4\pi e r^2}, t \right] \implies |e^*(t; r)| \leq \frac{1}{4r^2\pi e}. \quad (3.18)$$

If  $G$  is a measured space,  $\lambda$  a positive measure on  $G$  and  $q > 1$ ,  $M^q(G, \lambda)$  is the Marcinkiewicz space of measurable functions  $f : G \mapsto \mathbb{R}$  satisfying for some constant  $c > 0$  and all measurable set  $E \subset G$ ,

$$\int_E |f| d\lambda \leq c(\lambda(E))^{\frac{1}{p}}, \quad (3.19)$$

and

$$\|f\|_{M^q(G, \lambda)} = \inf\{c > 0 \text{ s.t. (3.19) holds}\}.$$

**Lemma 3.4** Assume  $\mu, \nu$  are bounded measure in  $\overline{\mathbb{R}_+}$  and  $\mathbb{R}_+$  respectively and  $u$  is the solution of (1.17) given by (1.18) and  $v_{\nu, \mu}$  is the solution of (1.17). Then

$$\|v_{\nu, \mu}\|_{M^3(Q_{\mathbb{R}_+}^T)} + \left\| v_{\nu, \mu} \lfloor \partial Q_{\mathbb{R}_+}^T \right\|_{M^2(\partial Q_{\mathbb{R}_+}^T)} \leq c \left( \|\mu\|_{\mathfrak{M}(\partial Q_{\mathbb{R}_+}^T)} + \|\nu\|_{\mathfrak{M}(Q_{\mathbb{R}_+}^T)} \right). \quad (3.20)$$

*Proof.* First we consider  $v_{0, \mu}$

$$v_{0, \mu}(t, x) = 2 \int_0^t E(t-s, x) d\mu(s).$$

If  $F \subset [0, T]$  is a Borel set, than for any  $\tau > 0$

$$\begin{aligned} \int_F E(t-s, 0) ds &= \int_{F \cap \{E \leq \tau\}} E(t-s, 0) ds + \int_{F \cap \{E > \tau\}} E(t-s, 0) ds \\ &\leq \tau |F| + \int_{\{E > \tau\}} E(t-s, 0) ds \\ &\leq \tau |F| - \int_{\tau}^{\infty} \lambda d|e^*(t, \lambda)| \\ &\leq \tau |F| + \int_{\tau}^{\infty} \lambda d|e^*(t, \lambda)| \\ &\leq \tau |F| + \frac{1}{4\pi e \tau}. \end{aligned}$$

If we choose  $\tau^2 = \frac{1}{4\pi e |F|}$ , we derive

$$\int_F E(t-s, 0) ds \leq \frac{|F|^{\frac{1}{2}}}{\sqrt{\pi e}}. \quad (3.21)$$

If  $F \subset (0, T)$  is a Borel set then

$$\left| \int_F v_{0, \mu}(t, 0) dt \right| = 2 \left| \int_0^t \int_F E(t-s, 0) dt d\mu(s) \right| \leq \frac{2|F|^{\frac{1}{2}}}{\sqrt{\pi e}} \|\mu\|_{\mathfrak{M}(\partial Q_{\mathbb{R}_+}^T)}.$$

This proves that

$$\left\| v_{0, \mu} \lfloor \partial Q_{\mathbb{R}_+}^T \right\|_{M^2(\partial Q_{\mathbb{R}_+}^T)} \leq c \|\mu\|_{\mathfrak{M}(\partial Q_{\mathbb{R}_+}^T)}. \quad (3.22)$$

Similarly, if  $G \subset [0, T] \times [0, \infty)$  is a Borel set, then

$$\int_G \tilde{E}(t-s, x, 0) ds \leq \frac{2|G|^{\frac{1}{3}}}{\sqrt{\pi e}}, \quad (3.23)$$

and

$$\|v_{0, \mu}\|_{M^3(Q_{\mathbb{R}_+}^T)} \leq c \|\mu\|_{\mathfrak{M}(\partial Q_{\mathbb{R}_+}^T)}. \quad (3.24)$$

