Schrödinger equation in moving domains

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Abstract

We consider the Schrödinger equation

$$i\partial_t u(t) = -\Delta u(t) \qquad \text{on } \Omega(t) \tag{$*$}$$

where $\Omega(t) \subset \mathbb{R}^N$ is a moving domain depending on the time $t \in [0,T]$. The aim of this work is to provide a meaning to the solutions of such an equation. We use the existence of a bounded reference domain Ω_0 and a specific family of unitary maps $h^\sharp(t) : L^2(\Omega(t), \mathbb{C}) \longrightarrow L^2(\Omega_0, \mathbb{C})$. We show that the conjugation by $h^\sharp$ provides a new equation of the form

$$i\partial_t v = h^\sharp(t) H(t) h_\sharp(t) v \qquad \text{on } \Omega_0 \tag{$**$}$$

where $h_\sharp = (h^\sharp)^{-1}$. The Hamiltonian $H(t)$ is a magnetic Laplacian operator of the form

$$H(t) = -(\text{div}_x + iA) \circ (\nabla_x + iA) - |A|^2$$

where A is an explicit magnetic potential depending on the deformation of the domain $\Omega(t)$. The formulation $(**)$ enables to ensure the existence of weak and strong solutions of the initial problem $(*)$ on $\Omega(t)$ endowed with Dirichlet boundary conditions. In addition, it also indicates that the correct Neumann type boundary conditions for $(*)$ are not the homogeneous but the magnetic ones

$$\partial_\nu u(t) + i\langle \nu | A \rangle u(t) = 0,$$

even though $(*)$ has no magnetic term. All the previous results are also studied in presence of diffusion coefficients as well as magnetic and electric potentials. Finally, we prove some associated byproducts as an adiabatic result for slow deformations of the domain and a time-dependent version of the so-called "Moser's trick". We use this outcome in order to simplify Equation $(**)$ and to guarantee the well-posedness for slightly less regular deformations of $\Omega(t)$.

Keywords: Schrödinger equation, PDEs on moving domains, well-posedness, magnetic Laplacian operator, Moser's trick, adiabatic result.

1 Main results

In this article, we study the well-posedness of the Schrödinger equation

$$i\partial_t u(t,x) = -\Delta u(t,x), \qquad t \in I, \ \ x \in \Omega(t) \tag{1.1}$$

where I is an interval of times and $t \in I \mapsto \Omega(t) \subset \mathbb{R}^N$ is a time-dependent family of bounded domains of $\mathbb{R}^N$ with $d \geq 1$. We consider the cases of Dirichlet boundary conditions and of suitable magnetic Neumann boundary conditions. This kind of problem is very natural when we consider a quantum particle confined in a structure which deforms in time.

The Schrödinger equation in moving domains has been widely studied in literature and an example is the classical article of Doescher and Rice [18]. For other references on the subject, we mention [3, 5, 8, 9, 17, 38, 40, 43, 47, 50]. In most of these references, (1.1) is studied in dimension $d = 1$ or in higher dimensions with symmetries as the radial case or the translating case. From this perspective, the purpose of this work is natural: we aim to study the well-posedness of (1.1) in a very general framework.

The difficulty of considering an equation in a moving domain as (1.1) is that the phase space $L^2(\Omega(t), \mathbb{C})$ and thus the operator $\Delta = \Delta(t)$ depend on the time. The usual method adopted in these kinds of problems consists of transforming $\Omega(t)$ in a bounded reference domain $\Omega_0 \subset \mathbb{R}^N$. Such transformation is then used in order to bring back the Schrödinger equation (1.1) in an equivalent equation in the phase space $L^2(\Omega_0, \mathbb{C})$, which does not depend on the time. To this purpose, one can introduce a family of diffeomorphisms $(h(t, \cdot))_{t \in I}$ such that for each $t \in I$, $h(t, \cdot)$ is a $\mathcal{C}^p-$diffeomorphism from $\overline{\Omega}_0$ onto $\overline{\Omega}(t)$ with $p \geq 1$ (see Figure 1). Assume in addition that the function $t \in I \mapsto h(t, y) \in \mathbb{R}^N$ is of class $\mathcal{C}^q$ with respect to the time with $q \geq 1$.
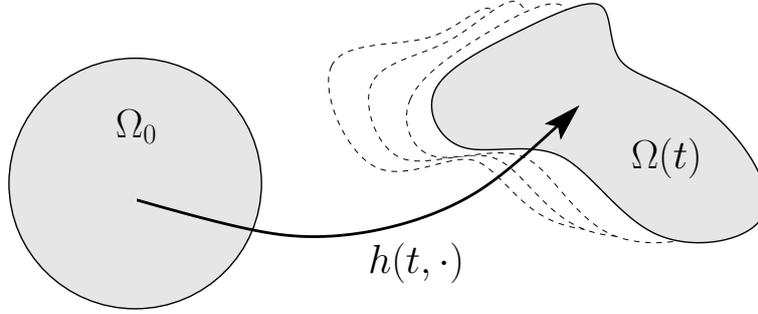

Figure 1: *The family of diffeomorphisms $(h(t, \cdot))_{t \in I}$ which enables to go back to a fixed domain Ω_0.*

In order to bring back the Schrödinger equation (1.1) in an equivalent equation in $L^2(\Omega_0, \mathbb{C})$, one can introduce the pullback operator

$$h^*(t) \ : \ \phi \in L^2(\Omega(t), \mathbb{C}) \ \longmapsto \ \phi \circ h = \phi(h(t, \cdot)) \in L^2(\Omega_0, \mathbb{C}) \tag{1.2}$$

and its inverse, the pushforward operator, defined by

$$h_*(t) \ : \ \psi \in L^2(\Omega_0, \mathbb{C}) \ \longmapsto \ \psi \circ h^{-1} = \psi(h^{-1}(t, \cdot)) \in L^2(\Omega(t), \mathbb{C}) \ . \tag{1.3}$$

If we compute the equation satisfied by $w = h^* u$ when u is solution of (1.1) at least in a formal sense, then we find that (1.1) becomes

$$i\partial_t w(t, y) \ = \ -\frac{1}{|J|} \operatorname{div}_y \left(|J| J^{-1} (J^{-1})^t \nabla_y w(t, y) \right) + i \langle J^{-1} \partial_t h(t, y) | \nabla_y w(t, y) \rangle, \qquad t \in I \ , \ y \in \Omega_0, \tag{1.4}$$

where $J = J(t, y) = D_y h(t, y)$ is the Jacobian matrix of h, $|J|$ stands for $|\det(J)|$ and $\langle \cdot | \cdot \rangle$ corresponds to the scalar product in $\mathbb{C}^N$ (see Remark 2.4 in Section 2 for the computations).

A possible way to give a sense to the Schrödinger equation (1.1) consists in proving that (1.4), endowed with some boundary conditions, generates a well-posed flow in $L^2(\Omega_0, \mathbb{C})$. This can be locally done by exploiting some specific properties of the Schrödinger equation with perturbative terms of order one (see [34, 35, 36, 39]). Nevertheless this method presents possible disadvantages for our purposes. Indeed, such approach does not provide the natural conservation of the L^2-norm and the Hamiltonian structure of the equation is lost in the sense that the new differential operator is no longer self-adjoint with respect to a natural time-independent hermitian structure. This fact represents an obstruction, not only to the proof of global existence of solutions, but also to the use of different techniques such as the adiabatic theory.

