



**HAL**  
open science

## Observer design via interconnections of second-order mixed sliding-mode/linear differentiators

Vincent Andrieu, Daniele Astolfi, Pauline Bernard

► **To cite this version:**

Vincent Andrieu, Daniele Astolfi, Pauline Bernard. Observer design via interconnections of second-order mixed sliding-mode/linear differentiators. *International Journal of Robust and Nonlinear Control*, 2020, 31 (9), pp.3631-3657. 10.1002/rnc.5301 . hal-02751929

**HAL Id: hal-02751929**

**<https://hal.science/hal-02751929v1>**

Submitted on 3 Jun 2020

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

**ARTICLE TYPE****Observer design via interconnections of second-order mixed sliding-mode/linear differentiators**Vincent Andrieu\*<sup>1</sup> | Daniele Astolfi<sup>1</sup> | Pauline Bernard<sup>2</sup><sup>1</sup>Univ Lyon, Université Claude Bernard  
Lyon 1, CNRS, LAGEPP UMR 5007,  
Villeurbanne, France<sup>2</sup>MINES ParisTech, Université PSL, Paris,  
France**Correspondence**\*V. Andrieu, Email:  
vincent.andrieu@gmail.com**Present Address**Université Claude Bernard Lyon 1, CNRS,  
LAGEPP UMR 5007, 43 boulevard du 11  
novembre 1918, F-69100, Villeurbanne,  
France**Abstract**

High-gain observers and sliding mode observers are two of the most common techniques to design observers (or differentiators) for lower triangular nonlinear dynamics. While sliding mode observers can handle globally bounded nonlinearities, high-gain linear techniques can deal with globally Lipschitz nonlinearities. To gain in generality and avoid the usual assumption that the plant's solutions are bounded with known bound, we propose here to mix both designs in the more general case where the nonlinearities satisfy a global incremental affine bound. We inspire from the recently-developed low-power high-gain observer technique, which relies on the interconnection of several second order high-gain observers. Adding sliding-mode correction terms into this low-power structure enables to guarantee global convergence of the estimation error in finite-time with gains depending only on the parameters of the incremental affine bound of the nonlinearities. The estimation error is also proved to be uniformly stable along solutions starting from any compact sets of initial conditions.

**KEYWORDS:**

Sliding mode observers, High-gain observers, Homogeneity

**1 | INTRODUCTION**

In this paper our aim is to design a state observer for a system in the form

$$\begin{cases} \dot{x}_1 = x_2 + \phi_1(x_1, t) \\ \dot{x}_2 = x_3 + \phi_2(x_1, x_2, t) \\ \vdots \\ \dot{x}_n = \phi_n(x_1, \dots, x_n, t) \\ y = x_1 \end{cases} \quad (1)$$

where  $x \in \mathbb{R}^n$  is the state,  $y \in \mathbb{R}$  is the measured output, and the functions  $\phi_i : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$  are continuous. This lower triangular form typically arises when considering (uniformly) observable (controlled) nonlinear systems (see<sup>[1]</sup> or more recently<sup>[2]</sup>). Designing an observer for this particular nonlinear dynamical system has been deeply studied in the last three decades. Two main approaches can be distinguished.

The first one assumes global Lipschitz bound conditions on the nonlinearities. More precisely, the functions  $\phi_i$  are supposed to satisfy for  $j = 2, \dots, n$

$$\left| \phi_j(y, x_2, \dots, x_j, t) - \phi_j(y, \hat{x}_2, \dots, \hat{x}_j, t) \right| \leq \ell_1 \sum_{i=2}^j |x_i - \hat{x}_i|, \forall (y, x, \hat{x}, t) \in \mathbb{R}^{2n+2}. \quad (2)$$

In this case, the very popular high-gain approach can be followed, see, for instance, [3], [4], [11], [5] and references therein. The observer gain is then composed of a linear correction term which is amplified by a high-gain parameter that is selected large enough compared to the Lipschitz constant. Following this route, the obtained observer is global and its convergence rate is exponential. An alternative design, based on the interconnection of two-order high-gain observers, has also been recently proposed in [6,7,8]. Note that if the plant's trajectories evolve in a compact set  $C$ , it is enough to check the bound (2) for  $x, \hat{x}$  in  $C$  and saturate  $\phi_j$  outside of  $C$  in the observer.

The second approach, initiated in [9], considers finite time differentiators, see also [10]. Employing set-valued homogeneous correction terms, in [9], a sliding mode observer for (1) is obtained in the particular case in which the functions  $\phi_i$  satisfy

$$\begin{cases} \phi_j(y, x_2, \dots, x_j, t) = \phi_j(y, t), j = 2, \dots, n-1 \\ |\phi_n(y, x_2, \dots, x_n, t) - \phi_n(y, \hat{x}_2, \dots, \hat{x}_n, t)| \leq \ell_0 \end{cases}, \forall (y, \hat{x}, x, t) \in \mathbb{R}^{2n+2}. \quad (3)$$

The observer gain is then a homogeneous set-valued vector field and allows to obtain convergence in finite time. The only constraint on the function  $\phi_n$  is that of being bounded. On the other hand, the other functions  $\phi_j, j < n$  must depend only on known variables<sup>1</sup>. If they depend on unknown state components, but satisfy

$$\left| \phi_j(y, x_2, \dots, x_j, t) - \phi_j(y, \hat{x}_2, \dots, \hat{x}_j, t) \right| \leq \ell_0, \forall (x, y, \hat{x}, t) \in C \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}, j = 2, \dots, n-1, \quad (4)$$

where  $C$  is a known compact set in  $\mathbb{R}^n$ , it is possible to follow the design proposed in [12], consisting of a cascade of second order sliding mode observers, each designed as in [9]. The resulting observer (see also [13,14,11]) has finite time convergence for state solutions of (1) remaining in the compact set  $C$ .

In this paper, we unify these three frameworks into a single global design, by considering the case in which the functions  $\phi_j$  satisfy the following assumption.

**Assumption 1.** *The function  $\phi_1$  is continuous and for each  $j$  in  $2, \dots, n$ , the function  $\phi_j$  is continuous and satisfies a lower triangular incremental affine bound. More precisely, there exist positive real numbers  $\ell_0$  and  $\ell_1$  such that*

$$\left| \phi_j(y, x_2, \dots, x_j, t) - \phi_j(y, \hat{x}_2, \dots, \hat{x}_j, t) \right| \leq \ell_0 + \ell_1 \sum_{i=2}^j |x_i - \hat{x}_i|, \quad \forall (y, x_2, \dots, x_j, \hat{x}_2, \dots, \hat{x}_j, t) \in \mathbb{R}^{2j}. \quad (5)$$

It can be checked that Assumption 1 encompasses nonlinearities satisfying (2), (3) or (4). But none of the existing observers can be applied under Assumption 1. Note that a simple motivation to consider such class of nonlinear systems comes from the fact that for controlled input affine systems, the observability canonical form usually leads to continuous non-lipschitz nonlinearities, see for instance [2].

The objective of this work is therefore that of proposing a new methodology for the state estimation of systems (1) in which each function  $\phi_j$  satisfies Assumption 1. For this, we will follow the interconnection design of second order high-gain observers, *low-power high-gain observer*, proposed in [6], in which each block will be replaced with a more general second order sliding-mode observer, also called *generalized super-twisting algorithm*, as proposed in [15], where both sliding-mode and linear corrections terms are mixed. The motivation is indeed to handle both constant and Lipschitz bounds in the nonlinearities according to (5). As explained in [15], mixing linear and sliding-mode correction terms destroys in general the homogeneity of the system and homogeneous Lyapunov functions can no longer be used, although some exception can be found in [11,16]. Some non-homogeneous Lyapunov functions have nevertheless been designed in [17,18] for the second-order generalized super-twisting algorithm. But in this paper, because of the presence of perturbations verifying (5) for each  $i = 1, \dots, n$ , we cannot follow the aforementioned approaches. In particular, the main proofs follow the *bi-homogeneity* paradigm developed in [19]. Indeed, the super-twisting algorithm somehow contains two homogeneities: the linear one, dominating and accelerating convergence when the error is large, and the sliding mode one, dominating and providing finite-time convergence when the error becomes small. In [19], this property is exploited in the so-called ‘‘homogeneity in the bi-limit framework’’ and provides a constructive design of

<sup>1</sup>This condition may be relaxed by supposing that each function  $\phi_j$  in (1) satisfies an incremental homogeneous bound, see [11]

robust Lyapunov functions. Such a Lyapunov function enables to prove the global finite-time asymptotic stability of the generalized second-order super-twisting algorithm in presence of nonlinearities verifying (5) through high-gain. A non-standard interconnection analysis between the second-order blocks then allows to build a global finite-time observer for the  $n$ th order system (1).

The paper is organized as follows. First, in Section 2, we analyze the generalized second-order high-gain super-twisting observer and we provide an ISS-result with respect to external disturbances, based on a bi-homogeneous Lyapunov function. Then, the observer for the  $n$ -th dimensional system is presented in Section 3 and the main results about convergence and stability of the proposed algorithm are discussed. The proofs are given in Section 4. A numerical example is presented in Section 5 and conclusions are derived in Section 6. Finally, some technical results are postponed to the Appendix.

## Notations/Definitions

$\mathbb{R}$  denotes the real numbers and  $\mathbb{R}_+ = [0, \infty)$ .

- $\text{Sign}$  is the set-valued map defined on  $\mathbb{R}$  by

$$\text{Sign}(s) = \begin{cases} \{-1\} & s < 0 \\ [-1, 1] & s = 0 \\ \{1\} & s > 0 \end{cases} . \quad (6)$$

- for  $(s, a)$  in  $\mathbb{R}^2$ ,  $[s]^a = \text{Sign}(s)|s|^a$ .

## 2 | OBSERVER FOR SECOND ORDER SYSTEMS

### 2.1 | Mixed sliding mode observers

In this section, an observer is designed for system (1) for  $n = 2$  when  $\phi_1$  and  $\phi_2$  satisfy Assumption 1. The system reads as

$$\begin{cases} \dot{x}_1 = x_2 + \phi_1(x_1, t) \\ \dot{x}_2 = \phi_2(x_1, x_2, t) \end{cases} , y = x_1 . \quad (7)$$

The observer we consider is in the form

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + \phi_1(y, t) - Lk_1(\hat{x}_1 - y) \\ \dot{\hat{x}}_2 \in \phi_2(y, \hat{x}_2, t) - L^2k_2(\hat{x}_1 - y) \end{cases} , \quad (8)$$

where  $k_1 : \mathbb{R} \mapsto \mathbb{R}$  is a continuous function,  $k_2 : \mathbb{R} \rightrightarrows \mathbb{R}$  is a set valued map which is outer semi-continuous with convex and compact values<sup>2</sup>, and  $L$  is a positive parameter to be selected large enough and referred to as high-gain parameter, according to standard nomenclature.

Depending on the mappings  $(k_1, k_2)$ , this observer may be the usual high-gain observer or the sliding mode observer.

- If we select

$$k_1(s) := \kappa s , k_2(s) := \kappa s , \forall s \in \mathbb{R} , \quad (9)$$

where  $\kappa$  is a positive real number<sup>3</sup>, this is the usual high-gain observer. Picking  $L$  sufficiently large compared to  $\ell_1$ , the system (8) is an observer for system (7) when  $\phi_2$  satisfies the Lipschitz bound (2).

- If we consider the standard sliding mode observer given by<sup>[9]</sup>, we select

$$k_1(s) := [\kappa s]^{\frac{1}{2}} , k_2(s) := \text{Sign}(s) , \forall s \in \mathbb{R} , \quad (10)$$

where  $\kappa$  is a positive real number selected sufficiently large. Picking  $L$  sufficiently large compared to  $\ell_2$ , (8) is an observer for system (7) when  $\phi_2$  satisfies the bounded assumption (3).

<sup>2</sup>We refer to the definition of semi-outer continuity given in<sup>[20]</sup>.

<sup>3</sup>Actually, a more general choice consists in selecting  $k_i(s) := \kappa_i s$ , with  $\kappa_1, \kappa_2 > 0$ . Without loss of generality from the conceptual point of view, and in order to simplify the proofs, here and in the following we select  $\kappa_1 = \kappa_2 = \kappa$ . Nevertheless, it has to be recalled that the choice of different  $\kappa$ 's may have an important impact on the sensitivity to measurement noise.

- Finally, we can design a mixed sliding mode observer, also denoted as generalized super-twisting algorithm<sup>[15]</sup>, by selecting  $k_1, k_2$  as

$$\begin{cases} k_1(s) := q(\kappa s) \\ k_2(s) := \text{Sign}(s) + q(\kappa s) \end{cases}, \quad q(s) := |s|^{\frac{1}{2}} + s. \quad (11)$$

The last approach is of particular interest since it addresses the case in which  $\phi_2$  satisfies Assumption 1, while the first two methodologies cannot be employed. From there, it yields the following theorem (which is similar to<sup>[21, Theorem 12.1.]</sup>).

**Theorem 1.** *Assume system (7) satisfies Assumption 1. There exists a positive real number  $\kappa^*$  such that for all  $\kappa > \kappa^*$ , there exists  $\underline{L}$ , such that for all  $L > \underline{L}$ , the observer (8)-(11) ensures finite time and stable estimation of system (7). More precisely, the set  $\{x = \hat{x}\}$  is globally and asymptotically stable and there exists a time  $T \geq 0$ , depending on the initial condition, such that*

$$x(t) = \hat{x}(t) \quad \forall t \geq T.$$

Before proving the previous result, we analyse in the following section a disturbed error system, and we give a proposition that is instrumental to the proof of Theorem 1 and also to the  $n$ -th order observer design in Section 3.

## 2.2 | Robustness analysis for a disturbed chain of integrator

In this subsection, the following system is considered

$$\begin{cases} \dot{e}_1 = e_2 - k_1(e_1 + w) + v_1 \\ \dot{e}_2 \in -k_2(e_1 + w) + v_2 \end{cases}, \quad (12)$$

where  $k_1$  and  $k_2$  are given in (11) and where  $v = (v_1, v_2) : \mathbb{R} \mapsto \mathbb{R}^2$ ,  $w : \mathbb{R} \mapsto \mathbb{R}$  are locally integrable time functions. System (12) is obtained by considering the error dynamics  $e = x - \hat{x}$ , with  $x, \hat{x}$  satisfying (7) and (8), respectively,  $L = 1$ ,  $\phi_2 = 0$ , and disturbances  $w, v$  acting on the measured output  $y$  and on the state-dynamics, respectively.

Consider the function  $V : \mathbb{R}^2 \rightarrow \mathbb{R}_+$  defined as

$$V(e_1, e_2) = |e_2|^3 + \int_{q^{-1}(e_2)}^{\kappa e_1} |h|^{\frac{1}{2}} - [q^{-1}(e_2)]^{\frac{1}{2}} + [h]^2 - [q^{-1}(e_2)]^2 dh, \quad (13)$$

where  $q^{-1}(s)$  is the  $C^1$  function satisfying

$$q^{-1}(q(s)) = s.$$

The function  $V$  is well-defined and  $C^1$  (see Section A.2). Moreover, in the following proposition it is shown that  $V$  is an ISS Lyapunov function for system (12).

