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BOUNDARY NULL-CONTROLLABILITY OF ONE-DIMENSIONAL COUPLED PARABOLIC SYSTEMS WITH KIRCHHOFF CONDITION

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Abstract. The main purpose of this paper is to investigate the boundary controllability of some 2×2 one-dimensional parabolic systems with both interior and boundary couplings: the interior coupling is chosen to be linear while the boundary one is considered by means of a Kirchhoff condition. We consider here the Dirichlet boundary control on either one of the two state components. In particular, we show that controllability properties change depending on which component of the system the control is being applied. Regarding this, we point out that the choices of interior coupling coefficient and the Kirchhoff parameter play a crucial role to deduce the positive or negative controllability results. Finally, we present a numerical implementation allowing us to illustrate our controllability results and extend the discussion to some other examples.

Key words. Boundary control, parabolic systems, Carleman estimate, moments method, spectral analysis, Kirchhoff condition

AMS subject classifications. 35K20 - 93B05 - 93B07 - 93B60

1. Introduction.

1.1. Motivation and the problem under study.

In this article, we discuss about the boundary null-controllability of some parabolic systems coupled in the interior as well as on the boundary with less number of control than the equations in dimension 1. One motivation for studying these kind of systems is coming from the following prototype of 2×2 general boundary controllability system,

$$\begin{cases} \partial_t y + \mathcal{A} y + \mathcal{M}_{coup}\, y = 0 & \text{in } (0,T)\times(0,1), \\ \mathcal{D}_0 y(t,0) + \mathcal{N}_0 \dfrac{\partial y}{\partial \nu_{\boldsymbol{\gamma}}}(t,0) = B v(t) & \text{in } (0,T), \\ \mathcal{D}_1 y(t,1) + \mathcal{N}_1 \dfrac{\partial y}{\partial \nu_{\boldsymbol{\gamma}}}(t,1) = 0 & \text{in } (0,T), \\ y(0,\cdot) = y_0(\cdot) & \text{in } (0,1), \end{cases} \tag{1.1}$$

where $y := (y_1, y_2)$ is the unknown and $y_0 := (y_{0,1}, y_{0,2})$ is the initial data from some suitable Hilbert space and $\mathcal{A}$ stands for some diffusion operator with its formal expression

$$\mathcal{A} := \begin{pmatrix} -\partial_x(\gamma_1 \partial_x) & 0 \\ 0 & -\partial_x(\gamma_2 \partial_x) \end{pmatrix}, \tag{1.2}$$

where the diffusion coefficients γ_1, γ_2 are chosen in such a way that

$$\gamma_i \in \mathcal{C}^1([0,1]) \text{ with } 0 < \gamma_{\min} \leq \gamma_i(x) \leq \gamma_{\max} < +\infty,\ \forall x \in [0,1],\ i = 1,2. \tag{1.3}$$

The precise form of the normal derivative on the boundary points $x \in \{0,1\}$ is $\frac{\partial y}{\partial \nu_{\boldsymbol{\gamma}}} = (\gamma_1 \frac{\partial y_1}{\partial \nu}, \gamma_2 \frac{\partial y_2}{\partial \nu})$, where ν is the normal vector.

We consider here the *interior coupling* by means of some 2×2 real matrix $\mathcal{M}_{coup}$ and the *boundary coupling* via the 2×2 real coefficient matrices $\mathcal{D}_j, \mathcal{N}_j$, for $j = 0, 1$ mentioned in (1.1). One may consider here v as a scalar control from some suitable space and B as some real vector.

We further make the following assumptions in our setting.

Assumption 1.1. *For each $j \in \{0,1\}$,*

1. *The 2×4 matrix $(\mathcal{D}_j, \mathcal{N}_j)$ has the maximal rank.*
2. *The matrix $\mathcal{D}_j \mathcal{N}_j^*$ is self-adjoint.*

The first assumption ensures the sufficient number of boundary conditions in (1.1), whereas the second one is important for the differential operator $\mathcal{A}$ defined by (1.2) to be self-adjoint in its domain

$$D(\mathcal{A}) := \Big\{ u \in (H^2(0,1))^2 \mid \mathcal{D}_0 u(0) + \mathcal{N}_0 \frac{\partial u}{\partial \nu_{\boldsymbol{\gamma}}}(0) = 0,\ \mathcal{D}_1 u(1) + \mathcal{N}_1 \frac{\partial u}{\partial \nu_{\boldsymbol{\gamma}}}(1) = 0 \Big\}. \tag{1.4}$$

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Note that the domain of the operator $\mathcal{A}$ considered in (1.1) is not exactly same as (1.4) due to presence of $Bv \neq 0$ on the boundary, but we keep the same notation $\mathcal{A}$ if there is no confusion.

Now, we must mention that studying a more general system like (1.1) is really intricate and widely open. Indeed there are some negative results also (even if Assumption 1.1 satisfied): it can be shown that a linear coupled system in the cascade form is not even approximately controllable for either $\mathcal{D}_j = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\mathcal{N}_j = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ or $\mathcal{D}_j = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\mathcal{N}_j = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $j = 1, 2$ and for $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; see for instance, [5, Remark 2.17].

In our present work, we will study some particular class of problems that fit in the framework of (1.1) from both the theoretical and numerical point of view; moreover, in Section 5.2 we shall provide a discrete setting for the general system (1.1).

We hereby choose the interior coupling

$$\mathcal{M}_{coup} = \mathcal{M}_a := \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}, \tag{1.5}$$

for some $a \in \mathbb{R}$, and two different kind of the boundary coefficient matrices:

$$\text{either} \quad \mathcal{D}_0 = I_{2\times 2}, \quad \mathcal{N}_0 = O_{2\times 2}, \tag{1.6a}$$

$$\text{or} \quad \mathcal{D}_0 = O_{2\times 2}, \quad \mathcal{N}_0 = I_{2\times 2}, \tag{1.6b}$$

where in both the cases, we assume

$$\mathcal{D}_1 = \begin{pmatrix} 1 & -1 \\ \alpha & 0 \end{pmatrix}, \quad \mathcal{N}_1 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \tag{1.7}$$

for some $\alpha \geq 0$. In what follows, we have the following coupled parabolic systems with *Dirichlet* or *Neumann* boundary control at left and the *Kirchhoff condition* at right which actually plays the role of boundary couplings.

- *Two Dirichlet control problems.* The problems of interest under (1.6a)–(1.7) are

$$\begin{cases} \partial_t y_1 - \partial_x(\gamma_1 \partial_x y_1) = 0 & \text{in } (0,T)\times(0,1), \\ \partial_t y_2 - \partial_x(\gamma_2 \partial_x y_2) + a y_1 = 0 & \text{in } (0,T)\times(0,1), \\ y_1(t,1) = y_2(t,1) & \text{in } (0,T), \\ \gamma_1(1)\partial_x y_1(t,1) + \gamma_2(1)\partial_x y_2(t,1) + \alpha\, y_1(t,1) = 0 & \text{in } (0,T), \\ y_1(0,\cdot) = y_{0,1}(\cdot), \quad y_2(0,\cdot) = y_{0,2}(\cdot) & \text{in } (0,1), \end{cases} \tag{1.8}$$

  with a Dirichlet control at the left end point either on the second or first component depending on the choices of $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, that is to say

$$\text{either} \quad y_1(t,0) = 0, \quad y_2(t,0) = v(t) \quad \text{in } (0,T), \tag{1.9a}$$

$$\text{or} \quad y_1(t,0) = v(t), \quad y_2(t,0) = 0 \quad \text{in } (0,T). \tag{1.9b}$$

- *Two Neumann control problems.* By taking into consideration (1.6b)–(1.7), we have two Neumann boundary control systems, that is the same system (1.8) along with the following two different kind of Neumann control at $x = 0$, depending on the choices of $B = \gamma_2(0)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ or $\gamma_1(0)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, that is

$$\text{either} \quad \partial_x y_1(t,0) = 0, \quad \partial_x y_2(t,0) = v(t) \quad \text{in } (0,T), \tag{1.10a}$$

$$\text{or} \quad \partial_x y_1(t,0) = v(t), \quad \partial_x y_2(t,0) = 0 \quad \text{in } (0,T). \tag{1.10b}$$

In this context, we mention that a null-controllability result has been proved in [7] for the linear Kuramoto-Sivashinsky equation on star-shaped trees with a Kirchhoff type boundary conditions. Of course in our setting, we can not assume a graph domain for any non-trivial interior coupling $\mathcal{M}_a$. Beside this, in [3, Remark 3.6] the authors mention a particular kind of boundary control system (with Dirichlet control) of type (1.8) when $a = 0$ and $\alpha = 0$; in particular, $a = 0$ immediately tells us that there is no interior coupling and in that case choosing a boundary control on either y_1 or y_2 does not make any difference in the system.

But as soon as $a \neq 0$, the two types of control systems (1.8)–(1.9a) or (1.9b) (also (1.8)–(1.10a) or (1.10b)) are certainly different in nature. In fact, we have the following two situations.

- **Case 1.** The boundary controllability of the system (1.8)–(1.9a), that is when we consider a control on the *second component* y_2, can be establish by means of global Carleman estimate (and then to find an observability inequality) for any interior coupling coefficient $a \in \mathbb{R}$ and boundary parameter $\alpha \geq 0$ which is precisely Theorem 2.6 (same tool can be adapted to the corresponding Neumann control case, that is to (1.8)–(1.10a)).
- **Case 2.** Surprisingly, when we consider our control to be acted on the *first component* y_1, it appears that the same tool can not be applied to the system (1.8)–(1.9b) (similarly, for the Neumann case (1.8)–(1.10b)), and in this situation a moments approach will be used. Moreover, we shall show in Theorem 2.7 that depending on the choices of quantities (α, a), the controllability issues significantly changes; indeed, in this situation we also find a class of negative results (see Remark 2.8) which is not alike the previous one.

1.2. Overview of the paper. For the theoretical part, we mainly study the two Dirichlet control systems under Case 1 and 2 above.

We prove the boundary controllability of the system (1.8)–(1.9a) in Section 3.1, where we establish a global boundary Carleman estimate to find an observability inequality for any $a \in \mathbb{R}$, $\alpha \geq 0$ and different diffusion coefficients γ_1, γ_2. But as we mentioned earlier, we can not apply the Carleman stuffs in the situation when the control is acting on y_1 instead of y_2, that is the system (1.8)–(1.9b) (see Remark 3.5 for details) and so in this situation, we take the advantage of applying the so-called moments technique to construct a control. In this case, we shall restrict ourselves to constant diffusion coefficients $\gamma_1 = \gamma_2 = 1$ to simplify the spectral properties of the adjoint to the corresponding elliptic operator, which we discuss in Section 4.1.2 in detail. This, together with the observation estimates in Section 4.2, we shall construct a control via moments method in Section 4.3. We also discuss the fact that how the controllability phenomena changes with respect to the choices of α and a.

To conclude, in Section 5 we introduce a discrete setting for the general control system (1.1). This will help to illustrate the controllability properties associated to systems (1.8) with boundary controls (1.9) or (1.10).

Notations. Throughout the paper we shall make use of following notations. The inner product and norm in the scalar space $L^2(0,1)$ will be simply denoted by $(\cdot,\cdot)_{L^2}$ and $\|\cdot\|_{L^2}$ respectively. We also denote the space $E := (L^2(0,1))^2$, its inner product and the norm by $(\cdot,\cdot)_E$ and $\|\cdot\|_E$ respectively. Moreover, we use the notation $\langle\cdot,\cdot\rangle_{X',X}$ to express the duality pair between a space X and its dual X'. Beside this, we sometimes write $\langle\cdot,\cdot\rangle_U$ with $U = \mathbb{R}^d$ or $\mathbb{C}^d$, $d \geq 1$, to specify the usual inner product in U.

Further, we declare $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ and $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$, where $\mathbb{R}^+$ denotes the set of all positive real numbers.

We use the letter C and subsequently $\widetilde{C}, C', C''$ to denote some positive constants (those may vary from line to line) which do possibly depend on $\gamma_1, \gamma_2, \alpha, a$ but not on T and y_0. Sometimes, we shall express some constants by $C_{p_1,p_2,\cdots,p_n}$ to specify its dependency on the quantities $p_1, p_2, \cdots, p_n$.

We often use the symbol M^* to denote the adjoint of a matrix or an operator M.

2. General settings and main results. In this section, we shall discuss briefly about the well-posedness of our systems (1.8)–(1.9a) and (1.8)–(1.9b) with L^2 boundary data. Also we will provide the main results concerning boundary null-controllability which are the main concerns of this paper.

2.1. Well-posedness of our systems.

2.1.1. The system with homogeneous Dirichlet data. Let us begin with the following coupled parabolic system with Kirchhoff condition at right end point and homogeneous Dirichlet conditions at left end point.

$$\begin{cases} \partial_t y_1 - \partial_x(\gamma_1 \partial_x y_1) = f_1 & \text{in } (0,T)\times(0,1), \\ \partial_t y_2 - \partial_x(\gamma_2 \partial_x y_2) + a y_1 = f_2 & \text{in } (0,T)\times(0,1), \\ y_1(t,0) = y_2(t,0) = 0 & \text{in } (0,T), \\ y_1(t,1) = y_2(t,1) & \text{in } (0,T), \\ \gamma_1(1)\partial_x y_1(t,1) + \gamma_2(1)\partial_x y_2(t,1) + \alpha\, y_1(t,1) = 0 & \text{in } (0,T), \\ y_1(0,\cdot) = y_{0,1}(\cdot), \quad y_2(0,\cdot) = y_{0,2}(\cdot) & \text{in } (0,1), \end{cases} \tag{2.1}$$

where regularity of $y_0 = (y_{0,1}, y_{0,2})$ and $f = (f_1, f_2)$ will be specified later.

We introduce the self-adjoint and positive elliptic operator $\mathcal{A}_\alpha$, corresponding to the above system

without interior coupling with its formal expression

$$\mathcal{A}_\alpha := \begin{pmatrix} -\partial_x(\gamma_1\partial_x) & 0 \\ 0 & -\partial_x(\gamma_2\partial_x) \end{pmatrix}, \tag{2.2a}$$

with its domain

$$D(\mathcal{A}_\alpha) := \Big\{ u=(u_1,u_2)\in (H^2(0,1))^2 \mid u_1(0)=u_2(0)=0,\ u_1(1)=u_2(1), \\ \gamma_1(1)u_1'(1)+\gamma_2(1)u_2'(1)+\alpha u_1(1)=0 \Big\}. \tag{2.2b}$$

Let us consider the space $\mathcal{H}_\alpha := D(\mathcal{A}_\alpha^{1/2})$ as a completion of $D(\mathcal{A}_\alpha)$ with respect to the norm

$$\|u\|_{\mathcal{H}_\alpha} := (\mathcal{A}_\alpha u,u)_E^{1/2} = \left(\sum_{i=1}^{2} \int_0^1 \gamma_i(x)|u_i'(x)|^2 dx + \alpha |u_1(1)|^2 \right)^{1/2}, \ \forall u\in D(\mathcal{A}_\alpha), \tag{2.3}$$

and one can prove that

$$\mathcal{H}_\alpha = \Big\{ u=(u_1,u_2)\in (H^1(0,1))^2 \mid u_1(0)=u_2(0)=0,\ u_1(1)=u_2(1) \Big\}. \tag{2.4}$$

Moreover, we denote the dual space of $\mathcal{H}_\alpha$ by $\mathcal{H}_{-\alpha}$ with respect to the pivot space E.

Recall the coupling matrix $\mathcal{M}_a$ defined in (1.5) and we further denote

$$\mathcal{A}_{\alpha,a} = \mathcal{A}_\alpha + \mathcal{M}_a, \quad \text{with the same domain } D(\mathcal{A}_{\alpha,a}) := D(\mathcal{A}_\alpha). \tag{2.5}$$

In particular, $\mathcal{A}_{\alpha,0} := \mathcal{A}_\alpha$.

By definition, it is clear that $\mathcal{A}_{\alpha,a}$ is not self-adjoint anymore, more precisely, $\mathcal{A}_{\alpha,a}$ has been obtained by a bounded perturbation $\mathcal{M}_a$ to the self-adjoint operator $\mathcal{A}_\alpha$.

PROPOSITION 2.1 (Existence of analytic semigroup). *The operator $(-\mathcal{A}_{\alpha,a}, D(\mathcal{A}_{\alpha,a}))$ defined by (2.5), generates an analytic semigroup in E.*

Proof. Let us first introduce the following densely defined bilinear form h; for all $u := (u_1,u_2)$, $\varphi := (\varphi_1,\varphi_2) \in \mathcal{H}_\alpha$ (defined by (2.4)), we consider

$$h(u,\varphi) = \sum_{i=1}^{2} \int_0^1 \gamma_i(x) u_i'(x)\varphi_i'(x)\,dx + a\int_0^1 u_1(x)\varphi_2(x)\,dx + \alpha u_1(1)\varphi_1(1). \tag{2.6}$$

It is clear that h is continuous in $\mathcal{H}_\alpha$ with

$$|h(u,\varphi)| \le \kappa_1 \|u\|_{\mathcal{H}_\alpha}\|\varphi\|_{\mathcal{H}_\alpha}, \quad \forall u,\varphi\in\mathcal{H}_\alpha,$$

where $\kappa_1>0$ depends on the diffusion coefficients γ_i, $i=1,2$, and the coupling coefficient a. On the other hand, we have

$$h(u,u) \ge \|u\|_{\mathcal{H}_\alpha}^2 - |a|\,\|u\|_E^2, \quad \forall u\in\mathcal{H}_\alpha.$$

Then, by [17, Proposition 1.51 and Theorem 1.52], the negative of the operator associated with h generates an analytic semigroup in E of angle $\big(\pi/2-\arctan\kappa_1\big)$. One can show that this operator is indeed $\mathcal{A}_{\alpha,a}$ with its domain $D(\mathcal{A}_{\alpha,a}) = D(\mathcal{A}_\alpha)$ (as defined in (2.5)). Henceforth, the proof is complete. ∎

PROPOSITION 2.2 (Regularity). *Let $f=(f_1,f_2)\in L^2(0,T;E)$ be any given source term.*

1. *For any given initial data $y_0=(y_{0,1},y_{0,2})\in E$, there exists a unique weak solution $y=(y_1,y_2)\in \mathcal{C}^0([0,T];E)\cap L^2(0,T;\mathcal{H}_\alpha)$ satisfying the following energy estimate*

$$\|y\|_{C^0([0,T];E)} + \|y\|_{L^2(0,T;\mathcal{H}_\alpha)} + \|\partial_t y\|_{L^2(0,T;\mathcal{H}_{-\alpha})} \le C_{T,a}\left(\|y_0\|_E + \|f\|_{L^2(0,T;E)}\right). \tag{2.7}$$

2. *For any initial data $y_0\in\mathcal{H}_\alpha$, the weak solution y belongs to the space $\mathcal{C}^0([0,T];\mathcal{H}_\alpha)\cap L^2(0,T;(H^2(0,1))^2)$ and satisfies*

$$\|y\|_{L^\infty(0,T;\mathcal{H}_\alpha)} + \|y\|_{L^2(0,T;(H^2(0,1))^2)} + \|\partial_t y\|_{L^2(0,T;E)} \\ \le C_{T,a}\left(\|y_0\|_{\mathcal{H}_\alpha} + \|f\|_{L^2(0,T;E)}\right). \tag{2.8}$$

Proof. 1. The existence of unique weak solution $y \in \mathcal{C}^0([0,T];E)$ to (2.1) for given data $y_0 \in E$ and source term $f \in L^2(0,T;E)$ can be concluded by Proposition 2.1.
Below, we provide the sketch of the proof for estimate (2.7). We shall prove the result with initial data $y_0 \in D(\mathcal{A}_\alpha)$ and the source term $f \in \mathcal{C}^1([0,T];E)$, which indeed gives us the existence of unique strong solution $y \in \mathcal{C}^1([0,T];E) \cap \mathcal{C}^0([0,T];D(\mathcal{A}_\alpha))$, and then by the usual density argument we deduce the final result in point 1.

- Let us test the first and second equations of (2.1) again y_1 and y_2 respectively, we obtain after an addition, for all $t \in [0,T]$,

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|_E^2 + (\mathcal{A}_\alpha y(t), y(t))_E + a(y_1(t), y_2(t))_{L^2} = (f(t), y(t))_E,$$

and finally

$$\frac{d}{dt}\|y(t)\|_E^2 + \|y(t)\|_{\mathcal{H}_\alpha}^2 \leq C_a\big(\|f(t)\|_E^2 + \|y(t)\|_E^2\big), \ \forall t \in [0,T]. \tag{2.9}$$

By applying Gronwall's lemma (see [10, Appendix B.2]) then by integration over $[0,T]$, we obtain the first two required estimates of (2.7).

- Next, to obtain the estimate of $\partial_t y \in L^2(0,T;\mathcal{H}_{-\alpha})$, let us pick any $\zeta := (\zeta_1, \zeta_2) \in \mathcal{H}_\alpha$ and observe that

$$\langle \partial_t y(t), \zeta\rangle_{\mathcal{H}_{-\alpha},\mathcal{H}_\alpha} + (\mathcal{A}_\alpha y(t), \zeta)_E + a(y_1(t), \zeta_2)_{L^2} = (f(t), \zeta)_E, \quad \forall t \in [0,T],$$

which implies

$$\big|\langle \partial_t y(t), \zeta\rangle_{\mathcal{H}_{-\alpha},\mathcal{H}_\alpha}\big| \leq C_a\big(\|y(t)\|_{\mathcal{H}_\alpha} + \|f(t)\|_E\big)\,\|\zeta\|_{\mathcal{H}_\alpha},$$

and the claim follows from the previous estimates.

2. We shall now prove the point 2 of our theorem, with the given data $y_0 \in D(\mathcal{A}_\alpha^2)$ and $f \in \mathcal{C}^1([0,T];D(\mathcal{A}_\alpha))$, and then again a density argument will give the required estimate (2.8) for any $y_0 \in \mathcal{H}_\alpha$ and $f \in L^2(0,T;E)$.