In the same way we prove that

$$\|v_{\nu,0}\|_{M^3(Q_{\mathbb{R}_+^T}^T)} + \left\| v_{\nu,0}|_{\partial Q_{\mathbb{R}_+^T}^T} \right\|_{M^2(\partial Q_{\mathbb{R}_+^T}^T)} \leq c \|\nu\|_{\mathfrak{M}(Q_{\mathbb{R}_+^T}^T)}. \quad (3.25)$$

This ends the proof.  $\square$

*Proof of Theorem 1.3*

*Uniqueness.* Assume  $u$  and  $\tilde{u}$  are solutions of (1.1), then  $w = u - \tilde{u}$  satisfies

$$\begin{aligned} w_t - w_{xx} &= 0 && \text{in } Q_{\mathbb{R}_+^T}^T \\ -w_x(\cdot, 0) + g(u(\cdot, 0)) - g(\tilde{u}(\cdot, 0)) &= 0 && \text{in } [0, T) \\ w(0, \cdot) &= 0 && \text{in } \mathbb{R}_+. \end{aligned} \quad (3.26)$$

Applying (3.9), we obtain

$$-\int_0^T \int_0^\infty (\zeta_t + \zeta_{xx})|w| dx dt + \int_0^\infty (g(u(\cdot, 0)) - g(\tilde{u}(\cdot, 0))) \text{sign}(w) \zeta(t, 0) dt \leq 0,$$

for any  $\zeta \in \mathbb{X}_{\mathbb{R}_+^T}^T$  with  $\zeta \geq 0$ . Let  $\theta \in C_c^1(Q_{\mathbb{R}_+^T}^T)$ ,  $\eta \geq 0$ , we take  $\zeta$  to be the solution of

$$\begin{aligned} -\zeta_t - \zeta_{xx} &= \theta && \text{in } (0, T) \times \mathbb{R}_+ \\ \zeta_x(t, 0) &= 0 && \text{in } (0, T) \\ \zeta(T, x) &= 0 && \text{in } (0, \infty). \end{aligned}$$

Then  $\zeta$  satisfies (3.14), hence

$$\int_0^T \int_0^\infty \theta |w| dx dt + \int_0^\infty (g(u(\cdot, 0)) - g(\tilde{u}(\cdot, 0))) \text{sign}(w) \zeta(t, 0) dt \leq 0,$$

This implies  $w = 0$ .

*Existence.* Let  $\{\nu_n\} \subset C_c(\mathbb{R}_+)$  and  $\{\mu_n\} \subset C_c([0, T])$  converging to  $\nu$  and  $\mu$  in the sense of measures and let  $u_n$  be the solution of (3.5). Then from (3.7)

$$\int_0^T \int_0^\infty |u_n| dx dt + \int_0^T |g(u_n(t, 0))| dt \leq T \int_0^\infty |\nu_n| dx + \int_0^T |\mu_n| dt. \quad (3.27)$$

Therefore  $u_n$  and  $g(u_n(\cdot, 0))$  remains bounded respectively in  $L^1(Q_{\mathbb{R}_+^T}^T)$  and in  $L^1(0, T)$ . Furthermore, by Lemma 3.4  $u_n$  remains bounded in  $M^3(Q_{\mathbb{R}_+^T}^T)$  and in  $M^2(\partial Q_{\mathbb{R}_+^T}^T)$ . We can also write  $u_n$  under the form

$$\begin{aligned} u_n(t, x) &= \int_0^\infty \tilde{E}(t, x, y) \mu_n(y) dy + 2 \int_0^t E(t-s, x) (\nu_n(t) - g(u_n(t, 0))) ds \\ &= A_n(t, x) + B_n(t, x). \end{aligned} \quad (3.28)$$

