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# A Two-Tiered Propositional Framework for Handling Multisource Inconsistent Information

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**Abstract.** This paper proposes a conceptually simple but expressive framework for handling propositional information stemming from several sources, namely a two-tiered propositional logic augmented with classical modal axioms (BC-logic), a fragment of the non-normal modal logic EMN, whose semantics is expressed in terms of two-valued monotonic set-functions called Boolean capacities. We present a theorem-preserving translation of Belnap logic in this setting. As special cases, we can recover previous translations of three-valued logics such as Kleene and Priest logics. Our translation bridges the gap between Belnap logic, epistemic logic, and theories of uncertainty like possibility theory or belief functions, and paves the way to a unified approach to various inconsistency handling methods.

## 1 Introduction

A number of works has been published proposing approaches that deal with inconsistent knowledge bases in such a way as to extract useful information from them in a non-explosive way [5, 18]. Inconsistency is often due to the presence of multiple sources providing information. Belnap 4-valued logic [4] is one of the earliest approaches to this problem. It is based on a very natural set-up where each source tentatively assigns truth-values to elementary propositions. The sets of truth-values thus collected for these propositions are summarized by so-called epistemic truth-values referring to whether sources are in conflict or not, informed or not. There are 4 such epistemic truth-values, two of which referring to ignorance and conflict. Truth tables for conjunction, disjunction, and negation are used to compute the epistemic status of other complex formulas. This approach underlies both Kleene three-valued logic (when no conflict between sources is observed) and the Priest three-valued logic of paradox [16] (sources are never ignorant and assign truth-values to all elementary propositions).

Besides, inconsistency and incompleteness are present in uncertainty theories as well, using monotonic set-functions called capacities with values in the unit interval, instead of logics. The simplest logical framework for incomplete

information is the two-tiered propositional logic MEL [3], that accounts for an all-or-nothing view of possibility theory [11], and borrows axioms K and D from modal epistemic logic. Replacing necessity measures by general inclusion-monotonic set-functions can account for the idea of conflicting sources of information. It leads to adopting a fragment of the non-normal modal logic EMN [7] as a general logical framework, which can encompass variants of probabilistic and belief function logics, for instance the logic of risky knowledge [15], where the adjunction rule is not valid. Here, we show that our two-tiered propositional setting related to EMN can encode Belnap 4-valued logic, namely that the four truth-values in Belnap logic are naturally expressed by means of capacities taking values on  $\{0, 1\}$ . Thus, we construct a bridge between Belnap logic and uncertainty theories. As special cases, we recover our previous translations of Kleene logic for incomplete information [8] and Priest logic of paradox [9]. Showing the possibility of this translation indicates that our logic has potential to support other inconsistency handling approaches as well. The paper is organized as follows. Section 2 presents the propositional logic of Boolean capacities BC and shows its capability to capture the notion of information coming from several sources. Section 3 recalls Belnap 4-valued logic from the point of view of its motivation, its syntactical inference and its semantics. Section 4 contains the main results pertaining to the translation of Belnap logic into BC. Most proofs are omitted due to length constraints.

## 2 The Logic of Boolean Capacities and Multisource Information Management

In this section, we consider an approach to the handling of pieces of incomplete and conflicting information coming from several sources. We show, following an intuition already suggested in [1, 12] that if we represent each body of information items supplied by each source by means of a set of possible states of affairs, the collective information supplied by the sources can be modelled in a lossless way by a monotonic set function (called a capacity) that takes value on  $\{0, 1\}$ . These set-functions can serve as natural multisource semantics for a simple flat non-regular modal logic that captures the four Belnap truth-values as already suggested in [10]. This logic looks rather uncommitting for handling multiple source information, while other approaches seem to put additional assumptions.

### 2.1 Boolean Capacities and Multisource Information

Consider a standard propositional language  $\mathcal{L}$  with variables  $V = \{a, b, c, \dots\}$  and connectives  $\wedge, \vee, \neg$ , for conjunction, disjunction and negation, respectively. We denote the propositional formulas of  $\mathcal{L}$  by letters  $p, q, \dots$ . Consider a set of states of affairs  $\Omega$  which is the set of interpretations of this language.

**Definition 1.** *A capacity (or fuzzy measure) is a mapping  $\gamma : 2^\Omega \rightarrow [0, 1]$  such that  $\gamma(\emptyset) = 0$ ;  $\gamma(\Omega) = 1$ ; and if  $A \subseteq B$  then  $\gamma(A) \leq \gamma(B)$ .*

The value  $\gamma(A)$  can be interpreted as the degree of support of a proposition  $p$  represented by the subset  $A = [p]$  of its models. A Boolean capacity (B-capacity, for short) is a capacity with values in  $\{0, 1\}$ . It can be defined from a usual capacity and any threshold  $\lambda > 0$  as  $\beta(A) = 1$  if  $\gamma(A) \geq \lambda$  and 0 otherwise.