- We begin by testing the first and second equations of (2.1) by $\partial_t y_1$ and $\partial_t y_2$, and by addition we observe that for any $t \in [0,T]$,

$$\|\partial_t y(t)\|_E^2 + \frac{1}{2}\frac{d}{dt}(\mathcal{A}_\alpha y(t), y(t))_E = -a(y_1(t), \partial_t y_2(t))_{L^2} + (f(t), \partial_t y(t))_E. \tag{2.10}$$

We now make use of Cauchy-Schwarz inequality to deduce

$$|a(y_1(t), \partial_t y_2(t))_{L^2} + (f(t), \partial_t y(t))_E| \leq \frac{1}{2}\|\partial_t y(t)\|_E^2 + C_a\big(\|y(t)\|_E^2 + \|f(t)\|_E^2\big).$$

Implementing this bound in (2.10) we respectively obtain the third and first estimate of (2.8).

- The $L^2(0,T;(H^2(0,1))^2)$ estimate for y simply follows from the bound

$$\|\partial_x^2 y\|_{L^2(0,T;E)} \leq \|f\|_{L^2(0,T;E)} + \|\partial_t y\|_{L^2(0,T;E)} + \|y\|_{L^2(0,T;E)},$$

and the previous two estimates. ∎

2.1.2. The system with non-homogeneous Dirichlet data. We consider here a similar coupled system as in the previous paragraph but with non-smooth Dirichlet boundary data; the system under study is the following

$$\begin{cases} \partial_t y_1 - \partial_x(\gamma_1 \partial_x y_1) = f_1 & \text{in } (0,T)\times(0,1), \\ \partial_t y_2 - \partial_x(\gamma_2 \partial_x y_2) + a\, y_1 = f_2 & \text{in } (0,T)\times(0,1), \\ y_1(t,0) = g_1 & \text{in } (0,T), \\ y_2(t,0) = g_2 & \text{in } (0,T), \\ y_1(t,1) = y_2(t,1) & \text{in } (0,T), \\ \gamma_1(1)\partial_x y_1(t,1) + \gamma_2(1)\partial_x y_2(t,1) + \alpha\, y_1(t,1) = 0 & \text{in } (0,T), \\ y_1(0,\cdot) = y_{0,1}(\cdot), \quad y_2(0,\cdot) = y_{0,2}(\cdot) & \text{in } (0,1), \end{cases} \tag{2.11}$$

In this context, it is worth introducing the adjoint of the operator $\mathcal{A}_{\alpha,a}$ (introduced in (2.5)), with its formal expression

$$\mathcal{A}^*_{\alpha,a} = \mathcal{A}^*_\alpha + \mathcal{M}^*_a = \begin{pmatrix} -\partial_x(\gamma_1\partial_x) & a \\ 0 & -\partial_x(\gamma_2\partial_x) \end{pmatrix}, \tag{2.12}$$

with its domain $D(\mathcal{A}^*_{\alpha,a}) = D(\mathcal{A}_{\alpha,a}) = D(\mathcal{A}_\alpha)$ (given by (2.2b)).

REMARK 2.3. *We could have replaced $\mathcal{A}^*_\alpha$ simply by $\mathcal{A}_\alpha$ as this operator is self-adjoint, but in order to be consistent with the non self-adjoint case (that is when $a \neq 0$), we decide to keep the notation $\mathcal{A}^*_\alpha$ in several places.*

Observe that the operator $-\mathcal{A}^*_{\alpha,a}$ also generates an analytic semigroup in E, thanks to Theorem 2.1 and we denote this semigroup by $\left(e^{-t\mathcal{A}^*_{\alpha,a}}\right)_{t\geq 0}$. Indeed, the solution to the adjoint system of (2.1), for any given $\zeta \in \mathcal{H}_\alpha$, satisfies the regularity result proved in point 2 of Theorem 2.2. Using this, one can classically obtain the well-posedness of the solution to (2.11) in a dual sense as in [8, 18].

PROPOSITION 2.4. *For any $y_0 := (y_{0,1}, y_{0,2}) \in \mathcal{H}_{-\alpha}$, $f := (f_1, f_2) \in L^2(0,T;E)$ and $g := (g_1, g_2) \in L^2(0,T;\mathbb{R}^2)$, there exists a unique $y \in \mathcal{C}^0([0,T];\mathcal{H}_{-\alpha}) \cap L^2(0,T;E)$, solution to (2.11), in the following sense: for any $t \in [0,T]$ and $\zeta := (\zeta_1, \zeta_2) \in \mathcal{H}_\alpha$, we have*

$$\langle y(t), \zeta\rangle_{\mathcal{H}_{-\alpha},\mathcal{H}_\alpha} = \langle y_0, e^{-t\mathcal{A}^*_{\alpha,a}}\zeta\rangle_{\mathcal{H}_{-\alpha},\mathcal{H}_\alpha} + \int_0^t (f(s), e^{-(t-s)\mathcal{A}^*_{\alpha,a}}\zeta)_E\, ds \\ - \int_0^t \left\langle g(s), \left(\frac{\partial}{\partial\nu_\gamma}\left(e^{-(t-s)\mathcal{A}^*_{\alpha,a}}\zeta\right)(x)\right)\Big|_{x=0}\right\rangle_{\mathbb{R}^2} ds.$$

2.2. Main results. We shall now formulate the null-control problems in terms of following proposition.

PROPOSITION 2.5. *Let $y_0 \in \mathcal{H}_{-\alpha}$, $a \in \mathbb{R}$, $\alpha \geq 0$ and any finite time $T > 0$ be given. Also recall the set U as defined in Proposition 2.4.*

1. *A function $v \in L^2(0,T;\mathbb{R})$ is a null-control for the problem (1.8)–(1.9a), if and only if it satisfies: for any $\zeta \in \mathcal{H}_\alpha$*

$$-\langle y_0, e^{-t\mathcal{A}^*_{\alpha,a}}\zeta\rangle_{\mathcal{H}_{-\alpha},\mathcal{H}_\alpha} = \gamma_2(0)\int_0^T v(t)\left\langle \begin{pmatrix}0\\1\end{pmatrix}, \left(\partial_x\left(e^{-(T-t)\mathcal{A}^*_{\alpha,a}}\zeta\right)(x)\right)\Big|_{x=0}\right\rangle_{\mathbb{R}^2} dt. \tag{2.13}$$

2. *A function $v \in L^2(0,T)$ is a null-control for the problem (1.8)–(1.9b), if and only if it satisfies: for any $\zeta \in \mathcal{H}_\alpha$*

$$-\langle y_0, e^{-t\mathcal{A}^*_{\alpha,a}}\zeta\rangle_{\mathcal{H}_{-\alpha},\mathcal{H}_\alpha} = \gamma_1(0)\int_0^T v(t)\left\langle \begin{pmatrix}1\\0\end{pmatrix}, \left(\partial_x\left(e^{-(T-t)\mathcal{A}^*_{\alpha,a}}\zeta\right)(x)\right)\Big|_{x=0}\right\rangle_{\mathbb{R}^2} dt. \tag{2.14}$$

Here, it is convenient to denote the observation operator (that does not depend on the quantities a or α) as follows

$$\mathcal{B}^*_1 : u = (u_1, u_2) \in (H^2(0,1))^2 \mapsto \gamma_2(0)u_2'(0), \tag{2.15a}$$

$$\mathcal{B}^*_2 : u = (u_1, u_2) \in (H^2(0,1))^2 \mapsto \gamma_1(0)u_1'(0). \tag{2.15b}$$

Now, we present the main theorems regarding the null-controllability issues for our problem (1.8) with both cases: the boundary-control to be applied on either y_1 or y_2; we shall also achieve some suitable estimates of the controls, depending on the coupling coefficient a, the boundary parameter α, as well as the diffusion coefficients γ_1, γ_2.

- **Case 1.** To show the boundary null-controllability of the problem (1.8)–(1.9a), that is when we consider the control applied on y_2, we prove a suitable observability inequality, and since we are in linear case so this will be obtained by a Carleman estimate, detailed in Section 3.1. Our main theorem is the following.

  THEOREM 2.6. *Let any $(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}$ and $T > 0$ be given. Then, for any $y_0 \in \mathcal{H}_{-\alpha}$, there exists a null-control $v \in L^2(0,T;\mathbb{R})$ for the problem (1.8)–(1.9a), that satisfies the estimate*

$$\|v\|_{L^2(0,T)} \leq Ce^{C/T}\|y_0\|_{\mathcal{H}_{-\alpha}},$$

  with the constant $C := C(\gamma_1, \gamma_2, \alpha, a) > 0$ which does not depend on $T > 0$ and y_0.

- **Case 2.** As we mentioned earlier, the above strategy of using Carleman estimate to prove the boundary controllability will no more be applicable for the problem (1.8) with boundary control on y_1, that is precisely (1.9b). This is because, the source integral due to the interior coupling in our Carleman estimate can not be observable, with our choices of weight functions. The exact technical point behind this will be specified later in Remark 3.5 in Section 3.1.
  Due to this obstacle, the next immediate idea is to investigate the spectral analysis of the adjoint to the corresponding elliptic operator and try to develop the moments method to construct a control by hand; here we shall restrict the diffusion coefficients $\gamma_1 = \gamma_2 = 1$ to ease the understanding of the spectrum. Indeed, by developing the spectral analysis, we will observe that the choices of coupling coefficient a and the boundary parameter α really have a crucial role for the controllability of (1.8)–(1.9b), which is not alike the case when we consider our control applied on the second component y_2, as per Theorem 2.6.
  Henceforth, it is reasonable not to find a good observability inequality using Carleman estimate in the case (1.8)–(1.9b), when the control input is assumed to be applied on the first component y_1. Let us state more precisely the controllability theorem concerning this case.

  THEOREM 2.7. *We fix $\gamma_1 = \gamma_2 = 1$. Then, there exists a set $\mathcal{R} \subset \mathbb{R}_0^+ \times \mathbb{R}^*$ such that*
  1. *for each pair $(\alpha, a) \notin \mathcal{R}$, there is a null-control to the problem (1.8)–(1.9b), for any given data $y_0 \in \mathcal{H}_{-\alpha}$,*
  2. *for each pair $(\alpha, a) \in \mathcal{R}$, there exists a subspace $\mathcal{Y}_{\alpha,a} \subset \mathcal{H}_{-\alpha}$ of co-dimension 1, such that there exists a null-control to the problem (1.8)–(1.9b), if and only if $y_0 \in \mathcal{Y}_{\alpha,a}$.*

  In addition, in the controllable cases we can choose such a null-control $v \in L^2(0,T;\mathbb{R})$ that satisfies the bound

$$\|v\|_{L^2(0,T)} \leq C_{\alpha,a}\, e^{C_{\alpha,a}/T} \|y_0\|_{\mathcal{H}_{-\alpha}}, \tag{2.16}$$

  where $C_{\alpha,a} > 0$ is independent on $T > 0$ and y_0.

  The set $\mathcal{R}$ and the space $\mathcal{Y}_{\alpha,a}$ will be specified later, namely in (4.29) and (4.31), while proving Lemma 4.7 in Section 4.2.1.

  REMARK 2.8. *In the case when $(\alpha, a) \in \mathcal{R}$, the problem (1.8)–(1.9b) is not even approximately controllable if we choose our initial data $y_0 \notin \mathcal{Y}_{\alpha,a}$.*

In the next sections, we develop the required results to prove the controllability of both the problems, namely the Theorem 2.6 and 2.7.

3. Boundary controllability of the system with control in the second component.

This section is devoted to prove the existence of a null-control of the coupled system (1.8)–(1.9a), in terms of finding a proper observability inequality, and so the Carleman estimate is the main ingredient to obtain.

3.1. A global boundary Carleman estimate.

Let us first write the adjoint system to (1.8)–(1.9a), with homogeneous Dirichlet conditions at the left end point.

$$\begin{cases} -\partial_t q_1 - \partial_x(\gamma_1 \partial_x q_1) + a\, q_2 = 0 & \text{in } (0,T)\times(0,1), \\ -\partial_t q_2 - \partial_x(\gamma_2 \partial_x q_2) = 0 & \text{in } (0,T)\times(0,1), \\ q_1(t,0) = q_2(t,0) = 0 & \text{in } (0,T), \\ q_1(t,1) = q_2(t,1) & \text{in } (0,T), \\ \gamma_1(1)\partial_x q_1(t,1) + \gamma_2(1)\partial_x q_2(t,1) + \alpha\, q_1(t,1) = 0 & \text{in } (0,T), \\ q_1(T,\cdot) = \zeta_1(\cdot),\ q_2(T,\cdot) = \zeta_2(\cdot) & \text{in } (0,1), \end{cases} \tag{3.1}$$

where the regularity of $\zeta := (\zeta_1, \zeta_2)$ will be imposed later when needed and for simplicity sometimes we shall use the notation $Q := (0,T)\times(0,1)$.

Now, we introduce the following space

$$\mathcal{Q} := \Big\{ q = (q_1, q_2) \in (\mathcal{C}^2(\overline{Q}))^2 \mid q_1(t,0) = q_2(t,0) = 0,\ q_1(t,1) = q_2(t,1), \\ \sum_{i=1}^{2} \gamma_i(1)\partial_x q_i(t,1) + \alpha q_1(t,1) = 0,\ \forall t \in (0,T) \Big\}.$$

Before introducing the main theorem regarding Carleman estimate, we define some standard weight functions which are the main ingredients to obtain the Carleman inequality.

Construction of the weight functions. Let $\mu_0 \in (0,1)$ close enough to 1 such that

$$\left[\frac{216\mu_0}{(1-\mu_0)^3}\gamma_2^2(1) - 7\gamma_1^2(1)\right] \geq 1. \tag{3.2}$$

We consider the following affine functions

$$\begin{cases} \beta_i(x) = 2 + c_i(x-1), \quad \forall x \in [0,1], \\ \text{with} \quad c_1 = 1, \quad c_2 = \dfrac{-6}{(1-\mu_0)}, \quad \text{for} \quad 0 < \mu_0 < 1, \end{cases} \tag{3.3}$$

that satisfy the following properties

$$\beta_2 \geq \beta_1 > 0, \text{ in } [0,1], \quad \beta_2(1) = \beta_1(1). \tag{3.4}$$

Remark 3.1. *We will see while proving a Carleman estimate (namely the Theorem 3.2 stated later), that the above assumption (3.2) is very sharp and crucial to absorb some unusual boundary integrals sitting in the right hand side of the Carleman estimate.*

Now, we assume that $\lambda > 1$ and $K = 2\max\{\|\beta_1\|_\infty, \|\beta_2\|_\infty\}$ and define the weight functions φ_i and η_i, for $i = 1, 2$, as follows

$$\varphi_i(t,x) = \frac{e^{\lambda\beta_i(x)}}{t(T-t)}, \quad \eta_i(t,x) = \frac{e^{\lambda K} - e^{\lambda\beta_i(x)}}{t(T-t)}, \quad \forall (t,x) \in Q. \tag{3.5}$$

From the properties of β_1 and β_2 in (3.4), we have that the functions φ_i and η_i are positive and satisfy

$$\varphi_1(t,1) = \varphi_2(t,1) \quad \text{and} \quad \eta_1(t,1) = \eta_2(t,1), \tag{3.6}$$

since $\beta_1(1) = \beta_2(1)$.

We also have the following relations in Q, for $i = 1, 2$,

$$\begin{cases} \partial_x\varphi_i = \lambda\varphi_i c_i, \quad \partial_x\eta_i = -\lambda\varphi_i c_i, \\ \partial_t\varphi_i = \varphi_i \dfrac{2t-T}{t(T-t)}, \quad \partial_t\eta_i = \eta_i \dfrac{2t-T}{t(T-t)}, \\ \partial_t^2\eta_i = \eta_i \dfrac{3(2t-T)^2 + T^2}{2t^2(T-t)^2}. \end{cases} \tag{3.7}$$

Now, we write the main theorem of this section concerning the Carleman estimate.

Theorem 3.2 (A Carleman estimate). *Let the weight functions φ_1, φ_2 and η_1, η_2 be defined as in (3.5). Then, there exists $\lambda_1 := \lambda_1(\gamma_1, \gamma_2, \alpha) > 0$, $s_1 := (T^2+T)\sigma_1 > 0$ with some $\sigma_1 := \sigma_1(\gamma_1, \gamma_2, \alpha) > 0$ and a constant $C' := C'(\gamma_1, \gamma_2, \alpha) > 0$, such that the following Carleman estimate holds true*

$$\begin{aligned} &s^3\lambda^4 \sum_{i=1}^{2} \int_0^T \int_0^1 e^{-2s\eta_i}\varphi_i^3 |q_i|^2 dx\, dt + s\lambda^2 \sum_{i=1}^{2} \int_0^T \int_0^1 e^{-2s\eta_i}\varphi_i |\partial_x q_i|^2 dx\, dt \\ &+ s^3\lambda^3 \int_0^T \varphi_1(t,1) e^{-2s\eta_1(t,1)} |q_1(t,1)|^2 dt \leq C' \Bigg[\sum_{i=1}^{2} \int_0^T \int_0^1 e^{-2s\eta_i} \left|\partial_t q_i + \partial_x(\gamma_i \partial_x q_i)\right|^2 dx\, dt \\ &\qquad + s\lambda \int_0^T \varphi_2(t,0) e^{-2s\eta_2(t,0)} |\partial_x q_2(t,0)|^2 dt \Bigg], \end{aligned} \tag{3.8}$$

for $s \geq s_1$, $\lambda \geq \lambda_1$ and for all $(q_1, q_2) \in \mathcal{Q}$.

Before going to the proof of the theorem above, we let any $s > 0$, $\lambda > 1$ and $(q_1, q_2) \in \mathcal{Q}$ and we write $f_i = \partial_t q_i + \partial_x(\gamma_i \partial_x q_i)$, then $f_i \in L^2(Q)$, for $i = 1, 2$. We also set

$$\psi_i(t,x) = e^{-s\eta_i(t,x)} q_i(t,x), \quad \forall (t,x) \in Q, \quad \text{for } i = 1, 2.$$

Observe that,

$$\psi_i(t,0) = 0, \ i = 1, 2, \quad \text{and} \quad \psi_1(t,1) = \psi_2(t,1), \tag{3.9}$$

using (3.6) and the properties of q_i, $i=1,2$ in $\mathcal{Q}$. Also look that

$$\partial_x\psi_i(t,x)=e^{-s\eta_i(t,x)}\partial_x q_i(t,x)+s\lambda\beta_i'(x)\varphi_i(t,x)\psi_i(t,x),\ \forall (t,x)\in Q, \tag{3.10}$$

so that we have

$$\sum_{i=1}^{2}\gamma_i(1)\partial_x\psi_i(t,1)=-\alpha\psi_1(t,1)+s\lambda\Big(\sum_{i=1}^{2}c_i\,\gamma_i(1)\Big)\varphi_1(t,1)\psi_1(t,1), \tag{3.11}$$

thanks to the boundary condition $\sum_{i=1}^{2}\gamma_i(1)\partial_x q_i(t,1)+\alpha q_1(t,1)=0$, the properties of φ_i in (3.6) and ψ_i in (3.9), and the fact that $\beta_i'=c_i$, for $i=1,2$.

Next, we see that the functions ψ_i satisfies the following relations in Q

$$M_1\psi_i+M_2\psi_i=F_i,\quad \text{for } i=1,2,$$

with

$$\begin{cases} M_1\psi_i=\partial_x(\gamma_i\partial_x\psi_i)+s^2\lambda^2c_i^2\varphi_i^2\gamma_i\psi_i+s(\partial_t\eta_i)\psi_i,\\ M_2\psi_i=\partial_t\psi_i-2s\lambda c_i\varphi_i(\gamma_i\partial_x\psi_i)-2s\lambda^2c_i^2\varphi_i\gamma_i\psi_i,\\ F_i=e^{-s\eta_i}f_i+s\lambda c_i\gamma_i'\varphi_i\psi_i-s\lambda^2c_i^2\varphi_i\gamma_i\psi_i. \end{cases} \tag{3.12}$$

We have for $i=1,2$,

$$\|M_1\psi_i\|^2_{L^2(Q)}+\|M_2\psi_i\|^2_{L^2(Q)}+2\big(M_1\psi_i,M_2\psi_i\big)_{L^2(Q)}=\|F_i\|^2_{L^2(Q)}. \tag{3.13}$$

Now, we present the following auxiliary lemma which is important to prove the main result in Theorem 3.2.

LEMMA 3.3. *Let the functions φ_i, η_i, ψ_i, $M_1\psi_i, M_2\psi_2$ in Q, for $i=1,2$, and the quantities c_1,c_2 be as introduced earlier. Then there exists $\lambda_0:=\lambda_0(\gamma_1,\gamma_2)>0$, $s_0:=(T^2+T)\sigma_0>0$ with some $\sigma_0:=\sigma_0(\gamma_1,\gamma_2)>0$ and a constant $C''=C''(\gamma_1,\gamma_2)>0$ such that we have the following inequality*

$$\begin{aligned} &\frac12\sum_{i=1}^{2}\|M_1\psi_i\|^2_{L^2(Q)}+\frac12\sum_{i=1}^{2}\|M_2\psi_i\|^2_{L^2(Q)}\\ &\qquad+\sum_{i=1}^{2}s^3\lambda^4\int_0^T\int_0^1\varphi_i^3|\psi_i|^2\,dx\,dt+\sum_{i=1}^{2}s\lambda^2\int_0^T\int_0^1\varphi_i|\partial_x\psi_i|^2\,dx\,dt\\ &\qquad+\sum_{i=1}^{2}\gamma_i(1)\int_0^T\partial_x\psi_i(t,1)\partial_t\psi_i(t,1)\,dt-s\lambda\sum_{i=1}^{2}c_i\int_0^T\varphi_i(t,1)\big|\gamma_i(1)\partial_x\psi_i(t,1)\big|^2dt\\ &+s\lambda\sum_{i=1}^{2}c_i\int_0^T\varphi_i(t,0)\big|\gamma_i(0)\partial_x\psi_i(t,0)\big|^2dt-2s\lambda^2\sum_{i=1}^{2}c_i^2\gamma_i^2(1)\int_0^T\varphi_i(t,1)\psi_i(t,1)\partial_x\psi_i(t,1)\,dt\\ &-s^3\lambda^3c_i^3\sum_{i=1}^{2}\gamma_i^2(1)\int_0^T\varphi_i^3(t,1)|\psi_i(t,1)|^2dt-s^2\lambda\sum_{i=1}^{2}c_i\gamma_i(1)\int_0^T\varphi_i(t,1)(\partial_t\eta_i)(t,1)|\psi_i(t,1)|^2dt\\ &\qquad\le C''\sum_{i=1}^{2}\|e^{-s\eta_i}f_i\|^2_{L^2(Q)}, \end{aligned} \tag{3.14}$$

for all $\lambda\ge\lambda_0$, $s\ge s_0$.