We are interested in studying the Schrödinger equation (1.1) by preserving its Hamiltonian structure. From this perspective, it is natural to introduce the following unitary operator $h^\sharp(t)$ defined by

$$h^\sharp(t) \ : \ \phi \in L^2(\Omega(t), \mathbb{C}) \ \longmapsto \ \sqrt{|J(\cdot, t)|} \, (\phi \circ h)(t) \in L^2(\Omega_0, \mathbb{C}) \ . \tag{1.5}$$

We also denote by $h_\sharp(t)$ its inverse

$$h_\sharp(t) = (h^\sharp(t))^{-1} \ : \ \psi \mapsto \left(\psi / \sqrt{|J(\cdot, t)|} \right) \circ h^{-1} \ . \tag{1.6}$$

Notice that the relation $\|h^\sharp(t) u\|_{L^2(\Omega_0)} = \|u\|_{L^2(\Omega(t))}$ enables to transport the conservative structure through the change of variables. A direct computation, provided in Section 3.1, shows that u solves (1.1)

if and only if $v = h^\sharp u$ is solution of

$$i\partial_t v(t,y) = -\frac{1}{\sqrt{|J|}}\operatorname{div}_y\left(|J|J^{-1}(J^{-1})^t\nabla_y\left(\frac{v(t,y)}{\sqrt{|J|}}\right)\right)$$

$$+\frac{i}{2}\frac{\partial_t\big(|J(t,y)|\big)}{|J|}v + i\sqrt{|J|}\langle J^{-1}\partial_t h(t,y)|\nabla_y\frac{v(t,y)}{\sqrt{|J|}}\rangle, \qquad t\in I\ ,\ \ y\in\Omega_0. \qquad (1.7)$$

Written as it stands, this equation is not easy to handle. For instance, it is unclear whether the equation is of Hamiltonian type and how to compute its spectrum. The central argument of this paper is to show that Equation (1.7) can be rewritten in the form

$$i\partial_t v(t,y) = -h^\sharp\Big[\big(\operatorname{div}_x +iA_h\big)\circ\big(\nabla_x + iA_h\big) + |A_h|^2\Big]h_\sharp v(t,y), \qquad t\in I,\ \ y\in\Omega_0, \qquad (1.8)$$

with $A_h(t,x) = -\frac{1}{2}(h_*\partial_t h)(t,x)$. Now, the operator appearing in the last equation is the conjugate with respect to $h^\sharp$ and $h_\sharp$ of an explicit magnetic Laplacian. Thus, its Hamiltonian structure becomes obvious and some of its properties, as the spectrum, may be easier to study. We refer to [10, 20, 21, 29, 30, 46] for different spectral results on magnetic Laplacian operators.

Using the unitary operator $h^\sharp$ rather than h^* is natural and it was already done in the literature in some very specific frameworks in [3, 5, 43]. The same idea was also adopted to study quantum waveguides in the time-independent framework, where the magnetic field A_h does not appear, see for instance [19, 26]. In [22, 23], the transformation h^* is used on manifolds with time-varying metrics. The authors assume h preserving the volumes which yields $h^* = h^\sharp$. They obtain an operator involving a magnetic field similar to the one in (1.8). From this perspective, the relation between motion and magnetic field is not surprising. Physically, the momentum $p = mv$ of a moving particle of mass m, velocity v and charge q in a magnetic field A must be replaced by $\widetilde{p} = mv + qA$. This corresponds to the magnetic field appearing in the equation (1.8) for the Galilean frames. An explicit example of the link between motion and magnetic field in our results can be seen in the boundary condition on a moving surface as in Figure 2 below. This is also related to the notion of "anti-convective derivative" of Henry, see [31].

The Dirichlet boundary condition

Once Equation (1.8) is obtained, the Cauchy problem is easy to handle by using classical results of existence of unitary flows generated by time-dependent Hamiltonians. In our work, we refer to Theorem A.1 presented in the appendix. In the case of the simple Laplacian operator with Dirichlet boundary condition, we obtain our first main result which states the following.

Theorem 1.1. *Let $I \subset \mathbb{R}$ be an interval of times and let $\{\Omega(t)\}_{t\in I} \subset \mathbb{R}^N$ with $N \in \mathbb{N}^*$ be a family of domains. Assume that there exist a bounded reference domain Ω_0 in $\mathbb{R}^N$ and a family of diffeomorphisms $(h(t,\cdot))_{t\in I} \in \mathcal{C}^2(I\times\overline{\Omega}_0,\mathbb{R}^N)$ such that $h(t,\overline{\Omega}_0) = \overline{\Omega}(t)$.*

Then, Equation (1.8) endowed with Dirichlet boundary conditions generates a unitary flow $\tilde{U}(t,s)$ on $L^2(\Omega_0)$ and we may define weak solutions of the Schrödinger equation

$$\begin{cases} i\partial_t u(t,x) = -\Delta_x u(t,x), & t\in I\ ,\ \ x\in\Omega(t) \\ u_{|\partial\Omega(t)} \equiv 0 \end{cases} \qquad (1.9)$$

by transporting this flow via $h_\sharp$ to a unitary flow $U(t,s): L^2(\Omega(s)) \to L^2(\Omega(t))$.

Assume in addition that the diffeomorphisms h are of class $\mathcal{C}^3$ with respect to the time and the space variable. Then, for any $u_0 \in H^2(\Omega(t_0)) \cap H^1_0(\Omega(t_0))$, the above flow defines a solution $u(t) = U(t,t_0)u_0$ in $\mathcal{C}^0(I, H^2(\Omega(t))\cap H^1_0(\Omega(t))) \cap \mathcal{C}^1(I, L^2(\Omega(t)))$ solving (1.9) in the L^2-sense.

Theorem 1.1 is consequence of the stronger result of Theorem 3.1 (Section 3.1) where we also include diffusion coefficients as well as magnetic and electric potentials. However, in this introduction, we consider the case of the free Laplacian to simplify the notations and to avoid further technicalities

Also notice that the above result does not require the reference domain Ω_0 to have any regularity. In particular, it may have corners such as a rectangle for example. Of course, all the domains $\Omega(t)$ have to be diffeomorphic to Ω_0. Hence, we cannot create singular perturbations such as adding or removing corners and holes. Nevertheless, $\Omega(t)$ may typically be a family of rectangles or cylinders with different proportions.

The magnetic Neumann condition

At first sight, one may naturally think to associate to the Schrödinger Equation (1.1) with the homogeneous Neumann boundary conditions $\partial_\nu u(t,x) = 0$ where $x \in \partial\Omega(t)$ and ν is the unit outward normal of $\partial\Omega(t)$. However, these conditions cannot generate a unitary evolution, which is problematic for quantum dynamics. Simply consider the solution $u \equiv 1$, whose norm depends on the volume of $\Omega(t)$. Even when the volume of $\Omega(t)$ is constant, if u solves (1.1) with homogeneous Neumann boundary conditions, then the computation (1.12) below shows that the evolution cannot be unitary, except when $h_*\partial_t h$ is tangent to $\partial\Omega(t)$ at all the boundary points, meaning that the shape of $\Omega(t)$ is in fact unchanged.

From this perspective, another interesting aspect of our result appears. The expression of Equation (1.8) indicates that the correct boundary conditions to consider are the ones associated with Neumann realization of the magnetic Laplacian operator, that are

$$\partial_\nu(h_\sharp v) + i\langle \nu | A_h \rangle (h_\sharp v) = 0 \quad \text{on } \partial\Omega(t). \tag{1.10}$$

If we denote by ν_0 the unit outward normal of $\partial\Omega_0$, then the last identity can be transposed in

$$\left\langle (J^{-1})^t \nu_0 \Big| (J^{-1})^t \sqrt{|J|} \nabla_y \big(\frac{v}{\sqrt{|J|}} \big) - \frac{i}{2}(\partial_t h)v \right\rangle = 0 \quad \text{on } \partial\Omega_0 \tag{1.11}$$

(see Remark 3.3 for further details on the computations). Even though the conditions seems complicated on Ω_0, they simply write as the classical magnetic Neumann boundary conditions for the original problem in $\Omega(t)$, see (1.13) below. In particular, they exactly correspond to the ones of a planar wave bouncing off the moving surface, as it is clear in the example of Figure 2. We also notice that they perfectly match with the preservation of the energy since if $u(t)$ solves (1.1) at least formally, then

$$\partial_t \int_{\Omega(t)} |u(t,x)|^2 \, \mathrm{d}x = \int_{\partial\Omega(t)} \langle \nu | h_*\partial_t h \rangle |u(t,x)|^2 \, \mathrm{d}x + 2\Re \int_{\Omega(t)} \partial_t u(t,x)\overline{u}(t,x) \, \mathrm{d}x$$

$$= \int_{\partial\Omega(t)} \langle \nu | h_*\partial_t h \rangle |u(t,x)|^2 \, \mathrm{d}x + 2\Re \left(i \int_{\partial\Omega(t)} \partial_\nu u(t,x)\overline{u}(t,x) \, \mathrm{d}x \right). \tag{1.12}$$

Once the correct boundary condition is inferred, we obtain the following result in the same way as the Dirichlet case.