**Proposition 1.** *There exists  $\kappa^* \geq 0$  such that, for all  $\kappa > \kappa^*$ , there exist positive real numbers  $\underline{c}, \bar{c}, c_V, c_w, c_{v_1}, c_{v_2}$ , such that the function  $V$  defined in (13) satisfies*

$$\underline{c} \left( |e_1|^{\frac{3}{2}} + |e_1|^3 + |e_2|^3 \right) \leq V(e) \leq \bar{c} \left( |e_1|^{\frac{3}{2}} + |e_1|^3 + |e_2|^3 \right), \quad (14)$$

and along the solutions<sup>4</sup> of (12)

$$\dot{V}(e, v, w) \leq -c_V \left[ V(e)^{\frac{2}{3}} + V(e) \right] + c_w [|w| + |w|^3] + c_{v_1} [|v_1|^2 + |v_1|^3] + c_{v_2} V(e)^{\frac{2}{3}} |v_2|. \quad (15)$$

The proof of this proposition is given in Appendix A and relies on the use of homogeneous in the bi-limit framework and the Lyapunov construction introduced in<sup>[22]</sup>. We see that the disturbances  $w, v_1$  and  $v_2$  are treated in a different manner in equation (15). Indeed, for  $c_{v_2} |v_2| < c_V$  the Lyapunov function is decreasing. This highlights the dead-zone property of the stability margin with respect to the disturbance  $|v_2|$ , coming from the sliding mode part of the observer. This crucial property confers to those observers their well-known robustness to bounded disturbance on their last line. However, as opposed to standard sliding mode observer design, the Lyapunov inequality (15) also establishes an ISS property with respect to all disturbances. Finally, note that Proposition 1 imposes a lower bound on  $\kappa$ . This constraints could be removed employing another Lyapunov function as it has been done in<sup>[23]</sup>. With the help of Proposition 1, we can now give the proof of Theorem 1.

<sup>4</sup>Here and all along the paper,  $\dot{V}(e, v, w)$  is the upper right Dini derivative of the function  $V(e)$  along any absolutely continuous solution of the associated dynamical system (here it is related to any solutions of (12)).

### 2.3 | Proof of Theorem 1

Let  $e_1 := \hat{x}_1 - x_1$  and  $e_2 := \hat{x}_2 - x_2$ . Along the solutions of (1)-(8),  $e$  is solution of

$$\begin{cases} \dot{e}_1 = e_2 - Lk_1(e_1), \\ \dot{e}_2 \in \phi_2(x_1, x_2 + e_2, t) - \phi_2(x_1, x_2, t) - L^2k_2(e_1). \end{cases} \quad (16)$$

Previous system can be rewritten as

$$\begin{cases} \frac{1}{L}\dot{e}_1 = \frac{e_2}{L} - k_1(e_1), \\ \frac{1}{L}\frac{\dot{e}_2}{L} \in -k_2(e_1) + \frac{v_2}{L^2}, \end{cases} \quad (17)$$

where

$$v_2 = \phi_2(y, x_2 + e_2, t) - \phi_2(y, x_2, t).$$

System (17) is exactly in the form of system (12) in the coordinates  $(e_1, \frac{e_2}{L})$ , with the perturbation  $v_2$  multiplied by  $L^{-2}$  and with the time scaled by  $\frac{1}{L}$ . Hence, let  $\kappa > \kappa^*$  given by Proposition 1. By using the Lyapunov function  $V$  defined in (13) and the inequality (15) of Proposition 1, we obtain

$$\frac{\dot{V}\left(e_1, \frac{e_2}{L}, v_2\right)}{L} \leq -c_V \left[ V\left(e_1, \frac{e_2}{L}\right)^{\frac{2}{3}} + V\left(e_1, \frac{e_2}{L}\right) \right] + c_{v_2} V\left(e_1, \frac{e_2}{L}\right)^{\frac{2}{3}} \frac{|v_2|}{L^2}. \quad (18)$$

With Assumption 1, it yields

$$\begin{aligned} \frac{|v_2|}{L^2} &\leq \frac{\ell_0}{L^2} + \frac{\ell_1}{L} \frac{|e_2|}{L}, \\ &\leq \frac{\ell_0}{L^2} + \frac{\ell_1}{L\bar{c}^{\frac{1}{3}}} V\left(e_1, \frac{e_2}{L}\right)^{\frac{1}{3}}. \end{aligned}$$

Hence, the former inequality becomes

$$\frac{\dot{V}\left(e_1, \frac{e_2}{L}, v_2\right)}{L} \leq -\left[ c_V - \frac{c_{v_2}\ell_0}{L^2} \right] V\left(e_1, \frac{e_2}{L}\right)^{\frac{2}{3}} - \left[ c_V - \frac{c_{v_2}\ell_1}{L\bar{c}^{\frac{1}{3}}} \right] V\left(e_1, \frac{e_2}{L}\right). \quad (19)$$

Let  $\underline{L} > 0$  be such that

$$\frac{c_{v_2}\ell_0}{\underline{L}^2} \leq \frac{c_V}{2}, \quad \frac{c_{v_2}\ell_1}{\underline{L}\bar{c}^{\frac{1}{3}}} \leq \frac{c_V}{2}.$$

It implies, for any  $L > \underline{L}$ ,

$$\dot{V}\left(e_1, \frac{e_2}{L}, v_2\right) \leq -\frac{c_V}{2} L \left[ V\left(e_1, \frac{e_2}{L}\right)^{\frac{2}{3}} + V\left(e_1, \frac{e_2}{L}\right) \right]. \quad (20)$$

Consequently, by standard Lyapunov arguments, see, e.g.,<sup>[24]</sup>, the result follows.

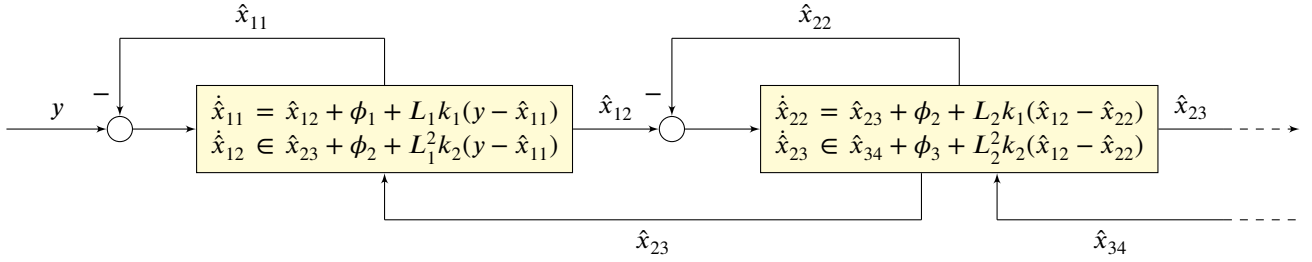
## 3 | OBSERVER FOR $N$ -TH ORDER SYSTEM

We are now interested in the design of an observer for the  $n^{\text{th}}$  order dynamical system (1) in triangular form. As it has been done in<sup>[12]</sup> in the semi-global case (i.e. the state  $x$  of system (1) is supposed to evolve in a known given compact set  $C$ ), the main idea is to employ a cascade of second order observers. However, in order to obtain a global result, we need to interconnect these observers as done in<sup>[6]</sup>. In particular, we propose the following interconnection of  $n - 1$  blocks of second order systems defined as

$$\begin{cases} \dot{\hat{x}}_{ii} &= \hat{x}_{i(i+1)} + \phi_i(y, \hat{x}_{12}, \hat{x}_{23}, \dots, \hat{x}_{(i-1)i}) - L_i k_1(\hat{x}_{ii} - \hat{x}_{(i-1)i}) \\ \dot{\hat{x}}_{i(i+1)} &\in \hat{x}_{(i+1)(i+2)} + \phi_{i+1}(y, \hat{x}_{12}, \hat{x}_{23}, \dots, \hat{x}_{i(i+1)}) - L_i^2 k_2(\hat{x}_{ii} - \hat{x}_{(i-1)i}) \end{cases}, \quad i = 1, \dots, n-1, \quad (21)$$

with the conventions  $\hat{x}_{01} = y$  and  $\hat{x}_{n(n+1)} = 0$ , and where the  $L_i$ 's are positive real numbers that will be selected later on. The overall state dimension of the observer is  $2n - 2$ . The indexes of the observer variables are selected with the convention<sup>5</sup> that  $\hat{x}_{ii}$  is the first state-component of the  $i$ -th block providing an estimate of the variable  $x_i$ , and  $\hat{x}_{i(i+1)}$  is the second state-component of the  $i$ -th block providing an estimate of the variable  $x_{i+1}$ . As a consequence, for each variable  $x_i$ , with  $i = 2, \dots, n - 1$ , we have two different estimates. The structure of observer (21) is depicted in Figure 1 below.

<sup>5</sup>Note that this convention is chosen differently from the one in<sup>[6], [7]</sup>.



**FIGURE 1** Block-diagram representation of the observer (21).

Note that we recover the “step-by-step sliding-mode” algorithm proposed by<sup>[12]</sup> when  $k_1, k_2$  are selected as in (10) and when we don’t put the feedback interconnection term  $\hat{x}_{(i+1)(i+2)}$  in the dynamics of  $\hat{x}_{i(i+1)}$ . On the other hand, we recover the “low-power high-gain observer” design proposed in<sup>[6]</sup>, when  $k_1, k_2$  are selected as in (9). In our context, observer (21) is composed of  $n - 1$  blocks of second order observers, with  $k_1$  a non locally Lipschitz function and  $k_2$  a set-valued correction term, as defined in (11). This set-valued map is outer semi-continuous with convex and compact values, thus ensuring well-posedness and sequential compactness of solutions, see<sup>[20]</sup>.

The two main theorems of this paper are stated and explained next.

### 3.1 | Global finite-time convergence

In the following theorem, it is shown that by selecting the gains  $L_i$  sufficiently large and in an appropriate way, observer (21) ensures finite time convergence of the estimate to the state of the system (1).

**Theorem 2.** *Consider system (1) and suppose Assumption 1 holds. There exist positive real numbers  $(\kappa, L_1, \dots, L_{n-1}) > 0$ , such that observer (21) ensures finite time estimation of system (1), namely there exists a time  $T \geq 0$  such that*

$$\hat{x}_{11}(t) = x_1(t), \hat{x}_{i(i+1)}(t) = \hat{x}_{(i+1)(i+1)}(t) = x_{i+1}(t), \quad i = 1, \dots, n-1, \quad \forall t \geq T. \quad (22)$$

**Proof:** See Section 4.2. □

As opposed to existing finite time results, see, e.g.,<sup>[12], [9]</sup>, we obtain a global finite-time observer. Indeed, no restriction is imposed on the set of initial conditions, nor on the set in which the system (1) is evolving, which may be, in this case, unbounded. Note however that the convergence time  $T$  is not uniform, namely it depends on the initial conditions  $x(0)$  of system (1). It is still an open question to achieve uniform finite time convergence since our approach fails to be applied in this context yet. The proof of Theorem 2 mainly combines the following two iterative arguments:

- Iterative construction of the  $L_i$ ’s starting from  $L_n$  going up to  $L_1$ , similarly to the proof of Proposition 3 in<sup>[8]</sup>, in order to ensure boundedness of solutions starting from  $\hat{x}_{(n-1)n} - x_n$  up to  $\hat{x}_{11} - x_1$ .
- Iterative convergence to zero of the errors starting from  $x_1 - \hat{x}_{11}$  down towards  $x_n - \hat{x}_{(n-1)n}$ .

However, since we have not provided a global Lyapunov function, Theorem 2 concerns only with the attractivity properties of the observer. In order to establish a stability result, a more detailed analysis is required, as detailed in the next section.

### 3.2 | Stability from compact sets of initial conditions

The previous section claimed global finite time convergence of observer (21) for some parameters  $(\kappa, L_1, \dots, L_{n-1})$  appropriately chosen. The following theorem shows that actually once those parameters are fixed, stability of the estimation error is also guaranteed when considering trajectories initialized in compact set of initial conditions.

**Theorem 3.** *Let parameters  $(\kappa, L_1, \dots, L_{n-1})$  be fixed as in Theorem 2, and consider compact sets  $\mathcal{X}_0 \subset \mathbb{R}^n, \hat{\mathcal{X}}_0 \subset \mathbb{R}^{2(n-1)}$ . Then, for any  $\varepsilon > 0$ , there exists  $\eta > 0$  such that for any solution  $x$  of (1) initialized in  $\mathcal{X}_0$  and any solution  $\hat{x}$  of (21) initialized in  $\hat{\mathcal{X}}_0$ , we have the implication*

$$|\hat{x}_{ii}(0) - x_i(0)| + |\hat{x}_{i(i+1)}(0) - x_{i+1}(0)| < \eta \quad \forall i = 1, \dots, n-1 \quad \Rightarrow \quad |\hat{x}_{ii}(t) - x_i(t)| + |\hat{x}_{i(i+1)}(t) - x_{i+1}(t)| < \varepsilon \quad \forall t. \quad (23)$$

**Proof:** See Section 4.3. □

The stability result given in Theorem 3 is stated according to the classical notion of  $\varepsilon$ - $\delta$  stability, see for instance<sup>[25]</sup>. A crucial remark is that the parameters  $(\kappa, L_1, \dots, L_{n-1})$  necessary for design of the observer (21) are independent from the compact sets of initial conditions: they have been fixed in Theorem 2. But because we don't have a global Lyapunov function of the error, we cannot obtain uniform stability of the entire zero-error set

$$\mathcal{E} = \{(x, \hat{x}) \in \mathbb{R}^n \times \mathbb{R}^{2(n-1)} : \hat{x}_{ii} = x_i, \hat{x}_{i(i+1)} = x_{i+1}\}. \quad (24)$$

Instead, we claim uniform stability for the set of solutions starting from compact sets of initial conditions<sup>6</sup>. Note that we do not require them to be bounded. The crucial argument is that for any given compact set of initial conditions the following facts hold.

- The time of convergence  $T$  given by Theorem 2 is uniform.
- Since the estimation error is zero after time  $T$ , we only need to show (23) on the compact time interval  $[0, T]$ .

Therefore, we can consider a compact set  $\mathcal{X}$  of  $\mathbb{R}^n$  such that for all  $x_0 \in \mathcal{X}_0, \hat{x}_0 \in \hat{\mathcal{X}}_0$ ,

$$x(t) \in \mathcal{X} \quad \text{and} \quad (x_1, \hat{x}_{12}, \hat{x}_{23}, \dots, \hat{x}_{(n-1)n})(t) \in \mathcal{X} \quad \forall t \in [0, T].$$

By continuity of  $\phi$ , we can then show that there exists a class- $\mathcal{K}$  function  $\gamma$ , bounded by  $\ell_0$ , such that

$$|\phi_j(y, x_2, \dots, x_j) - \phi_j(y, \hat{x}_2, \dots, \hat{x}_j)| \leq \gamma \left( \sum_{i=1}^j |x_i - \hat{x}_i| \right) + \ell_1 \sum_{i=1}^j |x_i - \hat{x}_i| \quad \forall (x, \hat{x}) \in \mathcal{X} \times \mathcal{X}$$

In other words, the conservative constant  $\ell_0$  in the global bound (5) is replaced in the analysis by  $\gamma \left( \sum_{i=1}^j |x_i - \hat{x}_i| \right)$ , that is bounded by  $\ell_0$ . Unlike  $\ell_0$ , this term encodes the fact that the quantity vanishes when  $(x, \hat{x}) \in \mathcal{E}$  defined in (24), thus rendering a stability proof possible. Finally, note that this proof doesn't follow standard technical arguments for the following main reasons.