The useful information in a B-capacity consists of its focal sets. A focal set  $E$  is such that  $\beta(E) = 1$  and  $\beta(E \setminus \{w\}) = 0, \forall w \in E$ . Let  $\mathcal{F}_\beta$  be the set of focal sets of  $\beta$ . They are minimal sets for inclusion such that  $\beta(E) = 1$ : we can check that  $\beta(A) = 1$  if and only if there is a subset  $E$  of  $A$  in  $\mathcal{F}_\beta$  with  $\beta(E) = 1$ .

Consider  $n$  sources providing information in the form of epistemic states modelled by non-empty sets  $E_i \subseteq \Omega$ : it is only known from source  $i$  that the real state of affairs  $s$  should lie in  $E_i$ . A capacity  $\beta$  can be built from these pieces of information then viewed as the set  $\mathcal{F}_\beta = \{E_1, E_2, \dots, E_n\}$  of its focal subsets. Then  $\beta([p]) = 1$  really means that there is at least one source  $i$  that believes that  $p$  is true (that is  $p$  is true in all states of affairs in  $E_i$ ). Note that this way of synthetizing information is not destructive: it preserves every initial piece of information. Given a proposition  $p$ , there are four epistemic statuses based on the information from sources, that can be described by the capacity:

- *Support of  $p$* :  $\beta([p]) = 1$  and  $\beta([\neg p]) = 0$ . Then  $p$  is asserted by at least one source and negated by no other one.
- *Rejection of  $p$* :  $\beta([\neg p]) = 1$  and  $\beta([p]) = 0$ . Then  $p$  is negated by at least one source and asserted by no other one.
- *Ignorance about  $p$* :  $\beta([p]) = \beta([\neg p]) = 0$ . No source supports nor negates  $p$ .
- *Conflict about  $p$* :  $\beta([p]) = \beta([\neg p]) = 1$ . Some sources assert  $p$ , some negate it.

Important special cases are

- when  $\beta$  is minitive, i.e.,  $\beta(A \cap B) = \min(\beta(A), \beta(B))$ . It is then a necessity measure and  $\mathcal{F}_\beta = \{E\}$ . There is only one source and its information is incomplete, but there is no conflict.
- when the focal sets are singletons  $\{e_i\}$ . Then  $\beta$  is maxitive, i.e.,  $\beta(A \cup B) = \max(\beta(A), \beta(B))$ . All sources have complete information, so there are conflicts, but no ignorance. Letting  $E = \{e_1, e_2, \dots, e_n\}$ , then  $\beta(A) = 1$  if and only if  $A \cap E \neq \emptyset$ , formally a possibility measure. But here  $E$  is a conjunction of non-mutually exclusive elements, not a possibility distribution.

## 2.2 The Logic BC

To provide a logical setting to the above situation, we build a higher level propositional language  $\mathcal{L}_\square$  on top of  $\mathcal{L}$ , whose formulas are denoted by Greek letters  $\phi, \psi, \dots$ , and defined by: if  $p \in \mathcal{L}$  then  $\square p \in \mathcal{L}_\square$ ; if  $\phi, \psi \in \mathcal{L}_\square$  then  $\neg\phi \in \mathcal{L}_\square$ ,  $\phi \wedge \psi \in \mathcal{L}_\square$ . Note that the language  $\mathcal{L}$  is not part of  $\mathcal{L}_\square$ , it is embedded in it, since atomic variables of  $\mathcal{L}_\square$  are of the form  $\square p, p \in \mathcal{L}$ . As usual  $\diamond p$  stands for  $\neg\square\neg p$ . It defines a very elementary fragment of a modal logic language [3].

A minimal logic for B-capacities has been proposed using the language  $\mathcal{L}_\square$  [12]. It is a two-tiered propositional logic plus some modal axioms:

1. All axioms of propositional logics for  $\mathcal{L}_\square$ -formulas.
2. The modal axioms:
  - (RM)  $\square p \rightarrow \square q$  if  $\vdash p \rightarrow q$  in propositional logic.
  - (N)  $\square p$ , whenever  $p$  is a propositional tautology.
  - (P)  $\diamond p$ , whenever  $p$  is a propositional tautology.