Since we can perform the even reflexion through  $y = 0$ , the mapping

$$(t, x) \mapsto A_n(t, x) := \int_0^\infty \tilde{E}(t, x, y) \mu_n(y) dy$$

is relatively compact in  $C_{loc}^m(\overline{Q_{\mathbb{R}_+^T}^T})$  for any  $m \in \mathbb{N}^*$ . Hence we can extract a subsequence  $\{u_{n_k}\}$  which converges uniformly on every compact subset of  $(0, T] \times [0, \infty)$ , hence a.e. on  $(0, T]$  for the 1-dimensional Lebesgue measure. Concerning the boundary term

$$(t, x) \mapsto B_n(t, x) := \int_0^t E(t-s, x)(\nu_n(t) - g(u_n(t, 0)))ds,$$

it is relatively compact on every compact subset of  $[0, T] \times (0, \infty)$ . If  $x = 0$ , then

$$B_n(t, 0) = \int_0^t (\nu_n(t) - g(u_n(t, 0))) \frac{ds}{\sqrt{\pi(t-s)}}$$

Since  $\|\nu_n(\cdot) - g(u_n(\cdot, 0))\|_{L^1(0, T)}$ ,  $t \mapsto B_n(t, 0)$  is uniformly integrable on  $(0, T)$ , hence relatively compact by the Theorem of Frechet-Kolmogorov. Therefore there exists a subsequence, still denoted by  $\{n_k\}$  such that  $B_{n_k}(t, 0)$  converges for almost all  $t \in (0, T)$ . This implies that the sequence of function  $\{u_{n_k}\}$  defined by (3.28) converges in  $Q_{\mathbb{R}_+^T}^T$  up to a set  $\Theta \cup \Lambda$  where  $\Theta \subset Q_{\mathbb{R}_+^T}^T$  is neglectable for the 2-dimensional Lebesgue measure and  $\Lambda \subset \partial_\ell Q_{\mathbb{R}_+^T}^T$  neglectable for the 1-dimensional Lebesgue measure.

Because of Lemma 3.4,  $(u_{n,k}|_{Q_{\mathbb{R}_+^T}^T}, u|_{\partial_\ell Q_{\mathbb{R}_+^T}^T})$  converges in  $L_{loc}^1(Q_{\mathbb{R}_+^T}^T) \times L^1(\partial_\ell Q_{\mathbb{R}_+^T}^T)$  and the convergence of each of the components holds also almost everywhere (up to a subsequence). Since  $u_{n,k}$  is a weak solution, it satisfies for any  $\zeta \in \mathbb{X}(Q_{\mathbb{R}_+^T}^T)$

$$\begin{aligned} - \int_0^T \int_0^\infty (\zeta_t + \zeta_{xx}) u_{n,k} dx dt + \int_0^T (g(u_{n,k})\zeta)(t, 0) dt \\ = \int_0^\infty \zeta \nu_{n,k}(x) dx + \int_0^T \zeta(t, 0) \mu_{n,k}(t) dt. \end{aligned} \quad (3.29)$$

In order to prove the convergence of  $g(u_{n,k}(t, 0))$ , we use Vitali convergence theorem and the assumption (1.19). Let  $F \subset [0, T]$  be a Borel set. Using the fact that  $u_{n,k} \leq v_{\nu_{n,k}, \mu_{n,k}}$  and the estimate of Lemma 3.4, we have for any  $\lambda > 0$ ,

$$\begin{aligned} \int_F |g(u_{n,k}(t, 0))| dt &\leq \int_{F \cap \{u_{n,k}(t, 0) \leq \lambda\}} |g(u_{n,k}(t, 0))| dt + \int_{\{u_{n,k}(t, 0) > \lambda\}} |g(u_{n,k}(t, 0))| dt \\ &\leq g(\lambda)|F| - \int_\lambda^\infty \sigma d|\{t : |g(u_{n,k}(t, 0))| > \sigma\}| \\ &\leq g(\lambda)|F| + c \int_\lambda^\infty |g(\sigma)| \sigma^{-3} ds, \end{aligned}$$

