

TOPOLOGICAL AND GEOMETRIC HYPERBOLICITY CRITERIA FOR POLYNOMIAL AUTOMORPHISMS OF \mathbb{C}^2

ERIC BEDFORD AND ROMAIN DUJARDIN

ABSTRACT. We prove that uniform hyperbolicity is invariant under topological conjugacy for dissipative polynomial automorphisms of \mathbb{C}^2 . Along the way we also show that a sufficient condition for hyperbolicity is that local stable and unstable manifolds of saddle points have uniform geometry.

1. INTRODUCTION

1.1. The main motivation of this note is to study the following problem:

Question 1.1. *Is uniform hyperbolicity a topological property for complex Hénon maps?*

We use the terminology “complex Hénon map” as a synonym for “polynomial automorphism of \mathbb{C}^2 with non-trivial dynamics”. By [FM] we can normalize such a map such that it is a product of actual Hénon maps $(z, w) \mapsto (aw + p(z), az)$. Hyperbolicity here is understood in the sense of [BS1], that is, we say that a complex Hénon map f is *hyperbolic* if its Julia set $J = J_f$ is a hyperbolic set, which must then be of saddle type. This was shown in [BS1] to have strong consequences on the global dynamics of f : the chain recurrent set consists of J together with finitely many periodic attractors, and f satisfies Smale’s Axiom A on \mathbb{C}^2 . (See [I] for a recent survey on complex Hénon dynamics, with an emphasis on hyperbolic maps.)

Let us recall a bit of standard notation. Given a complex Hénon map f , we denote by J^+ the forward Julia set, which is the locus of non-normality of forward iterates $(f^n)_{n \geq 0}$, or equivalently the boundary of the set K^+ of points with bounded forward orbits. We likewise define J^- and K^- for backward dynamics, and we set $J = J^+ \cap J^-$. We denote by J^* the closure of the set of saddle periodic orbits, which is contained in J . For hyperbolic maps, we have $J = J^*$; however, this equality is an open problem in the general case.

Question 1.1 was previously considered for rational maps on the Riemann sphere and for smooth Anosov diffeomorphisms of (real) compact manifolds. Let us start by briefly reviewing these cases.

1.2. One-dimensional rational maps. For polynomials and rational maps in one variable, the answer to the question is “yes” since there is a simple topological criterion for hyperbolicity: $\overline{\text{PC}(f)} \cap J = \emptyset$, where $\text{PC}(f)$ is the postcritical set. As a consequence, if f_0 and f_1 are rational maps in one variable such that f_0 is hyperbolic, and $\phi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is a topological conjugacy between f_0 and f_1 , then f_1 is hyperbolic. Actually the statement already holds locally near the Julia set:

Proposition 1.2. *Let f_0 and f_1 be rational maps on \mathbb{P}^1 , and assume that f_0 is hyperbolic. If there are neighborhoods $N(J_0)$ and $N(J_1)$ of J_0 and J_1 and a homeomorphism $\phi : N(J_0) \rightarrow N(J_1)$ which satisfies $\phi \circ f_0 = f_1 \circ \phi$ wherever these compositions make sense, then f_1 is hyperbolic.*

The result is not obvious since the conjugacy ϕ cannot detect that a point in $N(J_1)$ is post-critical. Here and throughout the paper we use indices 0 and 1 to label the dynamical objects (Julia set, etc.) respectively associated to f_0 and f_1 .

Proof. Notice first that the conjugacy ϕ sends periodic points to periodic points. The topological dynamics around a periodic point determines its type (attracting, repelling, neutral) so it follows that all periodic points of f_1 are hyperbolic. In particular f_1 has no parabolic points.

Since the Julia set is the accumulation set of periodic orbits we infer that $\phi(J_0) = J_1$. Now a rational map without parabolic points is hyperbolic if and only if its critical set is disjoint from the Julia set. This property holds for f_1 by the topological conjugacy, so the result follows. \square

Remark 1.3.

- (1) It is not enough in the proposition to assume that ϕ is a conjugating homeomorphism $J_0 \rightarrow J_1$. Indeed it is well known that $z^2 + \frac{1}{4}$ is topologically conjugate on its Julia set to any quadratic polynomial in the main cardioid (e.g. z^2). As we will comment below, a similar phenomenon holds for complex Hénon maps (see [RT]).
- (2) If we suppose a priori that $\deg(f_0) = \deg(f_1) = d$ we can relax the assumption on ϕ by assuming only that ϕ is any injective continuous map defined in a neighborhood N_0 of J_0 and such that $\phi \circ f_0 = f_1 \circ \phi$ wherever these compositions makes sense. Indeed by the invariance of domain theorem, $\phi(N_0)$ is an open subset of the plane. We only have to show that it contains $J(f_1)$. Indeed f_0 has only finitely many non-repelling periodic points, so J_0 contains $p_n \sim d^n$ repelling periodic points of period d for large n . Thus by the topological conjugacy, $\phi(J_0)$ contains p_n repelling periodic points of f_1 , which are equidistributed to the equilibrium measure μ_{f_1} whose support is $J(f_1)$. Therefore $\phi(J_0) = J_1$ and we are done.

1.3. Anosov diffeomorphisms. The problem of topological invariance of hyperbolicity in real dynamics has been popularized in particular by A. Katok. The answer is already quite subtle for the simplest case of Anosov diffeomorphisms of the 2-torus.

Indeed there exist examples of C^2 diffeomorphisms f of the 2-torus which are not hyperbolic but still globally topologically conjugate to a linear Anosov map. This can be done by either carefully deforming a linear Anosov map until some saddle fixed point becomes neutral by preserving the geometry of the stable and unstable foliations (see [K]), or by deforming the foliations until reaching a cubic heteroclinic tangency (see [E, BDV]).

If we now impose the conjugacy to be Hölder then there are different regimes depending on the precise Hölder regularity. First, it can be arranged that in the previous examples the conjugacy and its inverse are Hölder continuous [Go], and thus hyperbolicity is not invariant under Hölder conjugacy. On the other hand if the conjugacy is sufficiently close to being bi-Lipschitz – namely, the product of the Hölder exponents of ϕ and ϕ^{-1} is larger than $1/2$ – then f is Anosov [F] (see [Go]).

1.4. A conjecture. The most natural way to address Question 1.1 would be to find a topological criterion ensuring hyperbolicity for a complex Hénon map, in the spirit of the one-dimensional condition $J \cap \overline{\text{PC}}(f) = \emptyset$. Strictly speaking, a Hénon map admits no critical points; nevertheless there are ways to give a reasonable meaning to this condition – which are more differential-geometric than topological, though. For instance, in the dissipative regime, the condition that there are no critical points on J naturally corresponds to the existence of a dominated splitting and, provided dissipation is strong enough, a good analogue of the one-dimensional situation was achieved in [LP].

A variant is to translate the condition $J \cap \overline{\text{PC}(f)} = \emptyset$ into a regularity property of the geometry of the forward and backward Julia sets $J^{+/-}$ near J . In this respect it was shown in [BS8] that if in some neighborhood of J , J^+ and J^- are the supports of two Riemann surface laminations which are transverse along J , then f is hyperbolic. In §2 below we reprove and generalize this result in several ways.

Back to our initial problem, even if it is unclear how to design a purely topological criterion for hyperbolicity, one may ask whether hyperbolicity is invariant under topological conjugacy. Here is a precise analogue of Proposition 1.2 for complex Hénon maps:

Conjecture 1.4. *Let f_0 and f_1 be two polynomial automorphisms of \mathbb{C}^2 with non-trivial dynamics, and assume that f_0 is hyperbolic. Suppose that there exists respective neighborhoods N_0 and N_1 of $J_0 = J_0^*$ and J_1^* and a homeomorphism $\phi : N_0 \rightarrow N_1$ such that $\phi \circ f_0 = f_1 \circ \phi$ wherever these compositions makes sense. Then f_1 is hyperbolic.*

Here are some comments on this conjecture:

- (1) It was shown in [D2] (see also [GP]) that for a complex Hénon map hyperbolicity on J^* implies hyperbolicity. This explains why we can restrict to a neighborhood of J_0^* instead of J_0 , and opens the way to an analysis of hyperbolicity based on periodic points.
- (2) If we add the hypothesis that f_0 and f_1 have the same dynamical degree, then by using the equidistribution of periodic orbits from [BLS2] and arguing as in Remark 1.3 we can relax the assumption that $\phi(N(J_0^*))$ contains J_1^* .
- (3) As observed in Remark 1.3, the conjecture is false if the conjugating homeomorphism ϕ is only defined on $J_0 = J_0^*$ (see [RT]).
- (4) The conjecture is true if ϕ is obtained by deformation in the following sense: it was shown in [BD] that if there is a weakly stable holomorphic family (f_λ) connecting f_0 and f_1 , then f_1 is hyperbolic¹.