The only rule is modus ponens: If  $\psi$  and  $\psi \rightarrow \phi$  then  $\phi$ . This is a fragment of the non-regular logic EMN [7]. Note that the two dual modalities  $\square$  and  $\diamond$  play the same role. Namely the above axioms remain valid if we exchange  $\square$  and  $\diamond$ . So these modalities are not distinguishable. Semantics is usually expressed in terms of neighborhood semantics, but it can be equivalently expressed in terms of B-capacities on the set of interpretations  $\Omega$  of the language  $\mathcal{L}$ . We have indicated elsewhere [10] that the set of subsets  $A$  such that  $\beta(A) = 1$  is a special case of neighborhood family in the sense of neighborhood semantics [7]. This logic can thus be called the logic of Boolean Capacities (BC). A BC-model of an atomic formula  $\square p$  is a B-capacity  $\beta$ . The satisfaction of BC-formulas is defined as:

- $\beta \models \square p$ , if and only if  $\beta([p]) = 1$ ;
- $\beta \models \neg\phi$ ,  $\beta \models \phi \wedge \psi$  in the standard way.

Semantic entailment is defined classically, and syntactic entailment is classical propositional entailment taking RM, N, P as axioms:  $\Gamma \vdash_{BC} \phi$  if and only if  $\Gamma \cup \{\text{all instances of } RM, N, P\} \vdash \phi$  (classically defined). It has been proved that BC logic is sound and complete wrt B-capacity models [12]. In fact, axiom RM clearly expresses the monotonicity of capacities, and it is easy to realize that a classical propositional interpretation of  $\mathcal{L}_\square$  that respects the axioms of BC can be precisely viewed as a B-capacity.

As a B-capacity precisely encodes a set of sources each delivering incomplete information items in the form of an  $n$ -tuple of focal sets  $(E_1, E_2, \dots, E_n)$ , we can see that  $\beta \models \square p$  if and only if  $\exists i : E_i \subseteq [p]$ , so we may write  $(E_1, E_2, \dots, E_n) \models \varphi$  in place of  $\beta \models \varphi$ . Denoting by  $\square_i p$  the statement  $E_i \subseteq [p]$ , the formula  $\square p$  is of the form  $\square_1 p \vee \dots \vee \square_n p$  where  $\square_i$  is a standard KD modality in a regular modal logic. Likewise,  $\diamond p = \neg \square \neg p = \neg \square_1 \neg p \wedge \dots \wedge \neg \square_n \neg p = \diamond_1 p \wedge \dots \wedge \diamond_n p$  clearly means that no source is asserting  $\neg p$ . The four epistemic statuses of propositions in  $\mathcal{L}$  can then be expressed by means of modal formulas in  $\mathcal{L}_\square$  as follows [10]:

- *Support of  $p$* :  $(E_1, E_2, \dots, E_n) \models \square p \wedge \diamond p$
- *Rejection of  $p$* :  $(E_1, E_2, \dots, E_n) \models \square \neg p \wedge \diamond \neg p$
- *Ignorance about  $p$* :  $(E_1, E_2, \dots, E_n) \models \diamond p \wedge \diamond \neg p$
- *Conflict about  $p$* :  $(E_1, E_2, \dots, E_n) \models \square p \wedge \square \neg p$

Note that this framework is very cautious, in the sense that inferences made are minimal ones one can expect to make from multisource information. If we add axioms K and D of modal logics, then  $\beta$  is forced to be a necessity measure, and the conflict situation disappears: there is only one source with epistemic set  $E$  driving  $\beta$ . We get the logic MEL [3], a fragment of the logic KD. In case we restrict to capacities  $\beta$  whose focal sets are singletons, the  $\square$  modality has all properties of a KD possibility modality  $\diamond$ . It is a kind of mirror image of

logic MEL where conflict is taken into account but there is no ignorance. It can capture Priest logic of paradox [9]. The aim of this paper is to show that the general framework of BC-logic can encode Belnap logic as a special case.

### 3 Belnap 4-Valued Logic

Belnap [4] considers an artificial information processor, fed from a variety of sources, and capable of answering queries on propositions of interest. The basic assumption is that the computer receives information about atomic propositions in a cumulative way from outside sources, each asserting for each atomic proposition whether it is true, false, or being silent about it. The notion of *epistemic set-up* is defined as an assignment, of one of four values denoted by **T**, **F**, **C**, **U**, to each atomic proposition  $a, b, \dots$ :

1. Assigning **T** to  $a$  means the computer has only been told that  $a$  is true (1) by at least one source, and false (0) by none.
2. Assigning **F** to  $a$  means the computer has only been told that  $a$  is false by at least one source, and true by none.
3. Assigning **C** to  $a$  means the computer has been told at least that  $a$  is true by one source and false by another.
4. Assigning **U** to  $a$  means the computer has been told nothing about  $a$ .