- Since the gains  $(\kappa, L_1, \dots, L_{n-1})$  have already been fixed in Theorem 3, the same conditions need to be exploited for stability.
- The complex structure of the interconnections and the fact that we prove stability independently from attractivity, leads us to consider a very particular kind of interconnection for which an unusual ‘‘small-gain-like’’ theorem, see Lemma 2, needs to be established.

## 4 | PROOFS OF MAIN RESULTS

We provide here the full proofs of Theorems 2 and 3. To this end, we start by analysing the behavior of the estimation errors of each block  $i = 1, \dots, n - 1$  in (21).

### 4.1 | Robustness analysis of each error subsystems

The suggested observer gives several error dynamical (sub)systems which interact with each other. By defining  $e_i := (e_{ii}, e_{i(i+1)})$ , with  $e_{ii} := \hat{x}_{ii} - x_i$  and  $e_{i(i+1)} := \hat{x}_{i(i+1)} - x_{i+1}$  for  $i = 1, \dots, n - 1$ , gives

$$\begin{cases} \dot{e}_{ii} = e_{i(i+1)} - L_i k_1 (e_{ii} + w_i) + v_{i1} \\ \dot{e}_{i(i+1)} \in -L_i^2 k_2 (e_{ii} + w_i) + v_{i2} \end{cases}, \quad (25)$$

with

$$\begin{aligned} w_i &= -e_{(i-1)i}, \\ v_{i1} &= \phi_i(y, \dots, \hat{x}_{(i-1)i}) - \phi_i(y, \dots, x_i), \\ v_{i2} &= e_{(i+1)(i+2)} + \phi_{i+1}(y, \hat{x}_{12}, \hat{x}_{23}, \dots, \hat{x}_{i(i+1)}) - \phi_{i+1}(y, \dots, x_{i+1}). \end{aligned}$$

<sup>6</sup>Recall that in the definition of  $\varepsilon$ - $\delta$  stability for dynamical (time-varying) systems, the value of  $\delta$  may be not uniform in the initial time  $t_0$ . When  $\delta$  is independent of  $t_0$ , then we refer to uniform stability. Compare Definitions 7.1 and 7.2 in<sup>[25]</sup>. In a similar way, in our context, the value of  $\eta$  in Theorem 3 is not uniform in the initial conditions, namely it depends on the compact sets  $\mathcal{X}_0, \hat{\mathcal{X}}_0$ .



and the conventions  $e_{01} = 0$ ,  $e_{n(n+1)} = 0$ . Employing (5), it yields

$$|w_i| \leq |e_{(i-1)i}|, \quad (26a)$$

$$|v_{i1}| \leq \ell_0 + \ell_1 \sum_{j=1}^{i-1} |e_{j(j+1)}|, \quad (26b)$$

$$|v_{i2}| \leq |e_{(i+1)(i+2)}| + \ell_0 + \ell_1 \sum_{j=1}^i |e_{j(j+1)}|. \quad (26c)$$

We consider  $n - 1$  Lyapunov functions  $V_i(\mathbf{e}_i)$  defined as

$$V_i(\mathbf{e}_i) := V \left( e_{ii}, \frac{e_{i(i+1)}}{L_i} \right), \quad (27)$$

where  $V$  is defined in (13). With (14), it yields that the functions  $V_i$  satisfy

$$\underline{c} \left( |e_{ii}|^3 + |e_{ii}|^{\frac{3}{2}} + \frac{|e_{i(i+1)}|^3}{L_i^3} \right) \leq V_i(\mathbf{e}_i) \leq \bar{c} \left( |e_{ii}|^3 + |e_{ii}|^{\frac{3}{2}} + \frac{|e_{i(i+1)}|^3}{L_i^3} \right). \quad (28)$$

Applying Proposition 1 in each block  $i$  leads to the following result.

**Proposition 2.** *There exist positive real numbers  $\underline{L}_1, \dots, \underline{L}_{n-1}$ ,  $\tilde{c}_{v_{21}}$ , and  $c$  such that the following holds. For all  $i = 1, \dots, n - 1$ , and all  $L_i \geq \underline{L}_i$ , the time derivative of  $V_i$  along the solutions of (25) satisfies*

$$\begin{aligned} \frac{\dot{V}_i(\mathbf{e}, v_{i1}, v_{i2}, w)}{L_i} &\leq -\frac{c_V}{2} \left[ V_i(\mathbf{e}_i)^{\frac{2}{3}} + V_i(\mathbf{e}_i) \right] + L_{i-1}^3 \tilde{c}_w \left[ V_{i-1}(\mathbf{e}_{i-1})^{\frac{1}{3}} + V_{i-1}(\mathbf{e}_{i-1}) \right] + c c_{v_1} \left[ \frac{\ell_0^2}{L_i^2} + \frac{\ell_0^3}{L_i^3} \right] \\ &\quad + \tilde{c}_{v_{21}} \sum_{j=1}^{i-1} \frac{L_j^3}{L_i^2} \left( V_j(\mathbf{e}_j)^{\frac{2}{3}} + V_j(\mathbf{e}_j) \right) + \left( c_{v_2} \ell_1 \sum_{j=1}^{i-1} \frac{L_j}{L_i^2 \underline{c}^{\frac{1}{3}}} V_j(\mathbf{e}_j)^{\frac{1}{3}} + c_{v_2} \frac{L_{i+1}}{L_i^2 \underline{c}^{\frac{1}{3}}} V_{i+1}(\mathbf{e}_{i+1})^{\frac{1}{3}} \right) V_i(\mathbf{e}_i)^{\frac{2}{3}}, \end{aligned} \quad (29)$$

with the conventions  $V_0 \equiv 0$  and  $V_n \equiv 0$ . Moreover, if there exists a time  $T_{i-1} \geq 0$  such that

$$V_j(\mathbf{e}_j(t)) = 0, j = 1, \dots, i - 1 \quad \text{and} \quad V_{i+1}(\mathbf{e}_{i+1}(t)) < L_i^6 \frac{c_V \underline{c}}{8c_{v_2}^3 L_{i+1}^3}, \forall t \geq T_{i-1}, \quad (30)$$

then there exists  $T_i > T_{i-1}$  such that  $V_i(\mathbf{e}_i(t)) = 0$ ,  $\forall t \geq T_i$ .

The bound (29) describes how the error  $\mathbf{e}_i$  in the block  $i$  is impacted by the errors coming from other blocks. Now, when  $V_j(\mathbf{e}_j(t)) = 0$  for all  $j = 1, \dots, i - 1$ , inequality (29) gives in particular

$$\frac{\dot{V}_i(\mathbf{e}, v_{i1}, v_{i2}, w)}{L_i} \leq -\frac{c_V}{2} \left[ V_i(\mathbf{e}_i)^{\frac{2}{3}} + V_i(\mathbf{e}_i) \right] + c c_{v_1} \left[ \frac{\ell_0^2}{L_i^2} + \frac{\ell_0^3}{L_i^3} \right],$$

which is quite conservative, due to the presence of  $\ell_0$ . This is why we added the second part of the result: actually if (30) holds, we have the stronger property

$$\frac{\dot{V}_i(\mathbf{e}, v_{i1}, v_{i2}, w)}{L_i} \leq -\frac{c_V}{2} \left[ V_i(\mathbf{e}_i)^{\frac{2}{3}} + V_i(\mathbf{e}_i) \right],$$

showing finite-time convergence of  $V_i$ . In other words, if the blocks  $1, \dots, i - 1$  have converged and the error in the  $i$ -th block is sufficiently small, then the  $i$ -th block is also converging. This ‘‘cascade’’ property is crucial for the rest of the proof because the goal will be to show by recursion that (30) holds. This recursion provides also a constructive (but conservative) way to select the parameters  $L_1, \dots, L_{n-1}$ .

It is important to remark that the inequality inside of (30) has to be satisfied after a certain amount of time  $T_{i-1}$  which depends in general on the initial condition. We will show in Section 4.2 that this is indeed verified in order to prove Theorem 2.

**Proof:** First of all, the error system (25) may be rewritten as follows.

$$\begin{cases} \frac{1}{L_i} \dot{e}_{ii} = \frac{e_{i(i+1)}}{L_i} - k_1(e_{ii} + w_i) + \frac{v_{i1}}{L_i}, \\ \frac{1}{L_i} \frac{\dot{e}_{i(i+1)}}{L_i} \in -k_2(e_{ii} + w_i) + \frac{v_{i2}}{L_i^2}, \end{cases}, \quad (31)$$

Following the same steps of the proof of Theorem 1, it can be checked that system (31) is exactly in the form of system (12) in the coordinates  $(e_{ii}, \frac{e_{i(i+1)}}{L_i})$ , with the perturbation  $v_{i1}$  and  $v_{i2}$  multiplied by  $L_i^{-1}$  and  $L_i^{-2}$  and with the time scaled by  $\frac{1}{L_i}$ . Hence, from (15), the time derivative of  $V_i$  along the solutions of (31) satisfies, for all  $L_i > 0$ ,

$$\frac{\dot{V}_i(\mathbf{e}, v_{i1}, v_{i2}, w)}{L_i} \leq -c_V \left[ V_i(\mathbf{e}_i)^{\frac{2}{3}} + V_i(\mathbf{e}_i) \right] + c_w \left[ |w_i| + |w_i|^3 \right] + c_{v_1} \left[ \left( \frac{|v_{i1}|}{L_i} \right)^2 + \left( \frac{|v_{i1}|}{L_i} \right)^3 \right] + c_{v_2} V_i(\mathbf{e}_i)^{\frac{2}{3}} \frac{|v_{i2}|}{L_i^2}, \quad (32)$$

which gives

$$\begin{aligned} \frac{\dot{V}_i(\mathbf{e}, v_{i1}, v_{i2}, w)}{L_i} &\leq -c_V \left[ V_i(\mathbf{e}_i)^{\frac{2}{3}} + V_i(\mathbf{e}_i) \right] + c_w \left[ |e_{(i-1)i}| + |e_{(i-1)i}|^3 \right] \\ &+ c_{v_1} \left[ \left( \frac{\ell_0 + \ell_1 \sum_{j=1}^{i-1} |e_{j(j+1)}|}{L_i} \right)^2 + \left( \frac{\ell_0 + \ell_1 \sum_{j=1}^{i-1} |e_{j(j+1)}|}{L_i} \right)^3 \right] + c_{v_2} V_i(\mathbf{e}_i)^{\frac{2}{3}} \frac{|e_{(i+1)(i+2)}| + \ell_0 + \ell_1 \sum_{j=1}^i |e_{j(j+1)}|}{L_i^2}. \end{aligned} \quad (33)$$

From (28),  $\frac{|e_{(i-1)i}|}{L_i} \leq \frac{1}{\underline{c}^{\frac{1}{3}}} V_{i-1}(\mathbf{e}_{i-1})^{\frac{1}{3}}$ , so we get  $\tilde{c}_w$  such that for all  $L_{i-1} > 1$ ,

$$c_w \left[ |e_{(i-1)i}| + |e_{(i-1)i}|^3 \right] \leq L_{i-1}^3 \tilde{c}_w \left[ V_{i-1}(\mathbf{e}_{i-1})^{\frac{1}{3}} + V_{i-1}(\mathbf{e}_{i-1}) \right], \quad \forall \mathbf{e}_{i-1}.$$

Also, using  $\left( \sum_{j=1}^i a_j \right)^\ell \leq \mathfrak{c} \sum_{j=1}^i a_j^\ell$  for  $\ell = 2$  and  $\ell = 3$  for some positive real number  $\mathfrak{c} > 0$ , it yields

$$\begin{aligned} c_{v_1} \left[ \left( \frac{\ell_0 + \ell_1 \sum_{j=1}^{i-1} |e_{j(j+1)}|}{L_i} \right)^2 + \left( \frac{\ell_0 + \ell_1 \sum_{j=1}^{i-1} |e_{j(j+1)}|}{L_i} \right)^3 \right] \\ \leq \mathfrak{c} c_{v_1} \left[ \frac{\ell_0^2}{L_i^2} + \frac{\ell_0^3}{L_i^3} \right] + \mathfrak{c} c_{v_1} \sum_{j=1}^{i-1} \left[ \frac{\ell_1^2}{L_i^2} |e_{j(j+1)}|^2 + \frac{\ell_1^3}{L_i^3} |e_{j(j+1)}|^3 \right], \end{aligned} \quad (34)$$

and we have for  $L_j > 1, j = 1, \dots, i-1$

$$\mathfrak{c} c_{v_1} \sum_{j=1}^{i-1} \left[ \frac{\ell_1^2}{L_i^2} |e_{j(j+1)}|^2 + \frac{\ell_1^3}{L_i^3} |e_{j(j+1)}|^3 \right] \leq \tilde{c}_{v_1} \sum_{j=1}^{i-1} \frac{L_j^3}{L_i^2} \left( V_j(\mathbf{e}_j)^{\frac{2}{3}} + V_j(\mathbf{e}_j) \right).$$

Moreover,

$$c_{v_2} \frac{|e_{(i+1)(i+2)}| + \ell_0 + \ell_1 \sum_{j=1}^i |e_{j(j+1)}|}{L_i^2} \leq \frac{c_{v_2} \ell_0}{L_i^2} + c_{v_2} \ell_1 \sum_{j=1}^i \frac{L_j}{L_i^2 \underline{c}^{\frac{1}{3}}} V_j(\mathbf{e}_j)^{\frac{1}{3}} + c_{v_2} \frac{L_{i+1}}{L_i^2 \underline{c}^{\frac{1}{3}}} V_{i+1}(\mathbf{e}_{i+1})^{\frac{1}{3}}.$$

All together, it yields

$$\begin{aligned} \frac{\dot{V}_i(\mathbf{e}, v_{i1}, v_{i2}, w)}{L_i} &\leq -c_V \left[ V_i(\mathbf{e}_i)^{\frac{2}{3}} + V_i(\mathbf{e}_i) \right] + L_{i-1}^3 \tilde{c}_w \left[ V_{i-1}(\mathbf{e}_{i-1})^{\frac{1}{3}} + V_{i-1}(\mathbf{e}_{i-1}) \right] \\ &+ \mathfrak{c} c_{v_1} \left[ \frac{\ell_0^2}{L_i^2} + \frac{\ell_0^3}{L_i^3} \right] + \tilde{c}_{v_1} \sum_{j=1}^{i-1} \frac{L_j^3}{L_i^2} \left( V_j(\mathbf{e}_j)^{\frac{2}{3}} + V_j(\mathbf{e}_j) \right) \\ &+ \left( \frac{c_{v_2} \ell_0}{L_i^2} + c_{v_2} \ell_1 \sum_{j=1}^i \frac{L_j}{L_i^2 \underline{c}^{\frac{1}{3}}} V_j(\mathbf{e}_j)^{\frac{1}{3}} + c_{v_2} \frac{L_{i+1}}{L_i^2 \underline{c}^{\frac{1}{3}}} V_{i+1}(\mathbf{e}_{i+1})^{\frac{1}{3}} \right) V_i(\mathbf{e}_i)^{\frac{2}{3}}. \end{aligned} \quad (35)$$

Picking  $\underline{L}_i > 0$  sufficiently large, this yields for all  $L_i > \underline{L}_i$ , inequality (29) is satisfied.