1.5. Quasi-hyperbolicity. The methods in this paper are closely related to the notion of quasi-hyperbolicity. If p is a saddle point and $r > 0$, we let $W_r^{s/u}(p)$ denote the connected component of $W^{s/u}(p) \cap B(p, r)$ containing p . Following [BS8] map f is said *quasi-hyperbolic* if there exists positive constants r and B such that for every saddle periodic point p :

- (i) $W_r^{s/u}(p)$ is closed in $B(p, r)$ and
- (ii) the area of $W_r^{s/u}(p)$ is bounded by B .

If $\phi : N_0 \rightarrow N_1$ is a topological conjugacy as in Conjecture 1.4 then ϕ preserves stable and unstable manifolds, so if (i) holds for f_0 it will also hold for f_1 (after possibly shrinking r). It was shown in [BGS] that if f is quasi-hyperbolic then there exist stable and unstable manifolds $W^{s/u}(x)$ through each point $x \in J^*$. Furthermore f is uniformly hyperbolic (on J^* and thus J) if and only if there is no tangency between W^s and W^u . Thus if we know that f is already quasi-hyperbolic, then the additional condition of hyperbolicity is a topological invariant in the sense of the conjecture. At this stage, however, it remains an open question whether quasi-hyperbolicity is a topological property.

1.6. Results and outline. In §2 we establish several sufficient conditions for hyperbolicity based on the geometry of local stable and unstable manifolds of saddle periodic points. A first sufficient condition for hyperbolicity, which essentially follows from [BS8], is that these local stable and unstable manifolds have uniform size and the angle between them is uniformly bounded from below. We give a self-contained proof of this result (see Theorem 2.9). We further show that

¹The original statement in [BD] concerns hyperbolicity on J^* , so we further use [D2] to deduce hyperbolicity on J .

the transversality assumption is superfluous (Theorems 2.13 and 2.14), and that, as it might be expected, in the dissipative case it is enough to control the geometry of unstable manifolds (Theorem 2.19).

In §3 we prove Conjecture 1.4 in the case where f_1 is dissipative (Theorem 3.6). In the conservative case the conjecture holds provided ϕ is Hölder continuous (Theorem 3.8). The general case remains open².

2. GEOMETRIC CRITERIA FOR HYPERBOLICITY

2.1. Size of a submanifold at a point and u/s regularity. Endow \mathbb{C}^2 with the Euclidean metric. A *bidisk of size r* is the image of $D(0, r)^2$ under some affine isometry. A curve V in \mathbb{C}^2 is a graph over an affine line L if the orthogonal projection onto L is injective when restricted to V . Then there is a well-defined notion of slope of a holomorphic curve with respect to L .

Definition 2.1. *A curve V through p is said to have size r at p if there exists a neighborhood of p in V that is a graph of slope at most 1 over a disk of radius r in the tangent space $T_p V$.*

If Δ be a disk of size r at p , fixing orthonormal coordinates (x, y) so that $p = 0$ and $T_p V = \{y = 0\}$, we get that the connected component of Δ through p in the bidisk $D(0, r)^2$ is a graph $\{y = \varphi(x)\}$ over the first coordinate with $|\varphi'| \leq 1$ and $\varphi'(0) = 0$. In particular if Δ is immersed and has size r at p , then it is a submanifold in $B(0, r/\sqrt{2})$ (because a bidisk of size r contains a ball of radius $r/\sqrt{2}$).

We now recall a few concepts from [BD]. A point $x \in J^*$ is said *u-regular* (resp. *s-regular*) if there exists $r > 0$ and a sequence of saddle points (p_n) converging to x such that $W^u(p_n)$ (resp. $W^s(p_n)$) is of size r at p_n . In this case it can be shown that the sequence of disks $W_r^u(p_n)$ (resp. $W_r^s(p_n)$) converges in the C^1 topology to a (smooth) holomorphic disk of size r at x which we denote by $\mathcal{W}_r^u(x)$ (resp. $\mathcal{W}_r^s(x)$) (see [BD, Prop. 4.2]). This notation is meant to emphasize that at this stage $\mathcal{W}_r^{s/u}(x)$ need not be an stable/unstable manifold in the usual sense. We use the notation $\mathcal{W}_{loc}^{u/s}(x)$ for an unspecified neighborhood of x in $\mathcal{W}_r^u(x)$. We say that x is *regular* if it is u- and s-regular and $\mathcal{W}_{loc}^u(x)$ and $\mathcal{W}_{loc}^s(x)$ do not coincide, and *transverse regular* if they are transverse. In particular we have the implications

$$\text{u- and s-regular} \Leftarrow \text{regular} \Leftarrow \text{transverse regular.}$$

It is easy to see that if x is a saddle point, then x is regular (for instance because it generates homoclinic intersections, hence it belongs to a horseshoe) and $\mathcal{W}_{loc}^s(x)$ and $\mathcal{W}_{loc}^u(x)$ coincide with the classical local stable and unstable manifolds of x .

We define a local stable set

$$\dot{W}_{loc, \epsilon}^s(x) = \{y : \text{dist}(f^n(x), f^n(y)) < \epsilon \ \forall n \geq 0, \text{ and } \lim_{n \rightarrow \infty} \text{dist}(f^n(x), f^n(y)) = 0\}$$

Lemma 2.2. *Suppose that there is a complex disk Δ such that $x \in \Delta \subset \dot{W}_{loc, \epsilon}^s(x)$. If x is s-regular, then Δ coincides with $\mathcal{W}^s(x)$ locally at x . The analogous result holds for 's' replaced by 'u'.*

Lemma 2.3. *If x is Pesin regular, and x is u- and s-regular, then $\mathcal{W}_{loc}^{s/u}(x)$ agree locally at x with the Pesin manifolds $W_{Pesin}^{s/u}$. Further, $\mathcal{W}_{loc}^s(x) \neq \mathcal{W}_{loc}^u(x)$, so x is regular in the sense defined above, and in fact it is transverse regular.*

More generally we have:

²Notice that the Jacobian is not invariant under topological conjugacy: the Hénon map $(z, w) \mapsto (z^2 + c + aw, z)$ is conjugate to a horseshoe for any Jacobian a , when $|c| \gg |a|$.

Lemma 2.4. *If x is a Pesin regular point which is regular, then $\mathcal{W}_{\text{loc}}^s(x)$ and $\mathcal{W}_{\text{loc}}^u(x)$ coincide with the classical Pesin local stable and unstable manifolds of x .*

Proof. Denote temporarily the Pesin local unstable manifolds by $W_{\text{Pesin}}^u(x)$. If (p_n) is a sequence of saddle points converging to x , then $\mathcal{W}^u(p_n) = W_{\text{loc}}^u(p_n)$ must coincide with or be disjoint from $W_{\text{Pesin}}^u(x)$, and converge to $\mathcal{W}^u(x)$ in the C^1 topology. Since both $W_{\text{Pesin}}^u(x)$ and $\mathcal{W}^u(x)$ contain x , by the Hurwitz theorem we conclude that $W_{\text{Pesin}}^u(x)$ locally coincides with $\mathcal{W}^u(x)$. \square

We say that $x \in J^*$ *uniformly u-regular* if the uniform size property for local unstable manifolds holds for *any* sequence (p_n) converging to x . If required we can specify the size r in the terminology. Uniform s-regularity is defined similarly. We say that x is uniformly (resp. transverse) regular if it is uniformly u- and s- regular, and (resp. transverse) regular.

The following result will play an important role in this paper (of course it admits an identical s-regular version).