**Table 1.** Belnap disjunction, conjunction and negation

∨	<b>F</b>	<b>U</b>	<b>C</b>	<b>T</b>
<b>F</b>	<b>F</b>	<b>U</b>	<b>C</b>	<b>T</b>
<b>U</b>	<b>U</b>	<b>U</b>	<b>T</b>	<b>T</b>
<b>C</b>	<b>C</b>	<b>T</b>	<b>C</b>	<b>T</b>
<b>T</b>	<b>T</b>	<b>T</b>	<b>T</b>	<b>T</b>

∧	<b>F</b>	<b>U</b>	<b>C</b>	<b>T</b>
<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>	<b>F</b>
<b>U</b>	<b>F</b>	<b>U</b>	<b>F</b>	<b>U</b>
<b>C</b>	<b>F</b>	<b>F</b>	<b>C</b>	<b>C</b>
<b>T</b>	<b>F</b>	<b>U</b>	<b>C</b>	<b>T</b>

$a$	$\neg a$
<b>F</b>	<b>T</b>
<b>U</b>	<b>U</b>
<b>C</b>	<b>C</b>
<b>T</b>	<b>F</b>

If  $\{0, 1\}$  is the set of usual truth values (as assigned by the information sources), then the set  $\mathbb{V}_4 = \{\mathbf{T}, \mathbf{F}, \mathbf{C}, \mathbf{U}\}$  of epistemic truth values coincides with the power set of  $\{0, 1\}$ , letting  $\mathbf{T} = \{1\}$ ,  $\mathbf{F} = \{0\}$ . According to the convention initiated by Dunn [13], **U** represents the empty set and corresponds to no information received, while **C** =  $\{0, 1\}$  represents the presence of conflicting sources, expressing *true and false* at the same time. Belnap’s approach relies on two orderings in  $\mathbb{V}_4 = \{\mathbf{T}, \mathbf{F}, \mathbf{C}, \mathbf{U}\}$ , equipping it with two lattice structures:

- *The information ordering*,  $\sqsubset$  whose meaning is “less informative than”, such that  $\mathbf{U} \sqsubset \mathbf{T} \sqsubset \mathbf{C}$ ;  $\mathbf{U} \sqsubset \mathbf{F} \sqsubset \mathbf{C}$ . This ordering reflects the inclusion relation of the sets  $\emptyset, \{0\}, \{1\}$ , and  $\{0, 1\}$ .  $(\mathbb{V}_4, \sqsubset)$  is *the information lattice*.
- *The truth ordering*,  $<_t$ , representing “more true than” according to which  $\mathbf{F} <_t \mathbf{C} <_t \mathbf{T}$  and  $\mathbf{F} <_t \mathbf{U} <_t \mathbf{T}$ , each chain reflecting the truth-set of Kleene’s logic. In other words, ignorance and conflict play the same role with respect to **F** and **T** according to this ordering. It yields *the logical lattice*, based on

the truth ordering, and the interval extension of standard connectives  $\wedge$ ,  $\vee$  and  $\neg$  from  $\{0, 1\}$  to  $2^{\{0,1\}} \setminus \{\emptyset\}$ . In this lattice, the maximum of  $\mathbf{U}$  and  $\mathbf{C}$  is  $\mathbf{T}$  and the minimum is  $\mathbf{F}$ .

The syntax is the one of propositional logic. Connectives of negation, conjunction and disjunction are defined truth-functionally in Belnap 4-valued logic (see Table 1). Belnap 4-valued logic has no tautologies, but it has an inference system. It can be defined only via a set of inference rules, as those that can be found in [14,17]:

**Definition 2.** *Let  $a, b, c \in V$ . The inference system of Belnap 4-valued logic is defined by no axiom and the following set of rules*

$$\begin{aligned}
(R1) : \frac{a \wedge b}{a} & \quad (R2) : \frac{a \wedge b}{b} & (R3) : \frac{a}{a \wedge b} & \quad (R4) : \frac{a}{a \vee b} \\
(R5) : \frac{a \vee b}{b \vee a} & \quad (R6) : \frac{a \vee a}{a} & (R7) : \frac{a \vee (b \vee c)}{(a \vee b) \vee c} \\
(R8) : \frac{a \vee (b \wedge c)}{(a \vee b) \wedge (a \vee c)} & \quad (R9) : \frac{(a \vee b) \wedge (a \vee c)}{a \vee (b \wedge c)} & (R10) : \frac{a \vee c}{\neg \neg a \vee c} \\
(R11) : \frac{\neg(a \vee b) \vee c}{(\neg a \wedge \neg b) \vee c} & \quad (R12) : \frac{\neg(a \wedge b) \vee c}{(\neg a \vee \neg b) \vee c} & (R13) : \frac{\neg \neg a \vee c}{a \vee c} \\
(R14) : \frac{(\neg a \wedge \neg b) \vee c}{\neg(a \vee b) \vee c} & \quad (R15) : \frac{(\neg a \vee \neg b) \vee c}{\neg(a \wedge b) \vee c}
\end{aligned}$$