In this paper, we decide to omit the full proof for this auxiliary result in Lemma 3.3, as the computations we need to perform are more or less of standard fashion. We now focus on obtaining the Carleman estimate (3.8) which is the main concerned of this section.

Proof of Theorem 3.2.

The main idea to prove this theorem is to play with the boundary integrals of the inequality (3.14), so that we can absorb the lower order integrals by some leading terms and then to observe the proper observation term which will be eventually shifted in the right hand side.

We make use of the following notations: denote the all six boundary terms respectively by J_k, $1\le k\le 6$, by maintaining the same order as in (3.14).

- We have (using $\psi_2(t,1)=\psi_1(t,1)$)

$$J_1 := \int_0^T [\gamma_1(1)\partial_x\psi_1(t,1)+\gamma_2(1)\partial_x\psi_2(t,1)]\,\partial_t\psi_1(t,1)\,dt$$
$$= -\alpha\int_0^T \psi_1(t,1)\partial_t\psi_1(t,1)\,dt + s\lambda\Big(\sum_{i=1}^2 c_i\gamma_i(1)\Big)\int_0^T \varphi_1(t,1)\psi_1(t,1)\partial_t\psi_1(t,1)\,dt$$
$$= -\frac{s\lambda}{2}\Big(\sum_{i=1}^2 c_i\gamma_i(1)\Big)\int_0^T (\partial_t\varphi_1)(t,1)|\psi_1(t,1)|^2dt,$$

due to the condition (3.11) and the fact that $\psi_1(0,\cdot)=\psi_1(T,\cdot)=0$. Now, using $|\partial_t\varphi_1|\le T\varphi_1^2\le 2T^3\varphi_1^3$, we obtain

$$|J_1|\le \widetilde{C}s\lambda T^3\int_0^T \varphi_1^3(t,1)|\psi_1(t,1)|^2dt, \tag{3.15}$$

for some constant $\widetilde{C}>0$.

- Next, we write the second boundary term of (3.14) as $J_2 := J_{21}+J_{22}$, where we keep the second integral in the left hand side of (3.14) since

$$J_{22} := -s\lambda\, c_2\int_0^T \varphi_2(t,1)\big|\gamma_2(1)\partial_x\psi_2(t,1)\big|^2dt \ge 0 \tag{3.16}$$

due to the fact that $c_2<0$. Later, we will see that the integral J_{22} will be used to absorb some lower order terms.
On the other hand, the first integral of the second boundary term J_2 is

$$J_{21} := -s\lambda\, c_1\int_0^T \varphi_1(t,1)\big|\gamma_1(1)\partial_x\psi_1(t,1)\big|^2dt,$$

where $c_1>0$ and so $J_{21}\le 0$. So, we need to absorb those integrals by some higher order terms in the left hand side. Let us recall (3.11) to express

$$\gamma_1(1)\partial_x\psi_1(t,1) = -\alpha\psi_1(t,1)-\gamma_2(1)\partial_x\psi_2(t,1)+s\lambda\Big(\sum_{i=1}^2 c_i\gamma_i(1)\Big)\varphi_1(t,1)\psi_1(t,1),$$

so that we have the following,

$$|J_{21}|\le 3s\lambda\alpha^2c_1\int_0^T \varphi_1(t,1)|\psi_1(t,1)|^2dt + 3s\lambda c_1\int_0^T \varphi_1(t,1)\big|\gamma_2(1)\partial_x\psi_2(t,1)\big|^2dt$$
$$+ 6s^3\lambda^3c_1\Big(\sum_{i=1}^2 c_i^2\gamma_i^2(1)\Big)\int_0^T \varphi_1^3(t,1)|\psi_1(t,1)|^2dt := J_{21}^1+J_{21}^2+J_{21}^3, \tag{3.17}$$

with a simple observation (since $\varphi_1\le 4T^4\varphi_1^3$),

$$J_{21}^1\le \widetilde{C}s\lambda\alpha^2T^4\int_0^T \varphi_1^3(t,1)|\psi_1(t,1)|^2dt, \tag{3.18}$$

- Now, we look into the third boundary term, $J_3 := J_{31}+J_{32}$, where we have

$$J_{31} := s\lambda\, c_1\int_0^T \varphi_1(t,0)\big|\gamma_1(0)\partial_x\psi_1(t,0)\big|^2dt \ge 0,$$

since $c_1>0$ and the function $\varphi_1>0$, and so one can discard this term from the left hand side of (3.14).
On the other hand, we have

$$|J_{32}| := s\lambda\left|c_2\int_0^T \varphi_2(t,0)\big|\gamma_2(0)\partial_x\psi_2(t,0)\big|^2dt\right| \le \widetilde{C}s\lambda\int_0^T \varphi_2(t,0)e^{-2s\eta_2(t,0)}|\partial_x q_2(t,0)|^2dt, \tag{3.19}$$

following the expression of $\partial_x\psi_2$ given by (3.10) and using the fact that $\psi_2(t,0)=0$.

- Thereafter, we write the fourth boundary integral of (3.14) by $J_4 := J_{41} + J_{42}$, and we obtain the following

$$|J_{41}| = 2s\lambda^2 c_1^2\gamma_1^2(1)\int_0^T \varphi_1(t,1)\big|\psi_1(t,1)\partial_x\psi_1(t,1)\big|dt \tag{3.20}$$
$$\leq \widetilde{C}\epsilon\, s\lambda\int_0^T \varphi_1(t,1)\big|\gamma_1(1)\partial_x\psi_1(t,1)\big|^2 dt + \frac{\widetilde{C}}{\epsilon}\, s\lambda^3T^4\int_0^T \varphi_1^3(t,1)|\psi_1(t,1)|^2\, dt,$$

where we have used the Young's inequality and the fact $\varphi_1 \leq 2T^4\varphi_1^3$. Now, for the first integral in the right hand side of (3.20), we use the estimate for J_{21} given by (3.17) to obtain

$$|J_{41}| \leq \widetilde{C}\epsilon\, s\lambda\int_0^T \varphi_1(t,1)\big|\gamma_2(1)\partial_x\psi_2(t,1)\big|^2 dt \tag{3.21}$$
$$+\widetilde{C}\big(\epsilon\, s^3\lambda^3 + \epsilon\, s\lambda\alpha^2T^4 + \frac{1}{\epsilon}s\lambda^3T^4\big)\int_0^T \varphi_1^3(t,1)|\psi_1(t,1)|^2dt.$$

On the other hand, a similar computation as in (3.20) gives that

$$|J_{42}| \leq \widetilde{C}\epsilon\, s\lambda\int_0^T \varphi_2(t,1)\big|\gamma_2(1)\partial_x\psi_2(t,1)\big|^2 dt + \frac{\widetilde{C}}{\epsilon}s\lambda^3T^4\int_0^T \varphi_2^3(t,1)|\psi_2(t,1)|^2dt. \tag{3.22}$$

- The fifth and the leading boundary term in the left hand side of (3.14) is

$$J_5 = s^3\lambda^3\left(-c_2^3\gamma_2^2(1) - c_1^3\gamma_1^2(1)\right)\int_0^T \varphi_1^3(t,1)|\psi_1(t,1)|^2\, dt, \tag{3.23}$$

by writing $\varphi_2(t,1) = \varphi_1(t,1)$.
- Finally, the sixth boundary term J_6 satisfies

$$|J_6| \leq \widetilde{C}s^2\lambda T\int_0^T \varphi_1^3(t,1)|\psi_1(t,1)|^2\, dt, \tag{3.24}$$

due to the facts that $\varphi_2(t,1) = \varphi_1(t,1)$, $\partial_t\eta_2(t,1) = \partial_t\eta_1(t,1)$ and $|\partial_t\eta_1| \leq T\varphi_1^2$.
- Let us first try to show that the coefficient of the boundary integral $\int_0^T \varphi_1^3(t,1)|\psi_1(t,1)|^2\, dt$ is positive in the left hand side of the main inequality (3.14). To deduce this, recall the quantities J_5 from (3.23) and J_{21}^3 from (3.17) and take those quantities in the left hand side of the main inequality (3.14), we see

$$J_5 - J_{21}^3 = K_1 s^3\lambda^3\int_0^T \varphi_1^3(t,1)|\psi_1(t,1)|^2dt, \quad \text{with } K_1 = c_2^2\gamma_2^2(1)\left(|c_2| - 6c_1\right) - 7c_1^3\gamma_1^2(1). \tag{3.25}$$

Recall that we haven chosen $c_1 = 1, c_2 = \frac{-6}{(1-\mu_0)}$ (see (3.3)), so that we calculate

$$K_1 = \left[\frac{216\mu_0}{(1-\mu_0)^3}\gamma_2^2(1) - 7\gamma_1^2(1)\right] \geq 1,$$

thanks to the condition (3.4).
- Next, we recall J_{22} and J_{21}^2 respectively given by (3.16) and (3.17), use $\varphi_1(t,1) = \varphi_2(t,1)$, and we write the other leading boundary integral in the left hand side as follows

$$J_{22} - J_{21}^2 = K_2 s\lambda\int_0^T \varphi_2(t,1)\big|\gamma_2(1)\partial_x\psi_2(t,1)\big|^2 dt, \text{ with } K_2 = (|c_2| - 3c_1), \tag{3.26}$$

where we compute $K_2 = \frac{3(1+\mu_0)}{(1-\mu_0)} > 0$, using the values of c_1, c_2.
- Now, we gather the leading boundary terms $J_5 - J_{21}^3$ and $J_{22} - J_{21}^2$ given by (3.25) and (3.26) respectively, in the left hand side of our main inequality (3.14), and in the right hand side we consider all the estimates of the lower order terms namely, J_1 from (3.15), J_{21}^1 from (3.18), J_{31}

from (3.19), J_{41} from (3.21), J_{42} from (3.22) and J_6 from (3.24), so that the inequality (3.14) follows

$$
\begin{aligned}
(3.27)\quad & s^3\lambda^4\sum_{i=1}^{2}\int_0^T\int_0^1 \varphi_i^3|\psi_i|^2\,dx\,dt + +s\lambda^2\sum_{i=1}^{2}\int_0^T\int_0^1 \varphi_i|\partial_x\psi_i|^2\,dx\,dt \\
&\quad + K_1 s^3\lambda^3\int_0^T \varphi_1^3(t,1)|\psi_1(t,1)|^2dt + K_2 s\lambda\int_0^T \varphi_2(t,1)\big|\gamma_2(1)\partial_x\psi_2(t,1)\big|^2dt \\
&\quad \le \widetilde{C}\Big[\sum_{i=1}^{2}\int_0^T\int_0^1 e^{-2s\eta_i}|f_i|^2\,dx\,dt + s\lambda\int_0^T \varphi_2(t,0)e^{-2s\eta_2(t,0)}|\partial_x q_2(t,0)|^2dt\Big] \\
&\quad + \widetilde{C}\epsilon\, s^3\lambda^3\int_0^T \varphi_1^3(t,1)|\psi_1(t,1)|^2dt + 2\widetilde{C}\epsilon\, s\lambda\int_0^T \varphi_2(t,1)\big|\gamma_2(1)\partial_x\psi_2(t,1)\big|^2dt \\
&\qquad\qquad + \widetilde{C}_\epsilon\big(s\lambda T^3 + s\lambda\alpha^2T^4 + s\lambda^3T^4 + s^2\lambda T\big)\int_0^T \varphi_1^3(t,1)|\psi_1(t,1)|^2dt.
\end{aligned}
$$

By choosing $\epsilon > 0$ to be small enough, taking $\lambda \ge \lambda_1 := \lambda_1(\gamma_1,\gamma_2,\alpha)$ and $s \ge s_1 := (T^2+T)\sigma_1(\gamma_1,\gamma_2,\alpha) > 0$, where λ_1 and σ_1 are large enough so that all the boundary integrals, except the observation term, in right hand side can be absorbed by the corresponding integrals in left hand side of (3.27), which leads us

$$
\begin{aligned}
(3.28)\quad & s^3\lambda^4\sum_{i=1}^{2}\int_0^T\int_0^1 \varphi_i^3|\psi_i|^2\,dx\,dt + s\lambda^2\sum_{i=1}^{2}\int_0^T\int_0^1 \varphi_i|\partial_x\psi_i|^2\,dx\,dt \\
&\quad + s^3\lambda^3\int_0^T \varphi_1(t,1)|\psi_1(t,1)|^2dt \le C'\Big[\sum_{i=1}^{2}\int_0^T\int_0^1 e^{-2s\eta_i}|f_i|^2\,dx\,dt \\
&\qquad\qquad + s\lambda\int_0^T \varphi_2(t,0)e^{-2s\eta_2(t,0)}|\partial_x q_2(t,0)|^2dt\Big],
\end{aligned}
$$

for all $\lambda \ge \lambda_1$, $s \ge s_1$ with the constant $C' := C'(\gamma_1,\gamma_2,\alpha)$. Now, we recall the expression of $\partial_x\psi_i$ from (3.10), so that we have

$$
e^{-2s\eta_i}|\partial_x q_i|^2 \le 2|\partial_x\psi_i|^2 + 2s^2\lambda^2(c_i)^2\varphi_i^2|\psi_i|^2, \quad \text{for } i=1,2,
$$

and that implies

$$
\begin{aligned}
(3.29)\quad & s\lambda^2\sum_{i=1}^{2}\int_0^T\int_0^1 e^{-2s\eta_i}\varphi_i|\partial_x q_i|^2\,dx\,dt \\
&\qquad \le \widetilde{C}s\lambda^2\sum_{i=1}^{2}\int_0^T\int_0^1 \varphi_i|\partial_x\psi_i|^2\,dx\,dt + \widetilde{C}s^3\lambda^4\sum_{i=1}^{2}\int_0^T\int_0^1 \varphi_i^3|\psi_i|^2\,dx\,dt.
\end{aligned}
$$

Finally, combining (3.28) and (3.29), and replacing $f_i = \partial_t q_i + \partial_x(\gamma_i\partial_x q_i)$, $i=1,2$, we obtain the required Carleman inequality (3.8). ∎

3.2. Null-controllability in terms of a boundary observability inequality. The Carleman estimate indeed leads us to obtain the following observability inequality which is in fact a necessary and sufficient condition for null-controllability.

PROPOSITION 3.4 (observability inequality). *For any $\zeta := (\zeta_1,\zeta_2) \in \mathcal{H}_\alpha$, the associated solution $q := (q_1,q_2) \in \mathcal{C}^0([0,T];\mathcal{H}_\alpha)\cap L^2(0,T;(H^2(0,1))^2)$ to (3.1) satisfies the following observation estimate*

$$
\|q(0)\|^2_{\mathcal{H}_\alpha} \le Ce^{C/T}\int_0^T |\partial_x q_2(t,0)|^2dt,
$$

for some constant $C := C(\gamma_1,\gamma_2,\alpha,a) > 0$ that does not depend on $T > 0$ and ζ.

Proof. We shall prove the required observability inequality for $0 < T \le 1$ to show the existence of a control in $(0,T)$ for the system (1.8)–(1.9a); this will not loose the generality since for any time

$\widetilde{T} > 1$, a continuation of a control in $(0,1)$ by 0 in $(1,\widetilde{T})$ will do the job. Let us now focus on the proof.

For any given $\zeta \in \mathcal{H}_\alpha$, one can apply the Carleman inequality given by Theorem 3.2 to the solution q of (3.1), with $\partial_t q_1 + \partial_x(\gamma_1 \partial_x q_1) = a q_2$ and $\partial_t q_2 + \partial_x(\gamma_2 \partial_x q_2) = 0$ to deduce

$$
\begin{aligned}
(3.30)\quad & s^3\lambda^4 \sum_{i=1}^{2} \int_0^T \int_0^1 e^{-2s\eta_i} \varphi_i^3 |q_i|^2 dx\, dt + s\lambda^2 \sum_{i=1}^{2} \int_0^T \int_0^1 e^{-2s\eta_i} \varphi_i |\partial_x q_i|^2 dx\, dt \\
& \quad + s^3\lambda^3 \int_0^T \varphi_1(t,1) e^{-2s\eta_1(t,1)} |q_1(t,1)|^2 dt \le C' \Big[\int_0^T \int_0^1 e^{-2s\eta_1} |a q_2|^2\, dx\, dt \\
& \qquad + s\lambda \int_0^T \varphi_2(t,0) e^{-2s\eta_2(t,0)} |\partial_x q_2(t,0)|^2 dt \Big].
\end{aligned}
$$

Now, we use $1 \le 8T^6 \varphi_2^3$ to see the first term in right hand side of the above estimate as

$$
\begin{aligned}
(3.31)\quad \int_0^T \int_0^1 e^{-2s\eta_1} |a q_2|^2\, dx\, dt &\le 8a^2 T^6 \int_0^T \int_0^1 \varphi_2^3\, e^{-2s\eta_1} |q_2|^2\, dx\, dt \\
&\le 8a^2 T^6 \int_0^T \int_0^1 \varphi_2^3\, e^{-2s\eta_2} |q_2|^2\, dx\, dt =: \widetilde{X},
\end{aligned}
$$

since $\beta_2 \ge \beta_1$ and so $\eta_2 \le \eta_1$ by construction (see (3.5)) which implies $e^{-2s\eta_1} \le e^{-2s\eta_2}$ for any $s > 0$.

We see that the term $\widetilde{X}$ can be absorbed by the term $s^3\lambda^4 \int_0^T \int_0^1 e^{-2s\eta_2} \varphi_2^3 |q_2|^2\, dx\, dt$ in left hand side of the estimate (3.30) for some choice of $s \ge s_1 = (T^2 + T)\sigma_1$ in Theorem 3.2, possibly with some different $\sigma_1 > 0$, and also using the fact that $s^3\lambda^3 \ge s\lambda^2\alpha$, for any $\lambda \ge \lambda_1$ (may be with some larger $\lambda_1 := \lambda_1(\gamma_1, \gamma_2, \alpha)$), we obtain

$$
\begin{aligned}
(3.32)\quad & s\lambda^2 \sum_{i=1}^{2} \int_0^T \int_0^1 e^{-2s\eta_i} \varphi_i |\partial_x q_i|^2 dx\, dt + s\lambda^2\alpha \int_0^T \varphi_1(t,1) e^{-2s\eta_1(t,1)} |q_1(t,1)|^2 dt \\
& \qquad \le Cs\lambda \int_0^T \varphi_2(t,0) e^{-2s\eta_2(t,0)} |\partial_x q_2(t,0)|^2 dt,
\end{aligned}
$$

with some constant $C > 0$ that now depends on $\gamma_1, \gamma_2, \alpha$ and a.

Let us now restrict the integrals in left hand side in $(T/4, 3T/4) \times (0,1)$. We observe that for any $x \in [0,1]$, the minimum of the functions $\varphi_i(t,x) e^{-2s\eta_i(t,x)}$ exists at $t = T/4$, $i = 1, 2$, and the maximum of the function $\varphi_2(t,0) e^{-2s\eta_2(t,0)}$ exists at $t = T/2$; we see

$$
\begin{aligned}
\varphi_i\, e^{-2s\eta_i} &\ge \frac{16}{3T^2}\, e^{3\lambda \min_{[0,1]} \beta_i}\, e^{-(32s/3T^2)(e^{\lambda K} - e^{\lambda \min_{[0,1]} \beta_i})} \text{ in } (T/4, 3T/4) \times (0,1), \\
\varphi_2(t,0)\, e^{-2s\eta_2(t,0)} &\le \frac{4}{T^2}\, e^{\lambda \|\beta_2\|_\infty}\, e^{-(8s/T^2)(e^{\lambda K} - e^{\lambda \|\beta_2\|_\infty})} \text{ in } (0,T) \times (0,1).
\end{aligned}
$$

Implementing this in (3.32) and by fixing $\lambda = \lambda_1$, we deduce that

$$
\int_{T/4}^{3T/4} \Big(\|\partial_x q_1(t)\|_{L^2}^2 + \|\partial_x q_2(t)\|_{L^2}^2 + \alpha |q_1(t,1)|^2 \Big) dt \le C e^{Cs/T^2} \int_0^T \big|\partial_x q_2(t,0)\big|^2 dt,
$$

that implies

$$
(3.33)\quad \int_{T/4}^{3T/4} \|q(t)\|_{\mathcal{H}_\alpha}^2 \le C e^{Cs/T^2} \int_0^T \big|\partial_x q_2(t,0)\big|^2 dt.
$$

Now, thanks to the point 2 of Theorem 2 (which is also valid for the adjoint system (3.1) with source term $f = 0$), we have $\|q(0)\|_{\mathcal{H}_\alpha}^2 \le C_a \|q(t)\|_{\mathcal{H}_\alpha}^2$ for any $0 < t \le T (\le 1)$. Using this and by choosing $s = s_1 = (T^2 + T)\sigma_1 > 0$ the inequality (3.33) reduces to

$$
\|q(0)\|_{\mathcal{H}_\alpha}^2 \le C e^{C\sigma(T^2+T)/T^2} \int_0^T |\partial_x q_2(t,0)|^2 dt,
$$

which gives the required inequality in the proposition with the constant $C > 0$, independent on T and ζ. ∎

Proof of Theorem 2.6 (Null-controllability). Once we have the above observability estimate, then by some standard duality argument, see for instance [12], one can prove the existence of a boundary null-control $v \in L^2(0,T)$ for the problem (1.8)–(1.9a), and the estimation of the control cost $Ce^{C/T}$ follows from the sharp observability inequality in Proposition 3.4. ∎

REMARK 3.5. *For the other system, that is for (1.8)–(1.9b), the observation term is* $\mathcal{B}_2^* q(t) = \gamma_1(0)\partial_x q_1(t,0)$, *and so to obtain a good Carleman estimate one has to choose the functions* β_i, $i = 1,2$ *with* $c_1 = \frac{-6}{(1-\mu_0)}$ *and* $c_2 = 1$ *(which is opposite to the previous case), to construct the suitable weight functions.*

In that case, a Carleman estimate similar to (3.8) still holds with the observation integral

$$s\lambda \int_0^T \varphi_1(t,0) e^{-2s\eta_1(t,0)} |\partial_x q_1(t,0)|^2 dt,$$

but we cannot hope for a good observability inequality with observation term $\gamma_1(0)\partial_x q_1(t,0)$. *The reason behind this is the following: in this case, we have* $e^{-2s\eta_1} \geq e^{-2s\eta_2}$ *(since* $\beta_1 \geq \beta_2$ *now and consequently* $\eta_1 \leq \eta_2$*), which will prevent us from absorbing the source term* $\int_0^T \int_0^1 e^{-2s\eta_1} |aq_2|^2 \, dxdt$ *appeared in the right-hand side, by the term* $s^3\lambda^4 \int_0^T \int_0^1 e^{-2s\eta_2} \varphi_2^3 |q_2|^2 \, dxdt$ *in left-hand side.*

The above remark tells us that the Carleman trick is no more applicable to prove the boundary null-controllability of the system (1.8)–(1.9b).