Proposition 2.5. *The following assertions are equivalent:*

- (1) *Every point in J^* is uniformly u-regular.*
- (2) *There exists a uniform $r > 0$ such that for every saddle periodic point p , $W^u(p)$ has size r at p .*
- (3) *There exists a uniform $r > 0$ and a dense set D of saddle periodic points such that for every $p \in D$, $W^u(p)$ has size r at p .*
- (4) *There exists a lamination \mathcal{W}^u by Riemann surfaces in a neighborhood of J^* which extends the family of local unstable manifolds of saddle points.*

Proof. (1) implies (2) by a simple compactness argument. The implications (2) \Rightarrow (3) and (4) \Rightarrow (1) are obvious, and (3) \Rightarrow (4) follows from standard holomorphic motions techniques. Let us give some details on this last point for the reader's convenience (see also Prop. 4.2 and Lemma 5.3 in [BD]). We start with the following basic geometric idea: if Δ and Δ' flat disks in $B(0, r/10) \subset \mathbb{C}^2$ which both intersect $B(0, r/1000)$, and whose tangent vectors are 1/4-far apart (relative to the usual Fubini-Study metric on \mathbb{P}^1), then Δ and Δ' intersect and

$$\text{dist}(\Delta \cap \partial B(0, r/10), \Delta' \cap \partial B(0, r/10)) > \frac{1}{100}.$$

By the persistence of proper intersections, the same holds for $\tilde{\Delta}$ and $\tilde{\Delta}'$, whenever $\tilde{\Delta}$ and $\tilde{\Delta}'$ are holomorphic disks which are respectively 1/100 close to Δ and Δ' . Now if Δ is a disk of size r at x , by the Schwarz Lemma, $\Delta \cap B(0, r/10)$ remains 1/100 close to $T_x\Delta$. Taking the contrapositive we see that if Δ and Δ' are disks of size r respectively at x and x' , with $\text{dist}(x, x') < 1/1000$, then their tangent spaces must be 1/4-close to each other, in particular they are graphs over a disk of radius $r/4$ relative to the same orthogonal projection.

Now by (3), for every $x \in J^*$ there is a holomorphic disk $\mathcal{W}_r^u(x)$ of size r through x and these disks are either disjoint or locally coincide because local unstable manifolds of saddle points are disjoint. By the previous discussion, the disks $\mathcal{W}_{r/4}^u(y)$ are disjoint graphs over some direction for y close to x , so they form a lamination by the Lambda Lemma of [MSS]. Thus we get the desired lamination structure in the $r/5$ -neighborhood of J^* . \square

Remark 2.6. Under the assumptions of Proposition 2.5, there exists a neighborhood \mathcal{N} of J^* and a lamination \mathcal{W}^u of \mathcal{N} by Riemann surfaces which extends the family of local unstable manifolds of saddle points. Beware however that it does not *a priori* imply that $J^- \cap \mathcal{N}$ is laminated nor that it coincides with $\text{Supp}(\mathcal{W}^u)$: indeed, J^- is the closure of *global* unstable manifolds, which could recur to \mathcal{N} in a complicated fashion (this point is a main issue in [D2]).

2.2. Existence of invariant laminations and hyperbolicity. Recall that a complex Hénon map f is said to be *hyperbolic* if J is a hyperbolic set. As was noted above, by [D2] (see also [GP]) it is actually enough to check hyperbolicity on J^* : this opens the way to hyperbolicity criteria based on periodic points.

Theorem 2.7 ([D2]). *If J^* is a hyperbolic set for f , then f is hyperbolic.*

A geometric criterion for hyperbolicity based on the existence and transversality of unstable laminations was established in [BS8, Thm 8.3]. By incorporating the result of Theorem 2.7 it reads as follows.

Theorem 2.8 ([BS8]). *Let f be a complex Hénon map. Assume that there exists a neighborhood of J^* and Riemann surface laminations \mathcal{L}^\pm of J^\pm such that \mathcal{L}^+ and \mathcal{L}^- intersect transversally at all points of J^* . Then f is hyperbolic.*

It is convenient to formulate this result in the language of uniform regularity. The following is an essentially equivalent statement (see however Remark 2.6).

Theorem 2.9. *Let f be a complex Hénon map. If every point in J^* is uniformly regular and transverse then f is hyperbolic.*

Let us give a self-contained proof of this theorem, which basically follows the approach of [BS8, Thm 8.3]. First, recall from Proposition 2.5 that if every point in J^* is uniformly u -regular, then exists $r > 0$ and a lamination \mathcal{W}^u in the r -neighborhood of J^* , extending the unstable manifolds of saddle points. Recall also the dynamical Green function G^+ , defined by $G^+(x) = \lim_{n \rightarrow \infty} d^{-n} \log^+ \|f^n(x)\|$. It is a non-negative continuous psh function in \mathbb{C}^2 , with the property that $\{G^+ = 0\} = K^+$.

Proposition 2.10. *Let f be a complex Hénon map. Assume that every point in J^* is uniformly u -regular and that for every $x \in J^*$, $G^+|_{\mathcal{W}_{\text{loc}}^u(x)} \not\equiv 0$. Then f is uniformly expanding in the direction of $T\mathcal{W}^u$ along J^* .*

This condition on G^+ will be used several times in the sequel; it means that G^+ does not vanish identically on any neighborhood of x in $\mathcal{W}^s(x)$.

Proof. Let r be such that for any saddle point p , $W^u(p)$ has size $5r$ at p . Then by Proposition 2.5, \mathcal{W}^u defines a lamination in the r -neighborhood of J^* such that for every saddle point p , $\mathcal{W}^u(p)$ coincides with the local unstable manifold of p .

We have to show that f is uniformly expanding along $\mathcal{W}^u|_{J^*}$, that is, there exists $C > 0$ and $\lambda > 1$ such that for every $x \in J^*$, every $k \geq 1$ and $e \in T_x \mathcal{W}^u(x)$, $|Df_x^k(e)| \geq C\lambda^k |e|$ (where $|\cdot|$ denotes the Riemannian metric induced by the standard Hermitian structure of \mathbb{C}^2). By continuity it is enough to prove this property on the (dense) set \mathcal{S} of saddle periodic points. For this, we will construct a metric $|\cdot|'$ on $T\mathcal{W}^u|_{\mathcal{S}}$ which is equivalent to the ambient one (with uniform constants) and such that for every $p \in \mathcal{S}$, and $e \in T_p \mathcal{W}^u(p)$, $|Df_p(e)|' \geq \lambda |e|'$.

For every saddle point p , the global unstable manifold is biholomorphic to \mathbb{C} , so its uniformization $\psi_p^u : \mathbb{C} \rightarrow W^u(p)$ is unique up to a multiplicative factor at the source. In particular f is affine in these parameterizations, and there is a well-defined notion of a round disk in $W^u(p)$, which is f -invariant. For $e \in T_p W^u(p)$ and $\eta > 0$ we define $|e|_\eta$ in the style of the Kobayashi metric:

$$|e|_\eta = \frac{1}{\rho_\eta} \text{ where } \rho_\eta = \sup \left\{ \rho, \sup_{D(0,\rho)} G^+ \circ \psi_p^u \leq \eta \text{ where } \psi_p^u : \mathbb{C} \xrightarrow{\sim} W^u(p) \text{ and } (\psi_p^u)'(0) = e \right\}$$

For every $x \in J^*$ (not necessarily a saddle), $G^+|_{\mathcal{W}_{\text{loc}}^u(x)}$ is not identically 0 near x so we infer that for every $r > 0$, $\sup G^+|_{\mathcal{W}_r^u(x)} > 0$. From the continuity of the Green function, the compactness of J^* , and the lamination structure we infer the existence of constants $r > 0$ and $\eta'_1 > \eta_1 > 0$ such that for every $x \in J^*$,

$$\eta_1 \leq \sup G^+|_{\mathcal{W}_r^u(x)} \leq \eta'_1.$$

Recall that for near any $x \in J^*$, up to a unitary change of coordinates, \mathcal{W}^u is a union of graphs over a disk of size r and slope bounded by 1 (relative to some projection π). Thus if $p \in \mathcal{S}$ is close to x , and ψ_p^u is as above, it follows that $\pi \circ \psi_p^u|_{D(0, \rho_{\eta_1})}$ is a univalent holomorphic function. Set $\eta_2 = \eta_1/2$. The Koebe distortion theorem together with the uniform continuity of the Green function imply that for $\rho < \rho_{\eta_2}$, $\pi \circ \psi_p^u(D(0, \rho))$ is approximately a round disk (with uniform distortion bounds). From this uniformity, we infer that there exists $\lambda > 1$ such that for every $p \in \mathcal{S}$ and $e \in T_p \mathcal{W}^u(p)$, $|e|_{\eta_2/d} \geq \lambda |e|_{\eta_2}$. Set $\eta_3 = \eta_2/d$. The invariance relation of the Green function G^+ implies that $|df_p(e)|_{d\eta_3} = |e|_{\eta_3}$. From this we get that for every $p \in \mathcal{S}$ and $e \in T_p \mathcal{W}^u(p)$ $|df_p(e)|_{\eta_2} \geq \lambda |e|_{\eta_2}$. Finally, again from the uniform continuity of the Green function and bounded distortion, we get that $|\cdot|_{\eta_2}$ is (uniformly) equivalent to $|\cdot|$ on $T\mathcal{W}^u|_{\mathcal{S}}$ so the proof is complete. \square

Remark 2.11. By a standard procedure, up to reducing λ it is possible to construct a *continuous* metric $|\cdot|''$ on $T\mathcal{W}^u|_{J^*}$ such that $|Df_x(\cdot)|'' \geq \lambda |\cdot|''$. Indeed for $\varepsilon > 0$ and $e \in T_x \mathcal{W}^u(x)$, put

$$|e|'' = \sum_{n=0}^{\infty} (\lambda - \varepsilon)^n |Df_x^{-n}(e)|.$$

Then one easily checks that $|\cdot|''$ is well-defined, continuous, and satisfies

$$|Df_x^{-1}(e)|'' \leq (\lambda - \varepsilon)^{-1} |e|''.$$

The next result implies that if f is uniformly regular, then the second assumption of Proposition 2.10 holds.