where  $c$  depends of  $\|\mu\|_{\mathfrak{M}(\partial Q_{\mathbb{R}_+^T}^T)} + \|\nu\|_{\mathfrak{M}(Q_{\mathbb{R}_+^T}^T)}$ . For  $\epsilon > 0$  given, we chose  $\lambda$  large enough so that the integral term above is smaller than  $\epsilon$  and then  $|F|$  such that  $g(\lambda)|F| \leq \epsilon$ . Hence  $\{g(u_{n,k}(\cdot, 0))\}$  is uniformly integrable. Therefore up to a subsequence, it converges to  $g(u(\cdot, 0))$  in  $L^1(0, T)$ . Clearly  $u$  satisfies

$$\begin{aligned} - \int_0^T \int_0^\infty (\zeta_t + \zeta_{xx}) u dx dt + \int_0^T (g(u)\zeta)(t, 0) dt \\ = \int_0^\infty \zeta \nu(x) dx + \int_0^T \zeta(t, 0) \mu(t) dt, \end{aligned} \quad (3.30)$$



which ends the existence proof.

*Monotonicity.* If  $\nu \geq \tilde{\nu}$  and  $\mu \geq \tilde{\mu}$ ; we can choose the approximations such that  $\nu_n \geq \tilde{\nu}_n$  and  $\mu_n \geq \tilde{\mu}_n$ . It follows from (3.10) that  $u_{\nu_n, \mu_n} \geq u_{\tilde{\nu}_n, \tilde{\mu}_n}$ . Choosing the same subsequence  $\{n_k\}$ , the limits  $u, \tilde{u}$  are in the same order. The conclusion follows by uniqueness.  $\square$

### 3.3 The case $g(u) = |u|^{p-1}u$

Condition (1.19) is satisfied if  $p < 2$ . If this condition holds there exists a solution  $u_{\ell\delta_0} = u_{0, \ell\delta_0}$  and the mapping  $\ell \mapsto u_{\ell\delta_0}$  is increasing.

**Theorem 3.5** (i) If  $1 < p \leq \frac{3}{2}$ ,  $u_{\ell\delta_0}$  tends to  $\infty$  when  $k \rightarrow \infty$ .

(ii) If  $\frac{3}{2} < p < 2$ ,  $u_{\ell\delta_0}$  converges to  $U_{\omega_s}$  defined by  $U_{\omega_s}(t, x) = t^{-\frac{1}{2(p-1)}} \omega_s(\frac{x}{\sqrt{t}})$ , when  $k \rightarrow \infty$ .

*Proof.* By uniqueness and using (1.3), there holds

$$T_k[u_{\ell\delta_0}] = u_{\frac{2-p}{k^{p-1}} \ell \delta_0} \quad (3.31)$$

for any  $k, \ell > 0$ . Since  $\ell \mapsto u_{\ell\delta_0}$  is increasing, its limit  $u_\infty$  when  $\ell \rightarrow \infty$  satisfies

$$T_k[u_\infty] = u_\infty. \quad (3.32)$$

Hence  $u_\infty$  is a positive self-similar solution of (1.2), provided it exists. Hence  $u_\infty = U_{\omega_s}$  if  $\frac{3}{2} < p < 2$ . If  $1 < p \leq \frac{3}{2}$ ,  $u_{k\delta_0}$  admits no finite limit when  $k \rightarrow \infty$  which ends the proof.  $\square$