Theorem 1.2. *Let $I \subset \mathbb{R}$ be an interval of times and let $\{\Omega(t)\}_{t\in I} \subset \mathbb{R}^N$ with $N \in \mathbb{N}^*$ be a family of domains. Assume that there exist a bounded reference domain Ω_0 in $\mathbb{R}^N$ of class $\mathcal{C}^1$ and a family of diffeomorphisms $(h(t,\cdot))_{t\in I} \in \mathcal{C}^2(I \times \overline{\Omega}_0, \mathbb{R}^N)$ such that $h(t,\overline{\Omega}_0) = \overline{\Omega}(t)$.*

Then, Equation (1.8) endowed with the magnetic Neumann boundary conditions (1.10) (or equivalently (1.11)) generates a unitary flow $\tilde{U}(t,s)$ on $L^2(\Omega_0)$ and we may define weak solutions of the Schrödinger equation

$$\begin{cases} i\partial_t u(t,x) = -\Delta_x u(t,x), & t \in I, \quad x \in \Omega(t) \\ \partial_\nu u(t,x) - \frac{i}{2}\langle \nu | h_*\partial_t h(t,x) \rangle u(t,x) = 0, & t \in I, \quad x \in \partial\Omega(t) \end{cases} \tag{1.13}$$

by transporting this flow via $h_\sharp$ to a unitary flow $U(t,s) : L^2(\Omega(s)) \to L^2(\Omega(t))$.

Assume in addition that the diffeomorphisms h are of class $\mathcal{C}^3$ with respect to the time and the space variable. Then, for any $u_0 \in H^2(\Omega(t_0))$ satisfying the magnetic Neumann boundary condition of (1.13), the above flow defines a solution $u(t) = U(t,t_0)u_0$ in $\mathcal{C}^0(I, H^2(\Omega(t))) \cap \mathcal{C}^1(I, L^2(\Omega(t)))$ solving (1.13) in the L^2-sense and satisfying the magnetic Neumann boundary condition.

Gauge transformation

As it is well known, the magnetic potential has a gauge invariance. In particular, for any ϕ of class $\mathcal{C}^1$ in space, we have

$$e^{-i\phi(x)} \big[(\nabla_x + iA_h)^2 \big] e^{i\phi(x)} = \big(\nabla_x + i(A_h + \nabla_x \phi) \big)^2. \tag{1.14}$$

Thus, it is possible to delete the magnetic term $A_h = -\frac{1}{2}h_*\partial_t h$ by the change of gauge when there exists ϕ of class $\mathcal{C}^1$ such that

$$\forall t \in I, \quad \forall x \in \Omega(t), \quad (h_*\partial_t h)(t,x) = 2\nabla_x \phi(t,x).$$

In such context, the well-posedness of the equations (1.9) and (1.13) can be investigated by considering

$$w(t,y) = h^\sharp e^{-i\phi(t,x)} u := \sqrt{|J(t,y)|} e^{-i\phi(t,h(t,y))} u(t,h(t,y)).$$
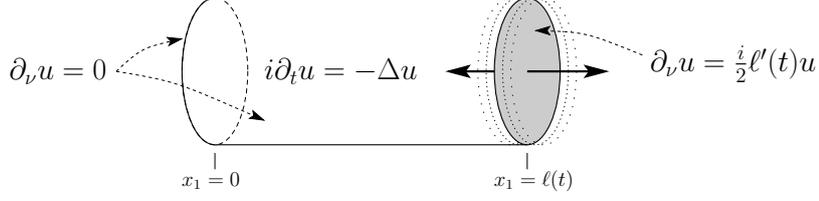
Figure 2: *The correct Neumann boundary conditions for the Schrödinger equation in a cylinder with a moving end. Notice the magnetic Neumann boundary condition at the moving surface, even though the equation has no magnetic term. See Section 5 for further details on the computations.*

and by studying the solution of the following equation endowed with the corresponding boundary conditions

$$i\partial_t w(t,y) = -\Big(h^\sharp \Delta_x h_\sharp + \frac{1}{4}|\partial_t h(t,y)|^2 - \partial_t (h^* \phi)(t,y)\Big) w(t,y), \quad t \in I, \quad y \in \Omega_0. \qquad (1.15)$$

The gauge transformation, not only simplifies Equation (1.8), but also yields a gain of regularity in the hypotheses on h adopted in the theorems 1.1 and 1.2. This fact follows as the main part of the new Hamiltonian in (1.15) does not contain $\partial_t h$ anymore. In details, if we consider $h \in \mathcal{C}_t^1(I, \mathcal{C}_x^2(\Omega_0, \mathbb{R}^N))$, then the existence of weak solutions of (1.9) and (1.13) can be guaranteed when ϕ is of class $\mathcal{C}^3$ in space and $W^{1,\infty}$ in time. The existence of strong solutions, instead, holds when ϕ is of class $\mathcal{C}^4$ in space and $W^{1,\infty}$ in time.

Also remark that the gauge transformation is not always possible to use. For example, if $\Omega(t)$ is a rotation of a square, $\partial_t h$ is not curl-free and cannot be rectified due to the presence of corners at which $h(t,y)$ is imposed (corners have to be send onto corners). Finally, we may also use the simpler gauge of the electric potential if some of the terms of (1.15) are constant, see Section 5.

Moser's trick

Another way to simplify Equation (1.8) is to use a family of diffeomorphisms $\tilde{h}(t)$ such that the determinant of the Jacobian is independent of y. In other words, when $\tilde{J} = D_y \tilde{h}$ satisfies the following identity

$$\forall t \in I, \quad \forall y \in \Omega_0, \quad |\tilde{J}(t,y)| := a(t). \qquad (1.16)$$

In this case, the multiplication for the Jacobian J of commutes with the spatial derivatives and then, Equations (1.7) and (1.8) can be simplified in the following expression, for $A_{\tilde{h}} = -\frac{1}{2}\tilde{h}_* \partial_t \tilde{h}$,

$$i\partial_t v(t,y) = -\operatorname{div}_y(\tilde{J}^{-1}(\tilde{J}^{-1})^t \nabla_y v) + \frac{i}{2}\frac{a'(t)}{a(t)}v + i\langle \tilde{J}^{-1}\partial_t \tilde{h}|\nabla_y v\rangle$$

$$= -\tilde{h}^* \big[(\operatorname{div}_x + iA_{\tilde{h}}) \circ (\nabla_x + iA_{\tilde{h}}) + |A_{\tilde{h}}|^2\big]\tilde{h}_* v(t,y). \qquad (1.17)$$

This strategy can be used, not only to simplify the equations, but also to gain some regularity since $|\tilde{J}|$ is now constant and thus smooth. Therefore, $\tilde{h}^\sharp$ maps $H^k(\Omega(t))$ into $H^k(\Omega_0)$ as soon as $\tilde{h}$ is of class $\mathcal{C}^k$ in space.

For t fixed, finding a diffeomorphism $\tilde{h}$ satisfying the identity (1.16) follows from a very famous work of Moser [42]. This kind of result called "Moser's trick" was widely studied in literature even in the case of moving domains (see Section 4.1). Nevertheless, most of these outcomes are not interested in studying the optimal time and space regularity as well as their proofs are sometimes simply outlined. For this purpose, in Section 4, we prove the following result by following the arguments of [16].

Theorem 1.3. *Let $k \geq 1$, and $r \in \mathbb{N}$ with $k \geq r \geq 0$. Let $\alpha \in (0,1)$ and let $\Omega_0 \subset \mathbb{R}^N$ be a connected bounded domain of class $\mathcal{C}^{k+2,\alpha}$. Let $I \subset \mathbb{R}$ an interval of times and assume that there exists a family $(\Omega(t))_{t\in I}$ of domains such that there exists a family $(h(t))_{t\in I}$ of diffeomorphisms*

$$h : y \in \overline{\Omega}_0 \longrightarrow h(t,y) \in \overline{\Omega(t)}$$

which are of class $\mathcal{C}^{k,\alpha}(\overline{\Omega}_0, \overline{\Omega(t)})$ with respect to y and of class $\mathcal{C}^r(\overline{\Omega}_0, \mathbb{R}^N)$ with respect to t.

Then, there exists a family $(\tilde{h}(t))_{t\in I}$ of diffeomorphisms from $\overline{\Omega}_0$ onto $\overline{\Omega(t)}$, with the same regularity as h, and such that $\det(D_y \tilde{h}(t))$ is constant with respect to y, that is that

$$\forall y \in \Omega_0, \quad \det(D_y \tilde{h}(t,y)) = \frac{\operatorname{meas}(\Omega(t))}{\operatorname{meas}(\Omega_0)} .$$

5

Even though results similar to Theorem 1.3 have already been stated, the fact that $h(t)$ may simply be continuous with respect to the time and that $\tilde{h}(t)$ has the same regularity of $h(t)$ seems to be new. There are some simple cases where we can define explicit $\tilde{h}$ as in dimension $N = 1$ or in the examples of Section 5, but this is not aways possible. In these last situations, Equation (1.17) may be difficult to use as $\tilde{h}$ is not explicit. However, Equation (1.17) yields a gain of regularity in the Cauchy problem which we resume in the following corollary.