These rules express that conjunction is idempotent and distributes over disjunction, disjunction is idempotent, associative and distributes over conjunction. Negation is involutive and De Morgan Laws are satisfied. It makes clear that the underlying algebra is a De Morgan algebra [17]. Applying these rules, formulas can be put in normal form as a conjunction of clauses, i.e.,  $p = p_1 \wedge \dots \wedge p_n$ , where the  $p_i$ 's are disjunctions of literals  $l_{ij} = a$  or  $\neg a$  where  $a \in V$ . For the semantics, consider again the four epistemic truth-values forming the set  $\mathbb{V}_4 = \{\mathbf{F}, \mathbf{U}, \mathbf{C}, \mathbf{T}\}$ . A Belnap valuation is a mapping  $vb : \mathcal{L} \mapsto \mathbb{V}_4$ . Let  $\Gamma \subseteq \mathcal{L}$  and  $p \in \mathcal{L}$ , then we define the consequence relation by means of the truth ordering  $\leq_t$  as

$$\Gamma \vDash_B p \quad \text{iff} \quad \exists p_1, \dots, p_n \in \Gamma, \forall vb \quad vb(p_1) \wedge \dots \wedge vb(p_n) \leq_t vb(p)$$

Now, let us consider the consequence relations  $\vDash_U, \vDash_C$  obtained by the designated values  $\{\mathbf{U}, \mathbf{T}\}$  or  $\{\mathbf{C}, \mathbf{T}\}$ , respectively defined as:

$$\begin{aligned}
\Gamma \vDash_U p : \forall vb \quad & \text{if } vb(p_i) \in \{\mathbf{U}, \mathbf{T}\}, \forall p_i \in \Gamma, \text{ then } vb(p) \in \{\mathbf{U}, \mathbf{T}\}; \\
\Gamma \vDash_C p : \forall vb \quad & \text{if } vb(p_i) \in \{\mathbf{C}, \mathbf{T}\}, \forall p_i \in \Gamma, \text{ then } vb(p) \in \{\mathbf{C}, \mathbf{T}\}.
\end{aligned}$$

Font [14] proves the following result:  $\Gamma \vDash_B p$  iff  $\Gamma \vDash_U p$  and  $\Gamma \vDash_C p$ . Moreover, due to the symmetric role that  $\{\mathbf{U}, \mathbf{T}\}$  and  $\{\mathbf{C}, \mathbf{T}\}$  play in Belnap's logic, the two relations  $\Gamma \vDash_U p$  and  $\Gamma \vDash_C p$  are equivalent:  $\Gamma \vDash_U p$  iff  $\Gamma \vDash_C p$ . The adequacy with the Hilbert-style axiomatization of Belnap logic and the above semantics is proved by Pynko [17] and Font [14]:

**Theorem 1.** *Belnap logic is sound and complete with respect to Belnap semantics, that is  $\Gamma \vdash_B p$  iff  $\Gamma \vDash_B p$  using the 15 rules  $Ri, i = 1, \dots, 15$ .*

Kleene logic has truth tables obtained from Belnap logic's by deleting the truth-value **C**, and has designated truth-value **T**. Priest logic is obtained by deleting the truth-value **U**, keeping **C**, **T** as designated. From a syntactic point of view, Kleene logic has one more inference rule than Belnap 4-valued logic, e.g.,  $q \wedge \neg q \vdash p \vee \neg p$ , while Priest logic is Belnap logic plus one axiom ( $p \vee \neg p$ , see [14,17]).

## 4 A Translation of Belnap Logic into BC

The above results, joined with the fact that Kleene logic and Priest logic can be translated into MEL [8,9] strongly suggest that Belnap logic can be expressed in BC. Formulas in BC can be related to Belnap truth-values **T**, **F**, **U**, **C** in an obvious way, provided that we restrict to atomic formulas. Let  $\mathcal{T}$  be the translation operation that changes a partial Belnap truth-value assignment  $vb(a) \in \Theta \subseteq \mathbb{V}_4$  to an atomic propositional formula  $a$ , into a modal formula, indicating its epistemic status w.r.t a set of sources. In agreement with the multisource semantics of the BC logic, we let  $\mathcal{T}(vb(a) \geq_t \mathbf{C}) = \Box a$  and  $\mathcal{T}(vb(a) \leq_t \mathbf{C}) = \Box \neg a$ . Likewise  $\mathcal{T}(vb(a) \geq_t \mathbf{U}) = \Diamond a, \mathcal{T}(vb(a) \leq_t \mathbf{U}) = \Diamond \neg a$ . Hence, we get the modal translation of the four Belnap epistemic values:

$$\begin{aligned} \mathcal{T}(vb(a) = \mathbf{T}) &= \Box a \wedge \Diamond a & \mathcal{T}(vb(a) = \mathbf{F}) &= \Box \neg a \wedge \Diamond \neg a \\ \mathcal{T}(vb(a) = \mathbf{U}) &= \Diamond a \wedge \Diamond \neg a & \mathcal{T}(vb(a) = \mathbf{C}) &= \Box a \wedge \Box \neg a \end{aligned}$$