*Remark.* As a consequence of this result, no a priori estimate of Brezis-Friedman type (parabolic Keller-Osserman) exists for a nonnegative function  $u \in C^{2,1}(Q_{\mathbb{R}_+}^\infty \setminus \{(0, 0)\})$  solution of

$$\begin{aligned} u_t - u_{xx} &= 0 && \text{in } Q_{\mathbb{R}_+}^\infty \\ -u_x(\cdot, 0) + |u|^{p-1}u(\cdot, 0) &= 0 && \text{for all } t > 0 \\ u(0, x) &= 0 && \text{for all } x > 0. \end{aligned} \quad (3.33)$$

when  $1 < p \leq \frac{3}{2}$ . When  $\frac{3}{2} < p < 2$  it is expected that

$$u(t, x) \leq \frac{c}{(|x|^2 + t)^{\frac{1}{2(p-1)}}}. \quad (3.34)$$

The type of phenomenon (i) in Theorem 3.5 is characteristic of fractional diffusion. It has already been observed in [5, Theorem 1.3] with equations

$$u_t + (-\Delta)^\alpha + t^\beta u^p = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^N \quad u((0, \cdot)) = k\delta_0 \quad \text{in } \mathbb{R}^N \quad (3.35)$$

when  $0 < \alpha < 1$  is small and  $p > 1$  is close to 1.

## 4 Extension and open problems

The natural extension is to replace a one dimensional domain by a multidimensional one. The main open problem is the question of a priori estimate as stated in the last remark above.

## 4.1 Self-similar solutions

Let  $\eta = (\eta_1, \dots, \eta_n)$  be the coordinates in  $\mathbb{R}^n$  and denote  $\mathbb{R}_+^n = \{\eta = (\eta_1, \dots, \eta_n) = (\eta', \eta_n) : \eta_n > 0\}$ . We set  $K(\eta) = e^{\frac{|\eta|^2}{4}}$  and  $K'(\eta') = e^{\frac{|\eta'|^2}{4}}$ . Similarly to Section 2 we define  $\mathcal{L}_K$  in  $C_0^2(\mathbb{R}^n)$  by

$$\mathcal{L}_K(\phi) = -K^{-1} \operatorname{div}(K \nabla \phi). \quad (4.1)$$

If  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ , we set  $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$ . We denote by  $\phi_1$  the function  $K^{-1}$ . Then the set of eigenvalues of  $\mathcal{L}_K$  is the set of numbers  $\{\lambda_k = \frac{n+k}{2} : k \in \mathbb{N}\}$  with corresponding set of eigenspaces

$$N_k = \operatorname{span} \{D^\alpha \phi_1 : |\alpha| = k\}.$$

The operators  $\mathcal{L}_K^{+,N}$  and  $\mathcal{L}_K^{+,D}$  are defined accordingly in  $H_K^1(\mathbb{R}_+^n)$  and  $H_K^{1,0}(\mathbb{R}_+^n)$  respectively and  $\sigma(\mathcal{L}_K^{+,N}) = \{\frac{n+k}{2} : k \in \mathbb{N}\}$  and  $\sigma(\mathcal{L}_K^{+,D}) = \{\frac{n+k}{2} : k \in \mathbb{N}^*\}$ . Furthermore

$$N_{k,N} = \ker \left( \mathcal{L}_K^{+,N} - \frac{n+k}{2} I_d \right) = \operatorname{span} \{D^\alpha \phi_1 : |\alpha| = k, \alpha_n = 2\ell \text{ with } \ell \in \mathbb{N}\}, \quad (4.2)$$

and

$$N_{k,D} = \ker \left( \mathcal{L}_K^{+,D} - \frac{n+k}{2} I_d \right) = \operatorname{span} \{D^\alpha \phi_1 : |\alpha| = k, \alpha_n = 2\ell + 1 \text{ with } \ell \in \mathbb{N}\}. \quad (4.3)$$

Since  $\mathcal{L}_K^{+,N}$  and  $\mathcal{L}_K^{+,D}$  are Fredholm operators,

$$H_K^1(\mathbb{R}_+^n) = \bigoplus_{k=0}^{\infty} N_{k,N} \quad \text{and} \quad H_K^{1,0}(\mathbb{R}_+^n) = \bigoplus_{k=1}^{\infty} N_{k,D}. \quad (4.4)$$