Corollary 1.4. *Let $I \subset \mathbb{R}$ be an interval of times and let $\{\Omega(t)\}_{t \in I} \subset \mathbb{R}^N$ with $N \in \mathbb{N}^*$ be a family of domains. Assume that there exist a bounded reference domain Ω_0 in $\mathbb{R}^N$ of class $\mathcal{C}^{4,\alpha}$ with $\alpha \in (0,1)$ and a family of diffeomorphisms $(h(t, \cdot))_{t \in I} \in \mathcal{C}_t^2(I, \mathcal{C}_x^1(\Omega_0, \mathbb{R}^N))$ such that $h(t, \overline{\Omega}_0) = \overline{\Omega}(t)$.*

Then, we may define as in Theorems 1.1 and 1.2 weak solutions of the above equations (1.9) and (1.13) by considering $v = \tilde{h}_\sharp u$, with $\tilde{h}$ as in Theorem 1.3, which solves the Schrödinger equation (1.17) with the corresponding boundary conditions and by transporting the flow of (1.17) via the above change of variable.

Assume in addition that h belongs to $\mathcal{C}_t^3(I, \mathcal{C}_y^2(\Omega_0, \mathbb{R}^N))$. Then, for any $u_0 \in H^2(\Omega(t_0)) \cap H_0^1(\Omega(t_0))$, the above flow defines a solution $u(t)$ in the space $\mathcal{C}^0(I, H^2(\Omega(t)) \cap H_0^1(\Omega(t))) \cap \mathcal{C}^1(I, L^2(\Omega(t)))$ solving (1.9) or (1.13) in the L^2−sense.

Corollary 1.4 follows from the same arguments leading to the theorems 1.1 or 1.2 (see Section 3). The only difference is the gain of regularity in space. Indeed, the term $|J(t, y)|$ appearing in $h^\sharp$ or $h_\sharp$ is replaced by $|\tilde{J}(t, y)| = \frac{\text{meas}(\Omega(t))}{\text{meas}(\Omega_0)} = a(t)$ which is constant in space and then smooth. To this end, we simply have to replace the first family of diffeomorphisms $h(t)$ by the one given by Theorem 1.3.

Notice that, in Corollary 1.4, we have to assume that the reference domain Ω_0 is smooth. If $\Omega(t)$ are simply of class $\mathcal{C}^1$ or $\mathcal{C}^2$, this is not a real restriction since we may choose a smooth reference domain and a not so smooth diffeomorphism h. In the case where $\Omega(t)$ has corner, as rectangles for example, then Corollary 1.4 do not formally apply. However, in the case of moving rectangles, finding a family of explicit diffeomorphisms $h(t)$ satisfying (1.16) is easy and the arguments behind Corollary 1.4 can be directly used, see the computations of Section 5.

An example of application: an adiabatic result

Consider a family of domains $\{\Omega(\tau)\}_{\tau \in [0,1]}$ of $\mathbb{R}^N$ such that there exist a bounded reference domain $\Omega_0 \subset \mathbb{R}^N$ and a family of diffeomorphisms $(h(\tau, \cdot))_{\tau \in [0,1]} \in \mathcal{C}^3([0,1] \times \overline{\Omega}_0, \mathbb{R}^N)$ such that $h(\tau, \overline{\Omega}_0) = \overline{\Omega}(\tau)$. Assume that for each $\tau \in [0,1]$, the Dirichlet Laplacian operator Δ on $\Omega(\tau)$ has a simple isolated eigenvalue $\lambda(\tau)$ with normalized eigenfunction $\varphi(\tau)$, associated with a spectral projector $P(\tau)$, all three depending continuously on τ. Following the classical adiabatic principle, we expect that if we start with a quantum state in $\Omega(0)$ close to $\varphi(0)$ and we deform very slowly the domain to the shape $\Omega(1)$, then the final quantum state should be close to $\varphi(1)$ up to a phase shift (see Figure 3). The slowness of the deformation is represented by a parameter $\epsilon > 0$ and we consider deformations between the times $t = 0$ and $t = 1/\epsilon$, that is the following Schrödinger equation

$$
\begin{cases}
i\partial_t u_\epsilon(t, x) = -\Delta u_\epsilon(t, x), & t \in [0, 1/\epsilon] \,, \quad x \in \Omega(\epsilon t) \\
u_\epsilon(t) \equiv 0, & \text{on } \partial\Omega(\epsilon t) \\
u_\epsilon(t = 0) = u_0 \in L^2(\Omega(0))
\end{cases}
\tag{1.18}
$$

Due to Theorem 1.1, we know how to define a solution of (1.18). Our main equation (1.8) provides the Hamiltonian structure required for applying the adiabatic theory, in contrast to the case of Equation (1.4). It also indicates that we should not consider the Laplacian operators Δ and its spectrum, but rather the magnetic Laplacian operators of Equation (1.8). With these hints, it is not difficult to adapt the classical adiabatic methods to obtain the following result (see Section 5).

Corollary 1.5. *Consider the above framework. Then, we have*

$$
\langle P(1) u_\epsilon(1/\epsilon) | u_\epsilon(1/\epsilon) \rangle \xrightarrow[\epsilon \longrightarrow 0]{} \langle P(0) u_0 | u_0 \rangle \,.
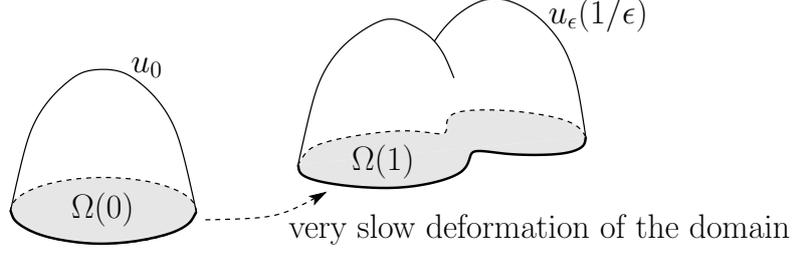$$

Figure 3: *The ground state of a domain $\Omega(0)$ can be transformed into the ground state of a domain $\Omega(1)$ if we slowly and smoothly deform $\Omega(0)$ to $\Omega(1)$ while the quantum state evolves following Schrödinger equation.*

2 Moving domains and change of variables

The study of the influence of the domain shape on a PDE problem has a long history, in particular in shape optimization. The interested reader may consider [31] or [44] for example. The basic strategy of these kinds of works is to bring back the problem in a fixed reference domain Ω_0 via diffeomorphisms. We thus have to compute the new differential operators in the new coordinates. The formulae presented in this section are well known (see [31] for example). We recall them for sake of completeness and also to fix the notations.

Let Ω_0 be a reference domain and let $h(t)$ be a $\mathcal{C}^2-$diffeorphism mapping Ω_0 onto $\Omega(t)$ (see Figure 1). In this section, we only work with fixed t. For this reason, we omit the time dependence when it is not necessary and forgot the question of regularity with respect to the time.

From now on, we denote by x the points in $\Omega(t)$ and by y the ones in Ω_0. For any matrix A, $|A|$ stands for $|\det(A)|$ and $\langle \cdot | \cdot \rangle$ is the scalar product in $\mathbb{C}^N$ with the convention

$$\langle v | w \rangle = \sum_{k=1}^{N} \overline{v_k}\, w_k \ , \qquad \forall v, w \in \mathbb{C}^N.$$

We use the pullback and pushforward operators defined by (1.2) and (1.3). We denote by $J = J(t,y) = D_y h(t,y)$ the Jacobian matrix of h. The basic rules of differential calculus give that

$$h^* D_x(h^{-1}) = (D_y h)^{-1} := J^{-1} \ , \qquad |J^{-1}| = |J|^{-1} \tag{2.1}$$

and

$$\int_{\Omega(t)} f(x)\,\mathrm{d}x \;=\; \int_{\Omega_0} |J(t,y)|(h^* f)(y)\,\mathrm{d}y \ . \tag{2.2}$$

The result below is due to the chain rule which implies the following identity

$$\partial_{y_j}(h^* f) = \langle \partial_{y_j} h | h^* \nabla_x f \rangle. \tag{2.3}$$

Proposition 2.1. *For every $g \in H^1(\Omega_0, \mathbb{C})$,*

$$(h^* \nabla_x h_*)g(y) = \left(J(t,y)^{-1} \right)^t \cdot \nabla_y g(y)$$

where $\left(J^{-1} \right)^t$ is the transposed matrix of $(D_y h)^{-1}$.