Proposition 2.12 (see [BD, Prop. 4.7]). *Let f be a complex Hénon map. If $x \in J^*$ is regular then $G^+|_{\mathcal{W}_{\text{loc}}^u(x)} \not\equiv 0$.*

Proof. If (p_n) is a sequence of distinct saddle points converging to x , then $W_r^s(p_n)$ is a sequence of disjoint submanifolds converging to $W_r^s(x)$. Since by assumption $W_r^u(x)$ and $W_r^s(x)$ are distinct, then for large n $W_r^s(p_n)$ must possess transverse intersection points with $W_r^u(x)$ close to x : if $W_r^u(x)$ and $W_r^s(x)$ are transverse this is clear, and if they are tangent this follows from [BLS1, Lemma 6.4]). Then the inclination lemma implies that $(f^n|_{\mathcal{W}_{\text{loc}}^u(x)})$ is not a normal family of holomorphic mappings, therefore G^+ is not harmonic on $\mathcal{W}_{\text{loc}}^u(x)$, thus not identically zero, and we are done. \square

Proof of Theorems 2.8 and 2.9. Under the assumptions of Theorem 2.9, it follows directly from Propositions 2.10 and 2.12 that f is uniformly expanding along $T\mathcal{W}^u|_{J^*}$ and contracting along $T\mathcal{W}^s|_{J^*}$, that is, J^* is a hyperbolic set. Then we conclude from Theorem 2.7 that f is hyperbolic.

To establish Theorem 2.8, it is enough to check that the existence of the transverse laminations \mathcal{L}^+ and \mathcal{L}^- imply uniform transverse regularity. We first observe that for any saddle point p , $W^s(p)$ locally coincides with the leaf $\mathcal{L}^+(p)$ of \mathcal{L}^+ through p , and likewise in the unstable direction. Indeed since the leaves of \mathcal{L}^+ are contained in J^+ , for every disk Δ contained in such a leaf, $(f^n|_{\Delta})_{n \geq 0}$ is a normal family. Now if $\mathcal{L}_{\text{loc}}^+(p) \neq W_{\text{loc}}^s(p)$ then either they are transverse and it follows from the inclination lemma that $(f^n|_{\mathcal{L}_{\text{loc}}^+(p)})_{n \geq 0}$ is not normal. Otherwise by [BLS1,

Lemma 6.4] for any $x \in J^*$ close to p , $\mathcal{L}_{\text{loc}}^+(x)$ is transverse to $W_{\text{loc}}^s(p)$ and similarly $(f^n|_{\mathcal{L}_{\text{loc}}^+(x)})_{n \geq 0}$ is not a normal family. In both cases we reach a contradiction. It then follows from Proposition 2.5 that every point in J^* is uniformly regular and transverse and we conclude as before. \square

It turns out that the transversality assumption in Theorem 2.8 is unnecessary, that is, uniform regularity rules out the possibility of tangencies.

Theorem 2.13. *Let f be a complex Hénon map. If every point in J^* is uniformly regular then f is uniformly hyperbolic.*

Proof. By Proposition 2.5 there exist laminations \mathcal{W}^u and \mathcal{W}^s in a neighborhood of J^* extending the family of local stable and unstable manifolds of periodic points, and by Proposition 2.10 and 2.12 we get that f is uniformly expanding along \mathcal{W}^u and f^{-1} is uniformly expanding along \mathcal{W}^s . To prove the theorem we thus have to show that these laminations are transverse at all points of J^* . Let \mathcal{T} be the tangency locus, that is the set of points $x \in J^*$ such that $\mathcal{W}^s(x)$ and $\mathcal{W}^u(x)$ are tangent at x . This is a closed invariant set. Assume by way of contradiction that it is non empty. Then it supports an ergodic invariant measure ν . Let

$$\chi^+ = \lim_{n \rightarrow +\infty} \frac{1}{n} \int \log \|df_x^n\| d\nu(x) \text{ and } \chi^- = \lim_{n \rightarrow +\infty} \frac{1}{n} \int \log \|df_x^{-n}\|^{-1} d\nu(x)$$

be the Lyapunov exponents of ν . Since f is uniformly expanding/contracting along $\mathcal{W}^{u/s}$ we infer that $\chi^- < 0 < \chi^+$. By Oseledets' theorem, there exists an associated invariant measurable decomposition $T_x\mathbb{C}^2 = E^-(x) \oplus E^+(x)$ defined ν -a.e. such that the growth rate of vectors in $E^\pm(x)$ is governed by χ^\pm . By Pesin's theory (see e.g. [FHY]) for ν -a.e. x there are local stable and unstable manifolds $W_{\text{Pesin}}^s(x)$ and $W_{\text{Pesin}}^u(x)$ respectively tangent to the characteristic directions associated to the negative and positive exponent. But by Lemma 2.4, $W_{\text{Pesin}}^{s/u}(x)$ locally coincides with $\mathcal{W}^{s/u}(x)$, so we infer that $E^+(x) = E^-(x)$ a.e. which contradicts the Oseledets theorem. This contradiction finishes the proof. \square

If f is not volume preserving we can further relax the previous criterion.

Theorem 2.14. *Let f be a complex Hénon map with $|\text{Jac}(f)| \neq 1$. If every point in J^* is uniformly u - and s -regular then f is uniformly hyperbolic.*

Proof. The difference with Theorem 2.13 is that \mathcal{T} can now contain local leaves so Proposition 2.10 does not apply. Let r is the uniform size of local s/u manifolds along J^* . Without loss of generality assume that $|\text{Jac}(f)| < 1$. Note that if $\mathcal{W}_{\text{loc}}^u(x) = \mathcal{W}_{\text{loc}}^s(x)$ then $\mathcal{W}_r^u(x) = \mathcal{W}_r^s(x)$. Denote by \mathcal{T}' the set of points $x \in J^*$ such that $\mathcal{W}_r^u(x) = \mathcal{W}_r^s(x)$. Then \mathcal{T}' is also closed and invariant. Indeed if $\mathcal{W}_r^u(x) = \mathcal{W}_r^s(x)$ then clearly $\mathcal{W}_{\text{loc}}^u(f(x)) = \mathcal{W}_{\text{loc}}^s(f(x))$, hence $\mathcal{W}_r^u(f(x)) = \mathcal{W}_r^s(f(x))$. Thus $f(\mathcal{T}') \subset \mathcal{T}'$, and the closedness of \mathcal{T}' follows directly from the continuity of $x \mapsto \mathcal{W}_r^{u/s}(x)$.

Assume by way of contradiction that \mathcal{T}' is non-empty. Then it supports an ergodic invariant measure ν . Since f is dissipative its Lyapunov exponents satisfy $\chi^- < 0 \leq \chi^+$. For every $x \in \mathcal{T}'$, $\mathcal{W}_r^u(x) = \mathcal{W}_r^s(x)$ is contained in $J^+ \cap J^-$ so it is a Fatou disk under forward and backward iteration. The following lemma relates these disks to the Oseledets decomposition.