In Belnap logic, though, sources provide information only on these elementary propositions, valuations for other propositions being obtained via truth-tables. The translation of Belnap truth-qualified formulas will be carried out using the truth-tables of the logic, which means that in all formulas of  $\mathcal{L}_\Box$  that can be reached via the translation, only literals appear in the scope of modalities.

Let us consider the fragment of BC language where we can only put a modality in front of literals:  $\mathcal{L}_\Box^\ell = \Box a | \Box \neg a | \neg \phi | \phi \wedge \psi | \phi \vee \psi$ . We can proceed to the translation of Belnap truth-tables into BC. First consider negation. It is easy to check that  $\mathcal{T}(vb(\neg p) = \mathbf{T}) = \mathcal{T}(vb(p) = \mathbf{F})$ ,  $\mathcal{T}(vb(\neg p) = \mathbf{x}) = \mathcal{T}(vb(p) = \mathbf{x})$ ,  $\mathcal{T}(vb(\neg p) \geq_t \mathbf{x}) = \mathcal{T}(vb(p) \leq_t \mathbf{x})$ ,  $\mathbf{x} \in \{\mathbf{U}, \mathbf{C}\}$ . On compound formulas built with conjunction and disjunction, it is clear that  $\mathcal{T}(vb(p \wedge q) = \mathbf{T}) = \mathcal{T}(vb(p) = \mathbf{T}) \wedge \mathcal{T}(vb(q) = \mathbf{T})$  but, due to the distributive lattice structure of  $\mathbb{V}_4$ , we have  $\mathcal{T}(vb(p \vee q) = \mathbf{T}) = \mathcal{T}(vb(p) = \mathbf{T}) \vee \mathcal{T}(vb(q) = \mathbf{T}) \vee (\mathcal{T}(vb(p) = \mathbf{U}) \wedge \mathcal{T}(vb(p) = \mathbf{C})) \vee (\mathcal{T}(vb(p) = \mathbf{C}) \wedge \mathcal{T}(vb(p) = \mathbf{U}))$ .

For elementary formulas  $\neg a, a \vee b, a \wedge b$  of Belnap logic, we get explicit translations using the truth-tables of Belnap logic, for instance:

$$\begin{aligned} \mathcal{T}(vb(\neg a) = \mathbf{T}) &= \mathcal{T}(vb(a) = \mathbf{F}) \\ \mathcal{T}(vb(\neg a) = \mathbf{U}) &= \mathcal{T}(vb(a) = \mathbf{U}); \mathcal{T}(vb(\neg a) = \mathbf{C}) = \mathcal{T}(vb(a) = \mathbf{C}) \\ \mathcal{T}(vb(\neg a) \geq_t \mathbf{C}) &= \mathcal{T}(vb(a) \leq_t \mathbf{C}) = \Box \neg a \end{aligned}$$

$$\begin{aligned}
\mathcal{T}(vb(a \wedge b) = \mathbf{T}) &= \Box a \wedge \Diamond a \wedge \Box b \wedge \Diamond b \\
\mathcal{T}(vb(a \vee b) = \mathbf{T}) &= (\Box a \wedge \Diamond a) \vee (\Box b \wedge \Diamond b) \vee (Ca \wedge Ub) \vee (Ua \wedge Cb) \\
\mathcal{T}(vb(a \wedge b) \geq_t \mathbf{U}) &= \Diamond a \wedge \Diamond b; \quad \mathcal{T}(vb(a \wedge b) \geq_t \mathbf{C}) = \Box a \wedge \Box b \\
\mathcal{T}(vb(a \vee b) \geq_t \mathbf{U}) &= \Diamond a \vee \Diamond b; \quad \mathcal{T}(vb(a \vee b) \geq_t \mathbf{C}) = \Box a \vee \Box b
\end{aligned}$$