Proof: We apply (2.3) with $f = h_* g$ in order to obtain the identity $\nabla_y g = J(t,y)^t h^*(\nabla_x f)$. $\qquad \square$

By duality, we obtain the corresponding property for the divergence.

Proposition 2.2. *For every vector field $A \in H^1(\Omega_0, \mathbb{C}^N)$,*

$$(h^* \operatorname{div}_x h_*)A(y) = \frac{1}{|J(t,y)|} \operatorname{div}_y \left(|J(t,y)| J(y,t)^{-1} A(y) \right).$$

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Let ν and σ respectively be the unit outward normal and the measure on the boundary of $\Omega(t)$. Let ν_0 and σ_0 respectively be the unit outward normal and the measure on the boundary of Ω_0. For every $A \in H^1(\Omega_0, \mathbb{C}^N)$ and $g \in H^1(\Omega_0, \mathbb{C})$,

$$\int_{\partial\Omega(t)} \langle \nu | (h_* A)(x) \rangle (h_* g)(x) \, d\sigma = \int_{\partial\Omega_0} |J(t,y)| \langle \nu_0 | J(t,y)^{-1} A(y) \rangle g(y) \, d\sigma_0.$$

Proof: Let $\varphi \in \mathcal{C}_0^\infty(\Omega_0, \mathbb{C})$ be a test function and let $A \in H^1(\Omega_0, \mathbb{C}^N)$. Applying the divergence theorem and the above formulas, we obtain

$$
\begin{aligned}
\int_{\Omega_0} |J(t,y)| (h^* \operatorname{div}_x h_* A)(y) \varphi(y) \, dy &= \int_{\Omega(t)} (\operatorname{div}_x(h_* A))(x)(h_*\varphi)(x) \, dx \\
&= -\int_{\Omega(t)} \langle h_* \overline{A}(x) | \nabla_x(h_*\varphi)(x) \rangle \, dx \\
&= -\int_{\Omega_0} |J(y,t)| \langle \overline{A}(y) | h^*(\nabla_x h_* \varphi)(y) \rangle \, dy \\
&= -\int_{\Omega_0} |J(y,t)| \langle \overline{A}(y) | (J(t,y)^{-1})^t \nabla_y \varphi(y) \rangle \, dy \\
&= -\int_{\Omega_0} \langle |J(y,t)| J(t,y)^{-1} \overline{A}(y) | \nabla_y \varphi(y) \rangle \, dy \\
&= \int_{\Omega_0} \operatorname{div}_y \left[|J(y,t)| J(t,y)^{-1} A(y) \right] \varphi(y) \, dy.
\end{aligned}
$$

The first statement follows from the density of $\mathcal{C}_0^\infty(\Omega_0, \mathbb{C})$ in $L^2(\Omega_0)$. The second one instead is proved by considering the border terms appearing when we proceed as above with $A \in H^1(\Omega(t), \mathbb{C}^N)$ and $g \in H^1(\Omega(t), \mathbb{C})$. In this context, we use twice the divergence theorem and we obtain that

$$
\begin{aligned}
\int_{\Omega_0} |J(t,y)| (h^* \operatorname{div}_x h_* A)(y) g(y) \, dy &= \int_{\partial\Omega(t)} \langle \nu | (h_* A)(x) \rangle (h_* g)(x) \, d\sigma \\
&\quad - \int_{\partial\Omega_0} |J(t,y)| \langle \nu_0 | J(t,y)^{-1} A(y) \rangle g(y) \, d\sigma_0 \\
&\quad + \int_{\Omega_0} \operatorname{div}_y \left(|J(y,t)| J(t,y)^{-1} A(y) \right) g(y) \, dy.
\end{aligned}
$$

The last relation yields the equality of the boundary terms since the first statement is valid in the L^2 sense. $\qquad\square$

By a direct application of the previous propositions, we obtain the operator associated with $\Delta_x = \operatorname{div}_x(\nabla_x \cdot)$.

Proposition 2.3. *For every $g \in H^2(\Omega_0, \mathbb{C})$,*

$$(h^* \Delta_x h_*) g(y) = \frac{1}{|J(t,y)|} \operatorname{div}_y \left(|J(t,y)| J(t,y)^{-1} (J(t,y)^{-1})^t \nabla_y g(y) \right).$$

Remark 2.4. *For an illustration, let us check, at least formally, that $u(t,x)$ solves the Schrödinger equation (1.1) in $\Omega(t)$ if and only if $w = h^* u$ solves (1.4). We have*

$$
\begin{aligned}
i\partial_t w(t,y) &= i\partial_t \big(u(t, h(t,y)) \big) \\
&= (i\partial_t u)(t, h(t,y)) + i\langle \partial_t h(t,y) | (\nabla_x u)(t, h(t,y)) \rangle \\
&= (-\Delta_x u)(t, h(t,y)) + i\langle \partial_t h(t,y) | (\nabla_x u)(t, h(t,y)) \rangle \\
&= -(h^* \Delta_x h_* w)(t,y) + i\langle \partial_t h(t,y) | (h^* \nabla_x h_* w)(t,y) \rangle \\
&= -\frac{1}{|J|} \operatorname{div}_y \left(|J| J^{-1} (J^{-1})^t \nabla_y g(t,y) \right) + i\langle \partial_t h(t,y) | (J^{-1})^t \nabla_y w(t,y) \rangle \\
&= -\frac{1}{|J|} \operatorname{div}_y \left(|J| J^{-1} (J^{-1})^t \nabla_y g(t,y) \right) + i\langle J^{-1} \partial_t h(t,y) | \nabla_y w(t,y) \rangle.
\end{aligned}
$$

3 Main results: The Cauchy problem

3.1 Proof of Theorem 1.1: the Dirichlet case

Let I be a time interval and let $\Omega(t) = h(t, \Omega_0)$ be a family of moving domains. In this section, we consider a second order Hamiltonian operator of the type

$$H(t) = -\Big[D(t,x)\nabla_x + iA(t,x)\Big]^2 + V(t,x) \tag{3.1}$$

where

$$\Big[D(t,x)\nabla_x + iA(t,x)\Big]^2 := \Big(\operatorname{div}_x\big(D(t,x)^t \cdot\big), + i\langle A(t,x)| \cdot \rangle\Big) \circ \Big(D(t,x)\nabla_x + iA(t,x)\Big) \tag{3.2}$$

and

- $D \in \mathcal{C}_t^2(I, \mathcal{C}_x^1(\mathbb{R}^N, \mathcal{M}_N(\mathbb{R})))$ are symmetric diffusions coefficients such that there exists $\alpha > 0$ satisfying
  $$\forall t \in I, \ \forall x \in \mathbb{R}^N, \ \forall \xi \in \mathbb{C}^N, \ \langle D(x,t)\xi | D(x,t)\xi\rangle \geq \alpha |\xi|^2 \ ;$$

- $A \in \mathcal{C}_t^2(I, \mathcal{C}_x^1(\mathbb{R}^N, \mathbb{R}^N))$ is a magnetic potential;

- $V \in \mathcal{C}_t^2(I, L_x^\infty(\mathbb{R}^N, \mathbb{R}))$ is an electric potential.