We define the following functional on  $H_K^1(\mathbb{R}_+^n)$

$$J(\phi) = \frac{1}{2} \int_{\mathbb{R}_+^n} \left( |\nabla \phi|^2 - \frac{1}{2(p-1)} \phi^2 \right) K d\eta + \frac{1}{p+1} \int_{\partial \mathbb{R}_+^n} |\phi|^{p+1} K' d\eta'. \quad (4.5)$$

The critical points of  $J$  satisfies

$$\begin{aligned} -\Delta \omega - \frac{1}{2} \eta \cdot \nabla \omega - \frac{1}{2(p-1)} \omega &= 0 & \text{in } \mathbb{R}_+^n \\ -\omega_{\eta_n} + |\omega|^{p-1} \omega &= 0 & \text{in } \partial \mathbb{R}_+^n \end{aligned} \quad (4.6)$$

If  $\omega$  is a solution of (4.6), the function

$$u_\omega(t, x) = t^{-\frac{1}{2(p-1)}} \omega\left(\frac{x}{\sqrt{t}}\right) \quad (4.7)$$

satisfies

$$\begin{aligned} u_{\omega t} - \Delta u_\omega &= 0 & \text{in } Q_{\mathbb{R}_+^n}^\infty := (0, \infty) \times \mathbb{R}_+^n \\ -u_{\omega x_n} + |u_\omega|^{p-1} u_\omega &= 0 & \text{in } \partial_\ell Q_{\mathbb{R}_+^n}^\infty := (0, \infty) \times \partial \mathbb{R}_+^n. \end{aligned} \quad (4.8)$$

Here we have set  $\mathbb{R}_+^n = \{x = (x_1, \dots, x_n) = (x', x_n) : x_n > 0\}$ . We denote by  $\mathcal{E}$  the subset  $H_K^1(\mathbb{R}_+^n) \cap L^p(\partial \mathbb{R}_+^n; d\eta')$  of solutions of (4.6) and by  $\mathcal{E}_+$  the subset of positive solutions. As for the case  $n = 1$  we have the following non-existence result

**Proposition 4.1** 1- If  $p \geq 1 + \frac{1}{n}$ , then  $\mathcal{E} = \{0\}$ .

2- If  $1 < p \leq 1 + \frac{1}{n+1}$ , then  $\mathcal{E}_+ = \{0\}$

The proof is similar to the one of Theorem 1.1. Hence the existence is to be found in the range  $1 + \frac{1}{n+1} < p < 1 + \frac{1}{n}$ . The surprising result which implies that the critical points of  $J$  cannot directly be looked for by minimization, is the following:

**Proposition 4.2** Assume  $1 + \frac{1}{n+1} < p < 1 + \frac{1}{n}$ , then the functional  $J$  is not bounded from below in  $H_K^1(\mathbb{R}_+^n) \cap L^p(\partial\mathbb{R}_+^n; d\eta')$ .

*Proof.* We set

$$\phi(\eta) = a\phi_1(\eta) + b\phi_{1\eta_n\eta_n}(\eta) = ae^{-\frac{|\eta|^2}{4}} + b\left(\frac{\eta_n^2}{4} - \frac{1}{2}\right)e^{-\frac{|\eta|^2}{4}}.$$

Since  $\phi_1 = K^{-1}$

$$\begin{aligned} J(\phi) &= \left(\frac{n}{4} - \frac{1}{4(p-1)}\right) a^2 \int_{\mathbb{R}_+^n} \phi_1 d\eta \\ &+ \left(\frac{n+2}{4} - \frac{1}{4(p-1)}\right) b^2 \int_{\mathbb{R}_+^n} \left(\frac{\eta_n^2}{4} - \frac{1}{2}\right)^2 \phi_1 d\eta + \frac{1}{p+1} \int_{\mathbb{R}^{n-1}} \left|a - \frac{b}{2}\right|^{p+1} \phi_1^p d\eta' \end{aligned}$$