Finally, assume there exists a time  $T_{i-1} \geq 0$  such that

$$V_j(\mathbf{e}_j(t)) = 0, \quad j = 1, \dots, i-1, \quad \forall t \geq T_{i-1}. \quad (36)$$

Then, going back in the computations, we actually have for all  $t \geq T_{i-1}$ ,

$$\begin{aligned} \frac{\dot{V}_i(\mathbf{e}, v_{i1}, v_{i2}, w)}{L_i} &\leq -c_V \left[ V_i(\mathbf{e}_i)^{\frac{2}{3}} + V_i(\mathbf{e}_i) \right] + c_{v_2} V_i(\mathbf{e}_i)^{\frac{2}{3}} \frac{|e^{(i+1)(i+2)}|}{L_i^2} \\ &\leq -c_V \left[ V_i(\mathbf{e}_i)^{\frac{2}{3}} + V_i(\mathbf{e}_i) \right] + c_{v_2} \frac{L_{i+1}}{L_i^2 \underline{c}^{\frac{1}{3}}} V_{i+1}(\mathbf{e}_{i+1})^{\frac{1}{3}} V_i(\mathbf{e}_i)^{\frac{2}{3}} \end{aligned}$$

and if besides,  $V_{i+1}(\mathbf{e}_{i+1}(t)) \leq L_i^6 \frac{c_V^3 \underline{c}}{8c_{v_2}^3 L_{i+1}^3}$ ,

$$\frac{\dot{V}_i(\mathbf{e}, v_{i1}, v_{i2}, w)}{L_i} \leq -\frac{c_V}{2} \left[ V_i(\mathbf{e}_i)^{\frac{2}{3}} + V_i(\mathbf{e}_i) \right].$$

Hence  $V_i$  converges in finite time to 0 with a time of convergence that depends on  $|e(T_{i-1})|$ .  $\square$

## 4.2 | Proof of Theorem 2 and selection of the parameters $L_i$ 's

Let  $W_\ell : \mathbb{R}^{2(n-\ell)} \rightarrow \mathbb{R}_+$  for  $\ell = 1, \dots, n-1$  be the  $C^1$  functions defined as

$$W_\ell(\mathbf{e}) = \sum_{i=\ell}^{n-1} L_i^3 V_i(\mathbf{e}_i), \quad \ell = 1, \dots, n-1. \quad (37)$$

This yields

$$\dot{W}_\ell(\mathbf{e}) \leq \sum_{i=\ell}^{n-1} -\frac{c_V}{2} L_i^4 \left[ V_i(\mathbf{e}_i)^{\frac{2}{3}} + V_i(\mathbf{e}_i) \right] + \sum_{k=1}^5 \Upsilon_k + \Omega_\ell, \quad (38)$$

where (skipping the dependence of  $\Upsilon$  in  $\ell$ )

$$\begin{aligned} \Upsilon_1 &= \sum_{i=\ell}^{n-1} L_i^2 \bar{c}_{v_2} L_{i+1} V_i(\mathbf{e}_i)^{\frac{2}{3}} V_{i+1}(\mathbf{e}_{i+1})^{\frac{1}{3}}, \\ \Upsilon_2 &= \sum_{i=\ell+1}^{n-1} L_i^4 L_{i-1}^3 \bar{c}_w \left[ V_{i-1}(\mathbf{e}_{i-1})^{\frac{1}{3}} + V_{i-1}(\mathbf{e}_{i-1}) \right], \\ \Upsilon_3 &= \sum_{i=\ell}^{n-1} L_i^2 \bar{c}_{v_1} (\ell_0^2 + \ell_0^3), \\ \Upsilon_4 &= \sum_{i=\ell}^{n-1} L_i^2 \bar{c}_{v_2} \sum_{j=\ell}^{i-1} L_j V_j(\mathbf{e}_j)^{\frac{1}{3}} V_i(\mathbf{e}_i)^{\frac{2}{3}}, \\ \Upsilon_5 &= \bar{c}_{v_1} \sum_{i=\ell}^{n-1} \sum_{j=\ell}^{i-1} L_i^2 L_j^3 \left( V_j(\mathbf{e}_j)^{\frac{2}{3}} + V_j(\mathbf{e}_j) \right). \end{aligned}$$

with  $L_0 = 0$ ,  $L_n = 0$ , and  $\Omega_\ell$  defined as

$$\Omega_\ell = \begin{cases} 0, & \ell = 1 \\ L_i^4 L_{i-1}^3 \bar{c}_w \left[ V_{\ell-1}(\mathbf{e}_{\ell-1})^{\frac{1}{3}} + V_{\ell-1}(\mathbf{e}_{\ell-1}) \right] + \bar{c}_{v_1} \sum_{i=\ell}^{n-1} \sum_{j=1}^{\ell-1} L_i^2 L_j^3 \left( V_j(\mathbf{e}_j)^{\frac{2}{3}} + V_j(\mathbf{e}_j) \right) + \sum_{i=\ell}^{n-1} L_i^2 \bar{c}_{v_2} \sum_{j=1}^{\ell-1} L_j V_j(\mathbf{e}_j)^{\frac{1}{3}} V_i(\mathbf{e}_i)^{\frac{2}{3}}, & \ell > 1 \end{cases}$$

Note that with Young inequality we have for some positive real numbers  $\bar{c}_{v_{12}}$ ,  $\bar{c}_w$ ,  $\bar{c}_{v_{22}}$ , independent of  $L_i$ 's, the following inequalities:

$$\Upsilon_1 \leq \sum_{i=\ell}^{n-1} \frac{c_V}{8} V_{i+1}(\mathbf{e}_{i+1}) + \sum_{i=\ell}^{n-1} \bar{c}_{v_2} V_i(\mathbf{e}_i) L_{i+1}^{\frac{3}{2}} L_i^3,$$

Also

$$\begin{aligned}
\Upsilon_2 &\leq \sum_{i=\ell+1}^{n-1} L_i^4 L_{i-1}^3 \tilde{c}_w V_{i-1}(\mathbf{e}_{i-1}) + \sum_{i=\ell+1}^{n-1} L_i^4 L_{i-1}^3 \tilde{c}_w V_{i-1}(\mathbf{e}_{i-1})^{\frac{1}{3}} \\
&\leq \sum_{i=\ell+1}^{n-1} L_i^4 L_{i-1}^3 \tilde{c}_w V_{i-1}(\mathbf{e}_{i-1}) + \sum_{i=\ell+1}^{n-1} L_i^4 L_{i-1}^3 \tilde{c}_w V_{i-1}(\mathbf{e}_{i-1})^{\frac{2}{3}} + \sum_{i=\ell+1}^{n-1} \frac{L_i^4}{4} L_{i-1}^3 \tilde{c}_w \\
&\leq \sum_{i=\ell+1}^{n-1} \tilde{c}_w L_i^4 L_{i-1}^3 \left[ V_{i-1}(\mathbf{e}_{i-1}) + V_{i-1}(\mathbf{e}_{i-1})^{\frac{2}{3}} \right] + \sum_{i=\ell+1}^{n-1} \frac{L_i^4}{4} L_{i-1}^3 \tilde{c}_w,
\end{aligned}$$

Also,

$$\begin{aligned}
\Upsilon_4 &= \tilde{c}_{v_2} \sum_{i=\ell}^{n-1} \sum_{j=\ell}^{i-1} L_j V_j(\mathbf{e}_j)^{\frac{1}{3}} L_i^2 V_i(\mathbf{e}_i)^{\frac{2}{3}} \\
&\leq \tilde{c}_{v_2} \sum_{i=\ell}^{n-1} \sum_{j=\ell}^{i-1} L_j^3 V_j(\mathbf{e}_j) + L_i^3 V_i(\mathbf{e}_i) \\
&\leq n \tilde{c}_{v_2} \sum_{i=\ell}^{n-1} L_i^3 V_i(\mathbf{e}_i)
\end{aligned}$$

Finally

$$\Upsilon_5 = \tilde{c}_{v_1} \sum_{i=\ell}^{n-1} \left( \sum_{j=i+1}^{n-1} L_j^2 \right) L_i^3 \left( V_i(\mathbf{e}_i)^{\frac{2}{3}} + V_i(\mathbf{e}_i) \right).$$

Hence,

$$\begin{aligned}
\dot{W}_\ell(\mathbf{e}) &\leq \sum_{i=\ell}^{n-1} \left( -\frac{c_V}{2} L_i^4 + \frac{c_V}{8} + \tilde{c}_{v_2} L_{i+1}^{\frac{3}{2}} L_i^3 + \tilde{c}_w L_{i+1}^4 L_i^3 + n \tilde{c}_{v_2} L_i^3 + \tilde{c}_{v_1} \left( \sum_{j=i+1}^{n-1} L_j^2 \right) L_i^3 \right) \left[ V_i(\mathbf{e}_i)^{\frac{2}{3}} + V_i(\mathbf{e}_i) \right] \\
&\quad + \sum_{i=\ell}^{n-1} L_i^2 \tilde{c}_{v_1} (\ell_0^2 + \ell_0^3) + \sum_{i=\ell+1}^{n-1} \frac{L_i^4}{4} L_{i-1}^3 \tilde{c}_w + \Omega_\ell, \quad (39)
\end{aligned}$$

We now define  $L_{n-1} > 1$  sufficiently large such that

$$-\frac{c_V}{2} L_{n-1}^4 + \frac{c_V}{8} + n \tilde{c}_{v_2} L_{n-1}^3 \leq -\frac{c_V}{4} L_{n-1}^4$$

and then, recursively,  $L_{n-2} \geq 1, L_{n-3} \geq 1, \dots, L_1 \geq 1$  sufficiently large such that

$$-\frac{c_V}{2} L_i^4 + \frac{c_V}{8} + L_{i+1}^{\frac{3}{2}} L_i^3 + \tilde{c}_w L_{i+1}^4 L_i^3 + n \tilde{c}_{v_2} L_i^3 + \tilde{c}_{v_1} \left( \sum_{j=i+1}^{n-1} L_j^2 \right) L_i^3 \leq -\frac{c_V}{4} L_i^4 \quad (40a)$$

$$\frac{8}{c_V} \left( \sum_{j=i}^{n-1} L_j^2 \tilde{c}_{v_1} (\ell_0^2 + \ell_0^3) + \sum_{j=i+1}^{n-1} \frac{L_j^4}{4} L_{j-1}^3 \tilde{c}_w \right) < L_i^6 \frac{c_V^3 c}{8 c_{v_2}^3} \quad (40b)$$

by exploiting the domination of  $L_i^4$  in (40a) and of  $L_i^6$  in (40b). Hence, it yields for each  $\ell = 1, \dots, n-1$ ,

$$\dot{W}_\ell(\mathbf{e}) \leq -\frac{c_V}{4} W_\ell(\mathbf{e}) + \sum_{i=\ell}^{n-1} L_i^2 \tilde{c}_{v_1} (\ell_0^2 + \ell_0^3) + \sum_{i=\ell}^{n-1} \frac{L_i^4}{2} L_{i-1}^3 + \Omega_\ell. \quad (41)$$

Take  $\ell > 1$  and assume there exists  $T_{\ell-1} > 0$  such that

$$V_1(\mathbf{e}_1(t)) = V_2(\mathbf{e}_2(t)) = \dots = V_{\ell-1}(\mathbf{e}_{\ell-1}(t)) = 0, \quad \forall t \geq T_{\ell-1} \quad (42)$$

this implies that  $\Omega_\ell = 0$  for all  $t \geq T_{\ell-1}$ . Consequently, this implies that along the solution

$$\dot{W}_\ell(\mathbf{e}) \leq -\frac{c_V}{4} W_\ell(\mathbf{e}) + \sum_{i=\ell}^{n-1} L_i^2 \tilde{c}_{v_1} (\ell_0^2 + \ell_0^3) + \sum_{i=\ell+1}^{n-1} \frac{L_i^4}{2} L_{i-1}^3. \quad (43)$$

With (40b), there exists a time  $\bar{T}_{\ell-1} > 0$  (depending on  $|\mathbf{e}(T_{\ell-1})|$  and the parameters) such that

$$W_{\ell}(\mathbf{e}(t)) \leq \frac{8}{c_V} \left( \sum_{i=\ell}^{n-1} L_i^2 \tilde{c}_{v_1} (\ell_0^2 + \ell_0^3) + \sum_{i=\ell+1}^{n-1} \frac{L_i^4}{4} L_{i-1}^3 \tilde{c}_w \right) \leq L_{\ell}^6 \frac{c_V^3 \ell}{8c_{v_2}^3}, \quad \forall t \geq \bar{T}_{\ell-1} > 0 \quad (44)$$

and therefore

$$V_{\ell+1}(\mathbf{e}_{\ell+1}(t)) \leq L_{\ell}^6 \frac{c_V^3 \ell}{8c_{v_2}^3 L_{\ell+1}^3}, \quad \forall t \geq \bar{T}_{\ell-1} > 0. \quad (45)$$

Consequently, with Proposition 2, there exists a time  $T_{\ell}$  (depending on  $|\mathbf{e}(\bar{T}_{\ell-1})|$  and the parameters) such that

$$V_1(\mathbf{e}_1(t)) = V_2(\mathbf{e}_2(t)) = \dots = V_{\ell}(\mathbf{e}_{\ell}(t)) = 0, \quad \forall t \geq T_{\ell}. \quad (46)$$

Observing that  $\Omega_1 = 0$ , and using the same reasoning for  $\ell = 1$ , we prove that  $V_1(\mathbf{e}_1(t)) = 0$  after some time, depending on  $|\mathbf{e}(0)|$  and the parameters  $(\kappa, L_1, \dots, L_{n-1})$ , and then by recursion, propagate to obtain  $V_i(\mathbf{e}_i(t)) = 0$  for all  $i = 1, \dots, n-1$  after a certain time, whose bound depends on the bound of  $|\mathbf{e}(0)|$  and the parameters  $(\kappa, L_1, \dots, L_{n-1})$ .  $\square$

### 4.3 | Proof of Theorem 3

From Theorem 2, there exists  $T$  such that for any initial conditions  $x_0 \in \mathcal{X}_0, \hat{x}_0 \in \hat{\mathcal{X}}_0$ ,

$$\hat{x}_{ii}(t) = x_i(t) \quad , \quad \hat{x}_{i(i+1)}(t) = x_{i+1}(t) \quad \forall t \geq T, \quad \forall i = 1, \dots, n-1$$

The uniformity of  $T$  with respect to the initial conditions comes from the fact that it only depends on  $\mathbf{e}(0)$  which is bounded for  $x_0 \in \mathcal{X}_0, \hat{x}_0 \in \hat{\mathcal{X}}_0$  and the given gains.