In this subsection, we assume that $H(t)$ is associated with Dirichlet boundary conditions and then its domain is $H^2(\Omega(t), \mathbb{C}) \cap H_0^1(\Omega(t), \mathbb{C})$. Of course, the typical example of such Hamiltonian is the Dirichlet Laplacian $H(t) = \Delta_x$ defined on $H^2(\Omega(t), \mathbb{C}) \cap H_0^1(\Omega(t), \mathbb{C})$. We consider the equation

$$\begin{cases} i\partial_t u(t,x) = H(t)u(t,x), & t \in I, \ x \in \Omega(t) \\ u_{|\partial\Omega(t)} \equiv 0 \end{cases} \tag{3.3}$$

We use the pullback and pushforward operators $h^\sharp$ and $h_\sharp$ defined by (1.5) and (1.6). We also use the notations of Section 2. Notice that $h^\sharp$ is an isometry from $L^2(\Omega(t))$ onto $L^2(\Omega_0)$. Moreover, if h is a class $\mathcal{C}^2$ with respect to the space, then $h^\sharp$ is an isomorphism from $H_0^1(\Omega(t))$ onto $H_0^1(\Omega_0)$. We set

$$v(t,y) := (h^\sharp u)(t,y) = \sqrt{|J(t,y)|}\, u(t, h(t,y)) \ . \tag{3.4}$$

At least formally, if u is solution of (3.3), then a direct computation shows that

$$i\partial_t v(t,y) = h^\sharp(i\partial_t u)(t,y) + \frac{i}{2}\frac{\partial_t\big(|J(t,y)|\big)}{\sqrt{|J(t,y)|}} u(t,y) + i\sqrt{|J(t,y)|}\langle \partial_t h(t,y) | h^*(\nabla_x u)\rangle$$

$$= \big(h^\sharp H(t) h_\sharp\big)v(t,y) + \frac{i}{2}\frac{\partial_t\big(|J(t,y)|\big)}{|J(t,y)|} v(t,y) + i\langle \partial_t h(t,y) | (h^\sharp \nabla_x h_\sharp)v(t,y)\rangle$$

$$= \big(h^\sharp H(t) h_\sharp\big)v(t,y) + \frac{i}{2}\frac{\partial_t\big(|J(t,y)|\big)}{|J(t,y)|} v(t,y) + h^\sharp\Big[i\langle (h_*\partial_t h)(t,x) | \nabla_x \cdot \rangle\Big] h_\sharp v(t,y). \tag{3.5}$$

The first operator $(h^\sharp H(t) h_\sharp)$ is a simple transport of the original operator $H(t)$ and it is clearly self-adjoint. Both other terms from the last relation come instead from the time derivative of $h^\sharp$. Since $h^\sharp$ is unitary, we expect that, their sum is also a self-adjoint operator. Notice that the last term may be expressed explicitly by Proposition 2.1 to obtain Equation (1.7). However, we would like to keep the conjugated form to obtain Equation (1.8). Due to Proposition A.2 in the appendix and (2.1), we have

$$\frac{\partial_t\big(|J(t,y)|\big)}{|J(t,y)|} = \operatorname{Tr}\Big(J(t,y)^{-1} \cdot \partial_t J(t,y)\Big) = \operatorname{Tr}\Big(\partial_t J(t,y) \cdot J(t,y)^{-1}\Big)$$

$$= h^* \operatorname{Tr}\Big(\big(\partial_t D_y h\big) \circ h^{-1} \cdot (D_y h)^{-1} \circ h^{-1}\Big)$$

$$= h^* \operatorname{Tr}\Big(\big(D_y \partial_t h\big) \circ h^{-1} \cdot D_x(h^{-1})\Big) = h^* \operatorname{Tr}\Big(D_x\big((\partial_t h) \circ h^{-1}\big)\Big)$$

$$= h^* \operatorname{div}_x\big(h_*(\partial_t h)\big) \ .$$

Since $h^\sharp$ differs from h^* by a multiplication, the conjugacy of a multiplicative operation by either h^* or $h^\sharp$ gives the same result. We obtain that

$$\frac{i}{2} \frac{\partial_t\big(|J(t,y)|\big)}{|J(t,y)|} v(t,y) = \frac{i}{2}\Big[h^* \operatorname{div}_x \big(h_*(\partial_t h) \big) \Big] v = \frac{i}{2} h^* \Big[\operatorname{div}_x \big(h_*(\partial_t h) \big) h_* v \Big]$$
$$= \frac{i}{2} h^\sharp \Big[\operatorname{div}_x \big(h_*(\partial_t h) \big) h_\sharp v \Big]. \tag{3.6}$$

We combine the result of (3.6) and half of the last term of (3.5) by using the chain rule

$$\frac{i}{2} \operatorname{div}_x \big(h_*(\partial_t h) \big) u + i\langle (h_* \partial_t h) | \nabla_x u \rangle = \frac{i}{2} \operatorname{div}_x \Big(\big(h_*(\partial_t h) \big) u \Big) + \frac{i}{2} \langle (h_* \partial_t h) | \nabla_x u \rangle.$$

We finally obtain

$$i\partial_t v(t,y) \; = \; h^\sharp \Big[H(t) \cdot \; - \; i \operatorname{div}_x \big(A_h(t,x) \cdot \big) \; - \; i\langle A_h(t,x) | \nabla_x \cdot \rangle \Big] h_\sharp v(t,y) \tag{3.7}$$

where $A_h(t,x) = -\frac{1}{2}(h_*\partial_t h)(t,x)$ is a magnetic field generated by the change of referential. The whole operator of (3.7) can be seen as a modification of the magnetic and electric terms of $H(t)$. Indeed, using (3.1), the equation (3.7) becomes

$$\begin{cases} i\partial_t v(t,y) \; = \; \big(h^\sharp \tilde{H}_h(t) h_\sharp \big) v(t,y), & t \in I \;, \; y \in \Omega_0 \\ v_{|\partial \Omega_0} \equiv 0 \end{cases} \tag{3.8}$$

where, using the same standard notation of (3.2),

$$\tilde{H}_h(t) = -\Big[D(t,x)\nabla_x + i\tilde{A}_h(t,x) \Big]^2 + \tilde{V}_h(t,x) \tag{3.9}$$

with

$$\tilde{A}_h(t,x) = A(t,x) + \big(D(t,x)^{-1} \big)^t \cdot A_h(t,x) \;, \qquad A_h = -\frac{1}{2} h_*(\partial_t h),$$

$$\tilde{V}_h(t,x) = V(t,x) - \big| \big(D(t,x)^{-1} \big)^t \cdot A_h(t,x) \big|^2 - 2\big\langle \big(D(t,x)^{-1} \big)^t \cdot A_h(t,x) \big| A(t,x) \big\rangle \;.$$

Notice that, in the simplest case of $H(t)$ being the Dirichlet Laplacian operator, we obtain Equation (1.8) discussed in the introduction. Similar computations are provided in [22, 23].

It is now possible to prove the main result of this section which generalizes Theorem 1.1 of the introduction.

Theorem 3.1. *Assume that $\Omega_0 \subset \mathbb{R}^N$ is a bounded domain (possibly irregular). Let $I \subset \mathbb{R}$ be an interval of times. Assume that $h(t) : \overline{\Omega}_0 \longrightarrow \overline{\Omega}(t) := h(t, \overline{\Omega}_0)$ is a family of diffeomorphisms which is of class $\mathcal{C}^2$ with respect to the time and the space variable. Let $H(t)$ be an operator of the form (3.1) with domain $H^2(\Omega(t), \mathbb{C}) \cap H_0^1(\Omega(t), \mathbb{C})$.*

Then, Equation (3.8) generates a unitary flow $\tilde{U}(t,s)$ on $L^2(\Omega_0)$ and we may define weak solutions of the Schrödinger equation (3.3) on the moving domain $\Omega(t)$ by transporting this flow via $h_\sharp$, that is setting $U(t,s) = h_\sharp \tilde{U}(t,s) h^\sharp$.

Assume in addition that the diffeomorphisms h are of class $\mathcal{C}^3$ with respect to the time and the space variable. Then, for any $u_0 \in H^2(\Omega(t_0)) \cap H_0^1(\Omega(t_0))$, the above flow defines a solution $u(t) = U(t,t_0)u_0$ in $\mathcal{C}^0(I, H^2(\Omega(t)) \cap H_0^1(\Omega(t))) \cap \mathcal{C}^1(I, L^2(\Omega(t)))$ solving (3.3) in the L^2-sense.