In fact, it is not a technical since, as we previously mentioned, the controllability property of the system will depend on the valuess of the coupling coefficient a and of the boundary parameter α. This will be investigated in the next section.

4. Boundary controllability of our system with control in the first component. This section is devoted to find a control for the prescribed problem (1.8)–(1.9b) with $\gamma_1 = \gamma_2 = 1$, using the moments technique; and as we know, the key point to develop and solve the moments problem is to obtain sharp estimates on spectral elements of the adjoint to the corresponding elliptic operator.

4.1. Description of spectrum of the underlying elliptic operator. In this section, we investigate some important spectral properties of the elliptic operator $\mathcal{A}^*_{\alpha,a}$ having the formal expression in (2.12) with $\gamma_1 = \gamma_2 = 1$.

REMARK 4.1. *For* $\gamma_1 = \gamma_2 = 1$ *also, we keep the same symbol* $\mathcal{A}_{\alpha,a}$ *and* $\mathcal{A}^*_{\alpha,a}$ *(for any* $a \in \mathbb{R}, \alpha \geq 0$*) to denote the corresponding elliptic operator and its adjoint respectively.*

Below, we present the eigenvalue problem $\mathcal{A}^*_{\alpha,a} u = \lambda u$, for $\lambda \in \mathbb{C}$, that is explicitly

$$\begin{cases} -u_1'' + au_2 = \lambda u_1 & \text{in } (0,1), \\ -u_2'' = \lambda u_2 & \text{in } (0,1), \\ u_1(0) = 0, \ u_2(0) = 0, \\ u_1(1) = u_2(1), \\ u_1'(1) + u_2'(1) + \alpha u_1(1) = 0. \end{cases} \tag{4.1}$$

We recall from (2.12) that for $a \neq 0$, the operator $\mathcal{A}^*_{\alpha,a}$ is no more self-adjoint and here we develop the spectral analysis of this operator (more precisely of its complex version) using some perturbation argument of linear operators which we discuss in Section 4.1.2. That's the reason why, we first need to describe the spectrum of the self-adjoint operator $\mathcal{A}^*_\alpha$, which we discuss in the subsequent section.

4.1.1. Spectrum of the self-adjoint operator $\mathcal{A}^*_\alpha$. We directly start with the eigenvalue problem $\mathcal{A}_\alpha u = \mathcal{A}^*_\alpha u = \lambda u$, $u \neq 0$, where one may assume that λ is real since $\mathcal{A}_\alpha$ is self-adjoint,

$$\begin{cases} -u_1'' = \lambda u_1 & \text{in } (0,1), \\ -u_2'' = \lambda u_2 & \text{in } (0,1), \\ u_1(0) = 0, \ u_2(0) = 0, \\ u_1(1) = u_2(1), \\ u_1'(1) + u_2'(1) + \alpha u_1(1) = 0. \end{cases} \tag{4.2}$$

Observe first that we necessarily have $\lambda > 0$. Indeed, multiplying the first and second equations by u_1 and u_2 respectively, then upon an integration by parts and using the boundary conditions, one

has

$$\int_0^1 (|u_1'(x)|^2 + |u_2'(x)|^2)\,dx + \alpha|u_1(1)|^2 = \lambda \int_0^1 (|u_1(x)|^2 + |u_2(x)|^2)\,dx,$$

which certainly tells us that $\lambda > 0$ since $\alpha \geq 0$. We shall set $\mu = \sqrt{\lambda}$.

Let us now solve (4.2). We start by observing that if $u_1 = 0$ then the equation for u_2 along with the boundary conditions gives $u_2 = 0$. Therefore, by using the boundary condition at $x = 0$, we expect the solution to be of the form

$$u_1(x) = C_1 \sin(\mu x), \quad u_2(x) = C_2 \sin(\mu x), \quad \forall x \in [0,1],$$

for some $C_1, C_2 \in \mathbb{R}$. On the other hand, the conditions $u_1(1) = u_2(1)$ and $u_1'(1) + u_2'(1) + \alpha u_1(1) = 0$ respectively provides

$$(C_1 - C_2)\sin\mu = 0 \quad \text{and} \tag{4.3a}$$

$$\mu C_1 \cos\mu + \mu C_2 \cos\mu + \alpha C_1 \sin\mu = 0. \tag{4.3b}$$

- First, when $\sin\mu \neq 0$, then $C_1 = C_2 \neq 0$ from (4.3a), so that from (4.3b) we end up with

$$2\mu\cos\mu + \alpha\sin\mu = 0. \tag{4.4}$$

  - If $\alpha = 0$, then $\mu_{k,1}^0 := (k + 1/2)\pi$ for $k \geq 0$ are the positive roots of the above equation. A first family of eigenvalues of (4.2) is thus given by $\lambda_{k,1}^0 := (k + 1/2)^2\pi^2$, $k \geq 0$.
  - If $\alpha > 0$, we rewrite the equation (4.4) as

$$g(\mu) := \tan\mu + \frac{2}{\alpha}\mu = 0.$$

    We calculate that $g'(\mu) = \sec^2\mu + 2/\alpha > 0$, and so in particular, $g'(\mu) > 0$ in $((k + 1/2)\pi, (k + 3/2)\pi)$, for any $k \geq 0$. Beside this, we have

$$\lim_{\mu \to ((k+1/2)\pi)^+} g(\mu) = -\infty, \quad g((k+1)\pi) = \frac{2}{\alpha}(k+1)\pi > 0.$$

    So, there exists exactly one root of g in $((k + 1/2)\pi, (k + 1)\pi)$, for each $k \geq 0$, and given $\alpha > 0$. Let us denote the roots of g by $\mu_{k,1}^\alpha$ and the eigenvalues by $\lambda_{k,1}^\alpha := (\mu_{k,1}^\alpha)^2$, for all $k \geq 0$ and given any $\alpha > 0$.

  Note that an associated set of normalized eigenfunctions is given by

$$\Phi_{\lambda_{k,1}^\alpha}(x) := \begin{pmatrix} \sin(\mu_{k,1}^\alpha x) \\ \sin(\mu_{k,1}^\alpha x) \end{pmatrix}. \tag{4.5}$$

- Assume now that $\sin\mu = 0$ (see (4.3a)), from which we deduce that $\mu = (k + 1)\pi$ for some $k \geq 0$. By (4.3b) we have $C_1 = -C_2$. Now, to be consistent with the notation, we shall denote this second set of eigenvalue-eigenfunction pairs by $\{\lambda_{k,2}^\alpha, \Phi_{\lambda_{k,2}^\alpha}\}_{k\geq 0}$, even if they are not depending on α, with $\lambda_{k,2}^\alpha = (k + 1)^2\pi^2$ and

$$\Phi_{\lambda_{k,2}^\alpha}(x) := \begin{pmatrix} \sin(k+1)\pi x \\ -\sin(k+1)\pi x \end{pmatrix}. \tag{4.6}$$

Remark 4.2. *Since the operator $\mathcal{A}_\alpha^*$ is self-adjoint, the family $\{\Phi_{\lambda_{k,1}^\alpha}, \Phi_{\lambda_{k,2}^\alpha}\}_{k\geq 0}$ indeed forms an orthonormal basis of E.*

Remark 4.3. *For any $\alpha \geq 0$, we deduce the following asymptotic formula of $\lambda_{k,1}^\alpha$,*

$$\lambda_{k,1}^\alpha = (k + \frac{1}{2})^2\pi^2 + \alpha + o_\alpha(1), \quad \textit{for } k \textit{ large},$$

To obtain the above asymptotic, we express $\mu_{k,1}^\alpha = (k + 1/2)\pi + \delta_{k,1}^\alpha$ with $\delta_{k,1}^\alpha \in (0, \pi/2)$ for $\alpha > 0$. Then we see from $g(\mu_{k,1}^\alpha) = 0$ that

$$\tan((k + 1/2)\pi + \delta_{k,1}^\alpha) + \frac{2}{\alpha}((k + 1/2)\pi + \delta_{k,1}^\alpha) = 0,$$

$$\textit{i.e.,} \quad \frac{\cos\delta_{k,1}^\alpha}{\sin\delta_{k,1}^\alpha} = \frac{2}{\alpha}((k + 1/2)\pi + \delta_{k,1}^\alpha), \tag{4.7}$$

of which, the right hand side goes to $+\infty$ *as* $k \to +\infty$*, and so, for any fixed* $\alpha > 0$*,* $\delta^\alpha_{k,1} \to 0^+$*. Consequently,* $\sin \delta^\alpha_{k,1} \sim \delta^\alpha_{k,1}$ *and* $\cos \delta^\alpha_{k,1} \sim 1$ *for large* k*. So, by taking into account* (4.7)*, we have*

$$\delta^\alpha_{k,1} \sim \frac{\alpha}{2k\pi}, \quad \textit{for } k \textit{ large.}$$

Henceforth, we deduce that

$$\begin{aligned}\lambda^\alpha_{k,1} - \big(k+\frac{1}{2}\big)^2\pi^2 &= \Big(\mu^\alpha_{k,1} - \big(k+\frac{1}{2}\big)\pi\Big)\Big(\mu^\alpha_{k,1} + \big(k+\frac{1}{2}\big)\pi\Big)\\ &= \delta^\alpha_{k,1}((2k+1)\pi + \delta^\alpha_{k,1})\\ &\xrightarrow[k\to\infty]{} \alpha.\end{aligned}$$

4.1.2. Spectrum of the main operator $\mathcal{A}^*_{\alpha,a}$. We begin with our main problem of interest, that is the system of odes (4.1). For our use, we first denote the set of all eigenvalues of $\mathcal{A}^*_{\alpha,a}$ by $\Lambda_{\alpha,a}$ for any $a \in \mathbb{R}$ and $\alpha \geq 0$.

Let us choose $a \in \mathbb{R}^*$ and $\alpha \geq 0$ and we pursue some detailed analysis step by step as follows.

– *Localization of the spectrum.*

We observe that

$$\Lambda_{\alpha,a} \subset \bigcup_{\lambda\in\Lambda_{\alpha,0}} D(\lambda, 2|a|), \tag{4.8}$$

where $\Lambda_{\alpha,0}$ is the set of all eigenvalues of the self-adjoint operator $\mathcal{A}_\alpha = \mathcal{A}_{\alpha,0}$.

Indeed, if $\xi \in \mathbb{C}$ is such that $|\xi - \lambda| \geq 2|a|$ for any $\lambda \in \Lambda_{\alpha,0}$, then in particular $\mathcal{A}^*_{\alpha,0} - \xi I$ is invertible and satisfies the resolvent estimate

$$\|(\mathcal{A}^*_{\alpha,0} - \xi I)^{-1}\| = \sup_{\lambda\in\Lambda_{\alpha,0}} \frac{1}{|\xi-\lambda|} \leq \frac{1}{2|a|}.$$

It follows that

$$\mathcal{A}^*_{\alpha,a} - \xi I = \mathcal{A}^*_{\alpha,0} - \xi I + \mathcal{M}^*_a = (\mathcal{A}^*_{\alpha,0} - \xi I)\left(I - (\mathcal{A}^*_{\alpha,0} - \xi I)^{-1}\mathcal{M}^*_a\right),$$

and thus ξ lies in the resolvent set of $\mathcal{A}^*_{\alpha,a}$ since

$$\|(\mathcal{A}^*_{\alpha,0} - \xi I)^{-1}\mathcal{M}^*_a\| \leq \|(\mathcal{A}^*_{\alpha,0} - \xi I)^{-1}\|\|\mathcal{M}^*_a\| \leq \frac{1}{2|a|}|a| < 1.$$

In particular, $\mathcal{A}^*_{\alpha,a}$ has compact resolvent since the self-adjoint operator $\mathcal{A}_{\alpha,0}$ has so, which ensures that the spectrum of $\mathcal{A}_{\alpha,a}$ is discrete.

– *Multiplicity.*

Observe now that all eigenvalues have geometric multiplicity 1. Assume that it is not the case, then we can find one associated eigenfunction $u = (u_1, u_2)$ such that $u_1(1) = 0$. By the boundary condition at $x = 1$, we also have $u_2(1) = 0$. Note that $u_2'(1) \neq 0$ since if it were not the case, we would have $u_2 = 0$ in $(0, 1)$ and then $u_1 = 0$ in $(0, 1)$ which is not possible.

From (4.1), we see that u_2 satisfies a second order ode with homogeneous Dirichlet boundary conditions from which we deduce that $\lambda = (k+1)^2\pi^2$ for some $k \geq 0$ and, up to a multiplicative constant, $u_2(x) = \sin(k+1)\pi x$, $\forall x \in [0, 1]$. In particular, λ is real.

Now, we multiply the differential equation of u_1 by u_2 and integrate, i.e.,

$$\int_0^1 -u_1''(x)u_2(x)\,dx + a\int_0^1 |u_2(x)|^2\,dx = \lambda\int_0^1 u_1(x)u_2(x)\,dx.$$

Performing integration by parts and $u_i(0) = u_i(1) = 0$ (for $i = 1, 2$), we get

$$\int_0^1 -u_2''(x)u_1(x)\,dx + a\int_0^1 |u_2(x)|^2\,dx = \lambda\int_0^1 u_1(x)u_2(x)\,dx.$$

Now, since $-u_2'' = \lambda u_2$ in $(0, 1)$, and $a \neq 0$, we deduce from the above equality that $u_2 = 0$ in $[0, 1]$ which is not possible as discussed above. The claim is proved.

– *The case* $\lambda = 0$.

We observe that $\lambda = 0$ is an eigenvalue if and only if $a + 3\alpha + 6 = 0$. Take $\lambda = 0$ in (4.1). Then solving the set of odes along with the homogeneous boundary condition at $x = 0$, one obtain $u_2(x) = c_1 x$ and $u_1(x) = c_1 \frac{ax^3}{6} + c_2 x$. Now, thanks to the Kirchhoff boundary condition at $x = 1$, we obtain $c_2 = c_1(1 - \frac{a}{6})$ and

$$c_1(a + 3\alpha + 6) = 0,$$

which shows that $c_1 = 0$ (consequently, $c_2 = 0$) provided $a + 3\alpha + 6 \neq 0$; in that case, $\lambda = 0$ is not an eigenvalue of $\mathcal{A}^*_{\alpha,a}$.

But, as soon as we have $a + 3\alpha + 6 = 0$ (then fix $c_1 = 1$), we see that $\lambda = 0$ is an eigenvalue with the eigenfunction

$$\Phi_0(x) = \begin{pmatrix} \frac{ax^3}{6} + \big(1 - \frac{a}{6}\big)x \\ x \end{pmatrix}, \quad \forall x \in [0, 1]. \tag{4.9}$$

– *The case* $\lambda \neq 0$.

As we have seen above we cannot have $u_2 = 0$ in $(0, 1)$. We take $\mu \in \mathbb{C}$ such that $\mu^2 = \lambda$ and we observe that the solution of (4.1) is necessarily of the form

$$\begin{cases} u_1(x) = \dfrac{aK_1 x}{2i\mu}(e^{i\mu x} + e^{-i\mu x}) + K_2(e^{i\mu x} - e^{-i\mu x}), \\ u_2(x) = K_1(e^{i\mu x} - e^{-i\mu x}), \end{cases} \tag{4.10}$$

for some $K_1, K_2 \in \mathbb{C}$. Thereafter, the two boundary conditions at $x = 1$ provides us the following two equations

$$\begin{cases} K_1(ac_\mu - 2i\mu s_\mu) + K_2(2i\mu s_\mu) = 0 \ \text{ and} \\ K_1(-2\mu^2 c_\mu + ai\mu s_\mu + ac_\mu + 2i\alpha\mu s_\mu) + K_2(-2\mu^2 c_\mu) = 0, \end{cases} \tag{4.11}$$

where we introduced $s_\mu := (e^{i\mu} - e^{-i\mu})$ and $c_\mu := (e^{i\mu} + e^{-i\mu})$. Now, for the existence of non-zero solution (K_1, K_2) of the above system, the following condition should necessarily be satisfied:

$$8\mu^2 i(s_\mu c_\mu) + 2\mu a(s_\mu^2 - c_\mu^2) - 2ai(s_\mu c_\mu) + 4\alpha\mu s_\mu^2 = 0,$$

which is actually the determinant of the coefficient matrix of system (4.11). Since we have assumed that $\mu \neq 0$, the condition above implies that $s_\mu \neq 0$.

Using the relations $s_\mu c_\mu = 2i \sin 2\mu$, $s_\mu^2 = -4 \sin^2 \mu$ and $s_\mu^2 - c_\mu^2 = -4$, the above equation simplifies as

$$(4\mu^2 - a) \sin 2\mu + 2a\mu + 4\alpha\mu \sin^2 \mu = 0. \tag{4.12}$$

Now, from the first equation of (4.11), we have $K_2 = K_1\Big(1 - \frac{ac_\mu}{2i\mu s_\mu}\Big) = K_1\left(1 + \frac{a}{2\mu}\frac{\cos\mu}{\sin\mu}\right)$. Next we fix $K_1 = 1/2i$ and deduce from (4.10) that the eigenfunction associated with $\lambda = \mu^2$ is given by

$$\Phi_\lambda(x) := \begin{pmatrix} -\frac{ax}{2\mu}\cos(\mu x) + \left(1 + \frac{a}{2\mu}\frac{\cos\mu}{\sin\mu}\right)\sin(\mu x) \\ \sin(\mu x) \end{pmatrix}, \tag{4.13}$$

as soon as $\mu \in \mathbb{C}$ satisfies (4.12).

– *Real solutions of the transcendental equation* (4.12).

We set

$$f(\mu) := (4\mu^2 - a) \sin 2\mu + 2a\mu + 4\alpha\mu \sin^2 \mu.$$

Our goal is to prove the following lemma.

LEMMA 4.4. *Let $a \in \mathbb{R}^*$ and $\alpha \geq 0$. There exists some $k_{\alpha,a} \in \mathbb{N} \cup \{0\}$, and $C_{\alpha,a} > 0$ such that for each $k \geq k_{\alpha,a}$, the function f has:*

- *one real root, denoted by $\mu^{\alpha,a}_{k,1}$, in the interval*

$$\big((k + 1/4)\pi, (k + 3/4)\pi\big),$$

and that satisfies

$$\mu^{\alpha,a}_{k,1} = (k + 1/2)\pi + \frac{2\alpha + a}{4k\pi} + o_{\alpha,a}(1/k), \tag{4.14}$$

- *one real root, denoted by $\mu_{k,2}^{\alpha,a}$, in the interval*

$$\big((k+3/4)\pi, (k+5/4)\pi\big),$$

and that satisfies

$$\mu_{k,2}^{\alpha,a} = (k+1)\pi - \frac{a}{4k\pi} + o_{\alpha,a}(1/k). \tag{4.15}$$

Proof. • Let $\epsilon = \mp\pi/4$. A straightforward computation gives

$$f((k+1/2)\pi+\epsilon) \sim -4\sin(2\epsilon)k^2\pi^2, \quad \text{for large } k.$$

Hence, for k large enough, $f((k+1/4)\pi)$ and $f((k+3/4)\pi)$ have different signs, which proves the existence of a root $\mu_{k,1}^{\alpha,a} \in \big((k+1/4)\pi, (k+3/4)\pi\big)$.
Let $\delta_k := \mu_{k,1}^{\alpha,a} - (k+1/2)\pi \in (-\pi/4, \pi/4)$. The equation $f(\mu_{k,1}^{\alpha,a}) = 0$ gives

$$\begin{aligned} -\big(4((k+1/2)\pi+\delta_k)^2 - a\big)\sin(2\delta_k) + 2a((k+1/2)\pi+\delta_k) \\ + 4\alpha((k+1/2)\pi+\delta_k)\cos^2\delta_k = 0, \end{aligned} \tag{4.16}$$

and in particular we get for some $C_{\alpha,a} > 0$,

$$|\sin(2\delta_k)| \le \frac{C_{\alpha,a}}{k},$$

which implies that $\delta_k \to 0$ as $k \to +\infty$ and coming back to (4.16), one can deduce that

$$\delta_k = \frac{(2\alpha+a)}{4k\pi} + O_{\alpha,a}(1/k^3).$$

- Similarly, we have

$$f((k+1)\pi+\epsilon) \sim 4\sin(2\epsilon)k^2\pi^2, \quad \text{for large } k,$$

and by the similar trick as previous we get the existence of a root $\mu_{k,2}^{\alpha,a} \in \big((k+3/4)\pi, (k+5/4)\pi\big)$.
Setting now $\delta_k := \mu_{k,2}^{\alpha,a} - (k+1)\pi \in (-\pi/4, \pi/4)$, the equation $f(\mu_{k,2}^{\alpha,a}) = 0$ gives

$$\begin{aligned} \big(4((k+1)\pi+\delta_k)^2 - a\big)\sin(2\delta_k) + 2a((k+1)\pi+\delta_k) \\ + 4\alpha((k+1)\pi+\delta_k)\sin^2\delta_k = 0, \end{aligned} \tag{4.17}$$

again from which we first deduce that δ_k tends to 0 (using the similar argument as before) and then

$$\delta_k = -\frac{a}{4k\pi} + O_{\alpha,a}(1/k^3). \qquad \square$$

COROLLARY 4.5. *For any $k \ge k_{\alpha,a}$, the operator $\mathcal{A}_{\alpha,a}^*$ has two real eigenvalues $\lambda_{k,1}^{\alpha,a}$ and $\lambda_{k,2}^{\alpha,a}$ that satisfy*

$$\lambda_{k,1}^{\alpha,a} = (k+1/2)^2\pi^2 + (\alpha + a/2) + o_{\alpha,a}(1),$$
$$\lambda_{k,2}^{\alpha,a} = (k+1)^2\pi^2 - a/2 + o_{\alpha,a}(1).$$

Moreover, for each $k \ge k_{\alpha,a}$ (possibly some larger $k_{\alpha,a}$ than earlier) and $i \in \{1,2\}$, $\lambda_{k,i}^{\alpha,a}$ is the unique eigenvalue of $\mathcal{A}_{\alpha,a}^*$ in the following disk of the complex plane

$$D(\lambda_{k,i}^{\alpha,0}, 2|a|),$$

where conventionally $\lambda_{k,i}^{\alpha,0} := \lambda_{k,i}^{\alpha}$, the eigenvalues of our self-adjoint operator $\mathcal{A}_\alpha$.