Let us now denote  $\mathcal{X}$  a compact set of  $\mathbb{R}^n$  such that for all  $x_0 \in \mathcal{X}_0, \hat{x}_0 \in \hat{\mathcal{X}}_0$ ,

$$x(t) \in \mathcal{X} \quad \text{and} \quad (x_1, \hat{x}_{12}, \hat{x}_{23}, \dots, \hat{x}_{(n-1)n})(t) \in \mathcal{X} \quad \forall t \in [0, T].$$

By continuity of  $\phi$ , there exists a class- $\mathcal{K}$  function  $\gamma_0$  such that

$$|\phi_j(y, x_2, \dots, x_j) - \phi_j(y, \hat{x}_2, \dots, \hat{x}_j)| \leq \gamma_0 \left( \sum_{i=1}^j |x_i - \hat{x}_i| \right) \quad \forall (x, \hat{x}) \in \mathcal{X} \times \mathcal{X}. \quad (47)$$

Therefore, we actually have

$$\begin{aligned} |\phi_j(y, x_2, \dots, x_j) - \phi_j(y, \hat{x}_2, \dots, \hat{x}_j)| &\leq \min \left\{ \ell_0, \gamma_0 \left( \sum_{i=1}^j |x_i - \hat{x}_i| \right) - \ell_1 \sum_{i=1}^j |x_i - \hat{x}_i| \right\} + \ell_1 \sum_{i=1}^j |x_i - \hat{x}_i| \\ &\leq \gamma \left( \sum_{i=1}^j |x_i - \hat{x}_i| \right) + \ell_1 \sum_{i=1}^j |x_i - \hat{x}_i| \quad \forall (x, \hat{x}) \in \mathcal{X} \times \mathcal{X} \end{aligned} \quad (48)$$

where  $\gamma$  is a class  $\mathcal{K}$  function that is globally bounded by  $\ell_0$ . Now reusing (32) with this new bound leads to

$$\begin{aligned} \frac{\dot{V}_i(\mathbf{e})}{L_i} &\leq -\frac{c_V}{2} \left[ V_i(\mathbf{e}_i)^{\frac{2}{3}} + V_i(\mathbf{e}_i) \right] + L_{i-1}^3 \tilde{c}_w \left[ V_{i-1}(\mathbf{e}_{i-1})^{\frac{1}{3}} + V_{i-1}(\mathbf{e}_{i-1}) \right] \\ &\quad + cc_{v_1} \left[ \frac{\gamma \left( \sum_{j=1}^{i-1} |e_{j(j+1)}| \right)^2}{L_i^2} + \frac{\gamma \left( \sum_{j=1}^{i-1} |e_{j(j+1)}| \right)^3}{L_i^3} \right] + \tilde{c}_{v_1} \sum_{j=1}^{i-1} \frac{L_j^3}{L_i^2} \left( V_j(\mathbf{e}_j)^{\frac{2}{3}} + V_j(\mathbf{e}_j) \right) \\ &\quad + \left( c_{v_2} \ell_1 \sum_{j=1}^{i-1} \frac{L_j}{L_i^2 c_{v_2}^{\frac{1}{3}}} V_j(\mathbf{e}_j)^{\frac{1}{3}} + c_{v_2} \frac{L_{i+1}}{L_i^2 c_{v_2}^{\frac{1}{3}}} V_{i+1}(\mathbf{e}_{i+1})^{\frac{1}{3}} \right) V_i(\mathbf{e}_i)^{\frac{2}{3}}, \end{aligned} \quad (49)$$

for all  $t \in [0, T]$ , namely

$$\begin{aligned} \frac{\dot{V}_i(\mathbf{e})}{L_i} &\leq -\frac{c_V}{2} \left[ V_i(\mathbf{e}_i)^{\frac{2}{3}} + V_i(\mathbf{e}_i) \right] + L_{i-1}^3 \tilde{c}_w \left[ V_{i-1}(\mathbf{e}_{i-1})^{\frac{1}{3}} + V_{i-1}(\mathbf{e}_{i-1}) \right] \\ &\quad + cc_{v_1} \left[ \frac{\gamma \left( \sum_{j=1}^{i-1} L_j V_j(\mathbf{e}_j)^{\frac{1}{3}} \right)^2}{L_i^2} + \frac{\gamma \left( \sum_{j=1}^{i-1} L_j V_j(\mathbf{e}_j)^{\frac{1}{3}} \right)^3}{L_i^3} \right] + \tilde{c}_{v_1} \sum_{j=1}^{i-1} \frac{L_j^3}{L_i^2} \left( V_j(\mathbf{e}_j)^{\frac{2}{3}} + V_j(\mathbf{e}_j) \right) \\ &\quad + \left( c_{v_2} \ell_1 \sum_{j=1}^{i-1} \frac{L_j}{L_i^2 c^{\frac{1}{3}}} V_j(\mathbf{e}_j)^{\frac{1}{3}} + c_{v_2} \frac{L_{i+1}}{L_i^2 c^{\frac{1}{3}}} V_{i+1}(\mathbf{e}_{i+1})^{\frac{1}{3}} \right) V_i(\mathbf{e}_i)^{\frac{2}{3}}. \end{aligned} \quad (50)$$

Hence, employing (40a), the function  $W_\ell$  defined in (37) satisfy for each  $\ell = 1, \dots, n-1$ , and for  $t \in [0, T]$

$$\begin{aligned} \dot{W}_\ell(\mathbf{e}) &\leq -\frac{c_V}{4} W_\ell(\mathbf{e}) + \sum_{i=\ell}^{n-1} L_i^2 \tilde{c}_{v_1} \left( \gamma \left( \sum_{j=1}^{i-1} L_j V_j(\mathbf{e}_j)^{\frac{1}{3}} \right)^2 + \gamma \left( \sum_{j=1}^{i-1} L_j V_j(\mathbf{e}_j)^{\frac{1}{3}} \right)^3 \right) \\ &\quad + \sum_{i=\ell+1}^{n-1} L_i^4 L_{i-1}^3 \tilde{c}_w \left( V_{i-1}(\mathbf{e}_{i-1})^{\frac{1}{3}} - V_{i-1}(\mathbf{e}_{i-1})^{\frac{2}{3}} \right) + \Omega_\ell(V_1(\mathbf{e}_1), \dots, V_{\ell-1}(\mathbf{e}_{\ell-1})) \\ &\leq -\frac{c_V}{4} W_\ell(\mathbf{e}) + \pi_\ell(V_1(\mathbf{e}_1), \dots, V_{n-2}(\mathbf{e}_{n-2})) + \Omega_\ell(V_1(\mathbf{e}_1), \dots, V_{\ell-1}(\mathbf{e}_{\ell-1})). \end{aligned} \quad (51)$$

where  $\pi_\ell$  is globally bounded by

$$\pi_\ell(V_1(\mathbf{e}_1), \dots, V_{n-2}(\mathbf{e}_{n-2})) \leq \sum_{i=\ell}^{n-1} L_i^2 c_{v_1} (\ell_0^2 + \ell_0^3) + \sum_{i=\ell+1}^{n-1} L_i^4 L_{i-1}^3 \frac{\tilde{c}_w}{4} < L_\ell^6 \frac{c_V^4 c}{64 c^3 v_{12}} \quad (52)$$

according to (40b), and where we recall that  $\Omega_\ell$  is a continuous map that vanishes when  $(V_1(\mathbf{e}_1), \dots, V_{\ell-1}(\mathbf{e}_{\ell-1})) = 0$  for  $\ell > 1$  and  $\Omega_1 = 0$ . Similarly, (50) can be rewritten as

$$\begin{aligned} \frac{\dot{V}_i(\mathbf{e})}{L_i} &\leq -\frac{c_V}{2} \left[ V_i(\mathbf{e}_i)^{\frac{2}{3}} + V_i(\mathbf{e}_i) \right] + c_{v_2} \frac{L_{i+1}}{L_i^2 c^{\frac{1}{3}}} V_{i+1}(\mathbf{e}_{i+1})^{\frac{1}{3}} V_i(\mathbf{e}_i)^{\frac{2}{3}} + \Omega'_i(V_1(\mathbf{e}_1), \dots, V_{i-1}(\mathbf{e}_{i-1})) \\ &\leq -\frac{c_V}{2} \left[ V_i(\mathbf{e}_i)^{\frac{2}{3}} + V_i(\mathbf{e}_i) \right] + c_{v_2} \frac{1}{L_i^2 c^{\frac{1}{3}}} W_i(\mathbf{e})^{\frac{1}{3}} V_i(\mathbf{e}_i)^{\frac{2}{3}} + \Omega'_i(V_1(\mathbf{e}_1), \dots, V_{i-1}(\mathbf{e}_{i-1})) \end{aligned} \quad (53)$$

where  $\Omega'_i$  is a continuous map that vanishes when  $(V_1(\mathbf{e}_1), \dots, V_{i-1}(\mathbf{e}_{i-1})) = 0$  for  $i > 1$  and  $\Omega'_1 = 0$ .

We will now recursively use the small-gain-like technical Lemma 2 on  $(W_i, V_i)$ , starting from  $(W_{n-2}, V_{n-2})$  up to  $(W_1, V_1)$ , in order to prove that  $W_1$  (and thus all the  $V_i$ ) is ISS with respect to the initial error  $\mathbf{e}(0)$ . From there, stability will follow.

Let us start with  $\ell = i = n-2$ . The key property that we are going to exploit is that the derivative of  $W_{n-2}$  in (51) depends only on  $V_i, i = 1, \dots, n-2$ , and not on  $V_{n-1}$ . More precisely, from (51), (52), (53), and by continuity on compact sets, there exist class- $\mathcal{K}$  functions  $\rho_{n-2}, \delta_{n-2}$  such that  $\delta_{n-2}$  verifies

$$\rho_{n-2}(s) < L_{n-2}^6 \frac{c_V^4 c}{64 c^3 v_2} \quad \forall s \quad (54)$$

and for all  $t \in [0, T]$ ,

$$\begin{aligned} \dot{W}_{n-2}(\mathbf{e}) &\leq -\frac{c_V}{4} W_{n-2}(\mathbf{e}) + \rho_{n-2}(V_{n-2}(\mathbf{e}_{n-2})) + \delta_{n-2} \left( \sum_{i=1}^{n-3} V_i(\mathbf{e}_i) \right) \\ \dot{V}_{n-2}(\mathbf{e}) &\leq -\frac{c_V}{2} L_{n-2} \left[ V_{n-2}(\mathbf{e}_{n-2})^{\frac{2}{3}} + V_{n-2}(\mathbf{e}_{n-2}) \right] + c_{v_2} \frac{1}{L_{n-2} c^{\frac{1}{3}}} W_{n-2}(\mathbf{e})^{\frac{1}{3}} V_{n-2}(\mathbf{e}_{n-2})^{\frac{2}{3}} + \delta_{n-2} \left( \sum_{i=1}^{n-3} V_i(\mathbf{e}_i) \right) \end{aligned}$$

To apply Lemma 2, the small-gain condition (B10) writes

$$c_{v_2} \frac{1}{L_{n-2} c^{\frac{1}{3}}} \left( \frac{8}{c_V} \rho_{n-2}(V_{n-2}(\mathbf{e}_{n-2})) \right)^{\frac{1}{3}} < \frac{c_V}{2} L_{n-2}$$

which holds according to (54). Therefore, according to Lemma 2, there exists a class- $\mathcal{K}$  function  $\alpha_{n-2}$  such that for all  $t \in [0, T]$ ,

$$W_{n-2}(\mathbf{e}(t)) \leq \alpha_{n-2} \left( |\mathbf{e}(0)| + \sup_{s \in [0, t]} \sum_{i=1}^{n-3} V_i(\mathbf{e}_i(s)) \right).$$

Now, by recursion, take  $1 \leq \ell \leq n-3$  and assume there exists a class- $\mathcal{K}$  function  $\alpha_{\ell+1}$  such that for all  $t \in [0, T]$ ,

$$W_{\ell+1}(\mathbf{e}(t)) \leq \alpha_{\ell+1} \left( |\mathbf{e}(0)| + \sup_{s \in [0, t]} \sum_{i=1}^{\ell} V_i(\mathbf{e}_i(s)) \right).$$

Then, since  $V_i(\mathbf{e}_i) \leq W_{\ell+1}(\mathbf{e})$  for all  $i \geq \ell+1$ , we deduce from (51), (52), (53), that there exist class- $\mathcal{K}$  functions  $\delta_\ell, \rho_\ell$  such that  $\rho_\ell$  verifies

$$\rho_\ell(s) < L_\ell^6 \frac{c_V^4 c}{64 c^3 v_2} \quad \forall s \in \mathbb{R}_+,$$

and for all  $t \in [0, T]$ ,

$$\begin{aligned} \dot{W}_\ell(\mathbf{e}) &\leq -\frac{c_V}{4} W_\ell(\mathbf{e}) + \rho_\ell \left( \sup_{s \in [0, t]} V_\ell(\mathbf{e}_\ell(s)) \right) + \delta_\ell \left( |\mathbf{e}(0)| + \sup_{s \in [0, t]} \sum_{i=1}^{\ell-1} V_i(\mathbf{e}_i(s)) \right), \\ \dot{V}_\ell(\mathbf{e}) &\leq -\frac{c_V}{2} L_\ell \left[ V_\ell(\mathbf{e}_\ell)^{\frac{2}{3}} + V_\ell(\mathbf{e}_\ell) \right] + c_{v_2} \frac{1}{L_\ell c^{\frac{1}{3}}} W_\ell(\mathbf{e})^{\frac{1}{3}} V_\ell(\mathbf{e}_\ell)^{\frac{2}{3}} + \delta_\ell \left( \sum_{i=1}^{\ell-1} V_i(\mathbf{e}_i) \right). \end{aligned}$$

Applying Lemma 2 again, we thus get that

$$W_\ell(\mathbf{e}(t)) \leq \alpha_\ell \left( |\mathbf{e}(0)| + \sup_{s \in [0, t]} \sum_{i=1}^{\ell-1} V_i(\mathbf{e}_i(s)) \right),$$

for some class- $\mathcal{K}$  function  $\alpha_\ell$ . By recursion, we thus obtain for all  $t \in [0, T]$

$$V_i(\mathbf{e}_i(t)) \leq \frac{1}{L_i^3} W_1(\mathbf{e}(t)) \leq \frac{1}{L_i^3} \alpha_1 (|\mathbf{e}(0)|) \quad \forall i = 1, \dots, n-1,$$

with  $\alpha_1$  of class- $\mathcal{K}$ . It follows that for any  $\varepsilon > 0$ , there exists  $\eta > 0$  such that

$$|\mathbf{e}(0)| \leq \eta \quad \Rightarrow \quad |\mathbf{e}(t)| \leq \varepsilon \quad \forall t \in [0, T].$$

Since  $\mathbf{e}(t) = 0$  for all  $t \geq T$ , we deduce the stability property.  $\square$

## 5 | NUMERICAL EXAMPLE

As an illustration, we consider a controlled two-mass spring damper system on a horizontal plane described by

$$\begin{aligned} \dot{p}_1 &= v_1 \\ m_1 \dot{v}_1 &\in \overline{\Psi}_1(v_1) + \sigma(p_2 - p_1) \\ \dot{p}_2 &= v_2 \\ m_2 \dot{v}_2 &\in \overline{\Psi}_2(v_2) + \sigma(p_1 - p_2) + u(t) \end{aligned} \tag{55}$$

in which  $p_i$  is the position of each system with respect to its equilibrium,  $v_i, m_i$  are its velocity and mass, respectively, and  $u(t)$  is a known external input force. We suppose that the masses are connected by a spring with stiffness coefficient  $\sigma$  and subject to Coulomb frictions described as in<sup>[26,27]</sup> by the set-valued maps  $s \rightrightarrows \overline{\Psi}_i(s)$ ,

$$\overline{\Psi}_i(v) := -\overline{F}_i \text{Sign}(v) - \gamma_i v, \tag{56}$$

in which  $\overline{F}_i$  is the static friction and  $\gamma_i s$  is the viscous contribution with viscous friction coefficient  $\gamma_i$ . More precisely, when  $v = 0$  and the spring force  $\sigma|p_2 - p_1| \leq \overline{F}_1$ , the dry friction compensates exactly the spring force which is in the interval  $[-\overline{F}_i, \overline{F}_i]$ , so that  $\dot{v}_1 = 0$  and the mass stays put.