Lemma 2.15. *Let f be a complex Hénon map and ν be an ergodic invariant measure whose Lyapunov exponents satisfy $\chi^- < 0 \leq \chi^+$, and $T_x\mathbb{C}^2 = E^-(x) \oplus E^+(x)$ be the associated measurable decomposition. If ν -a.e. point is u -regular then for ν -a.e. x , $\mathcal{W}_{\text{loc}}^u(x)$ is tangent to $E^+(x)$ at x .*

Assuming this result for the moment, let us conclude the proof. The contradiction hypothesis implies that for ν -a.e. x , $\mathcal{W}_{\text{loc}}^u(x) = \mathcal{W}_{\text{loc}}^s(x)$. By Pesin's theory a ν -generic point x admits a local strong stable manifold $W_{\text{Pesin}}^s(x)$, which is tangent to $E^-(x)$, and by Lemma 2.4 it coincides with $\mathcal{W}_{\text{loc}}^s(x)$. On the other hand, by Lemma 2.15, $\mathcal{W}_{\text{loc}}^u(x)$ is a.s. transverse to $E^-(x)$. This contradiction shows that \mathcal{T}' is empty. Therefore every point in J^* is regular and applying Theorem 2.13 finishes the proof. \square

Proof of Lemma 2.15. Since $\mathcal{W}_{\text{loc}}^u(x)$ is contained in J^- , $(f^{-n}|_{\mathcal{W}_{\text{loc}}^u(x)})$ is a normal family, so it follows from the Cauchy estimates that $\|df_x^{-n}(e^u(x))\|$ is bounded, where $e^u(x)$ is any tangent vector to $\mathcal{W}^u(x)$ at x . On the other hand the Oseledets theorem asserts that almost surely, if $e(x)$ is any non-zero vector such that $e(x) \notin E^+(x)$, $\|df_x^{-n}(e(x))\|$ grows exponentially at rate $|\chi^-|$. Hence $e^u(x) \in E^+(x)$ and we are done. \square

2.3. Unstable lamination, dominated splitting and hyperbolicity. It is natural to expect that in the dissipative setting, uniform u-regularity is enough to characterize hyperbolicity. Indeed, uniform u-regularity should provide uniform expansion along some field of directions, which, together with volume contraction yields uniform hyperbolicity. The basic technical tool needed to implement this idea is that of *dominated splitting*. Recall that a dominated splitting on some invariant set Λ is a splitting of the form $T\mathbb{C}^2|_{\Lambda} = E^s \oplus E^c$ for which there exists $C > 0$ and $\lambda < 1$ such that

$$\frac{\|df^n|_{E^s}\|}{\|df^n|_{E^c}\|} \leq C\lambda^n.$$

Then this splitting is automatically continuous, and if $|\text{Jac}(f)| \leq 1$ the direction E^s is contracting. The existence of a dominated splitting for f along J is a way to formalize the ‘‘absence of critical points’’ on J .

Our first result can be viewed as a version of [LP] in a (greatly) simplified setting.

Proposition 2.16. *Let f be a complex Hénon map with $|\text{Jac}(f)| \leq 1$. If every point in J^* is uniformly u-regular and if f admits a dominated splitting on J^* , then f is hyperbolic.*

Proof. Dominated splitting implies the existence of a strong stable lamination \mathcal{W}^s in a neighborhood of J^* , hence points of J^* are uniformly s-regular. Then if $|\text{Jac}(f)| < 1$, the result follows directly from Theorem 2.14. In the general case we just have to repeat the proof of Theorem 2.14, the only difference being that dissipativity was used there to show that ν has a negative exponent while here this follows from the dominated splitting assumption. \square

The idea of dominated splitting shows that hyperbolicity already holds under the assumptions of Proposition 2.10:

Proposition 2.17. *Let f be a complex Hénon map with $|\text{Jac}(f)| \leq 1$. If every point in J^* is uniformly u-regular and for every $x \in J^*$, $G^+|_{\mathcal{W}_{\text{loc}}^u(x)} \neq 0$ then f is hyperbolic.*

Applying Proposition 2.12 yields the following corollary, which generalizes (and gives a new approach to) Theorem 2.13.

Corollary 2.18. *Let f be a complex Hénon map with $|\text{Jac}(f)| \leq 1$. If every point in J^* is regular and uniformly u-regular then f is hyperbolic.*

Proof of Proposition 2.17. By the cone criterion for dominated splitting (see [S, Prop. 2.2]) it is enough to prove that for every $x \in J^*$ there exists a cone \mathcal{C}_x about $T_x\mathcal{W}^u(x)$ in $T_x\mathbb{C}^2$ such that the field of cones $(\mathcal{C}_x)_{x \in J^*}$ is strictly contracted by the dynamics. Then the result follows from Proposition 2.16. By Proposition 2.10 and Remark 2.11 there is a continuous Riemannian

metric on $T\mathcal{W}^u|_{J^*}$ which is immediately expanded by the dynamics. Let $(e_x)_{x \in J^*}$ be a field of tangent vectors to \mathcal{W}^u of unit norm relative to this metric, and f_x be orthogonal to e_x in $T_x\mathbb{C}^2$ (relative to the ambient Riemannian structure) and such that $\det(e_x, f_x) = 1$. For small ε , define a continuous field of cones $\mathcal{C}_x^\varepsilon \subset T_x\mathbb{C}^2$ by

$$\mathcal{C}_x^\varepsilon = \{ue_x + vf_x, |v| \leq \varepsilon|u|\}.$$

Working in the frame $\{(e_x, f_x), x \in J^*\}$, the matrix expression of df_x is of the form

$$\begin{pmatrix} \lambda_x & a(x) \\ 0 & \lambda_x^{-1}J \end{pmatrix},$$

where $|\lambda_x| \geq \lambda_0 > 1$ and J is the Jacobian, so $|J| \leq 1$. Since the frame (e_x, f_x) is continuous, $a(\cdot)$ is bounded. Then one checks easily that if ε is so small that $\lambda_0 - \varepsilon\|a\| > 1$, then

$$df_x(\mathcal{C}_x^\varepsilon) \subset \mathcal{C}_{f(x)}^{\lambda_0^{-1}\varepsilon}.$$

Hence the field of cones $(\mathcal{C}_x)_{x \in J^*}$ is strictly contracted by the tangent dynamics and we are done (note that a similar argument appears in [D1]). \square

The next result shows that uniform expansion can indeed be deduced from the geometric property of uniform u-regularity. Assume that every $x \in J^*$ is uniformly regular of size $4r$. Recalling the construction of global unstable manifolds from local ones, for $x \in J^*$ we define

$$(1) \quad \mathcal{W}^u(x) = \bigcup_{n \geq 0} f^n(\mathcal{W}_r^u(f^{-n}(x))).$$

It follows from this definition that $f^{-1}(\mathcal{W}^u(x)) = f^{-1}(\mathcal{W}_r^u(x)) \cup \mathcal{W}^u(f^{-1}(x))$, hence $f^{-1}(\mathcal{W}^u(x))$ contains $\mathcal{W}^u(f^{-1}(x))$ and it is not a priori clear that the $\mathcal{W}^u(x)$ define an invariant family of curves. However, if $\mathcal{W}^u(x)$ is biholomorphic to \mathbb{C} for every $x \in J^*$ then $f^{-1}(\mathcal{W}^u(x)) = \mathcal{W}^u(f^{-1}(x))$, for otherwise $\mathcal{W}^u(f^{-1}(x))$ would strictly contain $f^{-1}(\mathcal{W}^u(x))$, and it would be a complex submanifold of \mathbb{C}^2 biholomorphic to the Riemann sphere, which is contradictory. The following theorem confirms the expectation that the parabolicity of leaves in J^- is associated with expansion (compare e.g. [LM, §4]).

Theorem 2.19. *Let f be a dissipative complex Hénon map. If every point $x \in J^*$ is uniformly u-regular and in addition $\mathcal{W}^u(x)$ is biholomorphic to \mathbb{C} , then f is hyperbolic.*

Remark that the definition of $\mathcal{W}^u(x)$ in (1) a priori depends on r . The theorem shows that if these manifolds are biholomorphic to \mathbb{C} , this is actually not the case.

Proof. For every $x \in J^*$, fix a uniformization $\psi_x^u : \mathbb{C} \xrightarrow{\sim} \mathcal{W}^u(x)$ such that $\psi_x^u(0) = x$, which is normalized by $|(\psi_x^u)'(0)| = 1$. For $\eta > 0$, define $R_\eta(x)$ to be the maximal radius of a round disk in \mathbb{C} such that $G^+ \circ \psi_x^u|_{D(0, R_\eta(x))} \leq \eta$ (this is similar but not identical to the definition of ρ_η in Proposition 2.10). Since $\mathcal{W}^u(x)$ is an entire curve contained in J^- , $G^+|_{\mathcal{W}^u(x)}$ is unbounded so $R_\eta(x)$ is finite. We claim that for every $\eta > 0$, there exists $C_\eta > 0$ such that

$$(2) \quad \text{for every } x \in J^*, C_\eta^{-1} \leq R_\eta(x) \leq C_\eta.$$

Indeed, fix $x \in J^*$ and let us show that R_η is locally uniformly bounded from above and below in a neighborhood of x . Then by compactness these bounds will be uniform on J^* . Viewed in the unstable parameterizations f is affine so it maps circles to circles. Let $\Delta^u(x, R) = \psi_x^u(D(0, R))$. We first claim that there exists $k \geq 0$ such that $f^{-k}(\Delta^u(x, R_\eta(x)))$ is contained in $\mathcal{W}_r^u(f^{-k}(x))$. Indeed by definition of $\mathcal{W}^u(x)$, for every $x' \in \partial\Delta^u(x, 4R_\eta(x))$, there exists $k \geq 0$ such that $f^{-k}(x') \in \mathcal{W}_r^u(f^{-k}(x))$. As in the proof of Proposition 2.10, the Koebe distortion theorem implies that there is a coordinate $\pi : \mathcal{W}_{4r}^u(f^{-k}(x)) \rightarrow \mathbb{C}$ such that for $s \leq r$, $\pi(\Delta^u(f^{-k}(x), s))$ is

approximately a disk of radius s . Now $f^{-k}(\Delta^u(x, R_\eta(x)))$ is a round disk in the affine coordinate, and it possesses a boundary point in $\mathcal{W}_{r/2}^u(f^{-k}(x))$, so it follows that it is completely contained in $\mathcal{W}_r^u(f^{-k}(x))$. Therefore, replacing x by $f^{-k}(x)$ we can assume that $\psi_x^u(D(0, R_\eta)) \subset \mathcal{W}_r^u(x)$.