Since

$$\int_{\mathbb{R}_+^n} \phi_1 d\eta = \frac{(4\pi)^{\frac{n}{2}}}{2}$$

and, by integration by parts,

$$\int_{\mathbb{R}_+^n} \left(\frac{\eta_n^2}{4} - \frac{1}{2}\right)^2 \phi_1 d\eta = \frac{(4\pi)^{\frac{n}{2}}}{4}$$

Then

$$J(\phi) = \left(\left(2n - \frac{2}{p-1}\right) a^2 + \left(n + 2 - \frac{1}{p-1}\right) b^2\right) \frac{(4\pi)^{\frac{n}{2}}}{16} + \frac{1}{p+1} \int_{\mathbb{R}^{n-1}} \left|a - \frac{b}{2}\right|^{p+1} \phi_1^p d\eta'$$

If  $2a = b$ , the boundary term vanishes and

$$J(\phi) = \left(\left(2n - \frac{2}{p-1}\right) a^2 + \left(n + 2 - \frac{1}{p-1}\right) b^2\right) \frac{(4\pi)^{\frac{n}{2}}}{16}$$

Since  $2n - \frac{2}{p-1} < 0$ , the assertion follows by taking  $b^2 \leq \theta a^2$  for some  $\theta > 0$  small enough and letting  $a \rightarrow \infty$ .  $\square$

## 4.2 Problem with measure data

The method for proving Theorem 1.3 can be adapted to prove the following  $n$ -dimensional result

**Theorem 4.3** Let  $g : \mathbb{R} \mapsto \mathbb{R}$  be a nondecreasing continuous function such that  $g(0) = 0$  and

$$\int_1^\infty (g(s) - g(-s))s^{-\frac{2n+1}{n}} ds < \infty, \quad (4.9)$$

then for any bounded Radon measures  $\nu$  in  $\mathbb{R}_+^n$  and  $\mu$  in  $(0, T) \times \partial\mathbb{R}_+^n$ , there exists a unique Borel function  $u := u_{\nu, \mu}$  defined in  $Q_{T^+}^{\mathbb{R}_+^n} := [0, T] \times \mathbb{R}_+^n$  such that  $u \in L^1(Q_T^{\mathbb{R}_+^n})$ ,  $u|_{(0, T) \times \partial\mathbb{R}_+^n} \in L^1((0, T) \times \partial\mathbb{R}_+^n)$  and  $g(u) \in L^1((0, T) \times \partial\mathbb{R}_+^n)$  solution of

$$\begin{aligned} u_t - \Delta u &= 0 && \text{in } Q_{\mathbb{R}_+^n}^T \\ -u_{x_n} + g(u) &= \mu && \text{in } \partial_\ell Q_{\mathbb{R}_+^n}^T \\ u(0, \cdot) &= \nu && \text{in } \mathbb{R}_+^n, \end{aligned} \quad (4.10)$$

in the sense that

$$\iint_{Q_{\mathbb{R}_+^n}^T} (-\partial_t \zeta - \Delta \zeta) u dx dt + \iint_{\partial_\ell Q_{\mathbb{R}_+^n}^T} g(u) \zeta dx' dt = \int_{\mathbb{R}_+^n} \zeta d\nu + \iint_{\partial_\ell Q_{\mathbb{R}_+^n}^T} \zeta d\mu, \quad (4.11)$$

for all  $\zeta \in C_c^{1,2}(\overline{Q_{\mathbb{R}_+^n}^T})$  such that  $\zeta_{x_n} = 0$  on  $(0, T) \times \partial\mathbb{R}_+^n$  and  $\zeta(T, \cdot) = 0$ . Furthermore  $(\nu, \mu) \mapsto u_{\nu, \mu}$  is nondecreasing.

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