Proof. The solutions $\mu_{k,i}^{\alpha,a}$ of the transcendental equation $f(\mu) = 0$ are the square roots of the eigenvalues of our operator. Thus we can set $\lambda_{k,i}^{\alpha,a} = \big(\mu_{k,i}^{\alpha,a}\big)^2$.

Moreover, for k large enough, we have for $i = 1, 2$, that

$$d\left(\lambda_{k,i}^{\alpha,0},\, \Lambda_{\alpha,0} \setminus \{\lambda_{k,i}^{\alpha,0}\}\right) \ge 4|a|,$$

so that, we have the resolvent estimate

$$\|(\mathcal{A}^*_\alpha - \xi I)^{-1}\| \leq \frac{1}{2|a|}, \qquad \forall \xi \in \partial D(\lambda^{\alpha,0}_{k,i}, 2|a|),$$

and thus

$$\|\mathcal{M}^*_a\| \, \|(\mathcal{A}^*_\alpha - \xi I)^{-1}\| \leq \frac{1}{2}, \qquad \forall \xi \in \partial D(\lambda^{\alpha,0}_{k,i}, 2|a|).$$

From [14, Theorems IV–3.16 and 3.18, and V–§4.3], we know that the perturbed operator $\mathcal{A}^*_{\alpha,a}$ and the self-adjoint operator $\mathcal{A}^*_\alpha$ have the same number of eigenvalues in the disk $D(\lambda^{\alpha,0}_{k,i}, 2|a|)$. Therefore, in this disk, $\mathcal{A}^*_{\alpha,a}$ has only one eigenvalue which is $\lambda^{\alpha,a}_{k,i}$. □

– *Conclusion on the structure of* $\Lambda_{\alpha,a}$.

Using the fact given by (4.8) and the Corollary 4.5, we deduce that the spectrum of $\mathcal{A}^*_{\alpha,a}$ can be split into two disjoint parts

$$\Lambda_{\alpha,a} = \Lambda^0_{\alpha,a} \cup \Lambda^\infty_{\alpha,a}, \tag{4.18}$$

where $\Lambda^0_{\alpha,a}$ is finite, with possibly some complex eigenvalues, and satisfy

$$\Lambda^0_{\alpha,a} \subset \bigcup_{i=1,2} \bigcup_{0 \leq k < k_{\alpha,a}} D(\lambda^{\alpha,0}_{k,i}, 2|a|), \tag{4.19}$$

and $\Lambda^\infty_{\alpha,a} \subset (0, +\infty)$ and is defined by

$$\Lambda^\infty_{\alpha,a} := \left\{\lambda^{\alpha,a}_{k,1},\ k \geq k_{\alpha,a}\right\} \cup \left\{\lambda^{\alpha,a}_{k,2},\ k \geq k_{\alpha,a}\right\}. \tag{4.20}$$

The situation is illustrated in Figure 1.

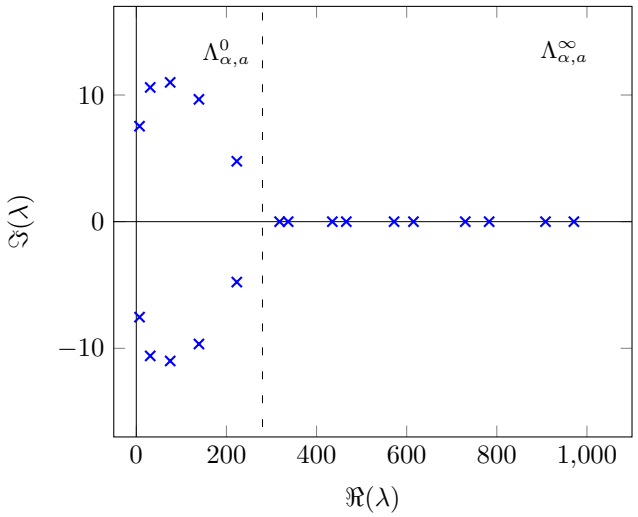


FIG. 1. *A numerical description of a part of the spectrum: for* $a = 30$, $\alpha = 0.1$

We can summarize the above analysis as follows

PROPOSITION 4.6. *Let* $a \in \mathbb{R}$ *and* $\alpha \geq 0$ *be any two parameters.*

- *The spectrum of the operator* $\mathcal{A}^*_{\alpha,a}$ *is discrete, made only of simple eigenvalues, and has the structure given in* (4.18).
- *Moreover, the associated family of eigenfunctions* $\{\Phi_\lambda\}_{\lambda \in \Lambda_{\alpha,a}}$ *is complete in* E *and* $\mathcal{H}_\alpha$.

Note that we considered here the complex version of the spaces E and $\mathcal{H}_\alpha$. Everything was proved above, except the completeness property of the eigenfunctions which comes as a consequence of a theorem of Keldysh, see for instance [15, Chapter 1–Theorem 4.3], since the perturbation $\mathcal{M}^*_a$ is bounded.

4.2. Observation estimates and bounds on norms of eigenfunctions . In this section, we analyze the size of the observation terms $|\mathcal{B}_2^*\Phi_\lambda|$ for $\lambda \in \Lambda_{\alpha,a}$ ($\mathcal{B}_2^*$ is defined by (2.15b)). If those quantities do not vanish then the approximate controllability of the problem (1.8)–(1.9b) will be guaranteed by means of Fattorini-Hautus test (see [11, 16]). Moreover, suitable lower bound for those quantities combined with upper bounds of $\|\Phi_\lambda\|_{\mathcal{H}_\alpha}$ will let us build and estimate a null-control in $L^2(0,T)$ via moments technique.

4.2.1. Approximate controllability. We prove the following lemma (recall that we have assumed the diffusion coefficients $\gamma_1 = \gamma_2 = 1$).

LEMMA 4.7. *Let any $a \in \mathbb{R}$ and $\alpha \geq 0$ be given. Then there exists a non-empty set $\mathcal{R} \subset \mathbb{R}_0^+ \times \mathbb{R}^*$, such that we have the following properties:*

1. *If $(\alpha, a) \notin \mathcal{R}$, the problem (1.8)–(1.9b) is approximately controllable at any time $T > 0$ in $\mathcal{H}_{-\alpha}$.*
2. *On the other hand, if $(\alpha, a) \in \mathcal{R}$, there exists a subspace $\mathcal{Y}_{\alpha,a} \subset \mathcal{H}_{-\alpha}$ of codimension one, such that the problem (1.8)–(1.9b) is approximately controllable at any time $T > 0$ if and only if the initial data belongs to $\mathcal{Y}_{\alpha,a}$.*

The set $\mathcal{R}$ and the spaces $\mathcal{Y}_{\alpha,a}$ are defined by (4.29) and (4.31) respectively inside the proof of this lemma.

Proof of Lemma 4.7.
We recall that the observation operator $\mathcal{B}_2^*$ is given in (2.15b).

- In the simplest case when $a = 0$, for any $\alpha \geq 0$, one can immediately see that the eigenfunctions in (4.5)–(4.6) satisfy

$$\mathcal{B}_2^*\Phi_{\lambda_{k,1}^\alpha} = \sqrt{\lambda_{k,1}^\alpha} \neq 0, \quad \mathcal{B}_2^*\Phi_{\lambda_{k,2}^\alpha} = \sqrt{\lambda_{k,2}^\alpha} = (k+1)\pi \neq 0, \quad \forall k \geq 0. \tag{4.21}$$

- The case $\lambda = 0$ can only happen if $a + 3\alpha + 6 = 0$ (so that in particular $a < 0$) and it follows from (4.9) that

$$\mathcal{B}_2^*\Phi_0 = 1 - \frac{a}{6} > 0.$$

- Let us assume that $a \neq 0$ and $\lambda \neq 0$ be an eigenvalue of $\mathcal{A}_{\alpha,a}^*$. The associated eigenfunction Φ_λ is given in (4.13).
  An immediate computation gives

$$\mathcal{B}_2^*\Phi_\lambda = -\frac{a}{2\mu} + \mu + \frac{a\cos\mu}{2\sin\mu}. \tag{4.22}$$

  From now on, we suppose that $\mathcal{B}_2^*\Phi_\lambda = 0$. Since $\mu \neq 0$ and $\sin\mu \neq 0$, this is equivalent to the relation

$$(2\mu^2 - a)\sin\mu + a\mu\cos\mu = 0. \tag{4.23}$$

  This equation has to be satisfied in addition to the transcendental equation (4.12). If we suppose that $\cos\mu = 0$, then (4.12) and (4.23) show that this can occur if and only if we have

$$a + 2\alpha = 0, \text{ and } \mu^2 = -\alpha, \tag{4.24}$$

  this last equation not being compatible with the condition $\cos\mu = 0$.
  Therefore, we can assume that $\cos\mu \neq 0$. Multiplying (4.23) by $\cos\mu$ and using straightforward trigonometry we obtain that the two equations (4.12) and (4.23) can be equivalently written as follows

$$\begin{pmatrix} (4\mu^2 - a) & 4\alpha\mu \\ (2\mu^2 - a) & -2a\mu \end{pmatrix} \begin{pmatrix} \sin 2\mu \\ \sin^2\mu \end{pmatrix} = \begin{pmatrix} -2a\mu \\ -2a\mu \end{pmatrix}. \tag{4.25}$$

  Denote the coefficient matrix in the left hand side of (4.25) by $M_\mu \in \mathcal{M}_{2\times 2}(\mathbb{C})$ and we calculate the determinant:

$$\det M_\mu = 2a\mu(a + 2\alpha) - 8\mu^3(a + \alpha).$$

– *Let us prove that* $\det M_\mu \neq 0$ *if* μ *satisfies* (4.25).
The claim is clear if $a+2\alpha=0$ or $a+\alpha=0$.
From now on we assume $a+2\alpha\neq 0$ and $a+\alpha\neq 0$. The determinant of M_μ cancels if and only if $\mu=\pm\sqrt{\frac{a(a+2\alpha)}{4(a+\alpha)}}\in\mathbb{C}$ and, if that happens, the matrix becomes

$$M_\mu=\begin{pmatrix} \frac{a\alpha}{a+\alpha} & \pm 2\alpha\sqrt{\frac{a(a+2\alpha)}{(a+\alpha)}} \\ -\frac{1}{2}\frac{a^2}{a+\alpha} & \mp a\sqrt{\frac{a(a+2\alpha)}{(a+\alpha)}}\end{pmatrix},$$

and a straightforward computation shows that

$$\operatorname{Ker} M_\mu^* = \operatorname{Span}\left\{\begin{pmatrix} a \\ 2\alpha\end{pmatrix}\right\}.$$

We deduce from (4.25) that $\begin{pmatrix}1\\1\end{pmatrix}$ belongs to the range of M_μ, which implies that it should be orthogonal to $\operatorname{Ker} M_\mu^*$, that is

$$\begin{pmatrix}1\\1\end{pmatrix}\perp\begin{pmatrix} a \\ 2\alpha\end{pmatrix}.$$

This is a contradiction since $a+2\alpha\neq 0$ and the claim is proved.

– *Solving* (4.25).
From the previous point, we know that M_μ is invertible, so that we can solve (4.25) to get

$$\begin{cases} \sin 2\mu = \dfrac{2a\mu(a+2\alpha)}{a(a+2\alpha)-4\mu^2(a+\alpha)}, \\ \sin^2\mu = \dfrac{-4a\mu^2}{2a(a+2\alpha)-8\mu^2(a+\alpha)}. \end{cases} \tag{4.26}$$

Using the standard trigonometric relation $\sin^2 2\mu = 4\sin^2\mu(1-\sin^2\mu)$, we can deduce from (4.26) that

$$4\mu^2 = a(a+2\alpha+2).$$

Since the sign of μ is unimportant, we conclude that this situation can only occur for the particular value

$$\mu=\mu^c_{\alpha,a}:=\frac{1}{2}\sqrt{a^2+2a\alpha+2a}.$$

To summarize, we have finally obtained that if $\mathcal{B}_2^*\Phi_\lambda=0$, then we necessarily have

$$\lambda=\lambda^c_{\alpha,a}:=\frac{a(a+2\alpha+2)}{4}. \tag{4.27}$$

This is a necessary condition and we still have to check whether or not this value of λ (or μ) does satisfy (4.26), that is to say if α and a satisfy

$$\sin(\sqrt{a^2+2a\alpha+2a}) = -\frac{(a+2\alpha)\sqrt{a^2+2a\alpha+2a}}{(a^2+a+3a\alpha+2\alpha^2)}, \tag{4.28a}$$

$$\sin^2\left(\frac{\sqrt{a^2+2a\alpha+2a}}{2}\right) = \frac{(a^2+2a\alpha+2a)}{2(a^2+a+3a\alpha+2\alpha^2)}. \tag{4.28b}$$

This leads us to introduce the critical set $\mathcal{R}$ as follows

$$\mathcal{R}:=\{(\alpha,a)\in\mathbb{R}_0^+\times\mathbb{R}^*,\ \text{s.t. (4.28) holds}\}. \tag{4.29}$$

The set $\mathcal{R}$ is the set of solutions to the two equations (4.28). We recall that those two equations were obtained from (4.26) by eliminating the value of μ and therefore, are not independent one from the other. Thus, we observe that any solution (4.28b) necessarily satisfies

$$\sin(\sqrt{a^2+2a\alpha+2a}) = \varepsilon_{\alpha,a}\frac{(a+2\alpha)\sqrt{a^2+2a\alpha+2a}}{(a^2+a+3a\alpha+2\alpha^2)}, \tag{4.30}$$

for $\varepsilon_{\alpha,a} \in \{-1, 1\}$. On any connected component of the set of solutions of (4.28b), we have either $\varepsilon_{\alpha,a} = -1$ (in which case (4.28a) is satisfied) or $\varepsilon_{\alpha,a} = 1$ (in which case (4.28a) is not satisfied).
We have plotted in Figure 2 the solution curves of (4.28b) in two colors: in blue the ones for which $\varepsilon_{\alpha,a} = -1$ and in red the ones for which $\varepsilon_{\alpha,a} = 1$. The set $\mathcal{R}$ is thus the union of the blue curves. The blue dot corresponds to the particular pair $(\alpha, a) = (1, 3.1931469)$ that is used in the numerical results of Section 5.3.

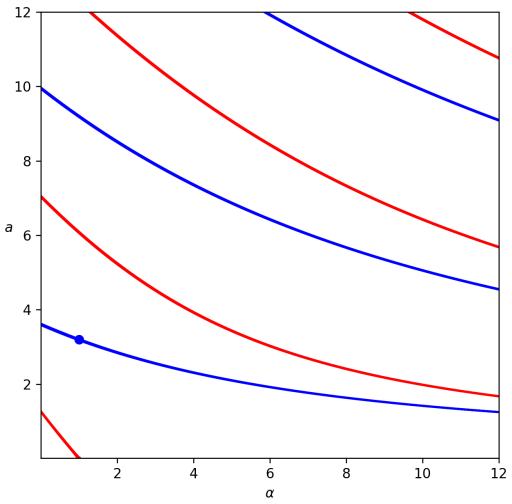


FIG. 2. *In blue: the set $\mathcal{R}$ of critical pairs (α, a). In red: The solutions to (4.28b) that are not solution of (4.28a).*

- To sum up the previous analysis, we have identified the set $\mathcal{R}$ of parameters (α, a) for which there exists a single critical eigenvalue $\lambda^c_{\alpha,a}$ given by (4.27) for which the associated eigenfunction is not observable, that is $\mathcal{B}_2^* \Phi_{\lambda^c_{\alpha,a}} = 0$.
  We can now find out the approximate controllability properties of our problem.
  1. For any given pair $(\alpha, a) \notin \mathcal{R}$ all the eigenfunctions of $\mathcal{A}^*_{\alpha,a}$ are observable, and henceforth, the Fattorini-Hautus criterion is satisfied (see [11, 16]) which implies the approximate controllability of the system in the space $\mathcal{H}_{-\alpha}$.
  2. If a given pair (α, a) belongs to $\mathcal{R}$, then the system (1.8)–(1.9b) can not be approximately controllable in the full space $\mathcal{H}_{-\alpha}$, since for the particular eigenvalue given in (4.27), we have $\mathcal{B}_2^* \Phi_{\lambda^c_{\alpha,a}} = 0$; thus the Fattorini-Hautus criterion fails.
     However, it is not difficult to observe that if the initial data belongs to the smaller space defined by

$$\mathcal{Y}_{\alpha,a} := \Big\{ y_0 \in \mathcal{H}_{-\alpha} \mid \langle y_0, \Phi_{\lambda^c_{\alpha,a}} \rangle_{\mathcal{H}_{-\alpha}, \mathcal{H}_\alpha} = 0 \Big\}, \tag{4.31}$$

     then the approximate controllability of the system holds true. ∎

4.2.2. Estimates on the eigenfunctions. We will gather here the estimates we need on the eigenfunctions, namely a bound from below for the observation terms $\mathcal{B}_2^* \Phi_\lambda$ and a bound from above for the norms $\|\Phi_\lambda\|_{\mathcal{H}_\alpha}$.

LEMMA 4.8. *Let $a \in \mathbb{R}$ and $\alpha \geq 0$ be given. Then, there exists some $C_{\alpha,a}$ such that we have*

$$\|\Phi_\lambda\|_{\mathcal{H}_\alpha} \leq C_{\alpha,a}(1 + \sqrt{|\lambda|}), \quad \forall \lambda \in \Lambda_{\alpha,a},$$

and moreover, the observation terms enjoy the following estimate

1. *When $(\alpha, a) \notin \mathcal{R}$, we have*

$$|\mathcal{B}_2^* \Phi_\lambda| \geq \frac{1}{C_{\alpha,a}}(1 + \sqrt{|\lambda|}), \quad \forall \lambda \in \Lambda_{\alpha,a}. \tag{4.32}$$

2. *On the other hand, for any pair* $(\alpha, a) \in \mathcal{R}$*, we have the same estimate* (4.32) *for all* $\lambda \in \Lambda_{\alpha,a} \setminus \{\lambda^c_{\alpha,a}\}$*, where* $\lambda^c_{\alpha,a}$ *is given in* (4.27).

Proof. We first observe that, thanks to the structure of the spectrum of our operator given in (4.18), it is enough to establish the required estimates for $\lambda \in \Lambda^\infty_{\alpha,a}$, in which case we can take advantage of the explicit asymptotic behavior of the eigenvalues that we have established above. Moreover we only treat here the case $a \neq 0$ since the case $a = 0$ can be treated easily in the very same way by using formulas (4.5)-(4.6) instead of (4.13).