where  $\Diamond a \wedge \Diamond \neg a$  is shortened as  $Ua$  and  $\Box a \wedge \Box \neg a$  as  $Ca$ . Belnap logic has two designated values:  $\mathbf{T}$  and  $\mathbf{C}$ . So, for inference purposes, we use translated semantic expressions  $\mathcal{T}(vb(p) \geq_t \mathbf{C})$ . According to this translation, Belnap logic reaches the following fragment of BC-language:  $\mathcal{L}_{\Box}^B = \Box a | \Box \neg a | \phi \wedge \psi | \phi \vee \psi$  without negation in front of  $\Box$ . Conversely, from the fragment  $\mathcal{L}_{\Box}^B$  we can go back to Belnap logic. Namely any formula in  $\mathcal{L}_{\Box}^B$  can be translated into a formula of the propositional logic language as follows:  $\Box a$  maps to  $a$  and  $\Box \neg a$  to  $\neg a$ ;  $\theta(\psi \wedge \phi)$  to  $\theta(\psi) \wedge \theta(\phi)$  and  $\theta(\psi \vee \phi)$  to  $\theta(\psi) \vee \theta(\phi)$ . We remark that  $\Box a \vee \Box \neg a$  is not a tautology in BC, and in general no tautologies can be expressed in the above fragment. This is coherent with the fact that Belnap logic has no theorems.

**Theorem 2.** *Let  $\phi/\psi$  be any of the 15 inference rules of Belnap logic. Then, the following inference rule is valid in BC:*

$$\frac{\mathcal{T}(vb(\phi) \geq_t \mathbf{C})}{\mathcal{T}(vb(\psi) \geq_t \mathbf{C})}$$

As a consequence we can mimic syntactic inference of Belnap logic in BC, more precisely restricting to formulas in  $\mathcal{L}_{\Box}^B$ . The restriction of the scope of modalities to literals also affects the set of B-capacities that can act as a semantic counterpart of the logic. We can check that semantic inference in Belnap logic can be expressed in the modal setting of BC by restricting the capacities that can be used as models of  $\mathcal{L}_{\Box}^B$  formulas. Namely, consider a Belnap set-up where each source  $i$  provides a set  $T_i$  of atoms considered true by this source, a set  $F_i$  of atoms considered false by this source, where  $T_i \cap F_i = \emptyset$ . It corresponds to a special kind of epistemic state with rectangular shape, namely:  $E_i = [(\bigwedge_{a \in T_i} a) \wedge (\bigwedge_{b \in F_i} \neg b)]$ .

As there are  $n$  sources of this kind, we can restrict to B-capacities  $\beta$  with such rectangular focal sets. In fact, as atoms of  $\mathcal{L}_{\Box}^B$  are of the form  $\Box \ell$  where  $\ell$  is a literal, and as we cannot put  $\Box$  in front of conjunctions nor disjunctions, it is enough to use capacities whose focal sets are of the form  $[a]$ ,  $a \in \cup_{i=1}^n T_i$  and  $[\neg b]$ ,  $b \in \cup_{i=1}^n F_i$  to interpret formulas in  $\mathcal{L}_{\Box}^B$ . We call such capacities *atomic*. Considering the Belnap valuation  $vb$  associated to the information supplied by  $n$  sources, there is a one-to-one correspondence between Belnap valuations and atomic B-capacities  $\alpha$  induced by this information:

**Proposition 1.** *For any B-capacity  $\beta$ , there is a single Belnap valuation  $vb_{\beta}$  such that  $\beta \models \phi$  if and only if  $vb_{\beta}(\theta(\phi)) \in \{\mathbf{C}, \mathbf{T}\}$ .*

The idea is to let  $vb_{\beta}(a) = \mathbf{T}$  if  $\beta([a]) = 1$  and  $\beta([\neg a]) = 0$ ,  $vb_{\beta}(a) = \mathbf{F}$  if  $\beta([a]) = 0$  and  $\beta([\neg a]) = 1$ , etc. In the other way around,

**Proposition 2.** *For each Belnap valuation  $vb$ , there exists a unique atomic B-capacity  $\alpha_{vb}$  such that  $vb \models p$  if and only if  $\alpha_{vb} \models \mathcal{T}(vb(p) \geq_t \mathbf{C})$ .*

Indeed, define  $T = \{a : vb(a) = \mathbf{T} \text{ or } \mathbf{C}\}$ ,  $F = \{a : vb(a) = \mathbf{F} \text{ or } \mathbf{C}\}$ , and let  $\alpha([a]) = 1$  if  $a \in T$ ,  $\alpha([\neg a]) = 1$  if  $a \in F$ . However there are several Belnap set-ups inducing a given Belnap valuation  $vb$ : for instance only two sources are enough to model the four values [6]. We thus introduce an equivalence relation on the set of B-capacities, whereby two of them are equivalent if they correspond to the same Belnap truth assignment:  $\beta \sim_B \beta'$  if and only if  $vb_\beta = vb_{\beta'}$ .