By uniform u-regularity, for y close to x , $\mathcal{W}_r^u(y)$ is a graph of slope at most 1 over a disk of size r relative to the projection π . Thus from Koebe distortion again, we infer that for y close to x , the distance induced by the normalized affine structure along the $\mathcal{W}_r^u(y)$ is equivalent to the ambient distance. In particular there exists a constant K depending only on r such that for y close to x and η as above,

$$K^{-1} \operatorname{dist}(y, \{G^+ = \eta\}) \leq R_\eta(y) \leq K \operatorname{dist}(y, \{G^+ = \eta\}).$$

Finally by the Hölder continuity of G^+ $\operatorname{dist}(y, \{G^+ = \eta\})$ is bounded from below by $C\eta^\theta$, and if $\operatorname{dist}(y, x) \leq r$ it is bounded from above by Cr . This completes the proof of (2).

Then, from the invariance relation of G^+ we have

$$f(\Delta^u(x, R_\eta(x))) = \Delta^u(f(x), R_{d_\eta}(f(x))) \supset \Delta^u(f(x), R_\eta(f(x))),$$

hence for every $n \geq 0$ we infer that $f^n(\Delta^u(x, R_\eta(x))) \supset \Delta^u(f^n(x), R_\eta(f^n(x)))$. In particular for every $x \in J^*$ and every $n \geq 1$ we have that $f^n(\Delta^u(x, C_\eta)) \supset \Delta^u(f^n(x), C_\eta^{-1})$. Again since f is affine in the unstable parameterizations we deduce that for every $t > 0$,

$$f^n(\Delta^u(x, tC_\eta)) \supset \Delta^u(f^n(x), tC_\eta^{-1}).$$

Finally taking the derivative at $t = 0$ we conclude that $\|Df_x^n|_{T_x \mathcal{W}^u}\| \geq (C_\eta)^{-2}$.

This bound in turns implies the existence of a dominated splitting along J^* . This follows from the criterion of Bochi-Gourmelon [BG, Thm A] (see also Yoccoz [Y]). Indeed since f has constant Jacobian, for $x \in J^*$ the singular values of Df_x^n are σ_n^+ and $\sigma_n^- = J^n/\sigma_n^+$, where $J = |\operatorname{Jac}(f)| < 1$, and $\sigma_n^+ \geq (C_\eta)^{-2}$. Therefore

$$\frac{\sigma_n^+}{\sigma_n^-} = \frac{(\sigma_n^+)^2}{J^n} \geq \frac{1}{C_\eta^4 J^n}$$

so [BG] applies and we get a dominated splitting on J^* . Applying proposition 2.16 concludes the proof. \square

Remark 2.20. If J^- is globally laminated (outside a finite set of periodic points, say) one might expect that the additional assumption that $\mathcal{W}^u(x) \simeq \mathbb{C}$ for every x in Theorem 2.19 would follow from the density of unstable manifolds of saddle points. Unfortunately there are examples of minimal Riemann surface laminations containing both parabolic and hyperbolic leaves (see [Gh, Thm 6.6])

2.4. Concluding remarks.

2.4.1. Uniform s-regularity on J^* does not imply hyperbolicity. Indeed there are examples of Hénon mappings with parabolic points and a dominated splitting on J^* (see [RT, LP]). It would be interesting to know whether uniform s-regularity on J^* implies the existence of a dominated splitting.

2.4.2. The only property of J^* that was used in the various hyperbolicity criteria in this section is that J^* is a closed invariant set in which saddle periodic points are dense. So in all these results we could with an arbitrary closed invariant set Λ , in which saddle points are dense. The notion of uniform u-regularity has to be replaced by uniform u-regularity along Λ , meaning that the uniform size of unstable manifolds holds only for sequences of saddle points in Λ , and likewise for s-regularity. Then there are statements analogous to Theorems 2.9, 2.13, 2.14 and 2.19, in

which uniform regularity is replaced by uniform regularity along Λ , and the conclusion is that Λ is a hyperbolic set.

3. A TOPOLOGICAL CRITERION FOR HYPERBOLICITY

In this section we work in the setting of Conjecture 1.4: We assume that f_0 and f_1 are two complex Hénon maps such that f_0 is hyperbolic, and that there exist respective neighborhoods N_0 and N_1 of $J_0 = J_0^*$ and J_1^* and a conjugating homeomorphism $\phi : N_0 \rightarrow N_1$. Our purpose is to show that f_1 is hyperbolic on J_1^* .

3.1. Periodic points and their (un)stable manifolds.

Proposition 3.1. *Let f_0 and f_1 be as in Conjecture 1.4. Then $\phi(J_0) = J_1^*$. If f_1 is dissipative then all periodic points of f_1 on J_1^* are saddles. If f_1 is conservative the same holds provided ϕ is Hölder continuous.*

Proof. The first assertion is a direct consequence of the equidistribution of periodic orbits. Indeed the topological conjugacy shows that f_0 and f_1 have the same entropy, hence the same dynamical degree. Since periodic orbits equidistribute towards the maximal entropy measure, we get that $\phi_*\mu_0 = \mu_1$. Since $\text{Supp}(\mu_1) = J_1^*$, we infer that $\phi(J_0) = J_1^*$. (On the other hand it is unclear at this stage whether $J_1^* = J_1$.)

Any periodic point on J_1^* admits a neighborhood in which it is topologically conjugate to a saddle. Let $p \in J_1^*$ be some periodic point which we may suppose fixed. Assume that f_1 is dissipative. Then if p is not a saddle it is semi-attracting. By the hedgehog theory of [FLRT, LRT] there exists in some neighborhood of p a non-trivial totally invariant set \mathcal{H} made of points which do not converge to p under backward nor forward iteration: indeed there is a subsequence q_n such that $f^{q_n} \rightarrow \text{id}$ on \mathcal{H} . This is not compatible with the local conjugacy to a saddle fixed point, therefore we conclude that p is a saddle.

If f is conservative and p is not a saddle, then it is neutral. Since ϕ is Hölder, then points in $\phi(W_{\text{loc}}^s(\phi^{-1}(p)))$ converge to the origin exponentially fast. On the other hand for every $\varepsilon > 0$ there exists a norm on \mathbb{C}^2 for which

$$(1 - \varepsilon) \|v\| \leq \|df_p(v)\| \leq (1 + \varepsilon) \|v\|.$$

Indeed if df_p is diagonalizable this is clear since the eigenvalues have modulus 1, and otherwise we can make df_p triangular with the off-diagonal term as small as we wish, and take an adapted norm. Then if x is close to p and $f_1^n(x) \rightarrow p$ we infer that $\|f_1^n(x) - f_1^n(p)\| \geq (1 - 2\varepsilon)^n \|x - p\|$ which is contradictory if ε is small enough. Thus again we conclude that all periodic points on J_1^* are saddles. \square

For a saddle point p , we now denote by $W_r^s(p)$ the component of $W^s(p) \cap B(p, r)$ containing p . We fix r_0 such that for every $x \in J_0$, $W_{r_0}^s(x)$ (resp. $W_{r_0}^u(x)$) is a properly embedded holomorphic disk with the property that there exist uniform $C > 0$ and $0 < \lambda < 1$ such that for every $x' \in W_{r_0}^s(x)$ (resp. $W_{r_0}^u(x)$), $\text{dist}(f^n(x), f^n(x')) \leq C\lambda^{|n|}$ when $n \rightarrow \infty$ (resp. $n \rightarrow -\infty$). We also assume for further reference that f has product structure in the $2r_0$ -neighborhood of J_0^* .