In our context, we suppose that the position of the first mass  $p_1$  is measured and we are interested in estimating the position and the velocity of both objects. Due to the presence of the sign and the differential inclusion,  $p_2, v_2$  in (55) are not observable for all solutions evolving in the set

$$\mathcal{N} = \left\{ (p_1, v_1, p_2, v_2) \in \mathbb{R}^4 : v_1 = 0 \right\},$$

$m_1 = 1$	$\gamma_1 = 0.2$	$\bar{F}_1 = 0.2$	$\alpha_1 = 0.01$	$\sigma = 1$
$m_2 = 1$	$\gamma_2 = 0.4$	$\bar{F}_2 = 0.2$	$\alpha_2 = 0.01$	
$\kappa = 4$	$L_1 = 3$	$L_2 = 2$	$L_3 = 1$	

**TABLE 1** Values of the parameters of model (58) and observer (21) with  $n = 4$ .

since for all  $(p_1, v_1, p_2, v_2) \in \mathcal{N}$  we have  $\dot{v}_1$  constantly equal to zero, although  $p_2$  and  $v_2$  may be different from zero. Therefore, from  $v_1$  we cannot observe  $p_2, v_2$ . Hence, in the following we use an approximate model of the function  $\Psi$  in (56) that does not destroy the observability properties of the system, given by

$$\Psi_i(v) := -\bar{F}_i \text{sat}_1(|v|^{\alpha_i}) - \gamma_i v \quad (57)$$

with  $\alpha_i \in (0, 1)$ , and

$$\text{sat}_1(s) = \max(\min(s, 1), -1).$$

Note that for  $\alpha_i \rightarrow 0$ , we recover the friction model (56). By selecting  $x \in \mathbb{R}^4 = (p_1, v_1, \frac{\sigma}{m_1} p_2, \frac{\sigma}{m_1} v_2)$ , model (55), (57), reads therefore

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 + \phi_2(x_1, x_2) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \phi_4(x_1, x_3, x_4, t) \\ y &= x_1 \end{aligned} \quad (58)$$

with

$$\phi_2(x_1, x_2) := \frac{1}{m_1} [\Psi_1(x_2) - \sigma x_1], \quad \phi_4(x_1, x_3, x_4, t) := \frac{\sigma}{m_1 m_2} \left[ \Psi_2\left(\frac{m_1}{\sigma} x_4\right) + \sigma x_1 - m_1 x_3 + u(t) \right].$$

Due to the definition of  $\Psi_i$  in (57), functions  $\phi_2, \phi_4$  are continuous and satisfy Assumption 1 with

$$\ell_0 = \max \left\{ \frac{2\bar{F}_1}{m_1}, \frac{2\sigma\bar{F}_2}{m_1 m_2} \right\}, \quad \ell_1 = \max \left\{ \frac{\gamma_1}{m_1}, \frac{\gamma_2}{m_2}, \frac{\sigma}{m_2} \right\}.$$

It can be noticed that for the class of system (58) none of existing techniques<sup>[9,22,11,6,16,14]</sup> can be employed when  $\alpha_1 > \frac{n-2}{n-1} = \frac{2}{3}$  since the system (58) is not homogeneous. The approach of<sup>[12]</sup>, on the other hand, would now allow to achieve global convergence. We follow therefore the design proposed in (21) obtaining an observer of total dimension  $2n-2 = 6$ . Table 1 summarizes the values of parameters of system (58) and that of observer (21) used in the simulations. For the simulations, we considered the initial conditions of the plant (58) as  $x(0) = (3, -4, 6, -8)$ , with input  $u$  selected as  $u(t) = 3$  for  $t \in [1, 3]$  and  $u(t) = 0$  otherwise. Figure 2 shows the behaviour of the plant (55) with (56). The observer (21) is initialized at the origin. Simulations have been done in Matlab-Simulink 2017b with fixed-step size  $1e-6$  and solver ‘‘ode8’’. Figure 3 shows the behaviour of the estimation errors for the first 5 seconds, while Table 2 summarizes the convergence time and asymptotic values of the estimation errors. It can be noticed that due to the filtering properties of the cascade induced by the linear gains, the effect of the chattering induced by the  $\text{Sign}(\cdot)$  function in the first-block is attenuated through the successive two blocks. As a result, for the states  $x_2, x_3$ , the estimate  $\hat{x}_{22}$ , respectively  $\hat{x}_{33}$ , has a better asymptotic behaviour compared to  $\hat{x}_{12}$ , respectively  $\hat{x}_{23}$ . By increasing the values of  $L_i$ , the convergence time can be reduced, but the chattering effect is amplified. Simulations suggests that a further development of this work is to investigate how to reduce the effect of chattering for large systems.

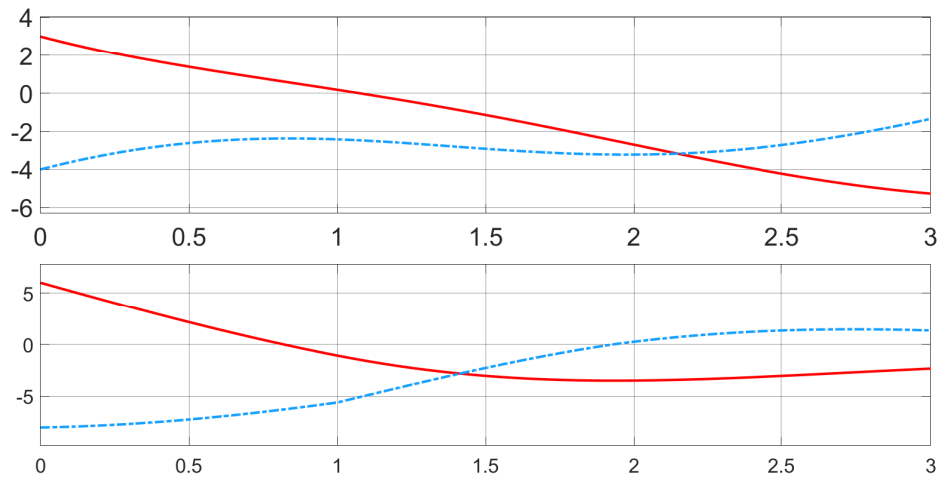
## 6 | CONCLUSIONS

In this paper, we presented a new observer design for lower-triangular systems. The proposed design allows to obtain a global finite time convergent observer when the nonlinearity are affinely bounded. The proposed design generalizes a certain number of usual observer design techniques, since its structure combines standard high-gain observers, sliding mode observers (denoted also as generalized super-twisting algorithm) and interconnection of second-order observers (denoted as low-power high-gain observers). A future development is that of quantifying the effect of measurement noise and chattering in the proposed algorithm.

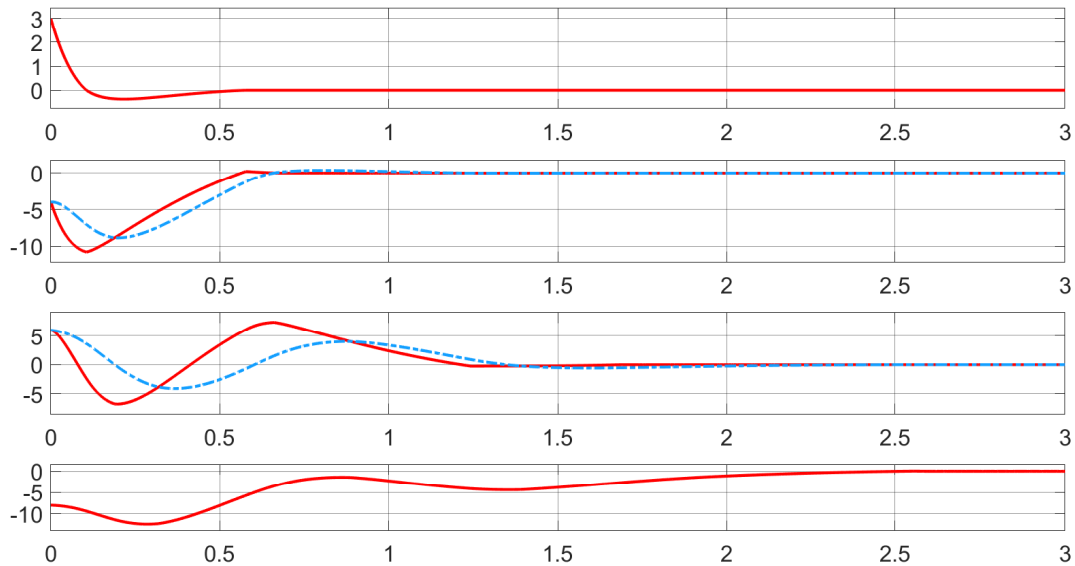


Estimate	Convergence Time	Asymptotic Error
$ x_1 - \hat{x}_{11} $	0.67s	$1e-11$
$ x_2 - \hat{x}_{12} $	0.67s	$1e-5$
$ x_2 - \hat{x}_{22} $	2.09s	$1e-7$
$ x_3 - \hat{x}_{23} $	2.09s	$3e-4$
$ x_3 - \hat{x}_{33} $	2.59s	$1e-4$
$ x_4 - \hat{x}_{34} $	2.59s	$5e-3$

**TABLE 2** Values of the convergence time and asymptotic estimation errors of observer (21) of dimension 6. The estimation error is computed as  $|x|_\infty = \limsup_{t \rightarrow \infty} |x(t)|$  and is induced by the chattering phenomenon. The



**FIGURE 2** Behaviour of system (56). First graph:  $p_1$  (in red) and  $v_1$  (in dash-dotted blue). Second graph:  $p_2$  (in red) and  $v_2$  (in dash-dotted blue).



**FIGURE 3** Convergence of the estimation errors of observer (21). First graph:  $x_1 - \hat{x}_{11}$ . Second graph:  $x_2 - \hat{x}_{12}$  (in red) and  $x_2 - \hat{x}_{22}$  (in dash-dotted blue). Third graph:  $x_3 - \hat{x}_{23}$  (in red) and  $x_3 - \hat{x}_{33}$  (in dash-dotted blue). Fourth graph:  $x_4 - \hat{x}_{34}$ .

**How to cite this article:** Andrieu V., D. Astolfi, and P. Bernard (2021), Observer design via interconnections of second-order mixed sliding-mode/linear differentiators, *International Journal of Robust and Nonlinear Control*, 2021;00:1–∞.

## APPENDIX

### A HOMOGENEOUS IN THE BI-LIMIT CORRECTION TERMS

The objective of this appendix is to establish Proposition 1. This proof is based on the use of the homogeneous in the bi-limit framework and is obtained in several steps.

#### A.1 Homogeneous in the bi-limit framework

The particular feature of the bound (5) is that, for small values of the error, it is bounded by a constant, but for large values, it is Lipschitz. Homogeneity in the bi-limit is a property that has been introduced in<sup>[22]</sup>. It characterizes functions which have two (homogeneous) distinct behaviors at infinity and around the origin. Typically, the set-valued mappings  $k_1$  and  $k_2$  given in (11) have two different behaviors. Around the origin,  $(k_1, k_2)$  are (almost) equal to

$$k_{1,0}(s) = [\kappa s]^{\frac{1}{2}}, \quad k_{2,0}(s) = \text{Sign}(s)$$

and at infinity  $(k_1, k_2)$  are (almost) equal to

$$k_{1,\infty}(s) = \kappa s, \quad k_{2,\infty}(s) = \kappa s.$$

$k_0 = (k_{1,0}, k_{2,0})$  and  $k_\infty = (k_{1,\infty}, k_{2,\infty})$  are homogeneous approximating vector fields corresponding to the sliding mode observer and the high-gain observer respectively, and  $k = (k_1, k_2)$  is a homogeneous in the bi-limit vector field. More precisely, we say that  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$  is bi-homogeneous (or homogeneous in the bi-limit) with weights  $r_0 \in \mathbb{N}^n$  and  $r_\infty \in \mathbb{N}^n$ , degrees  $d_0$  and  $d_\infty$ , and approximating functions  $\phi_0$  and  $\phi_\infty$  if

- $\phi_0$  and  $\phi_\infty$  are homogeneous with weights  $r_0$  and  $r_\infty$ , and degrees  $d_0$  and  $d_\infty$  respectively.
- for every compact set  $C$  which doesn't contain the origin, every  $\epsilon > 0$ , there exists  $\lambda_0 > 0$  and  $\lambda_\infty > 0$  such that for all  $x \in C$ ,

$$\begin{aligned} \left| \frac{\phi(\lambda^{r_0} \cdot x)}{\lambda^{d_0}} - \phi_0(x) \right| &\leq \epsilon && \forall \lambda \in (0, \lambda_0] \\ \left| \frac{\phi(\lambda^{r_\infty} \cdot x)}{\lambda^{d_\infty}} - \phi_\infty(x) \right| &\leq \epsilon && \forall \lambda \in [\lambda_\infty, +\infty) \end{aligned}$$

where we denote  $\lambda^r \cdot x = (\lambda^{r_1} x_1, \dots, \lambda^{r_n} x_n)$ . To simplify we say that  $\phi$  is bi-homogeneous with triples  $(r_0, d_0, \phi_0)$  and  $(r_\infty, d_\infty, \phi_\infty)$ . As for a vector field  $f = \sum f_i \frac{\partial}{\partial x_i}$ , it is bi-homogeneous with triples  $(r_0, d_0, f_0)$  and  $(r_\infty, d_\infty, f_\infty)$  if each  $f_i$  is bi-homogeneous with triples  $(r_0, d_0 + r_{0,i}, f_{0,i})$  and  $(r_\infty, d_\infty + r_{\infty,i}, f_{\infty,i})$ . Homogeneity in the bi-limit has been studied in<sup>[22]</sup> only when there are functions with homogeneous degree larger than 0. So the case of Sign (set-valued) function has not been considered. However, by extending these tools, it is possible to include this case.