**Proposition 3.** *For any B-capacity  $\beta$ , there exists an atomic B-capacity  $\alpha$  such that  $\beta \sim_B \alpha$ .*

Indeed, consider  $\beta$  with focal sets  $E_1, \dots, E_n$ . Let  $T_i = \{a \in V : E_i \subseteq [a]\}$  and  $F_i = \{b \in V : E_i \subseteq [\neg b]\}$ . The focal sets of  $\alpha$  are based on such literals and form the family

$$\mathcal{F}_\alpha = \{[a] : a \in \cup_{i=1}^n T_i\} \cup \{[\neg b] : b \in \cup_{i=1}^n F_i\}.$$

From Proposition 3 we can conclude that for any B-capacity  $\beta$ , there exists an atomic B-capacity  $\alpha \sim_B \beta$  such that  $\beta \models \phi \in \mathcal{L}_\square^B$  if and only if  $\alpha \models \phi$ . We then can prove that our translation of Belnap logic into BC is consequence-preserving:

**Theorem 3.** *Let  $\Gamma$  be a set (conjunction) of formulas in propositional logic interpreted in Belnap logic, and  $p$  be another such formula. Then  $\Gamma \vdash_B p$  if and only if  $\{\mathcal{T}(vb(q) \geq_t \mathbf{C}) : q \in \Gamma\} \vdash_{BC} \mathcal{T}(vb(p) \geq_t \mathbf{C})$ .*

*Proof.* Suppose  $\Gamma \vdash_B p$ . Then from Theorem 2, all inference rules of Belnap logic become valid inferences in BC using the translations of their premises and conclusions. So the inference can be made in BC. Conversely, by completeness of BC, suppose  $\forall \beta$ , if  $\beta \models \mathcal{T}(vb(q) \geq_t \mathbf{C}), \forall q \in \Gamma$  then  $\beta \models \mathcal{T}(vb(p) \geq_t \mathbf{C})$ . Using Proposition 3, for all B-capacities  $\beta$ ,  $\exists \alpha \sim_B \beta$ , where  $\alpha$  is atomic, such that  $\forall q \in \Gamma$ ,  $\alpha \models \mathcal{T}(vb(q) \geq_t \mathbf{C})$  if and only if  $\beta \models \mathcal{T}(vb(q) \geq_t \mathbf{C})$  and  $\alpha \models \mathcal{T}(vb(p) \geq_t \mathbf{C})$  if and only if  $\beta \models \mathcal{T}(vb(p) \geq_t \mathbf{C})$ . Then, we have that if  $vb(q) \geq_t \mathbf{C}, \forall q \in \Gamma$  then  $vb(p) \geq_t \mathbf{C}$  for the Belnap valuation  $vb$  associated to  $\alpha$ . So  $\Gamma \models_B p$ . By completeness of Belnap logic,  $\Gamma \vdash_B p$  follows.  $\square$

We can recover our previous translations of three-valued Kleene logic and the logic of paradox into the logic MEL [8,9], from our translation of Belnap logic into BC, by translating into BC the properties added to Belnap logic to recover these logics. Namely Kleene logic is obtained by adding the inference rule  $q \wedge \neg q \vdash p \vee \neg p$  to Belnap logic, which comes down to adding inference rule (KL) :  $\square q \wedge \square \neg q \vdash \square p \vee \square \neg p$  to BC. A simpler approach is to add axiom D ( $\square p \rightarrow \diamond p$ ) to BC. To recover Priest logic from Belnap's, axiom  $p \vee \neg p$  must be added, which means adding to BC the (unusual) axiom  $\square p \vee \square \neg p$  [9].

## 5 Conclusion

In this paper, we have pursued our work regarding a class of many-valued logics dealing with inconsistent or incomplete information processing [8,9]. Just like Kleene logic and Priest's logic of paradox in MEL, we can capture Belnap 4-valued logics in a simple two-tiered propositional logic couched in the language of

modal logic EMN involving only depth-1 formulas. The natural semantics for this propositional logic is in terms of all-or-nothing set-functions that model Belnap set-ups and capture both incomplete and inconsistent pieces of information. The use of set-functions clarifies the connection between Belnap 4-valued logic and uncertainty modeling. The use of set-functions beyond possibility and necessity measures is in agreement with the fact that propositions in Belnap 4-valued logics cannot be viewed as S5-like beliefs. The logic BC is cautious enough to be a general setting for modeling incomplete and inconsistent logical information. It subsumes Belnap 4-valued logic, doing away with the restriction to literals, and accounting for generalized Belnap set-ups considered by Avron et al. [2]. It seems that our framework may be used to capture various approaches to inconsistent and incomplete information handling; for instance, the one based on maximal consistent subsets could be obtained by considering B-capacities such that  $\beta(A \cap B) = \min(\beta(A), \beta(B))$  if  $A \cap B \neq \emptyset$ . Moreover it can be extended to handling degrees of support. This is to be explored in the future.

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