Proof: Assume that I is compact, otherwise it is sufficient to cover I with compact intervals and to glue the unitary flows defined on each one of them. We simply apply Theorem A.1 in appendix. We notice $A_h = -\frac{1}{2} h_*(\partial_t h)$ is a bounded term in $\mathcal{C}_t^1(I, \mathcal{C}_x^2(\overline{\Omega}(t), \mathbb{R}))$. First, the assumptions on $H(t)$ and the regularity of the diffeomorphisms $h(t, \cdot)$ imply that $\tilde{H}_h(t)$ defined by (3.9) is a well defined self-adjoint operator on $L^2(\Omega(t))$ with domain $H^2(\Omega(t)) \cap H_0^1(\Omega(t))$. Second, it is of class $\mathcal{C}^1$ with respect to the time and, for every $u \in H^2 \cap H^1(\Omega(t))$,

$$\big\langle \tilde{H}_h(t)u | u \big\rangle_{L^2(\Omega(t))} = \int_{\Omega(t)} \big| (D(t,x)\nabla_x + i\tilde{A}_h) u(x) \big|^2 + \tilde{V}_h(t,x)|u(x)|^2 \, \mathrm{d}x$$

$$\geq \gamma \|u\|_{H^1(\Omega(t))}^2 - \kappa \|u\|_{L^2(\Omega(t))}^2$$

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for some $\gamma > 0$ and $\kappa \in \mathbb{R}$. Let $\mathcal{H}(t) = h^\sharp(t) \circ \tilde{H}_h(t) \circ h_\sharp(t)$. Since $h^\sharp$ is an isometry from $L^2(\Omega(t))$ onto $L^2(\Omega_0)$ continuously mapping $H^1(\Omega(t))$ into $H^1(\Omega_0)$, the above properties of $\tilde{H}_h(t)$ are also valid for $\mathcal{H}(t)$ because for every $v \in H^2(\Omega_0) \cap H^1_0(\Omega_0)$

$$\left\langle \mathcal{H}(t)v | v \right\rangle_{L^2(\Omega_0)} = \left\langle h^\sharp \tilde{H}_h(t) h_\sharp v | v \right\rangle_{L^2(\Omega_0)} = \left\langle \tilde{H}_h(t) h_\sharp v | h_\sharp v \right\rangle_{L^2(\Omega(t))}.$$

The only problem concerns the regularities. Indeed, the domain of $\mathcal{H}(t)$ is

$$D(\mathcal{H}(t)) = \{v \in L^2(\Omega_0) \ , \ h_\sharp(t)v \in H^2(\Omega(t)) \cap H^1_0(\Omega(t))\}$$

and if h is only $\mathcal{C}^2$ with respect to the space, then $D(\mathcal{H}(t))$ is not necessarily $H^2(\Omega_0) \cap H^1_0(\Omega_0)$ due to the presence of the Jacobian of h in the definition (1.6) of $h_\sharp$. However, if h is of class $\mathcal{C}^2$, then $h_\sharp$ transports the $\mathcal{C}^1-$regularity in space as well as the boundary condition and

$$D(\mathcal{H}(t)^{1/2}) = \{v \in L^2(\Omega_0) \ , \ h_\sharp(t)v \in H^1_0(\Omega(t))\} = H^1_0(\Omega_0)$$

does not depend on the time. We can apply Theorem A.1 in the appendix and obtain the unitary flow $\tilde{U}(t,s)$. The flow $U(t,s) = h_\sharp \tilde{U}(t,s) h^\sharp$ on $L^2(\Omega(t))$ is then well defined but corresponds to solutions of (3.3) only in a formal way.

Assume finally that the diffeomorphisms h are of class $\mathcal{C}^3$ with respect to the time and the space variable. Then, we have no more problems with the domains and

$$D(\mathcal{H}(t)) = \{v \in L^2(\Omega_0) \ , \ h_\sharp(t)v \in H^2(\Omega(t)) \cap H^1_0(\Omega(t))\} = H^2(\Omega_0) \cap H^1_0(\Omega_0)$$

which does not depend on the time. Since $H(t)$ is now of class $\mathcal{C}^2$ with respect to the time, we may apply the second part of Theorem A.1 and obtain strong solutions of the equation on Ω_0, which are transported to strong solutions of the equation on $\Omega(t)$. $\qquad \square$

3.2 Proof of Theorem 1.2: the magnetic Neumann case

In the previous section, the well-posedness of the dynamics of (3.8) is ensured by studying the self-adjoint operator $\tilde{H}_h(t)$ defined in (3.9) and with domain $H^2(\Omega(t)) \cap H^1_0(\Omega(t))$. This operator corresponds to the following quadratic form in $H^1_0(\Omega(t))$

$$q(\psi) := \int_{\Omega(t)} \left| D(t,x)\nabla\psi(x) + i\tilde{A}_h(t,x)\psi(x) \right|^2 \mathrm{d}x + \int_{\Omega(t)} \tilde{V}_h(t,x)|\psi(x)|^2 \, \mathrm{d}x, \qquad \forall \psi \in H^1_0(\Omega(t))$$

which is the Friedrichs extension of the quadratic form q defined on $\mathcal{C}_0^\infty(\Omega(t))$. Now, it is natural to consider the Friedrichs extension of q defined in $H^1(\Omega(t))$. This corresponds to the Neumann realization of the magnetic Laplacian $\tilde{H}_h(t)$ defined by (3.9) and with domain

$$D\big(\tilde{H}_h(t)\big) = \left\{ u \in H^2(\Omega(t), \mathbb{C}) \ : \ \left\langle \nu \, | \, D\nabla_x u + i\tilde{A}_h u \right\rangle(t,x) = 0, \ \ \forall x \in \partial\Omega(t) \right\} \tag{3.10}$$

(see [20] for example). Such as the well-posedness of the Schrödinger equation on moving domains can be achieve when it is endowed with Dirichlet boundary conditions, the same result can be addressed in this new framework. Indeed, the operators $\tilde{H}_h(t)$ endowed with the domain (3.10) are still self-adjoint and bounded from below. The arguments developed in the previous section lead to the well-posedness in Ω_0 of the equation

$$\begin{cases} i\partial_t v(t,y) = \big(h^\sharp \tilde{H}_h(t) h_\sharp\big) v(t,y), & t \in I \ , \ y \in \Omega_0 \\ \big(h^\sharp \left\langle \nu \, | \, D\nabla_x(h_\sharp v) + i\tilde{A}_h(h_\sharp v) \right\rangle\big)(t,y) = 0, & t \in I \ , \ x \in \partial\Omega_0 \end{cases} \tag{3.11}$$

Going back to the original moving domain $\Omega(t)$, (3.11) becomes

$$\begin{cases} i\partial_t u(t,x) = H(t)u(t,x), & t \in I \ , \ x \in \Omega(t) \\ \left\langle \nu \, | \, D\nabla_x u + i\tilde{A}_h u \right\rangle(t,x) = 0, & t \in I \ , \ x \in \partial\Omega(t). \end{cases} \tag{3.12}$$

It is noteworthy that in this case, the modified magnetic potential $\tilde{A}_h$ appears in the original equation. In particular, notice that the boundary condition of (3.12) is not the natural one associated with $H(t)$ for t fixed.

We resume in the following theorem the well-posedness of the dynamics when the Neumann magnetic boundary conditions are considered. The result is a generalization of Theorem 1.2 in the introduction.

Theorem 3.2. *Assume that $\Omega_0 \subset \mathbb{R}^N$ is a bounded domain (possibly irregular). Let $I \subset \mathbb{R}$ be an interval of times. Assume that $h(t) : \overline{\Omega}_0 \longrightarrow \overline{\Omega}(t) := h(t, \overline{\Omega}_0)$ is a family of diffeomorphisms which is of class $\mathcal{C}^2$ with respect to the time and the space variable. Let $H(t)$ be a of the form (3.1) and with domain (3.10).*

Then, Equation (3.11) generates a unitary flow $\tilde{U}(t, s)$ on $L^2(\Omega_0)$ and we may define weak solutions of the Schrödinger equation (3.12) on the moving domain $\Omega(t)$ by transporting this flow via $h_\sharp$, that is setting $U(t, s) = h_\sharp \tilde{U}(t, s) h^\sharp$.

Assume in addition that the diffeomorphisms h are of class $\mathcal{C}^3$ with respect to the time and the space variable. Then, for any $u_0 \in H^2(\Omega(t_0)) \cap H_0^1(\Omega(t_0))$, the above flow defines a solution $u(t) = U(t, t_0)u_0$ in $\mathcal{C}^0(I, H^2(\Omega(t)) \cap H_0^1(\Omega(t))) \cap \mathcal{C}^1(I, L^2(\Omega(t)))$ solving (3.12) in the L^2-sense.

Proof: *Mutatis mutandis*, the proof is the same as the one of Theorem 3.1. $\qquad\square$

Remark 3.3. *In (3.11), the boundary conditions satisfied by v are not explicitly stated in terms of v but rather in terms of $u = h_\sharp v$. Even if it not necessary for proving Theorem 3.2, it could be interesting to compute them completely. The second statement of Proposition 2.2 implies*

$$|J| \langle \nu_0 \, | \, J^{-1} \cdot h^*(D \cdot \nabla_x(h_\sharp v)) + i\big(J^{-1} \cdot h^*(\tilde{A}_h(h_\sharp v))\big) \rangle = 0 \quad on \; \partial\Omega_0 \;.$$

Now, $h_\sharp v = h_* \frac{1}{\sqrt{|J|}} v$. Thanks to Proposition 2.1, the fact that $|J|$ never vanishes yields

$$\Big\langle (J^{-1})^t \nu_0 \Big| (h^* D) \cdot (J^{-1})^t \cdot \sqrt{|J|} \nabla_y\big(\frac{v}{\sqrt{J}}\big) + i(h^* \tilde{A}_h)v \Big\rangle = 0 \quad on \; \partial\Omega_0 \;.$$

When $\tilde{A}_h$ corresponds to $A_h = -\frac{1}{2} h_*(\partial_t h)$ and $D(t, x) = Id$, we obtain the boundary condition presented in (1.11).