- For the case $\lambda = 0$, (which is possible only when $a + 3\alpha + 6 = 0$) it is easy to see from (4.9) that there exists some $C_{\alpha,a} > 0$ such that
$$\|\Phi_0\|_{\mathcal{H}_\alpha} \leq C_{\alpha,a}.$$
- Next, we suppose $\lambda \neq 0$ and observe that
$$\begin{aligned}\|\Phi_\lambda\|^2_{\mathcal{H}_\alpha} &= (\mathcal{A}_{\alpha,0}\Phi_\lambda, \Phi_\lambda)_E \\ &= (\mathcal{A}^*_{\alpha,a}\Phi_\lambda, \Phi_\lambda)_E - (\mathcal{M}^*_a\Phi_\lambda, \Phi_\lambda)_E \\ &= \lambda\|\Phi_\lambda\|^2_E - (\mathcal{M}^*_a\Phi_\lambda, \Phi_\lambda)_E \\ &\leq (|a| + |\lambda|)\|\Phi_\lambda\|^2_E.\end{aligned}$$
Therefore, we are reduced to find a uniform estimate of the norm in E of Φ_λ.
Using the explicit expression (4.13) of the eigenfunction Φ_λ we get that
$$\|\Phi_\lambda\|_E \leq C_a\left(1 + \frac{1}{|\mu \sin \mu|}\right),$$
and so we finally simply need to show that
$$\sup_k \|\Phi_{\lambda^{\alpha,a}_{k,1}}\|_E + \sup_k \|\Phi_{\lambda^{\alpha,a}_{k,2}}\|_E < +\infty.$$
  - Concerning the first family of eigenvalues, by the asymptotics (4.14), we have that $|\sin(\mu^{\alpha,a}_{k,1})|$ is close to 1 for k large enough. Therefore it is clear that $\|\Phi_{\lambda^{\alpha,a}_{k,1}}\|_E$ is a bounded quantity when k goes to infinity.
  - For the second family of eigenvalues, using (4.15) we see that $\sin(\mu^{\alpha,a}_{k,2})$ is now close to 0 for k large. However, the precise asymptotics shows that the product
$$\left|\mu^{\alpha,a}_{k,2} \sin(\mu^{\alpha,a}_{k,2})\right|,$$
is close to $|a|/4$ for k large, and thus $\|\Phi_{\lambda^{\alpha,a}_{k,2}}\|_E$ is also bounded.
- Concerning the observation terms, we start from (4.22) and separate again the study for the two families of eigenvalues.
  - By the same argument as before, we see that $|\sin(\mu^{\alpha,a}_{k,1})|$ is close to 1 and $|\cos(\mu^{\alpha,a}_{k,1})|$ is close to 0, so that we get
$$\mathcal{B}^*_2\Phi_{\lambda^{\alpha,a}_{k,1}} \underset{+\infty}{\sim} \mu^{\alpha,a}_{k,1} = \sqrt{\lambda^{\alpha,a}_{k,1}}.$$
  - Concerning the second family of eigenvalues, we need to carefully study the last term in (4.22), that gives
$$\frac{a\cos(\mu^{\alpha,a}_{k,2})}{2\sin(\mu^{\alpha,a}_{k,2})} \underset{+\infty}{\sim} \frac{a(-1)^{k+1}}{2(-1)^{k+1}\left(\frac{-a}{4k\pi}\right)} = -2k\pi \underset{+\infty}{\sim} -2\mu^{\alpha,a}_{k,2}.$$
Coming back to (4.22), we conclude that
$$\mathcal{B}^*_2\Phi_{\lambda^{\alpha,a}_{k,2}} \underset{+\infty}{\sim} -\mu^{\alpha,a}_{k,2} = -\sqrt{\lambda^{\alpha,a}_{k,2}}.$$

This, along with Lemma 4.7 gives the required results in points 1 and 2 of our Lemma. ∎

4.3. Null-controllability. We now focus on obtaining a null-control for the system (1.8)–(1.9b). We recall again that the diffusion coefficients are $\gamma_1 = \gamma_2 = 1$.

4.3.1. The moments problem. The set of eigenfunctions $\{\Phi_\lambda\}_{\lambda\in\Lambda_{\alpha,a}}$ of $\mathcal{A}^*_{\alpha,a}$ is a complete family in $\mathcal{H}_\alpha$ on account of Proposition 4.6, so it is enough to check the controllability equation (2.14) for Φ_λ for each $\lambda\in\Lambda_{\alpha,a}$. This indeed tells us that, for any $y_0\in\mathcal{H}_{-\alpha}$, the input $v\in L^2(0,T;\mathbb{C})$ is a null-control for (1.8)–(1.9b) if and only if we have

$$-\overline{e^{-\lambda T}}\langle y_0,\Phi_\lambda\rangle_{\mathcal{H}_{-\alpha},\mathcal{H}_\alpha}=\overline{\mathcal{B}_2^*\Phi_\lambda}\int_0^T v(t)\,\overline{e^{-\lambda(T-t)}}\,dt,\quad\forall\lambda\in\Lambda_{\alpha,a}. \tag{4.33}$$

where we used the fact that $e^{-t\mathcal{A}^*_{\alpha,a}}\Phi_\lambda=e^{-t\lambda}\,\Phi_\lambda$, for any $\lambda\in\Lambda_{\alpha,a}$.

The above set of equations are the moments problem in our case, that we need to solve.

4.3.2. Existence of bi-orthogonal family. From the set of moments problem, we shall find a control v built upon a suitable bi-orthogonal family to the time-dependent exponential functions. In this context, it is worth mentioning [2, Theorem 1.5] where the authors proved the existence of bi-orthogonal families to $\{t^je^{-\sigma_kt}\}_{k\geq0,0\leq j\leq\eta}$ ($\eta\in\mathbb{N}$) for a complex sequence $\{\sigma_k\}_{k\geq0}$ with non-decreasing modulus. This proof is based on a proper gap condition of $|\sigma_k-\sigma_n|$ for all $k\neq n$ and some property of the counting function associated with $\{\sigma_k\}_{k\geq0}$. In fact, concerning this hypothesis on the counting function, a slightly modified version has been introduced in [1, Remark 4.3] and we indeed make use of this fact in our case.

Here, we show that the set of eigenvalues $\Lambda_{\alpha,a}$ satisfies all the assumptions of [2, Theorem 1.5]. First, that theorem needs all the elements of $\Lambda_{\alpha,a}$ to have positive real part. So, if needed, one could choose some $m_{\alpha,a}>0$, such that $(\lambda+m_{\alpha,a})$ has positive real part for all $\lambda\in\Lambda_{\alpha,a}$, and we shall then focus on finding the bi-orthogonal family for $\big\{e^{-(\lambda+m_{\alpha,a})(T-t)}\big\}_{\lambda\in\Lambda_{\alpha,a}}$.

1. We know that the set of eigenvalues $\Lambda_{\alpha,a}$ is discrete, and that only a finite number of them are possibly non-real, see (4.18). Therefore, we clearly have the estimate
$$\big|\Im(\lambda)\big|\leq C_{\alpha,a}\sqrt{\Re(\lambda)+m_{\alpha,a}},\quad\forall\lambda\in\Lambda_{\alpha,a}.$$
2. *The gap condition.*
Since the set of eigenvalues $\Lambda_{\alpha,a}$ is discrete, one has $|\lambda-\widetilde{\lambda}|\neq0$ for any two elements $\lambda\neq\widetilde{\lambda}$ of $\Lambda_{\alpha,a}$. So, it is enough to obtain a proper gap condition for large eigenvalues where we can take the advantage of having only real eigenvalues.
In this context, it is important to recall the set of all real eigenvalues defined in (4.20) and for simplicity one may re-denote the sequence $\big\{\lambda^{\alpha,a}_{k,i}\big\}_{k\geq k_{\alpha,a},i=1,2}$ by $\{\lambda_{2k+i}\}_{k\geq k_{\alpha,a},i=1,2}$ ($k_{\alpha,a}$ has been introduced in Corollary 4.5), in increasing order as follows
$$\lambda_{2k_{\alpha,a}+1}<\lambda_{2k_{\alpha,a}+2}<\lambda_{2(k_{\alpha,a}+1)+1}<\cdots,$$
with
$$\lambda_{2k+1}:=\lambda^{\alpha,a}_{k,1},\quad\lambda_{2k+2}:=\lambda^{\alpha,a}_{k,2},\quad\forall k\geq k_{\alpha,a}.$$
For the re-defined sequence above, we start with the index $2k_{\alpha,a}+1$, since we have that the set $\Lambda^0_{\alpha,a}\subset\Lambda_{\alpha,a}$ (see (4.19)) consists of exactly $2k_{\alpha,a}$ eigenvalues.
Let us take into account the asymptotic formulas given by Corollary 4.5 and compute the following,
$$\begin{aligned}\lambda_{2k+2}-\lambda_{2k+1}=\lambda^{\alpha,a}_{k,2}-\lambda^{\alpha,a}_{k,1}&=(k+1)^2\pi^2-(k+\frac12)^2\pi^2+O_{\alpha,a}(1)\\&\geq\widetilde{c}_{\alpha,a}\,k\pi^2,\quad\text{for large } k\geq k_{\alpha,a},\\&\geq\widetilde{c}_{\alpha,a}\,\frac{\pi^2}{7}\big[(2k+2)^2-(2k+1)^2\big],\end{aligned}$$
as also,
$$\begin{aligned}\lambda_{2(k+1)+1}-\lambda_{2k+2}=\lambda^{\alpha,a}_{k+1,1}-\lambda^{\alpha,a}_{k,2}&=(k+1+\frac12)^2\pi^2-(k+1)^2\pi^2+O_{\alpha,a}(1)\\&\geq\widetilde{c}_{\alpha,a}\,(k+1)\pi^2,\quad\text{for large } k\geq k_{\alpha,a},\\&\geq\widetilde{c}_{\alpha,a}\,\frac{\pi^2}{5}\big[(2(k+1)+1)^2-(2k+2)^2\big],\end{aligned}$$

for some constant $\widetilde{c}_{\alpha,a} > 0$. Now, from the above two inequalities, it is not difficult to obtain

$$\lambda_{2k+i} - \lambda_{2n+j} \geq \rho_{\alpha,a}\left[(2k+i)^2 - (2n+j)^2\right], \ \forall \begin{cases} k > n \geq \tilde{k}_{\alpha,a},\ i,j \in \{1,2\}, \\ k = n \geq \tilde{k}_{\alpha,a},\ i > j, \end{cases} \tag{4.34}$$

with $\rho_{\alpha,a} = \widetilde{c}_{\alpha,a}\,\pi^2/7$, independent of the choices of eigenvalues. Hence, the requirement of gap condition satisfies.

3. *The counting function.*
Let $\mathcal{N}$ be the counting function associated with the set of eigenvalues $\Lambda_{\alpha,a}$, defined by

$$\mathcal{N}(r) = \#\big\{\lambda \in \Lambda_{\alpha,a} : |\lambda| \leq r\big\}, \quad \forall r > 0.$$

We have that, the function $\mathcal{N}$ is piecewise constant and non-decreasing in the interval $[0, +\infty)$. Also for every $r \in (0, +\infty)$ we have $\mathcal{N}(r) < +\infty$ and $\lim_{r\to+\infty} \mathcal{N}(r) = +\infty$.

Without loss of generality, one can start with some sufficiently large number $\overline{r} > 0$, such that $\forall r \geq \overline{r}$, the eigenvalue $\lambda_{\mathcal{N}(r)}$ is real. Assuming this $\mathcal{N}(r)$ to be an odd number, we have, from the definition of $\mathcal{N}$, that

$$\begin{aligned} &\mathcal{N}(r) = 2k+1 \Longleftrightarrow \lambda_{2k+1} \leq r \text{ and } \lambda_{2k+2} > r, \quad \text{for } k \geq k_{\alpha,a}, \\ &\text{i.e., } \sqrt{\lambda_{2k+1}} \leq \sqrt{r} < \sqrt{\lambda_{2k+2}}, \quad \text{for } k \geq k_{\alpha,a}, \end{aligned}$$

which yields, by Lemma 4.4,

$$(k + \frac{1}{4})\pi \leq \sqrt{r} < (k + \frac{5}{4})\pi, \quad \text{for } k \geq k_{\alpha,a} \text{ and } \forall r \geq \overline{r} > 0.$$

Replacing k by $\frac{(\mathcal{N}(r)-1)}{2}$, we determine that

$$\frac{2}{\pi}\sqrt{r} - \frac{3}{2} < \mathcal{N}(r) \leq \frac{2}{\pi}\sqrt{r} + \frac{1}{2}, \quad \forall r \geq \overline{r} > 0. \tag{4.35}$$

Similarly, for even $\mathcal{N}(r)$, we shall have similar estimate for $\mathcal{N}(r)$, possibly with different constants in both sides.

Now, for smaller $0 < r < \overline{r}$, it is obvious that there always exists some constant $\widehat{c}_{\alpha,a} > 0$, sufficiently large and independent of $0 < r < \overline{r}$ such that

$$\mathcal{N}(r) \leq \widehat{c}_{\alpha,a}(\sqrt{r} + 1), \tag{4.36}$$

since $\mathcal{N}$ is bounded function in $(0, \overline{r})$.
The above inequalities (4.35) and (4.36) are the required conditions for counting functions.

So, by virtue of [2, Theorem 1.5], we can ensure the existence of a family $\{q_\lambda\}_{\lambda\in\Lambda_{\alpha,a}} \subset L^2(0,T;\mathbb{C})$, bi-orthogonal to $\big\{e^{-(\lambda+m_{\alpha,a})(T-t)}\big\}_{\lambda\in\Lambda_{\alpha,a}}$, $t \in (0,T)$, that is to say

$$\int_0^T q_\lambda(t)\,\overline{e^{-(\widetilde{\lambda}+m_{\alpha,a})(T-t)}}\,dt = \delta_{\lambda,\widetilde{\lambda}}, \quad \forall \lambda, \widetilde{\lambda} \in \Lambda_{\alpha,a}.$$

In addition, this family satisfies the following estimate

$$\|q_\lambda\|_{L^2(0,T)} \leq C_{\alpha,a}\, e^{C_{\alpha,a}\left(\sqrt{\Re(\lambda)+m_{\alpha,a}}+\frac{1}{T}\right)}, \quad \forall \lambda \in \Lambda_{\alpha,a}, \tag{4.37}$$

for some $C_{\alpha,a} > 0$ which only does depend the constants obtained in the point 1, 2, 3 in the above discussions but definitely not on the eigenvalues $\lambda \in \Lambda_{\alpha,a}$.

4.3.3. Existence of a control. Now we are in the situation to prove the null-controllability result, typically the following proof.

Proof of Theorem 2.7.
Without loss of generality, we prove the theorem for given time $0 < T \leq 1$. Since for any time $\widetilde{T} > 1$, we know that a continuation by 0 of a control in $(0,1)$ will still be a control in $(0,\widetilde{T})$.

- We first suppose $(\alpha, a) \notin \mathcal{R}$ and consider

$$v(t) = \sum_{\lambda \in \Lambda_{\alpha,a}} v_\lambda(t), \quad \forall t \in (0,T), \quad \text{with} \tag{4.38a}$$

$$v_\lambda(t) = -\frac{\overline{e^{-\lambda T}}}{\overline{\mathcal{B}_2^* \Phi_\lambda}} \langle y_0, \Phi_\lambda \rangle_{\mathcal{H}_{-\alpha}, \mathcal{H}_\alpha} q_\lambda(t), \quad \forall t \in (0,T), \tag{4.38b}$$

for $\lambda \in \Lambda_{\alpha,a}$, any given $y_0 \in \mathcal{H}_{-\alpha}$ and any $0 < T \leq 1$. The above construction of v_λ is well-defined since we have, by Lemma 4.7, that $\mathcal{B}_2^* \Phi_\lambda \neq 0$, $\forall \lambda \in \Lambda_{\alpha,a}$.
With this choice of v, we can observe that the set of moments problem (4.33) is formally satisfied. It remains to show the convergence of the series, and then we need to find the $L^2(0,T)$ norm of v_λ for each $\lambda \in \Lambda_{\alpha,a}$. We see that

$$\begin{aligned} \|v_\lambda\|_{L^2(0,T)} &\leq \frac{|e^{-\lambda T}|}{|\mathcal{B}_2^* \Phi_\lambda|} \|y_0\|_{\mathcal{H}_{-\alpha}} \|\Phi_\lambda\|_{\mathcal{H}_\alpha} \|q_\lambda\|_{L^2(0,T)} \\ &\leq C_{\alpha,a}\, e^{\frac{C_{\alpha,a}}{T}}\, e^{-T\Re(\lambda)}\, e^{C_{\alpha,a}\sqrt{\Re(\lambda)+m_{\alpha,a}}} \times \|y_0\|_{\mathcal{H}_{-\alpha}} \frac{\|\Phi_\lambda\|_{\mathcal{H}_\alpha}}{|\mathcal{B}_2^* \Phi_\lambda|}, \end{aligned} \tag{4.39}$$

thanks to the estimate of bi-orthogonal family in (4.37).
Thereafter, an application of Cauchy-Schwarz inequality gives

$$C_{\alpha,a}\sqrt{\Re(\lambda)+m_{\alpha,a}} \leq \frac{T}{2}\big(\Re(\lambda)+m_{\alpha,a}\big) + \frac{C_{\alpha,a}^2}{2T},$$

so that one has

$$\begin{aligned} e^{-T\Re(\lambda)}\, e^{C_{\alpha,a}\sqrt{\Re(\lambda)+m_{\alpha,a}}} &\leq e^{\frac{C_{\alpha,a}}{T}+\frac{T}{2}m_{\alpha,a}}\, e^{-\frac{T}{2}\Re(\lambda)} \\ &\leq C_{\alpha,a}\, e^{\frac{C_{\alpha,a}}{T}-\frac{T}{2}\Re(\lambda)} \quad \forall \lambda \in \Lambda_{\alpha,a}, \end{aligned} \tag{4.40}$$

where we have used that $0 < T \leq 1$ to write $e^{\frac{T}{2}m_{\alpha,a}} \leq C_{\alpha,a}$ for some constant $C_{\alpha,a} > 0$ (which may differ from the previous one).
Next, we use the estimates of the eigenfunctions from Lemma 4.8 to deduce

$$\frac{\|\Phi_\lambda\|_{\mathcal{H}_\alpha}}{|\mathcal{B}_2^* \Phi_\lambda|} \leq C_{\alpha,a}, \quad \forall \lambda \in \Lambda_{\alpha,a}.$$

Now, taking the sum over $\lambda \in \Lambda_{\alpha,a}$ in (4.39), using the above bounds and applying (4.40), we get

$$\sum_{\lambda \in \Lambda_{\alpha,a}} \|v_\lambda\|_{L^2(0,T)} \leq C_{\alpha,a}\, e^{\frac{C_{\alpha,a}}{T}} \|y_0\|_{\mathcal{H}_{-\alpha}} \sum_{\lambda \in \Lambda_{\alpha,a}} e^{-\frac{T}{2}\Re(\lambda)}$$

We finally get that

$$\|v\|_{L^2(0,T)} \leq C_{\alpha,a}\, e^{\frac{C_{\alpha,a}}{T}} \|y_0\|_{\mathcal{H}_{-\alpha}}, \tag{4.41}$$

with a constant $C_{\alpha,a} > 0$ does not depend on T.
- On the other hand, when $(\alpha, a) \in \mathcal{R}$, we consider our control as

$$v(t) = \sum_{\lambda \in \Lambda_{\alpha,a} \setminus \{\lambda_{\alpha,a}^c\}} v_\lambda(t), \quad \forall t \in (0,T), \tag{4.42}$$

with the same formulation of v_λ as prescribed in (4.38b).
Since we have assumed that $y_0 \in \mathcal{Y}_{\alpha,a}$ (the space $\mathcal{Y}_{\alpha,a}$ has been defined in (4.31)), we see that the moments problem (4.33) is actually satisfied for any eigenvalue (in the case $\lambda = \lambda_{\alpha,a}^c$, both sides of the equality are zero).
The L^2-bound of this control alike (4.41) can be then obtained by a similar approach as previous. ∎

5. Some numerical studies. We devote this section to illustrate numerically the controllability results shown in the previous sections. We begin by presenting some facts about the classical penalized Hilbert Uniqueness Method (see e.g. [13] and [6]) and then we will introduce a general methodology to incorporate the effect of the boundary conditions into the discretization of the problem. We conclude by presenting several controllability experiments.

5.1. Preliminaries about the penalized HUM. Following the well-known penalized HUM approach, we shall look for the control v minimizing the primal functional given by

$$F_\epsilon(v) := \frac{1}{2}\int_0^T |v(t)|^2 \mathrm{d}t + \frac{1}{2\epsilon}\|y(T)\|^2_{\mathcal{H}_{-\alpha}}, \tag{5.1}$$

where we used the same notation $y = (y_1, y_2)$ to denote the unique weak solution to the system (1.8) either with the boundary conditions (1.9a) or (1.9b). For the sake of exposition, we assume in what follows that (1.9b) are satisfied.

Observe that, for any $\epsilon > 0$, the functional (5.1) has a unique minimizer in $L^2(0,T;\mathbb{R})$ since F_ϵ is continuous, strictly convex and coercive. Hereafter, we denote this minimizer by v_ϵ.

Using Fenchel-Rockafellar theory (see, for instance [9]), we can identify an associated dual functional, more precisely, for any $\epsilon > 0$, consider

$$J_\epsilon(\zeta) := \frac{1}{2}\int_0^T |\partial_x q_1(t,0)|^2 \,\mathrm{d}t + \frac{\epsilon}{2}\|\zeta\|^2_{\mathcal{H}_\alpha} + (y_0, q(0))_E, \tag{5.2}$$

for given initial data $y_0 \in E$, where $q = (q_1, q_2)$ is the solution to the adjoint system (3.1) with given data $q(T) = \zeta = (\zeta_1, \zeta_2) \in \mathcal{H}_\alpha$.

For any $\epsilon > 0$, the dual functional (5.2) also has a unique minimizer, that we denote by ζ^ϵ. Note that, in this case the coercivity comes from the term $\frac{\epsilon}{2}\|\zeta\|^2_{\mathcal{H}_\alpha}$ which corresponds, by duality, to the penalty term introduced in F_ϵ.

Also, by following the arguments in [13, Sections 2.1–2.2], we can obtain the following result relating the corresponding minimizers of F_ϵ and J_ϵ.

PROPOSITION 5.1. *For any $\epsilon > 0$, the minimizers v_ϵ and ζ^ϵ of the functionals F_ϵ and J_ϵ respectively, are related through the formula*

$$v_\epsilon(t) = \partial_x q_1^\epsilon(t,0), \tag{5.3}$$

$$y^\epsilon(T,\cdot) = -\epsilon \mathcal{A}_\alpha \zeta^\epsilon, \tag{5.4}$$

where $q^\epsilon = (q_1^\epsilon, q_2^\epsilon)$ is the solution to (3.1) with given data ζ^ϵ, and y^ϵ stands for the solution to (1.8)–(1.9b) with control v_ϵ. Consequently, we have

$$\inf_{L^2(0,T;\mathbb{R})} F_\epsilon = F_\epsilon(v_\epsilon) = -J_\epsilon(\zeta^\epsilon) = -\inf_{\mathcal{H}_\alpha} J_\epsilon. \tag{5.5}$$

The following result allows us to relate the controllability properties of system (1.8)–(1.9b) with the behavior of the minimizers shown above. More precisely, we write the following theorem.