Proposition 3.2. *Let f_0 and f_1 be as in Conjecture 1.4. There exists $r_1 > 0$ such that for any saddle periodic point p for f_1 , $W_{r_1}^s(p)$ (resp. $W_{r_1}^u(p)$) is a submanifold of $B(p, r_1)$.*

Proof. Without loss of generality we treat the case of stable manifolds and assume that p is fixed. If r_0 is as above there exists $r_1 > 0$ such that for every $y \in J_1^*$, $B(y, r_1) \subseteq \phi(B(\phi^{-1}(y), r_0))$. We claim that for every saddle fixed point p for f_1 ,

$$(3) \quad W_{r_1}^s(p) \subset \phi(W_{r_0}^s(\phi^{-1}(p))) \subset W^s(p).$$

The right inclusion is obvious since belonging to $W^s(p)$ is characterized by the topological property that $f^n(y) \rightarrow p$. For the left inclusion, just observe that $\phi^{-1}(W_{r_1}^s(p))$ is a connected subset of $W^s(\phi^{-1}(p)) \cap B(\phi^{-1}(p), r_0)$ containing p , hence it is contained in $W_{r_0}^s(\phi^{-1}(p))$.

To show that it is properly embedded, we first observe that there exists $r = r(p)$ such that $W_r^s(p)$ is properly embedded in $B(p, r)$. By the invariance of domain theorem $\phi^{-1}(W_r^s(p))$ is a neighborhood of $\phi^{-1}(p)$ in $W^s(\phi^{-1}(p))$. Thus it follows that there exists $n = n(p)$ such that $f_0^n(W_{r_0}^s(\phi^{-1}(p))) \subset \phi^{-1}(W_r^s(p))$. Then from (3) we get that $f_1^n(W_{r_1}^s(p)) \subset W_r^s(p)$, so $W_{r_1}^s(p) \subset f_1^{-n}W_r^s(p)$. From this we conclude that $W_{r_1}^s(p)$ is properly embedded in $B(p, r_1)$, as desired. \square

Remark 3.3. At this stage we know that stable manifolds are properly embedded in a ball of uniform size, but since in the last argument the quantities n and r are a priori not uniform in p , we have no uniformity for the geometry of $W_{r_1}^s(p)$. Obtaining such a uniformity will be the purpose of the forthcoming arguments.

3.2. Tube argument.

Lemma 3.4 (Tubular neighborhood lemma). *If Δ is a subvariety in $B(0, 2r)$ of size r at 0 then there exists $\eta = \eta(r)$ such that if Δ' is a subvariety in $B(0, 2r)$ such that $d_H(\Delta, \Delta') < \eta$ in $B(0, 2r)$, then Δ' is a branched cover over Δ in $B(0, r/2)$. (Here d_H denotes the Hausdorff distance.)*

Proof. After a unitary change of coordinates, Δ is a graph $y = \psi(x)$ of slope at most 1 in the bidisk $D(0, r)^2$. Since $\psi'(0) = 0$ by the Schwarz lemma we have $|\psi'(x)| \leq |x|/r$ so actually $|\psi(x)| \leq r/2$. It follows that if $\eta < r/4$ and $d_H(\Delta, \Delta') < \eta$ in $D(0, r)^2$, Δ' is horizontal in this bidisk. Thus it is a branched cover over the first coordinate, hence over Δ . \square

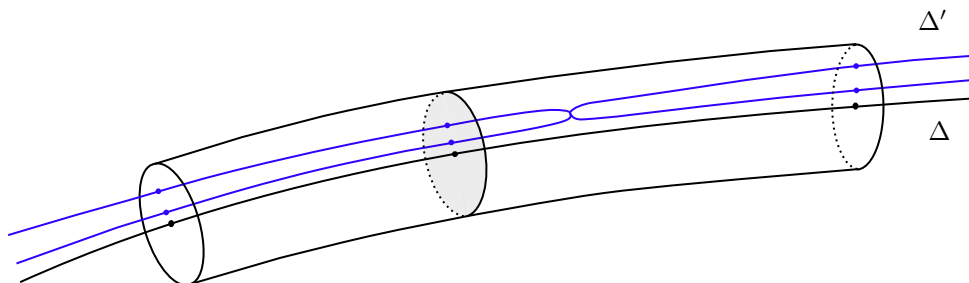


FIGURE 1. Tubular neighborhood lemma

The “tube argument” alluded to in the title consists in applying the previous lemma to construct invariant laminations in the setting of the conjecture. Here is a sample statement.

Proposition 3.5. *Let f_0 and f_1 be as in Conjecture 1.4. Then every saddle periodic point p for f_1 is uniformly transverse regular.*

Proof. Let p be as in the statement of the proposition, and assume that $W^s(p)$ has size r at p . Reducing r if necessary, we may assume that $\phi^{-1}(W_r^s(p))$ is contained in a flow box of the stable lamination of f_0 . Without loss of generality we may also assume that $r < r_1/4$, where r_1 is as in Proposition 3.2. We will show that there exists a neighborhood V of p such that if $q \in V$ is another periodic point, then $W_r^s(q)$ is a graph over $W^s(p)$ in $B(0, r/2)$. This implies that p

is uniformly s-regular. Uniform u-regularity is proven in the same way, and the transversality property is obvious since p is a saddle.

We know that for any saddle point $q \in J_1^*$, the stable manifold $W_{r_1}^s(q)$ is properly embedded in $B(q, r_1)$. In addition, by (3) it is contained in $\phi(W_{r_0}^s(\phi^{-1}(q)))$. By the uniform continuity of ϕ there exists a neighborhood V of p such that for every saddle point $q \in V$, $W_r^s(q)$ is η -close to $W_r^s(p)$ in $B(p, 2r)$, where η is as in the tubular neighborhood lemma. Thus $W_r^s(q)$ is a branched cover over $W_r^s(p)$ in $B(p, r/2)$, and to conclude the proof it remains to show that this cover has degree 1. By the product structure of f_0 in $(J_0^*)_{2r_0}$ we have that $W_{r_0}^s(\phi^{-1}(p)) \cap W_{r_0}^u(\phi^{-1}(p)) = \{\phi^{-1}(p)\}$ hence $W_{r_1}^s(p) \cap W_{r_1}^u(p) = \{p\}$. Thus, reducing η if necessary, to compute the degree of this branched cover it is enough to count the number of intersection points, with multiplicity, between $W_r^s(q)$ and $W_{r_1}^u(p)$. Applying the product structure again we get that $W_r^s(q) \cap W_{r_1}^u(p)$ is a single point. Furthermore it is well-known that the order of contact between two smooth complex curves in \mathbb{C}^2 is a topological invariant. Indeed if we consider two smooth curves C and D with an isolated intersection at $0 \in \mathbb{C}^2$ and intersect them with a small sphere \mathbb{S} about 0, then $C \cap \mathbb{S}$ winds n times about $D \cap \mathbb{S}$ where n is the intersection multiplicity. So we conclude that the intersection $W_r^s(q) \cap W_{r_1}^u(p)$ is transverse and we are done. \square

3.3. Proof of Conjecture 1.4 in the dissipative case.

Theorem 3.6. *Let f_0 and f_1 be two polynomial automorphisms of \mathbb{C}^2 with non-trivial dynamics, and assume that f_0 is hyperbolic and that f_1 is dissipative.*

Suppose that there exists respective neighborhoods N_0 and N_1 of $J_0 = J_0^$ and J_1^* and a homeomorphism $\phi: N_0 \rightarrow N_1$ such that $\phi \circ f_0 = f_1 \circ \phi$ where these compositions makes sense. Then f_1 is hyperbolic.*

To prove the theorem, let W_0^s and W_0^u be the stable and unstable laminations in N_0 and \mathcal{L}_1^s and \mathcal{L}_1^u be their respective images under ϕ . At this stage \mathcal{L}_1^s and \mathcal{L}_1^u are topological laminations by topological disks in N_1 .

Define Ω to be the set of points $x \in J_1^*$ such that there exists a neighborhood V of x in J_1^* such that for every $y \in V$, $\mathcal{L}_1^s(y)$ and $\mathcal{L}_1^u(y)$ are holomorphic and of uniform size in V . Note that they must be transverse by the topological invariance of the order of contact between smooth curves. By construction, Ω is open in J_1^* and completely invariant (i.e. $f(\Omega) = \Omega$). Proposition 3.5 shows that Ω contains all saddle points.