#### A.2 The function $V$ is $C^1$

In this section, it is shown that the function  $V$  defined in (13) is  $C^1$ . Note that this is obtained if the function  $e_2 \mapsto [q^{-1}(e_2)]^{\frac{1}{2}}$  is  $C^1$ . First of all, this function is  $C^1$  every where expect maybe at 0. By<sup>[22, Proposition 2.11]</sup>,  $q^{-1}$  is homogeneous in the 0-limit with homogeneous approximating function  $e_2 \mapsto e_2^2$ . Moreover, by definition of  $q$ , we have

$$\frac{dq^{-1}}{de_2}(e_2) = \frac{1}{q'(q^{-1}(e_2))} = \frac{2\sqrt{|q^{-1}(e_2)|}}{1 + 2\sqrt{|q^{-1}(e_2)|}}, \quad \forall e_2 \neq 0. \quad (\text{A1})$$

Moreover, for all  $e_2 \neq 0$ , it yields

$$\frac{d [q^{-1}(e_2)]^{\frac{1}{2}}}{de_2} = \frac{1}{1 + 2\sqrt{|q^{-1}(e_2)|}}, \quad \forall e_2 \neq 0. \quad (\text{A2})$$

By extension by continuity, this implies that  $e_2 \mapsto [q^{-1}(e_2)]^{\frac{1}{2}}$  is  $C^1$ .

### A.3 Finite time stability for the error system

In this section we consider the set valued vector field

$$F(e) = \begin{bmatrix} e_2 - k_1(e_1) \\ -k_2(e_1) \end{bmatrix} \quad (\text{A3})$$

where  $k_1$  and  $k_2$  are given in (11). We show in the following proposition that when  $\kappa$  is selected sufficiently large, the function  $V$  defined in (13) is a Lyapunov function for this vector field (see also<sup>[16]</sup> for the homogeneous case).

**Proposition 3.** *There exists a positive real number  $\kappa^*$  such that, for all  $\kappa > \kappa^*$ , there exists a positive real number  $c_{V,0}$  such that the function  $V$  given in (13) satisfies*

$$\max \left\{ \frac{\partial V}{\partial e}(e)F(e) \right\} \leq -c_{V,0} \left[ V(e) + V(e)^{\frac{2}{3}} \right], \quad \forall e \in \mathbb{R}^2. \quad (\text{A4})$$

**Proof:** The proof is inspired from Theorem 3.1 in<sup>[22]</sup>. Note that

$$\frac{\partial V}{\partial e}(e)F(e) \subset \{T_1(\kappa e_1, e_2) + \kappa T_2(\kappa e_1, e_2)\}, \quad (\text{A5})$$

where

$$T_1(v, e_2) = -(\text{Sign}(q(v)) + q(v)) \left( 3 [e_2]^2 + \int_{q^{-1}(e_2)}^v \left( [q^{-1}(e_2)]^{\frac{1}{2}} + [q^{-1}(e_2)]^2 \right)' dh \right)$$

and

$$T_2(v, e_2) = (e_2 - q(v)) \left( [v]^{\frac{1}{2}} - [q^{-1}(e_2)]^{\frac{1}{2}} + [v]^2 - [q^{-1}(e_2)]^2 \right).$$

Furthermore, that there exists a continuous single-valued map  $\tilde{T}_1 : \mathbb{R}^3 \rightarrow \mathbb{R}$  such that

$$T_1(v, e_2) = \{\tilde{T}_1(v, e_2, s), s \in \text{sign}(q(v))\}.$$

Define  $r_0 = (2, 1)$ ,  $r_\infty = (1, 1)$ .  $T_2$  and  $\tilde{T}_1$  are both homogeneous in the bi-limit with weights  $r_0, r_\infty$  and  $(r_0, 0), (r_\infty, 0)$  respectively, with same degrees  $d_0 = 2, d_\infty = 3$  and with homogeneous approximations

$$\begin{aligned} \tilde{T}_{1,0}(v, e_2, s) &= -s \left( 3 [e_2]^2 + v - [e_2]^2 \right), \\ \tilde{T}_{1,\infty}(v, e_2, s) &= -v \left( 3 [e_2]^2 + [v]^2 - [e_2]^2 \right), \end{aligned}$$

and

$$\begin{aligned} T_{2,0}(s, e_2) &= - \left( e_2 - [v]^{\frac{1}{2}} \right)^2, \\ T_{2,\infty}(s, e_2) &= (e_2 - v) \left( [v]^2 - [e_2]^2 \right). \end{aligned}$$

Moreover,  $q$  is an increasing function and  $T_2 \leq 0$ , with  $T_2 = 0$  only if  $e_2 = q(v)$ . Note also that if  $e_2 = q(v)$ ,  $T_1(v, e_2) = -3[e_2]^2 - 3[e_2]^3 < 0$ . The same holds for the homogeneous approximation functions given above when  $s \in \text{Sign}(q(v))$ . Employing the technical Lemma 1, it yields the existence of  $\kappa^* > 0$  such that for all  $\kappa > \kappa^*$ ,

$$\max_{s \in \text{sign}(q(v))} \tilde{T}_1(v, e_2, s) + \kappa T_2(v, e_2) < 0, \quad \forall (v, e_2) \in \mathbb{R}^2 \setminus \{0\}.$$

It thus follows that

$$\max \left\{ \frac{\partial V}{\partial e}(e)F(e) \right\} < 0, \quad \forall (e_1, e_2) \in \mathbb{R}^2 \setminus \{0\},$$

and the same for its homogeneous approximation. Following the proof of Corollary 2.15 in<sup>[22]</sup> employed with Lemma 1, it implies that there exists  $c_V > 0$  such that

$$\max \left\{ \frac{\partial V}{\partial e}(e)F(e) \right\} \leq -c_V V(e_1, e_2) - c_V V(e_1, e_2)^{\frac{2}{3}}.$$

□

## A.4 Proof of Proposition 1

Let  $\kappa > \kappa^*$  where  $\kappa^*$  is given in Proposition 3. Along any solution of system (12), we have

$$\dot{V}(e, v, w) \leq \eta(e, v, w) + \frac{\partial V}{\partial e_2}(e)v_2$$

with

$$\eta(e, v, w) = \max \left\{ \frac{\partial V}{\partial e}(e) \begin{pmatrix} e_2 + k_1(e_1 + w) + v_1 \\ k_2(e_1 + w) \end{pmatrix} \right\}.$$

Attributing to  $w$  the same homogeneous weights as  $e_1$ , and to  $v_1$  the same homogeneous weights as  $e_2$ ,  $\eta$  is homogeneous in the bi-limit and, according to Proposition 3, we obtain

$$\eta(e, 0, 0) + \frac{c_V}{2} \left[ V(e)^{\frac{2}{3}} + V(e) \right] < 0 \quad \forall e.$$

Applying the technical Lemma 1 with

$$\gamma(w, v_1) = [|w| + |w|^3] + [|v_1|^2 + |v_1|^3],$$

shows that there exists  $c > 0$  such that

$$\eta(e, w, v_1) + \frac{c_V}{2} \left[ V(e)^{\frac{2}{3}} + V(e) \right] - c\gamma(w, v_1) < 0.$$

Finally, we can observe that  $\frac{\partial V}{\partial e_2}(e)$  is homogeneous of degree 2, so that there exists  $c_{v_2} > 0$  such that

$$\left| \frac{\partial V}{\partial e_2}(e) \right| \leq c_{v_2} V(e)^{\frac{2}{3}},$$

concluding the proof.

## B TECHNICAL LEMMAS

**Lemma 1.** Let  $\eta : \mathbb{R}^n \rightarrow \mathbb{R}$  be homogeneous in the bi-limit, with weights  $r_0$  and  $r_\infty$ , degrees  $d_0$  and  $d_\infty$ , of the form

$$\eta(x) = \max_{s \in \text{sign}(f(x))} \tilde{\eta}(x, s)$$

for some continuous maps  $\tilde{\eta} : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ , and  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  such that, for all  $x \in \mathbb{R}^n$ , for all  $\lambda > 0$

$$\text{Sign}(f(\lambda^{r_0} \cdot x)) = \text{Sign}(f(x)) \tag{B6}$$

$$\text{Sign}(f(\lambda^{r_\infty} \cdot x)) = \text{Sign}(f(x)), \tag{B7}$$

and such that  $\tilde{\eta}$  is homogeneous in the bi-limit, with weights  $(r_0, 0)$  and  $(r_\infty, 0)$ , degrees  $d_0$  and  $d_\infty$ , and approximating functions  $\tilde{\eta}_0$  and  $\tilde{\eta}_\infty$ . Consider a continuous function  $\gamma : \mathbb{R}^n \rightarrow \mathbb{R}_+$  that is homogeneous in the bi-limit, with same weights and degrees and with approximating functions  $\gamma_0$  and  $\gamma_\infty$  such that  $\forall x \in \mathbb{R}^n \setminus \{0\}$ ,  $\forall s \in \text{Sign}(f(x))$

$$\begin{cases} \gamma(x) = 0 \Rightarrow & \tilde{\eta}(x, s) < 0, \\ \gamma_0(x) = 0 \Rightarrow & \tilde{\eta}_0(x, s) < 0, \\ \gamma_\infty(x) = 0 \Rightarrow & \tilde{\eta}_\infty(x, s) < 0. \end{cases}$$

Then there exists a real number  $c^*$  such that, for all  $c \geq c^*$ , and for all  $x$  in  $\mathbb{R}^n \setminus \{0\}$

$$\eta(x) - c\gamma(x) < 0. \tag{B8}$$

**Proof:** First by homogeneity of the approximations, according to<sup>[11, Lemma 4]</sup>, there exist  $c_0^* > 0$  and  $c_\infty^* > 0$ ,  $\epsilon_0^* > 0$  and  $\epsilon_\infty^* > 0$ , such that for all  $c_0 \geq c_0^*$  and  $c_\infty \geq c_\infty^*$ , and for all  $x$  in  $\mathbb{R}^n \setminus \{0\}$ , and for all  $s \in \text{Sign}(f(x))$

$$\tilde{\eta}_0(x, s) - c_0\gamma_0(x) < -\epsilon_0, \quad \tilde{\eta}_\infty(x, s) - c_\infty\gamma_\infty(x) < -\epsilon_\infty.$$

Define  $c_1 = \max\{c_0, c_\infty\}$  and  $\epsilon_1 = \min\{\epsilon_0, \epsilon_\infty\}$ . Reproducing arguments of<sup>[22, Appendix C]</sup>, we next prove that there exists a compact set  $C$  such that for all  $c \geq c_1$ ,

$$\{x \in \mathbb{R}^n, \exists s \in \text{Sign}(f(x)), \tilde{\eta}(x, s) - c\gamma(x) \geq 0\} \subseteq C. \tag{B9}$$

Indeed, the bi-homogeneity of  $\tilde{\eta}$  and  $\gamma$  means that there exist  $\lambda_0 > 0$ ,  $\lambda_\infty > 0$  such that, denoting the homogeneous norm  $|x|_{r,d} = \left( \sum_{i=1}^n |x_i|^{\frac{d}{r_i}} \right)^{\frac{1}{d}}$ , we have the following properties.

- For all  $\lambda \in (0, \lambda_0]$ , for all  $x$  such that  $|x|_{r_0, d_0} = 1$ , and for all  $s \in \text{Sign}(f(x))$ ,

$$\begin{aligned} |\tilde{\eta}(\lambda^{r_0} \cdot x, s) - \lambda^{d_0} \tilde{\eta}_0(x, s)| &\leq \lambda^{d_0} \frac{\epsilon}{4}, \\ |\gamma(\lambda^{r_0} \cdot x) - \lambda^{d_0} \gamma_0(x, s)| &\leq \lambda^{d_0} \frac{\epsilon}{4}, \end{aligned}$$

which implies for  $c \geq c_1$

$$\begin{aligned} \tilde{\eta}(\lambda^{r_0} \cdot x, s) - c\gamma(\lambda^{r_0} \cdot x) &\leq \lambda^{d_0} (\tilde{\eta}_0(\lambda^{r_0} \cdot x, s) - c\gamma_0(\lambda^{r_0} \cdot x)) + \lambda^{d_0} \frac{\epsilon}{2} \\ &\leq -\frac{\epsilon}{2}, \end{aligned}$$

using (B6), and therefore,

$$\tilde{\eta}(x, s) - c\gamma(x) < 0 \quad \forall 0 < |x|_{r_0, d_0} \leq \lambda_0.$$

- For all  $\lambda \in [\lambda_\infty, +\infty)$ , for all  $x$  such that  $|x|_{r_\infty, d_\infty} = 1$ , and for all  $s \in \text{Sign}(f(x))$ ,

$$\begin{aligned} |\tilde{\eta}(\lambda^{r_\infty} \cdot x, s) - \lambda^{d_\infty} \tilde{\eta}_\infty(x, s)| &\leq \lambda^{d_\infty} \frac{\epsilon}{4}, \\ |\gamma(\lambda^{r_\infty} \cdot x) - \lambda^{d_\infty} \gamma_\infty(x, s)| &\leq \lambda^{d_\infty} \frac{\epsilon}{4}, \end{aligned}$$

which implies for  $c \geq c_1$  in the same way

$$\tilde{\eta}(x, s) - c\gamma(x) < 0 \quad \forall |x|_{r_\infty, d_\infty} \geq \lambda_\infty.$$

Therefore, by defining the compact set

$$C := \{x \in \mathbb{R}^n, |x|_{r_0, d_0} \geq \lambda_0, |x|_{r_\infty, d_\infty} \leq \lambda_\infty\},$$

we indeed have (B9).

Finally, assume that for all  $c \geq c_1$ , there exists  $x \neq 0$  such that  $\eta(x) - c\gamma(x) \geq 0$ . Then, we can build a sequence  $(x_k, s_k)$  of elements of  $C \times [-1, 1]$  such that

$$\tilde{\eta}(x_k, s_k) - k\gamma(x_k) \geq 0$$

with  $s_k \in \text{Sign}(f(x_k))$  for all  $k \in \mathbb{N}^*$ . Since  $C$  is compact, there exists a subsequence which converges to  $(x^*, s^*) \in C \times [-1, 1]$ . Taking the limit and using the continuity of  $\tilde{\eta}$ , necessarily implies that  $\gamma(x^*) = 0$  and  $\tilde{\eta}(x^*, s^*) \geq 0$ . This is impossible if  $s^* \in \text{Sign}(f(x^*))$ . But either  $f(x^*) \neq 0$ , and  $s^* = \text{Sign}(f(x^*))$  by continuity of  $f$ , or  $f(x^*) = 0$  and  $s^* \in [-1, 1] = \text{Sign}(f(x^*))$ . This concludes the proof.  $\square$

**Lemma 2.** Let  $(a_W, a_V)$ , two positive real numbers and  $(\rho_W, \rho_V)$  two class- $\mathcal{K}$  functions verifying

$$\rho_V \left( \frac{2}{a_W} \rho_W(s) \right) < a_V \quad \forall s > 0. \quad (\text{B10})$$

There exists a class- $\mathcal{K}$  function  $\alpha$  such that for any two locally integrable functions  $\delta_W : [0, T] \rightarrow \mathbb{R}$  and  $\delta_V : [0, T] \rightarrow \mathbb{R}$ , any two continuous functions  $W : [0, T] \rightarrow \mathbb{R}_+$ , and  $V : [0, T] \rightarrow \mathbb{R}_+$  whose upper right Dini derivative (denoted  $D^+$ ) satisfies for all  $t$  in  $[0, T)$

$$D^+W(t) \leq -a_W W(t) + \rho_W \left( \sup_{s \in [0, t]} V(s) \right) + \delta_W(t), \quad (\text{B11})$$

$$D^+V(t) \leq -a_V \left( V(t) + V(t)^{\frac{2}{3}} \right) + \rho_V(W(t)) V(t)^{\frac{2}{3}} + \delta_V(t) \quad (\text{B12})$$

the following inequality is satisfied

$$W(t) \leq \alpha \left( W(0) + V(0) + \sup_{s \in [0, t]} |\delta_W(s)| + |\delta_V(s)| \right), \quad \forall t \in [0, T].$$

Lemma 2 looks like a small-gain theorem but with the following two differences:

- due to the presence of  $\sup_{s \in [0, t]} V(s)$  instead of  $V(t)$  in (B11), we do not prove ISS but simply that  $W$  is “stable” with respect to the initial conditions and the maximal perturbations ;

- in (B12), taking  $\delta_V = 0$ , we have  $D^+V(t) < 0$  whenever  $a_V > \rho_V(W(t))$ , which is not of the form  $V(t) > \alpha(W(t))$  with  $\alpha$  of class- $\mathcal{K}$  as in the standard small-gain theorem.