THEOREM 5.2. *Let v_ϵ and y^ϵ be as in Proposition 5.1. Then we have the following.*

- *System (1.8)–(1.9b) is approximately controllable at time T if and only if*

$$y^\epsilon(T) \to 0, \quad \text{as} \quad \epsilon \to 0. \tag{5.6}$$

- *System (1.8)–(1.9b) is null controllable at time T if and only if*

$$\mathcal{M}^2 := 2\sup_{\epsilon>0}\left(\inf_{L^2(0,T;\mathbb{R})} F_\epsilon\right) < +\infty. \tag{5.7}$$

In this case, we have

$$\|v_\epsilon\|_{L^2(0,T;\mathbb{R})} \le \mathcal{M}, \quad \|y^\epsilon(T)\|_{\mathcal{H}_{-\alpha}} \le \mathcal{M}\sqrt{\epsilon}. \tag{5.8}$$

The proof of such result follows from an adaptation of [6, Theorem 1.7]. Let us remark that the supremum in (5.7) corresponds actually to the limit as $\epsilon \to 0$ of $\inf_{L^2(0,T;\mathbb{R})} F_\epsilon$.

The main relevance of this theorem is that allow us to recover the controllability results presented in the previous sections using the constructive approach of the penalized HUM instead of other more involved arguments. At the numerical level this will be important since we expect that upon discretization the corresponding system maintains its controllability properties and Theorem 5.2 will help to conclude and illustrate this fact.

Now, we discuss some details about the implementation we follow to obtain the controls for problem (1.8)–(1.9b). A straightforward computation yields

$$\nabla J_\epsilon(\zeta) = \Lambda\zeta + \mathcal{A}_\alpha\zeta + \mathring{y}(T), \tag{5.9}$$

where $\mathring{y} := (\mathring{y}_1, \mathring{y}_2)$ is the free solution to (1.8)–(1.9b) that is the solution when we consider $v = 0$ in (1.9b) and Λ stands for the Grammian operator defined by

$$\begin{array}{rcl} \Lambda : \mathcal{H}_\alpha & \to & \mathcal{H}_{-\alpha} \\ \zeta & \mapsto & w(T), \end{array}$$

where $w(T) = (w_1(T), w_2(T))$ that is actually the solution w at time T to the following system: for given $\zeta \in \mathcal{H}_\alpha$, we first solve the adjoint system (3.1) and then

$$\begin{cases} \partial_t w_1 - \partial_x(\gamma_1 \partial_x w_1) = 0 & \text{in } (0,T)\times(0,1), \\ \partial_t w_2 - \partial_x(\gamma_2 \partial_x w_2) + a\, w_1 = 0 & \text{in } (0,T)\times(0,1), \\ w_1(t,1) = w_2(t,1) & \text{in } (0,T), \\ \gamma_1(1)\dfrac{\partial w_1}{\partial x}(t,1) + \gamma_2(1)\dfrac{\partial w_2}{\partial x}(t,1) + \alpha\, w_1(t,1) = 0 & \text{in } (0,T), \\ w_1(0,\cdot) = w_2(0,\cdot) = 0 & \text{in } (0,1), \end{cases}$$

along with the conditions at $x = 0$ as

$$w_1(t,0) = \partial_x q_1(t,0), \quad w_2(t,0) = 0 \quad \text{in } (0,T).$$

In this way, the control we are looking for, can be obtained as follows: for any given $\epsilon > 0$, we compute $\zeta^\epsilon = (\zeta_1^\epsilon, \zeta_2^\epsilon)$, which is solution to the linear problem

$$\epsilon\mathcal{A}_\alpha\zeta + \Lambda\zeta = -\mathring{y}(T) \tag{5.10}$$

and then compute the solution to the adjoint equation with this initial data. Since Λ is a symmetric (w.r.t the duality product between $\mathcal{H}_\alpha$ and $\mathcal{H}_{-\alpha}$), positive semi-definite operator, the conjugate gradient algorithm is a good candidate to solve the linear problem (5.10). We refer to [13, Section 2.2] for the implementation of such algorithm.

Once we have computed the minimizer, we use formula (5.3) to obtain the desired result and by means of Theorem 5.2, the expected controllability properties can be tested by analyzing the involved quantities with respect to the parameter ϵ.

5.2. Numerical implementation for the general system (1.1). For the numerical tests, the systems (1.1) and its adjoint are discretized in time by using a standard implicit Euler scheme with a uniform time step given by $\delta t = T/M$ where M is the number of steps on the mesh. The PDEs are discretized in space by a standard finite-difference scheme, adapted to the corresponding boundary conditions, with a constant discretization step of size $h = 1/(N+1)$, where N is the chosen number of steps. More precisely, we consider fully discrete systems of the form

$$\begin{cases} \dfrac{y^{n+1} - y^n}{\delta t} + \mathcal{A}_h\, y^{n+1} = \mathcal{B}_h v^{n+1}, \quad n \in \{0, \dots M-1\}, \\ y^0 = y_h^0, \end{cases} \tag{5.11}$$

where $y_h^0 \in \mathbb{R}^{2N}$ is an approximation of the given initial data $y(0,\cdot)$, $\mathcal{A}_h \in \mathbb{R}^{2N\times 2N}$ is a suitable approximation of the elliptic operator $\mathcal{A}$ and $\mathcal{B}_h \in \mathbb{R}^{2N}$ stands for the corresponding approximation of the control operator.

As usual, we denote by y_j $(j = 1, 2)$, each of the components of system (1.1).

1. Using a standard finite-difference method, we construct the matrix $\mathcal{A}_{h,D} \in \mathbb{R}^{2N\times 2N}$, which is composed by two tridiagonal matrix coming from the discretization of the operator $-\gamma_j\partial_x^2$,

$j = 1, 2$, with homogeneous Dirichlet boundary conditions and $\gamma_j > 0$ is a constant diffusion coefficient, i.e.,

$$\mathcal{A}_{h,D} = \begin{pmatrix} \mathcal{A}_h^1 & 0 \\ 0 & \mathcal{A}_h^2 \end{pmatrix}, \tag{5.12}$$

where for each $j = 1, 2$, $(\mathcal{A}_h^j y_j)_i = -\frac{\gamma_j}{h^2}(y_{j,i+1} - 2y_{j,i} + y_{j,i-1})$, $i = 1, \dots, N$. At this point, we impose that $y_{j,0} = y_{j,N+1} = 0$. In the subsequent steps we will compute and add the contribution of the boundary conditions to the discretization scheme.

2. To incorporate the effect of the boundary condition at the left point, we compute

$$\mathcal{A}_{h,0} = -\left(-\mathcal{N}_0 + h\mathcal{D}_0\right)^{-1}\mathcal{N}_0. \tag{5.13}$$

This corresponds to writing the boundary unknowns $y_{j,0}$ in terms of the values of $y_{j,1}$ and yields a 2×2 matrix. The result will be then used to construct the auxiliary matrix:

$$\widetilde{\mathcal{A}}_h = \mathcal{A}_{h,D} - \frac{1}{h^2}\mathcal{A}_{h,0} \otimes \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}_{N\times N} \tag{5.14}$$

where $\otimes$ denotes the Kronecker product, i.e., for matrices $S \in \mathbb{R}^{m\times n}$ and $T \in \mathbb{R}^{p\times q}$, the product $S \otimes T$ is the $mp \times nq$ matrix given by

$$S \otimes T = \begin{pmatrix} s_{11}T & \cdots & s_{1n}T \\ \vdots & \ddots & \vdots \\ s_{1m}T & \cdots & s_{mn}T \end{pmatrix} \tag{5.15}$$

3. In a similar fashion, for adding the contribution of the boundary at $x = 1$, we compute

$$\mathcal{A}_{h,1} = (\mathcal{N}_1 + h\mathcal{D}_1)^{-1}\mathcal{N}_1. \tag{5.16}$$

This will give the coefficients obtained by expressing $y_{j,N+1}$ in terms of the values $y_{j,N}$. We add the resulting matrix to the one obtained in the previous step as follows

$$\widehat{\mathcal{A}}_h = \widetilde{\mathcal{A}}_h - \frac{1}{h^2}\mathcal{A}_{h,1} \otimes \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}_{N\times N} \tag{5.17}$$

4. To conclude, we need to add the internal coupling terms. This can be easily done by computing

$$\mathcal{A}_h = \widehat{\mathcal{A}}_h + \mathcal{M}_{coup} \otimes I_{N\times N}. \tag{5.18}$$

Observe that in our theoretical results, we have considered the simple case where the control v is applied to one of the equations of system (1.8) through the boundary conditions (1.9a) or (1.9b). However, observe that in the general system (1.1), the control can be applied in fact to any linear combination of boundary values. To take into account this in our discretization, we propose the following:

1. We obtain the auxiliary vector

$$\widetilde{\mathcal{B}}_h = h(-\mathcal{N}_0 + h\mathcal{D}_0)^{-1}B, \tag{5.19}$$

where one might consider B as the canonical vector $(1, 0)$ or $(0, 1)$, depending on which equation the control is being applied.

2. We obtain the control operator by setting

$$\mathcal{B}_h = \frac{1}{h^2}\widetilde{\mathcal{B}}_h \otimes \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_N. \tag{5.20}$$

REMARK 5.3. *Some remarks are in order.*

- *In the general case, under the conditions of Assumption 1.1, the invertibility of the matrices shown in formulas (5.13), (5.16) and (5.19) is guaranteed for any $h > 0$ small enough, see Lemma A.1 in appendix.*
- *The discretization of system (1.8) with either boundary conditions (1.9a), (1.9b) is a particular case of the scheme presented above. Indeed, we readily see that for such cases we have $\mathcal{A}_{h,0} = 0_{2\times 2}$,*

$$\mathcal{A}_{h,1} = \begin{pmatrix} \frac{\gamma_1}{\gamma_1+\gamma_2+\alpha h} & \frac{\gamma_2}{\gamma_1+\gamma_2+\alpha h} \\ \frac{\gamma_1}{\gamma_1+\gamma_2+\alpha h} & \frac{\gamma_2}{\gamma_1+\gamma_2+\alpha h} \end{pmatrix}, \tag{5.21}$$

$\mathcal{M}_{coup} = \mathcal{M}_a$ and $\widetilde{B}_h = \begin{pmatrix} 0 & 1 \end{pmatrix}^{\mathsf{T}}$ for the boundary condition (1.9a) (resp. $\begin{pmatrix} 1 & 0 \end{pmatrix}^{\mathsf{T}}$ for (1.9b)). In this case, we note that since $\alpha \geq 0$, (5.21) holds for any value of $h > 0$.

We denote by E_h, $\mathcal{H}_{-\alpha,h}$, U_h and $L^2_{\delta t}(0,T;U_h)$ the discrete spaces associated to E, $\mathcal{H}_{-\alpha}$, $\mathbb{R}$ and $L^2(0,T;\mathbb{R})$, respectively. We denote by $F_\epsilon^{h,\delta t}$ the discretization of the functional F_ϵ, $v^{\epsilon,h,\delta t}$ the corresponding minimizer and $y^{\epsilon,h,\delta t} = (y_1^{\epsilon,h,\delta t}, y_2^{\epsilon,h,\delta t})$ the associated controlled solution.

As usual in this context, to connect the discretization to the control problem, we use the penalization parameter $\epsilon = \phi(h) = h^4$. This choice is consistent with the order of approximation of the finite difference scheme. We refer the reader to [6, Section 4] for a more detailed discussion on the selection of the function $\phi(h)$ in the context of the null-controllability of some parabolic problems and its implications.

To concentrate on the dependency of the numerical experiments with respect to the mesh size h, in the following we will always set $M = 4000$. This is due to the fact that the results do not depend too much on the time step (as soon as it is chosen to ensure at least the same accuracy as the space discretization). This was observed in [6] and the same still applies here.

5.3. Numerical experiments.

5.3.1. Dirichlet boundary control.

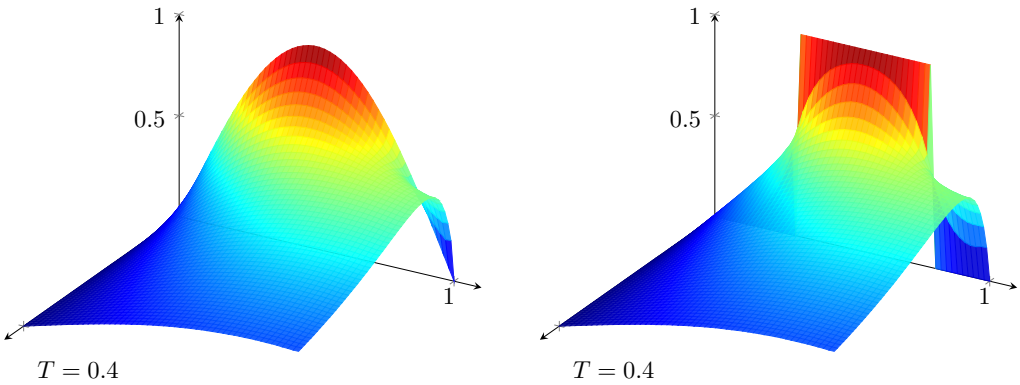
The case $a = 0$. Using our computational tool, we begin by obtaining the solution to system (1.8)–(1.9b) without any control. We consider the set of parameters

$$T = 0.4, \quad \gamma_1 = \gamma_2 = 1, \tag{5.22}$$

$$a = 0, \quad \alpha = 1, \tag{5.23}$$

$$y_{0,1}(x) = \sin(\pi x), \quad y_{0,2}(x) = \mathbf{1}_{(0.3,0.8)}(x) \tag{5.24}$$

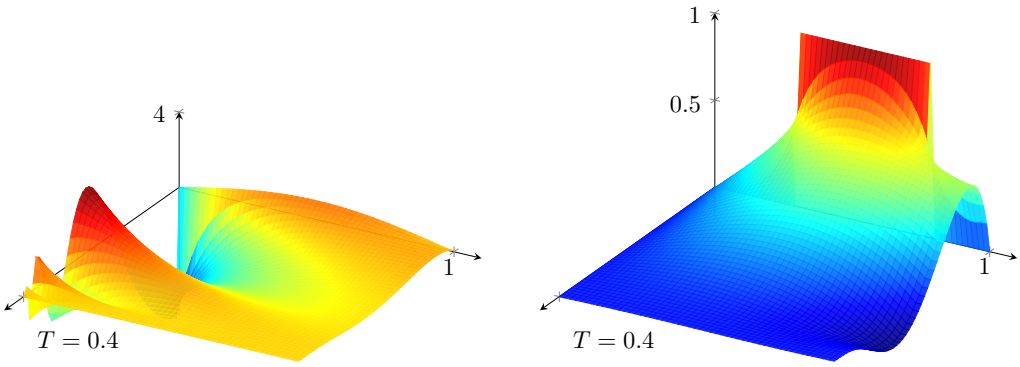
and plot the time evolution of the uncontrolled system in Figure 3. We observe that the solution of both components is damped over time, however they are far from the desired null target.



(a) The state $(t,x) \mapsto y_1(t,x)$ (b) The state $(t,x) \mapsto y_2(t,x)$

FIG. 3. *Evolution in time of the uncontrolled solution of system (1.8)–(1.9b).*

In Figure 4, we show the solution (y_1, y_2) obtained after applying the HUM control $v(t)$ (see Figure 5) computed by algorithm (5.9)–(5.10). We observe, that due to this action, both components reach zero at the prescribed time $T = 0.4$. Notice that, since we have chosen $a = 0$ in (1.8)–(1.9b), the action of the control acts indirectly on the second just by means of the boundary coupling. Intuitively, this



(a) The state $(t, x) \mapsto y_1(t, x)$ (b) The state $(t, x) \mapsto y_2(t, x)$

FIG. 4. *Evolution in time of the controlled solution of system* (1.8)–(1.9b).

problem is harder to solve than other classical problems where the coupling is made in the internal domain.

As far as the asymptotic of the method, we present in Figure 6 the behavior of various quantities of interest as the mesh size goes to 0. We observe that the control cost $\|v^{h,\delta t}_{\phi(h)}\|_{L^2_{\delta t}(0,T;U_h)}$ (—◆—) as well as the optimal energy $\inf F^{h,\delta t}_{\phi(h)}$ (—●—) remain bounded as the mesh size h tends to 0. Also, we see that the norm of the state $\|(y_1^{h,\delta t}(T), y_2^{h,\delta t}(T))\|_{\mathcal{H}_{-\alpha,h}}$ (—■—) behaves like $\sim C\sqrt{\phi(h)} = Ch^2$. This behavior is in agreement with Theorem 5.2 and illustrates our null controllability result.

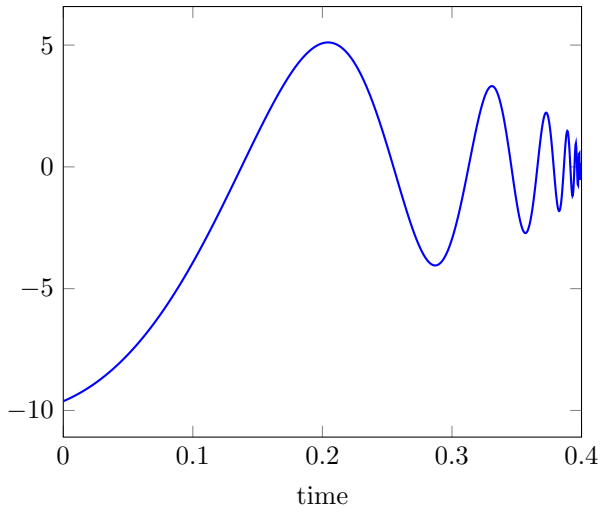


FIG. 5. *Control function* $v(t)$.

The case $a \neq 0$**.** According to our main controllability results, Theorems 2.6 and 2.7, the controllability of system (1.8) is guaranteed depending on the selection of the parameters (α, a) and the way the control enters the system. When the control enters through the first equation, that is, when (1.9b) is verified, we know from Lemma 4.7 that there exist values of (α, a) for which system (1.8) is not even approximately controllable, this is described by means of the set $\mathcal{R}$ defined in (4.29). We illustrate this fact below. By using a numerical algorithm, we can determine that the approximate pair $(\alpha_0, a_0) = (1, 3.1931469)$ belongs to $\mathcal{R}$ (see Figure 2) and corresponds to the critical eigenvalue $\lambda^c_{\alpha_0,a_0} \approx 5.7421936$. Therefore, the eigenfunction $\Phi_{\lambda^c_{\alpha_0,a_0}}$ fails to verify the Fattorini-Hautus criterion. The next figure will elaborate this phenomena.

In Figure 7, we plot the eigenfunction corresponding to the critical eigenvalue $\lambda^c_{\alpha_0,a_0}$. We observe that the first component of the eigenfunction, that is the one in blue color, is almost flat as it approaches to the boundary point $x = 0$ and in fact, numerically we can compute the size of the normal derivative which is of order 10^{-5}. We expect that this is somehow reflected during the penalized HUM procedure.

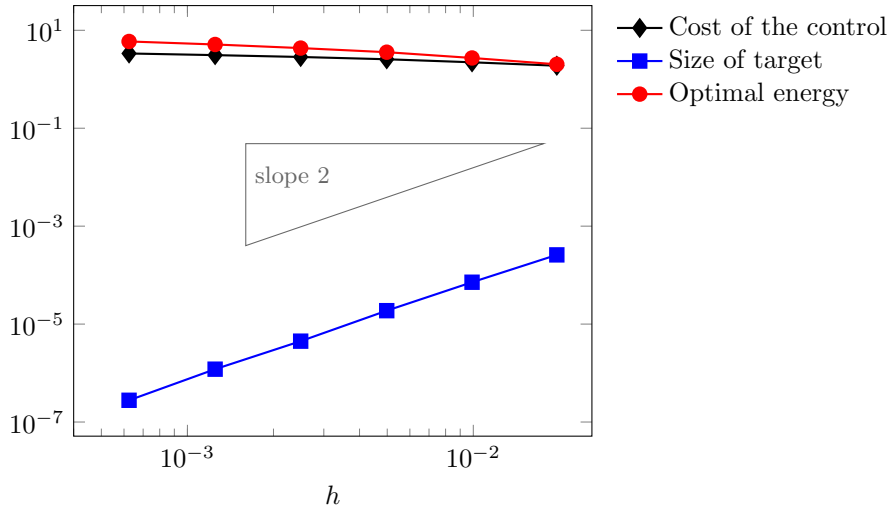


FIG. 6. *Convergence properties of the HUM method.*

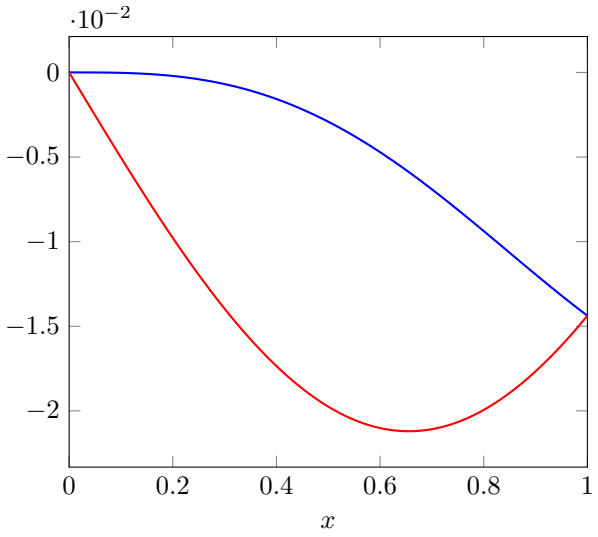


FIG. 7. *Eigenfunctions corresponding to the critical eigenvalue* $\lambda^c_{\alpha_0,a_0}$

We set the parameters

$$\begin{aligned} T = 0.3, \quad \gamma_1 = \gamma_2 = 1, \\ a_0 = 3.1931469, \quad \alpha_0 = 1, \\ y_{0,1}(x) = 10\sin(2\pi x)^3, \quad y_{0,2}(x) = 5 \times \mathbf{1}_{(0.3,0.8)}(x), \end{aligned} \tag{5.25}$$

and apply our computational tool to obtain boundary controls. In Figure 8 we observe the asymptotic behavior of the algorithm. Unlike the previous case, we observe that the optimal energy (−●−) blows up as $\phi(h)^{-1} = h^4$ while the size of the target (−■−) remains constant. This indicates that for the selection of the initial data, system (1.8)–(1.9b) is neither null-controllable or approximately controllable, which is in accordance with our theoretical results.