The main step of the proof is the following:

Lemma 3.7. *Let f_0 and f_1 be as in Theorem 3.6. Then any invariant measure supported on J_1^* gives full mass to Ω .*

Theorem 3.6 follows easily. Indeed, if non-empty, the complement of Ω in J_1^* is a closed invariant set hence if it is non-empty it supports an invariant measure ν . By the lemma, $\nu(\Omega) = 1$ hence the contradiction. Therefore we conclude that $\Omega = J_1^*$, in particular all points in J_1^* are uniformly regular, and the result follows from Theorem 2.9. \square

Proof of Lemma 3.7. The method is to adapt the ‘‘tube argument’’ to Pesin stable manifolds. Under the assumptions of the theorem, let ν be any invariant measure for f_1 supported on J_1^* . Then by Oseledets’ Theorem for ν -a.e. x there exist Lyapunov exponents $\chi_1(x) \leq \chi_2(x)$ satisfying $\chi_1(x) + \chi_2(x) = \log |\text{Jac}(f)|$. In addition since ν is not concentrated on a periodic orbit we have $\chi_2(x) \geq 0$ a.e. hence $\chi_1(x) < 0$ since $|\text{Jac}(f)| < 1$. By the Pesin stable manifold theorem, for ν -a.e. x , there exists a local stable manifold $W_{\text{loc}}^s(x)$ which can be characterized as the set of points y sufficiently close to x such that $\limsup \frac{1}{n} \log \text{dist}(f_1^n(y), f_1^n(x)) < 0$. Pick any

point x such that $W_{\text{loc}}^s(x)$ exists. We will show that *both* \mathcal{L}_1^s and \mathcal{L}_1^u are laminations by Riemann surfaces near x .

Observe first that by hyperbolicity of f_0 , the local stable manifold of $\phi^{-1}(x)$ is the set of points z near $\phi^{-1}(x)$ such that $\text{dist}(f_0^n(z), f_0^n(\phi^{-1}(x))) \rightarrow 0$ as $n \rightarrow +\infty$. Hence $\phi^{-1}(W_{\text{loc}}^s(x)) \subset W_{\text{loc}}^s(\phi^{-1}(x))$. Since ϕ^{-1} is continuous and injective, by the invariance of domain theorem, $\phi^{-1}(W_{\text{loc}}^s(x))$ is neighborhood of $\phi^{-1}(x)$ in $W_{\text{loc}}^s(\phi^{-1}(x))$. Thus $W_{\text{loc}}^s(x)$ coincides with $\mathcal{L}_1^s(x)$ in a neighborhood of x .

Let r be so small that $W_{\text{loc}}^s(x)$ has size r at x and $W_{\text{loc}}^s(x) = \mathcal{L}_1^s(x)$ in $B(x, 2r)$. Then after a unitary change of coordinates as in Lemma 3.4, $W_{\text{loc}}^s(x)$ is a graph of the form $y = \psi(x)$ over $D(0, r)$. Denote this graph by Δ_r^s . For small $\eta > 0$, we define

$$\text{Tub}_\eta = \text{Tub}_\eta(\Delta_r^s) = \{(x, y), |x| < r, |y - \psi(x)| < \eta\}.$$

We say that a submanifold M of Tub_η (which extends to some neighborhood of $\overline{\text{Tub}_\eta}$) is *horizontal* if

$$M \cap \partial\text{Tub}_\eta \subset \{(x, y) \in \overline{\text{Tub}_\eta}, |x| = r\}$$

and similarly it is *vertical* if $M \cap \partial\text{Tub}_\eta \cap \{|x| = r\} = \emptyset$. As already observed, if M is horizontal it is a branched covering over the first coordinate, and similarly if it is vertical the restriction of $(x, y) \mapsto y - \psi(x)$ to $M \cap \text{Tub}_\eta$ is a branched covering over $D(0, \eta)$.

Exactly as in Proposition 3.5, if $q \in J^*$ is a saddle point sufficiently close to x , $W_{\text{loc}}^s(q)$ is horizontal in Tub_η . Now by the transversality of \mathcal{W}_0^s and \mathcal{W}_0^u , there exists a neighborhood N of $\phi^{-1}(x)$ such that for any $z \in N$, the distance between $\mathcal{W}_0^s(z)$ and $\phi^{-1}(\partial\Delta_r^s)$ is bounded from below by a uniform positive constant. By continuity of ϕ , for any $y = \phi(z) \in \phi(N)$ and reducing η if necessary we get that $\text{dist}(\partial\Delta_r^s, \mathcal{L}_1^u(y)) > 2\eta$. By Proposition 3.2, if q is a saddle point close to x , $W_{r_1}^u(q) = \mathcal{L}_1^u(q) \cap B(q, r_1)$ is a submanifold in $B(q, r_1)$ for a uniform r_1 (which we may assume to be large with respect to r and η). So we conclude that it is a submanifold in a neighborhood of $\overline{\text{Tub}_\eta}$, which must be vertical in Tub_η . Thus we have shown that if q is a saddle periodic point sufficiently close to x , the local stable and unstable manifolds of q are respectively horizontal and vertical in Tub_η , with a single transverse intersection point (for transversality again we use the topological invariance of the order of contact). Hence both have covering degree 1 respectively over the horizontal and vertical directions in Tub_η , i.e. they are graphs. Then by the Schwarz Lemma they have uniformly bounded geometry. So we conclude that x belongs to Ω , and the proof is complete. \square

3.4. The conservative case. In the conservative case we can only prove Conjecture 1.4 in the case of a Hölder conjugacy.

Theorem 3.8. *Let f_0 and f_1 be two polynomial automorphisms of \mathbb{C}^2 with non-trivial dynamics, and assume that f_0 is hyperbolic and f_1 is conservative.*

Suppose that there exists respective neighborhoods N_0 and N_1 of $J_0 = J_0^$ and J_1^* and a Hölder continuous homeomorphism $\phi : N_0 \rightarrow N_1$ such that $\phi \circ f_0 = f_1 \circ \phi$ where these compositions makes sense. Then f_1 is hyperbolic.*

The proof is identical to that of Theorem 3.8, the only difference is that in Lemma 3.7 we need a different argument to show that any ergodic invariant measure ν for f_0 admits a negative Lyapunov exponent (this issue already appeared in the proof of Proposition 3.1). So Theorem 3.8 follows from:

Lemma 3.9. *Let f_0 and f_1 be as in Theorem 3.8. Then all measures invariant under f_1 are hyperbolic.*

Proof. This follows from standard Pesin-theoretic considerations. Let ν be an invariant measure for f_1 . Without loss of generality we can assume that ν is ergodic so it admits two Lyapunov exponents $\chi_1 \leq \chi_2$ with $\chi_1 + \chi_2 = 0$. Assume by way of contradiction that $\chi_1 = \chi_2 = 0$. The Oseledets-Pesin reduction theorem (see [KM, Thm. S.2.10], note that it does not require ν to be hyperbolic) asserts that for every $\varepsilon > 0$ there exists a measurable cocycle C_ε with values in $\mathrm{GL}_2(\mathbb{C})$ such that for ν -a.e. x , the matrix $A_\varepsilon(x) := C_\varepsilon(f_1(x))^{-1} \cdot (Df_1)_x \cdot C_\varepsilon(x)$ satisfies $e^{-\varepsilon} \leq \|A_\varepsilon(x)\| \leq e^\varepsilon$ and $e^{-\varepsilon} \leq \|(A_\varepsilon(x))^{-1}\| \leq e^\varepsilon$. Then, the Pesin theorem on existence of regular neighborhoods (see [KM, Thm. S.3.1]) implies that there is a measurable function q such that for ν -a.e. x , f behaves like $(Df_1)_x$ on $B(x, q(x))$ and furthermore $e^{-\varepsilon} < q(f_1(x))/q(x) < e^\varepsilon$. More precisely there is a change of coordinates Ψ_x defined on $B(x, q(x))$ such that $\Psi_{f_1(x)} \circ f \circ \Psi_x$ is ε C^1 -close to its differential at x , which equals $A_\varepsilon(x)$.

Now by the Hölder conjugacy to f_0 , for every x there exists y close to x such that $\mathrm{dist}(f_1^n(y), f_1^n(x))$ decreases like $e^{-\alpha n}$ for some $\alpha > 0$. If we pick ε small as compared to α , then for generic x we have that $f_1^n(y) \in B(f_1^n(x), q(f_1^n(x)))$ for every large n . It follows that for large k

$$\mathrm{dist}(f_1^{n+k}(y), f_1^{n+k}(x)) \geq C e^{-2\varepsilon k} \mathrm{dist}(f_1^n(y), f_1^n(x)),$$

which is contradictory, and the proof is complete. \square

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INSTITUTE FOR MATHEMATICAL SCIENCES STONY BROOK UNIVERSITY STONY BROOK, NY 11794

Email address: ebedford@math.stonybrook.edu

SORBONNE UNIVERSITÉ, CNRS, LABORATOIRE DE PROBABILITÉS, STATISTIQUES ET MODÉLISATIONS (LPSM),
F-75005 PARIS, FRANCE

Email address: romain.dujardin@sorbonne-universite.fr