Hence, the small-gain proof needs to be adapted.

**Proof:** Let  $\rho$  be a class- $\mathcal{K}$  function that is  $C^1$  on  $(0, +\infty)$  and verifies

$$\rho'(s) > 0 \quad \forall s > 0 \quad , \quad \rho_W(s) \leq \frac{a_W}{2} \rho(s) \quad \forall s \geq 0 \quad , \quad \rho_V \circ \rho(s) \leq a_V \quad \forall s \geq 0$$

which is always possible thanks to (B10). Define

$$U(t) = \max \left\{ W(t), \sup_{s \in [0, t]} \rho(V(s)) \right\} , \quad \forall t.$$

Let  $t$  be in  $[0, T)$ , we have

- if  $W(t) > \rho(\sup_{s \in [0, t]} V(s))$ , then  $U(t) = W(t)$ , and

$$D^+U(t) \leq -a_W W(t) + \rho_W \left( \sup_{s \in [0, t]} V(s) \right) + \delta_W \tag{B13}$$

$$\leq -\frac{a_W}{2} W(t) - \frac{a_W}{2} \rho \left( \sup_{s \in [0, t]} V(s) \right) + \rho_W \left( \sup_{s \in [0, t]} V(s) \right) + \delta_W(t) \tag{B14}$$

$$\leq -\frac{a_W}{2} W(t) + \delta_W(t) \tag{B15}$$

$$\leq -\frac{a_W}{2} U(t) + \delta_W(t) \tag{B16}$$

- if  $W(t) < \rho(\sup_{s \in [0, t]} V(s))$ , then  $V(t) \neq 0$ ,  $U(t) = \rho(\sup_{s \in [0, t]} V(s))$ , and we have the following cases :

- if  $\sup_{s \in [0, t]} V(s) \neq V(t)$ , then, by continuity of  $V$  there exists  $t_1$  such that  $V(\tau) < \sup_{s \in [0, t]} V(s)$ , for all  $\tau$  in  $[t, t_1]$ . Consequently,

$$\sup_{s \in [0, \tau]} V(s) = \sup_{s \in [0, t]} V(s) , \quad \forall \tau \in [t, t_1]$$

and thus,  $D^+U(t) = 0$ .

- if  $\sup_{s \in [0, t]} V(s) = V(t)$  and  $D^+V(t) \leq 0$ , to show that  $D^+ \sup_{s \in [0, t]} V(s) \leq 0$ , let us assume the opposite and the existence of  $\epsilon > 0$  such that  $D^+ \sup_{s \in [0, t]} V(s) > \epsilon > 0$ .

$$D^+ \sup_{s \in [0, t]} V(s) = \limsup_{h \searrow 0} \frac{\sup_{s \in [0, t+h]} V(s) - \sup_{s \in [0, t]} V(s)}{h} \tag{B17}$$

$$= \limsup_{h \searrow 0} \frac{\sup_{s \in [t, t+h]} V(s) - V(t)}{h} \tag{B18}$$

$$= \lim_{h \searrow 0} \sup_{\tau \in (0, h]} \frac{\sup_{s \in [0, \tau]} V(t+s) - V(t)}{\tau} > \epsilon \tag{B19}$$

Hence, this implies that for all  $(h_i)_{i \in \mathbb{N}}$  there exists  $s_i \leq \tau_i \leq h_i$  such that

$$\frac{V(t+s_i) - V(t)}{\tau_i} > \frac{\epsilon}{2}$$

This implies that for all  $h_i$  there exists  $s_i \leq h_i$

$$\frac{V(t+s_i) - V(t)}{s_i} = \frac{V(t+s_i) - V(t)}{\tau_i} \frac{\tau_i}{s_i} > \frac{\epsilon}{2}$$

which contradicts  $D^+V(t) \leq 0$ . Consequently, if  $D^+V(t) \leq 0$  then  $D^+ \sup_{s \in [0, t]} V(s) \leq 0$  and since  $\rho$  is an increasing function  $D^+ \rho(\sup_{s \in [0, t]} V(s)) \leq 0$ .

- if  $\sup_{s \in [0, t]} V(s) = V(t)$  and  $D^+V(t) > 0$ . By definition of Dini derivative, for each  $\epsilon > 0$ , there exists  $h^* > 0$ , such that for all  $0 < h \leq h^*$ ,

$$\frac{V(t+h) - V(t)}{h} \leq D^+V(t) + \epsilon .$$

So for all  $h \leq h^*$

$$\sup_{\tau \in (0, h]} \frac{V(t+\tau) - V(t)}{\tau} \leq D^+V(t) + \epsilon$$

Moreover, for all  $h \leq h^*$  such that  $\max_{\tau \in [0, h]} V(t + \tau) - V(t) > 0$ ,

$$0 < \frac{\max_{\tau \in [0, h]} V(t + \tau) - V(t)}{h} \leq \sup_{\tau \in (0, h]} \frac{V(t + \tau) - V(t)}{\tau} \leq D^+ V(t) + \epsilon .$$

And for all  $h \leq h^*$  such that  $\max_{\tau \in [0, h]} V(t + \tau) - V(t) \leq 0$ , we trivially have

$$\frac{\max_{\tau \in [0, h]} V(t + \tau) - V(t)}{h} \leq D^+ V(t) + \epsilon$$

since  $D^+ V(t) > 0$ . Therefore, the previous inequality holds for all  $h \leq h^*$ . So since  $\max_{s \in [0, t]} V(s) = V(t)$ , we get for all  $h \leq h^*$

$$\frac{\max_{s \in [0, t+h]} V(s) - \max_{s \in [0, t]} V(s)}{h} \leq D^+ V(t) + \epsilon ,$$

and thus

$$D^+ \max_{s \in [0, t]} V(s) \leq D^+ V(t) + \epsilon$$

This property being true for all  $\epsilon > 0$ , it yields  $D^+ \sup_{s \in [0, t]} V(s) \leq D^+ V(t)$ . Thus, since  $\rho$  is an increasing function,

$$\begin{aligned} D^+ U(t) &\leq \rho'(V(t)) D^+ V(t) \\ &\leq \rho'(V(t)) \left[ -a_V \left( V(t) + V(t)^{\frac{2}{3}} \right) + \rho_V(W(t)) V(t)^{\frac{2}{3}} + \delta_V(t) \right] \\ &\leq \rho'(V(t)) \left[ -a_V \left( V(t) + V(t)^{\frac{2}{3}} \right) + \rho_V \circ \rho \left( \sup_{s \in [0, t]} V(s) \right) V(t)^{\frac{2}{3}} + \delta_V(t) \right] \\ &\leq \rho'(V(t)) \left[ -a_V V(t) + \delta_V(t) \right] \\ &\leq \rho'(\rho^{-1}(U(t))) \left[ -a_V \rho^{-1}(U(t)) + \delta_V(t) \right] \end{aligned}$$

- if  $W(t) = \rho \left( \sup_{s \in [0, t]} V(s) \right)$ , we have

$$\begin{aligned} D^+ U(t) &= \limsup_{h \searrow 0} \frac{U(t+h) - U(t)}{h} \\ &= \limsup_{h \searrow 0} \frac{\max \{ W(t+h), \sup_{s \in [0, t+h]} \rho(V(s)) \} - U(t)}{h} \\ &= \limsup_{h \searrow 0} \max \left\{ \frac{W(t+h) - W(t)}{h}, \frac{\sup_{s \in [0, t+h]} \rho(V(s)) - \sup_{s \in [0, t]} \rho(V(s))}{h} \right\} \\ &= \max \left\{ D^+ W(t), D^+ \left( \sup_{s \in [0, t]} \rho(V(s)) \right) \right\}, \end{aligned}$$

and we use the previous two steps.

It follows that

$$D^+ U(t) \leq 0 \quad \text{if } U(t) \geq \max \left\{ \frac{2}{a_W} |\delta_W(t)|, \rho \left( \frac{1}{a_V} |\delta_V(t)| \right) \right\}$$

with the convention that  $\rho^{-1}(U) = +\infty$  if  $U \notin \rho(\mathbb{R}_{\geq 0})$ . According to the following lemma, we conclude

$$U(t) \leq \max \left\{ U(0), \max_{s \in [0, t]} \frac{2}{a_W} |\delta_W(s)|, \max_{s \in [0, t]} \rho \left( \frac{1}{a_V} |\delta_V(s)| \right) \right\} .$$

□

**Lemma 3.** Let  $g, f : [0, T] \rightarrow \mathbb{R}_{\geq 0}$  be two continuous functions, such that for all  $t$

$$D^+ f(t) \leq 0 \quad \text{if } f(t) \geq g(t) .$$

Then,

$$f(t) \leq \max \left\{ f(0), \max_{s \in [0, t]} g(s) \right\} \quad \forall t \in [0, T] .$$

**Proof:** Assume there exists  $t_1$  such that the former property is violated. This implies that

$$f(t_1) > \max \left\{ f(0), \max_{s \in [0, t_1]} g(s) \right\}.$$

The function  $f$  being continuous,

$$\max \left\{ f(0), \max_{s \in [0, t_1]} g(s) \right\} \in f([0, t_1]).$$

Hence, we can define  $t_0$  in  $[0, t_1]$  as

$$t_0 = \max \left\{ f^{-1} \left( \max \left\{ f(0), \max_{s \in [0, t_1]} g(s) \right\} \right) \right\}.$$

In other words,  $f(t_0) = \max\{f(0), \max_{s \in [0, t_1]} g(s)\}$ , and

$$f(t) > \max \left\{ f(0), \max_{s \in [0, t_1]} g(s) \right\} \geq \max \left\{ f(0), \max_{s \in [0, t]} g(s) \right\}$$

for all  $t \in [t_0, t_1]$ . This implies

$$f(t_0) < f(t_1) \text{ \& } D^+ f(t) \leq 0, \forall t \in [t_0, t_1]$$

which contradicts<sup>[28, Theorem 2.1, Appendix I, p 347]</sup>.

□

## References

1. Gauthier JP, Kupka I. *Deterministic observation theory and applications*. Cambridge University Press . 2001.
2. Bernard P, Praly L, Andrieu V, Hammouri H. On the triangular canonical form for uniformly observable controlled systems. *Automatica* 2017; 85: 293–300.
3. Tornambè A. Use of asymptotic observers having-high-gains in the state and parameter estimation. In: IEEE. ; 1989: 1791–1794.
4. Emelyanov S, Korovin S, Nikitin S, Nikitina M. Observers and output differentiators for nonlinear systems. *Doklady Akademii Nauk* 1989.
5. Khalil HK, Praly L. High-gain observers in nonlinear feedback control. *International Journal of Robust and Nonlinear Control* 2014; 24(6): 993–1015.
6. Astolfi D, Marconi L. A high-gain nonlinear observer with limited gain power. *IEEE Transactions on Automatic Control* 2015; 60(11): 3059–3064.
7. Wang L, Astolfi D, Marconi L, Su H. High-gain observers with limited gain power for systems with observability canonical form. *Automatica* 2017; 75: 16–23.
8. Astolfi D, Marconi L, Praly L, Teel AR. Low-power peaking-free high-gain observers. *Automatica* 2018; 98: 169–179.
9. Levant A. Robust exact differentiation via sliding mode technique. *automatica* 1998; 34(3): 379–384.
10. Shtessel Y, Edwards C, Fridman L, Levant A. *Sliding mode control and observation*. Springer . 2014.
11. Bernard P, Praly L, Andrieu V. Observers for a non-Lipschitz triangular form. *Automatica* 2017; 82: 301–313.
12. Floquet T, Barbot JP. Super twisting algorithm-based step-by-step sliding mode observers for nonlinear systems with unknown inputs. *International journal of systems science* 2007; 38(10): 803–815.
13. Bejarano F, Poznyak A, Fridman L. Hierarchical second-order sliding-mode observer for linear time invariant systems with unknown inputs. *International Journal of Systems Science* 2007; 38(10): 793–802.
14. Fridman L, Levant A, Davila J. Observation of linear systems with unknown inputs via high-order sliding-modes. *International Journal of systems science* 2007; 38(10): 773–791.



15. Moreno J. A linear framework for the robust stability analysis of a generalized super-twisting algorithm. *IEEE Conference on Electrical Engineering, Computing Science and Automatic Control* 2009.
16. Cruz-Zavala E, Moreno JA. Homogeneous high order sliding mode design: a Lyapunov approach. *Automatica* 2017; 80: 232–238.
17. Moreno J. On strict Lyapunov functions for some non-homogeneous super-twisting algorithms. *Journal of the Franklin Institute* 2014; 351(4): 1902–1919.
18. Castillo I, Fridman L, Moreno JA. Super-twisting algorithm in presence of time and state dependent perturbations. *International Journal of Control* 2018; 91(11): 2535–2548.
19. Andrieu V, Praly L. On the existence of Kazantzis-Kravaris / Luenberger Observers. *SIAM Journal on Control and Optimization* 2006; 45(2): 432–456.
20. Filippov A. *Differential equations with discontinuous right-hand sides*. Mathematics and its Applications Kluwer Academic Publishers Group . 1988.
21. Moreno JA. On discontinuous observers for second order systems: properties, analysis and design. In: Springer. 2013 (pp. 243–265).
22. Andrieu V, Praly L, Astolfi A. Homogeneous approximation, recursive observer design and output feedback. *SIAM Journal on Control and Optimization* 2008; 47(4): 1814–1850.
23. Moreno JA. Lyapunov approach for analysis and design of second order sliding mode algorithms. In: Springer. 2011 (pp. 113–149).
24. Bhat SP, Bernstein DS. Geometric homogeneity with applications to finite-time stability. *Mathematics of Control, Signals and Systems* 2005; 17(2): 101–127.
25. Yoshizawa T. *Stability theory by Liapunov's second method*. 1966.
26. Armstrong-Hélouvry B, Dupont P, De Wit CC. A survey of models, analysis tools and compensation methods for the control of machines with friction. *Automatica* 1994; 30(7): 1083–1138.
27. Beerens R, Bisoffi A, Zaccarian L, Heemels W, Nijmeijer H, Wouw v. dN. Reset integral control for improved settling of PID-based motion systems with friction. *Automatica* 2019; 107: 483–492.
28. Rouche N, Habets P, Laloy M. *Stability theory by Liapunov's direct method*. 4. Springer . 1977.