A further validation of this result can be done by adapting [6, Theorem 1.11], which gives a hint of the general behavior of the penalized HUM method in the limit. In our case, it can be shown that as $h \to 0$, $\mathcal{A}^{-1}_{\alpha,h} y^h(T)$ should converge towards a nonzero function which belongs to the space of unobservable modes. As we have seen in Section 4, this space consists only one element which is the eigenfunction associated to the critical eigenvalue. Thus, we expect to see this at the numerical level.

In Figure 9, it can be seen that as N increases (and therefore $h \downarrow 0$) the target is converging towards some function instead of going to zero. In this case, it is clear that the target converges to the critical eigenfunction (up to a constant) shown in Figure 7 which validates the discussion above.

At this point, we shall mention that the approximation of the critical parameter a_0 plays an

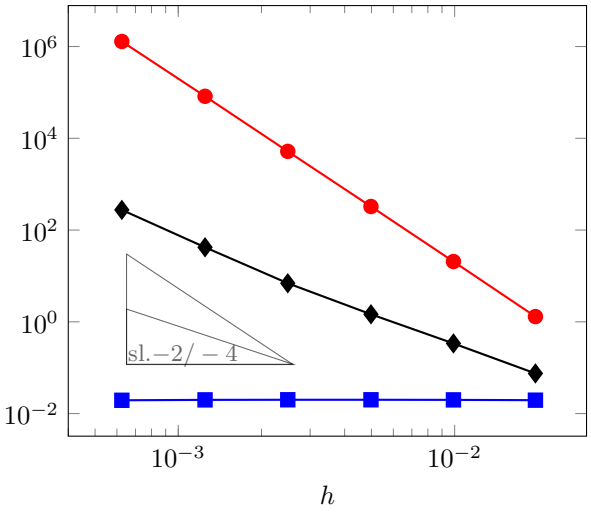
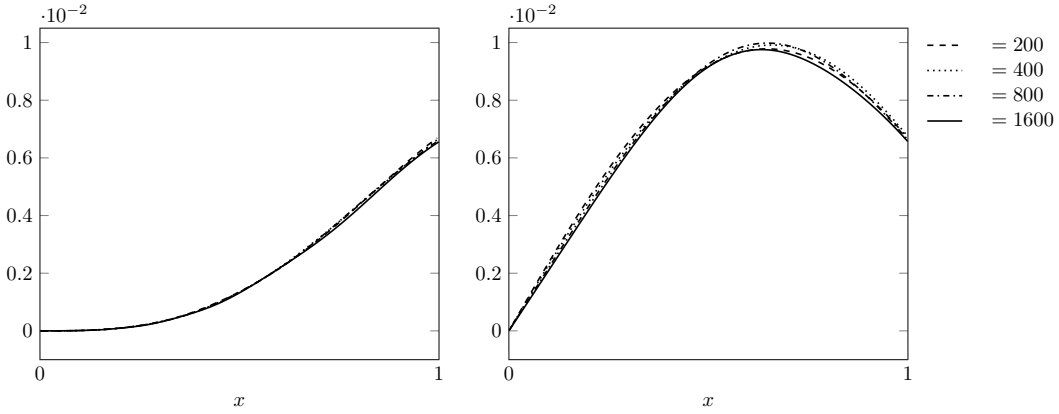


FIG. 8. *Convergence properties of the HUM method for the critical case. Same legend as in Figure 6.*



(a) $y_1(T)$ (b) $y_2(T)$

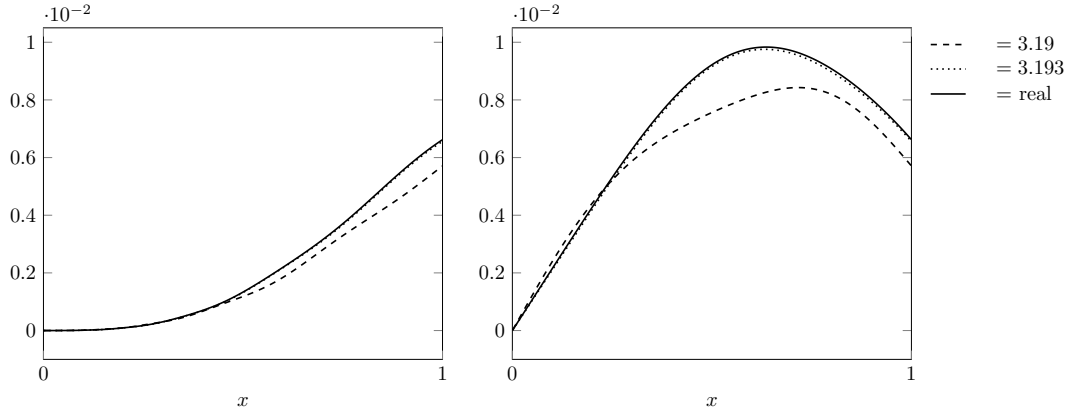
FIG. 9. *Convergence of the* $\mathcal{A}_{\alpha,h}^{-1} y^h(T)$ *as* $h \downarrow 0$.

important role in the numerical experiments. In Figure 10 we present a series of experiments where the parameter a_0 is approximated by truncating up to a certain number of decimals. For a fixed value of h, We see that for a rough approximation (two decimals) the convergence of the target is not as good as for the finer ones (in the experiments shown $h = 1/1600$). We recall that the critical parameters come from obtaining a simultaneous solution to (4.28), therefore the non-controllability result is very sensitive to even small changes of such values. The behavior shown in Figure 10 is therefore consistent with this fact.

We finish the discussion here by emphasizing that the behavior shown in Figure 8 comes from the fact that the control is placed on the boundary of the first component, namely the condition (1.9b). If instead we consider the boundary control on the second component as (1.9a), Theorem 2.6 indicates that regardless the choice of $(\alpha, a) \in \mathbb{R}_0^+ \times \mathbb{R}$, system 1.8–(1.9a) is null-controllable at any time T. We illustrate this fact in Figure 11, where consider the same parameters as in (5.25) with the difference that the control is applied on the boundary of the second equation. We observe that as $h \to 0$ the size of the target decreases as $\sqrt{\phi(h)} = h$ and both the control cost and the optimal energy remain bounded, which is in concordance with the theoretical controllability result.

5.3.2. Neumann boundary control. The goal of this section is to show that our computation tool can be used to illustrate other cases not covered in the theoretical results presented in this paper. This is possible thanks to the general methodology we introduced in Section 5.2.

We will discuss about the controllability of system (1.8) in the case when the boundary conditions



(a) $y_1(T)$ (b) $y_2(T)$

FIG. 10. *Convergence of the target for different approximations of the critical value* a_0.

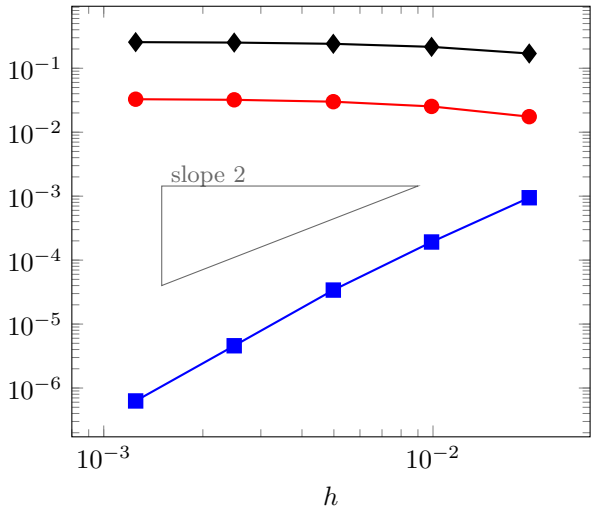


FIG. 11. *Convergence properties of the HUM method with critical values but control applied on the second equation. Same legend as in Figure 6.*

at $x = 0$ are replaced by the Neumann conditions

$$\partial_x y_1(t,0) = v(t), \quad \partial_x y_2(t,0) = 0, \quad \text{in } (0,T). \tag{5.26}$$

Even though this change is quite simple to understand, our results require several non straightforward adaptations for this case, especially the ones at the heart of the moment's method technique.

However, at the numerical level, using the discretization scheme shown in Section 5.2, we just have to set the matrices

$$\mathcal{N}_0 = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}, \quad \mathcal{D}_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.27}$$

$$\mathcal{N}_1 = \begin{pmatrix} 0 & 0 \\ \gamma_1 & \gamma_2 \end{pmatrix}, \quad \mathcal{D}_1 = \begin{pmatrix} 1 & -1 \\ \alpha & 0 \end{pmatrix}, \tag{5.28}$$

and compute the formulas given in (5.12)–(5.20). This simple idea actually allows to test for many configurations and test for different values of a and α. We consider the following simulation parameters

$$\begin{gathered} T = 0.5, \quad \gamma_1 = \gamma_2 = 1, \\ a = 2, \quad \alpha = 4, \\ y_{0,1}(x) = \sin(\pi x), \quad y_{0,2}(x) = \mathbf{1}_{(0.3,0.8)}(x), \end{gathered} \tag{5.29}$$

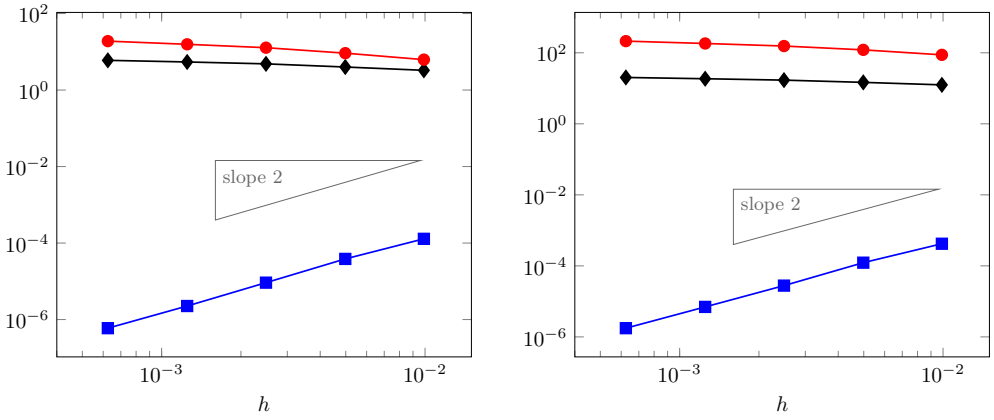
and use our tool to obtain numerical results for two different configurations. In Figure 12a, we show the convergence result for the case where v is applied on the first equation, that is, (5.26). We can see

that as h tends to zero, the size of the target decreases as $\sqrt{\phi(h)} = h^2$ and both the optimal energy and the control cost remain bounded.

On the other hand, we show in Figure 12b the result by changing the control to the second equation, i.e., we consider

$$\partial_x y_1(t,0) = 0, \quad \partial_x y_2(t,0) = v(t), \quad \text{in } (0,T). \tag{5.30}$$

We observe that the behavior of the convergence of the method is exactly the same as in the previous example.



(a) Control applied on the first equation (b) Control applied on the second equation

FIG. 12. *Convergence properties of the HUM method with Neumann control applied in different equations. Same legend as in Figure 6.*

Both simulations point toward a positive null controllability result, nevertheless one should be cautious with such conclusion. In fact, in the case where the control is applied on the second equation, some adaptations can be made to our Carleman estimate presented in Theorem 3.2 to deduce a result for the Neumann condition and thus we can expect null controllability for any a and α.

On the other hand, as we have seen in Section 4.1, a detailed analysis of the spectral behavior of the underlying operator is required when the control is applied on the first equation and the answer of whether the system is null-controllable or not in the whole L^2 space is far from obvious.

Numerical evidence presented in Figure 13 shows that as in the Dirichlet case, there exists at least one couple (α, a) for which the observation of one eigenfunction is zero. We can approximate numerically this pair to $(\alpha_c, a_c) = (0.1, 1.2369289)$. In Figure 14 we are plotting the first eigenfunction associated to this pair and from there, it is clear that such eigenfunction is non observable.

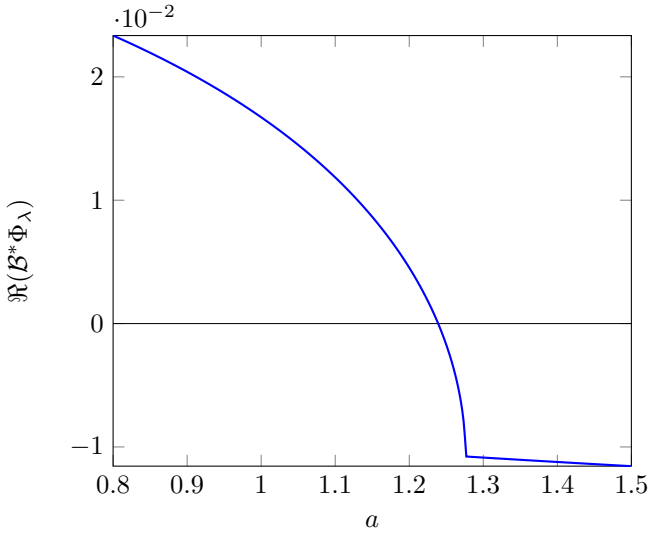


FIG. 13. *Size of the observation in the Neumann control case (5.26) for $\alpha = 0.1$ and $a \in [0.8, 3]$.*

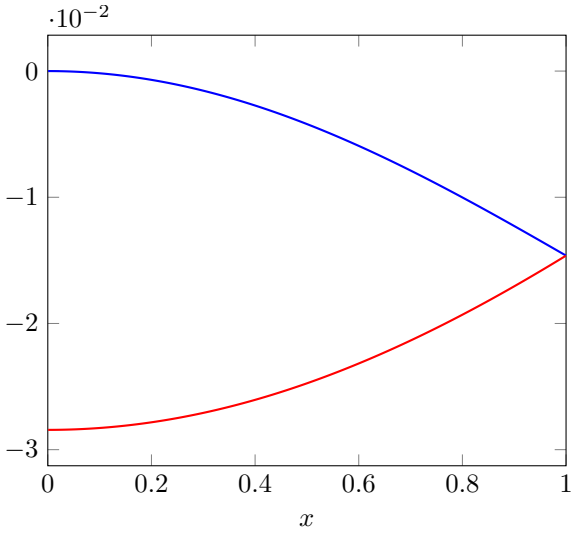


FIG. 14. *Eigenfunctions corresponding to the critical eigenvalue* $\lambda^c_{\alpha,a}$ *(Neumann case).*

Using the new couple (α, a) for simulation purposes, we can use our computational tool to test for controllability. In Figure 15 we present the convergence of the method and as in the Dirichlet case we observe that the size of the target is not decreasing while the optimal energy is blowing up. This points towards a non-controllable result.

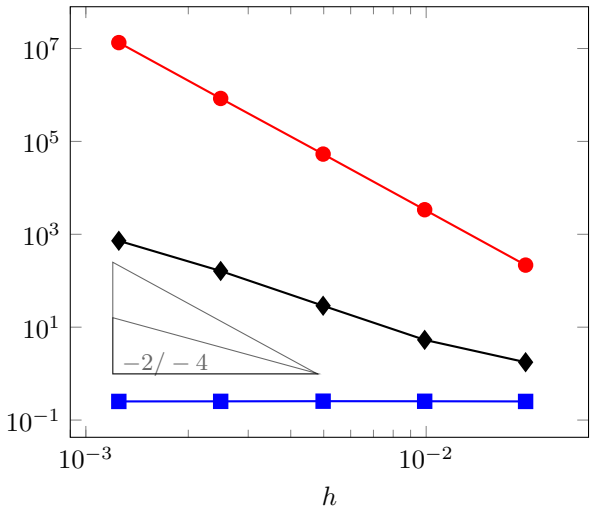


FIG. 15. *Convergence properties of the HUM method for the critical value in the Neumann case. Same legend as in Figure 6.*

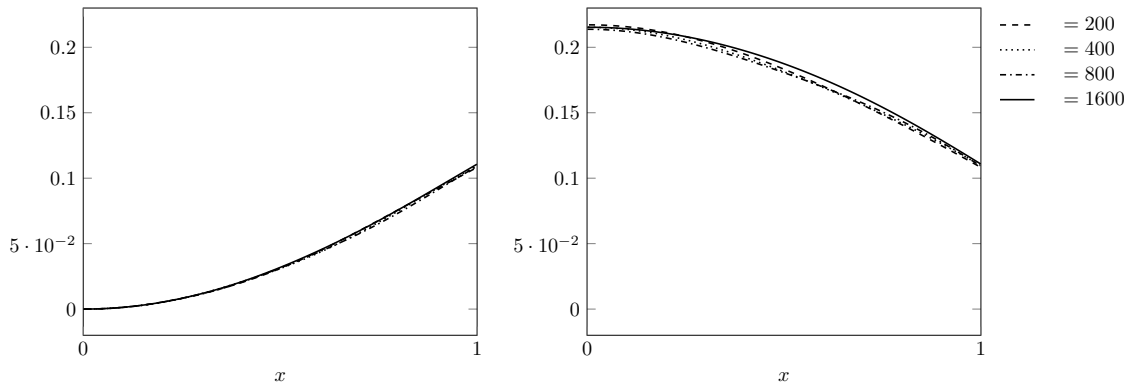
Following with the discussion of the Dirichlet case, we see in Figure 16 that the target is indeed converging towards the critical eigenfunction (up to some constant) which is consistent with the lack of controllability. In view of these results, a deeper study of the Neumann control case is needed to conclude.

We would like to finish this section by emphasizing that as in the Dirichlet case, we need a good approximation of the critical parameter a_c to observe the lack of controllability of the system. In Figure 17, we see the convergence of the target for $h = 1/1600$ and different approximations of a_c. This experiment seems to be more sensitive than the previous case since we need a four decimal approximation of the parameter to obtain a good convergence of the target for the given value of h.

Appendix A. An intermediate result.

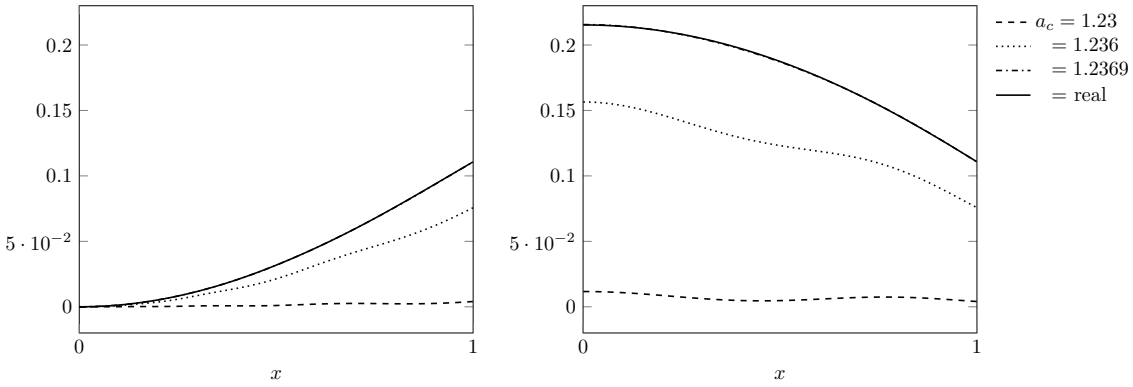
LEMMA A.1. *Let* $\mathcal{D}$ *and* $\mathcal{N}$ *be two real* $d \times d$ *matrices such that*

$$(\mathcal{D}, \mathcal{N}) \text{ is full rank,} \tag{A.1}$$



(a) $y_1(T)$ (b) $y_2(T)$

FIG. 16. *Convergence of the* $\mathcal{A}_{\alpha,h}^{-1} y^h(T)$ *as* $h \downarrow 0$ *in the Neumann case.*



(a) $y_1(T)$ (b) $y_2(T)$

FIG. 17. *Convergence of the target for different approximations of the critical value* a_c

and

$$\mathcal{D}\mathcal{N}^* \text{ is self adjoint,} \tag{A.2}$$

then $\mathcal{N} + t\mathcal{D}$ *is invertible for any* $t \in \mathbb{R}$ *except perhaps for a finite number of values of* t*.*

Proof. We follow the same computations as in [4, Theorem 1.4.4]. More precisely, we first observe that, under the assumptions of the lemma, we have that $\mathcal{D} + i\mathcal{N}$ is invertible. Indeed,

- by (A.1), we know that $(\ker \mathcal{D}^*) \cap (\ker \mathcal{N}^*) = \{0\}$,
- by (A.2), for any $x \in \mathbb{C}^2$ we have

$$\|(\mathcal{D}^* - i\mathcal{N}^*)x\|^2 = \|\mathcal{D}^* x\|^2 + \|\mathcal{N}^* x\|^2,$$

so that $\ker(\mathcal{D}^* - i\mathcal{N}^*) \subset (\ker \mathcal{D}^*) \cap (\ker \mathcal{N}^*) = \{0\}$ and the claim is proved.

We can now can define $\mathcal{U} = -(\mathcal{D} + i\mathcal{N})^{-1}(\mathcal{D} - i\mathcal{N})$ (which is actually a unitary matrix but we don't need this fact here). It satisfies

$$2(\mathcal{D} + i\mathcal{N})^{-1}\mathcal{D} = (\mathcal{D} + i\mathcal{N})^{-1}(\mathcal{D} + i\mathcal{N} + \mathcal{D} - i\mathcal{N}) = (I - \mathcal{U}),$$

$$2(\mathcal{D} + i\mathcal{N})^{-1}\mathcal{N} = -i(\mathcal{D} + i\mathcal{N})^{-1}\big((\mathcal{D} + i\mathcal{N}) - (\mathcal{D} - i\mathcal{N})\big) = -i(I + \mathcal{U}).$$

If we assume that $t \in \mathbb{R}$ is such that $\mathcal{N} + t\mathcal{D}$ is not invertible, then there exists $x \in \mathbb{R}^d, x \neq 0$ such that $(\mathcal{N} + t\mathcal{D})x = 0$. Left-multiplying this equality by $(\mathcal{D} + i\mathcal{N})^{-1}$ an using the above relations we end up with

$$\left(\frac{t-i}{t+i} I - \mathcal{U}\right) x = 0,$$

which proves that $(t - i)/(t + i)$ is an eigenvalue of $\mathcal{U}$. This can only happen for a finite number of values of t. ∎

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