4 Moser's trick

The aim of this section is to prove the following result.

Theorem 4.1. *Let $k \geq 1$ and $r \geq 0$ with $k \geq r$ and let $\alpha \in (0, 1)$. Let Ω_0 be a bounded connected open $\mathcal{C}^{k+2,\alpha}$ domain of $\mathbb{R}^N$. Let $I \subset \mathbb{R}$ be an interval and let $f \in \mathcal{C}^r(I, \mathcal{C}^{k-1,\alpha}(\overline{\Omega}_0, \mathbb{R}_+^*))$ be such that $\int_{\Omega_0} f(t, y) \, dy = \text{meas}(\Omega_0)$ for all $t \in I$. Then, there exists a family $(\varphi(t))_{t \in I}$ of diffeomorphisms of $\overline{\Omega}_0$ of class $\mathcal{C}^r(I, \mathcal{C}^{k,\alpha}(\overline{\Omega}_0, \overline{\Omega}_0))$ satisfying*

$$\begin{cases} \det(D_y \varphi(t, y)) = f(t, y), & y \in \Omega_0 \\ \varphi(t, y) = y, & y \in \partial\Omega_0. \end{cases} \tag{4.1}$$

As explained in Section 1, an interesting change of variables for a PDE with moving domains would be a family of diffeomorphisms having Jacobian with constant determinant. This is the goal of Theorem 1.3, which is a direct consequence of Theorem 4.1.

Proof of Theorem 1.3: Let Theorem 4.1 be valid. We set $\tilde{h}(t) = h(t) \circ \varphi^{-1}(t)$ and we compute

$$\det\big(D_y \tilde{h}(t, y)\big) = \det\big(D_y h(t, \varphi^{-1}(y))\big) \det\big(D_y(\varphi^{-1})(t, y)\big)$$
$$= \det\big(D_y h(t, \varphi^{-1}(y))\big) \det\big((D_y \varphi)^{-1}(t, \varphi^{-1}(y))\big)$$
$$= \left(\frac{\det(D_y h)}{\det(D_y \varphi)}\right)(t, \varphi^{-1}(y)).$$

Since we want to obtain a spatially constant right-hand side, the dependence in $\varphi^{-1}(y)$ is not important. Thus,

$$\forall y \in \Omega_0 \;, \quad \det(D_y \tilde{h}(t, y)) = \frac{\text{meas}(\Omega(t))}{\text{meas}(\Omega_0)}$$

if and only if

$$\forall y \in \Omega_0 \;, \quad \det(D_y \varphi(t, y)) = \frac{\text{meas}(\Omega_0)}{\text{meas}(\Omega(t))} \det(D_y h(t, y)) \;.$$

To obtain such a diffeomorphism φ, it remains to apply Theorem 4.1 with $f(t, y)$ being the above right-hand side. $\qquad\square$

4.1 The classical results

Theorem 4.1 is an example of a family of results aiming to find diffeomorphisms having prescribed determinant of the Jacobian. Such outcomes are often referred as "Moser's trick" because they were originated by the famous work of Moser [42]. A lot of variants can be found in the literature, depending on the needs of the reader. We refer to [15] for a review on the subject. The following result comes from [14, 16], see also [15].

Theorem 4.2. Dacorogna-Moser (1990)
Let $k \geq 1$ and $\alpha \in (0,1)$. Let Ω_0 be a bounded connected open $\mathcal{C}^{k+2,\alpha}$ domain. Then, the following statements are equivalent.

(i) The function $f \in \mathcal{C}^{k-1,\alpha}(\overline{\Omega}_0, \mathbb{R}_+^*)$ satisfies $\int_{\Omega_0} f = \mathrm{meas}(\Omega_0)$.

(ii) There exists $\varphi \in \mathrm{Diff}^{k,\alpha}(\overline{\Omega}_0, \overline{\Omega}_0)$ satisfying

$$\begin{cases} \det(D_y \varphi(y)) = f(y), & y \in \Omega_0 \\ \varphi(y) = y, & y \in \partial\Omega_0. \end{cases}$$

Furthermore, if $c > 0$ is such that $\max(\|f\|_\infty, \|1/f\|_\infty) \leq c$ then there exists a constant $C = C(c, \alpha, k, \Omega_0)$ such that

$$\|\varphi - \mathrm{id}\|_{\mathcal{C}^{k,\alpha}} \leq C\|f - 1\|_{\mathcal{C}^{k-1,\alpha}}.$$

A complete proof of theorem 4.2 can be found in [15]. There, a discussion concerning the optimality of the regularity of the diffeomorphisms is also provided. In particular, notice that the natural gain of regularity from $f \in \mathcal{C}^{k-1,\alpha}$ to $\varphi \in \mathcal{C}^{k,\alpha}$ was not present in the first work of Moser [42] and cannot be obtained through the original method. Cases with other space regularity are studied in [48, 49].

For t fixed, Theorem 4.1 corresponds to Theorem 4.2. Our main goal is to extend the result to a time-dependent measure $f(t, y)$. In the original proof of [42], Moser uses a flow method and constructs φ by a smooth deformation starting at $f(t = 0) = \mathrm{id}$ and reaching $f(t = 1) = f$. In this proof, the smooth deformation is a linear interpolation $f(t) = (1 - t)\mathrm{id} + tf$ and one of the main steps consists in solving an ODE with a non-linearity as smooth as $\partial_t f(t)$. The linear interpolation is of course harmless in such a context. But when we consider another type of time-dependence, we need to have enough smoothness to solve the ODE. Typically, at least $\mathcal{C}^1$−smoothness in time is required to use the arguments from the original paper [42] of Moser. Thus, the original proof of [42] does not provide an optimal regularity, neither in space or time. In particular, in Theorem 1.3, this type of proof provides a diffeomorphism $\tilde{h}$ with one space regularity less than h. We refer to [1, 4, 25] for other time-dependent versions of Theorem 4.2.

Our aim is to obtain a better regularity in space and time as well as to provide a complete proof of a time-dependent version of Moser's trick. To this end, we follow a method coming from [15, 16], which is already known for obtaining the optimal space regularity. This method uses a fixed point argument, which can easily be parametrized with respect to time. Nevertheless, the fixed point argument only provides a local construction which is difficult to extend by gluing several similar construction (equations as (4.1) have an infinite number of solutions and the lack of uniqueness makes difficult to glue smoothly the different curves). That is the reason why we follow a similar strategy to the ones adopted in [15, 16]. First, at the cost of losing some regularity, we prove the global result with the flow method. Second, we exploit a fixed point argument in order to ensure the result with respect to the optimal regularity.

4.2 The flow method

By using the flow method of the original work of Moser [42], we obtain the following version of Theorem 4.1, where the statements on the regularities are weakened.

Proposition 4.3. *Let $k \geq 2$ and $r \geq 1$ with $k \geq r$. Let Ω_0 be a bounded connected open $\mathcal{C}^{k,\alpha}$ domain of $\mathbb{R}^N$ for some $\alpha \in (0,1)$. Let $I \subset \mathbb{R}$ be an interval of times and let $f \in \mathcal{C}^r(I, \mathcal{C}^{k-1}(\overline{\Omega}_0, \mathbb{R}_+^*))$ be such that $\int_{\Omega_0} f(t, y) \,\mathrm{d}y = \mathrm{meas}(\Omega_0)$ for all $t \in I$. Then, there exists a family $(\varphi(t))_{t \in I}$ of diffeomorphisms of $\overline{\Omega}_0$ of class $\mathcal{C}^r(I, \mathcal{C}^{k-1}(\overline{\Omega}_0, \overline{\Omega}_0))$ satisfying (4.1).*
Moreover, there exist continuous functions $M \mapsto C(M)$ and $M \mapsto \lambda(M)$ such that, if

$$\forall t \in I, \quad \|f(t, \cdot)\|_{\mathcal{C}^{k-1}} \leq M \text{ and } \min_{y \in \Omega_0} f(t, y) \geq \frac{1}{M},$$

then

$$\forall t \in I \ , \quad \|\varphi(t,\cdot)\|_{\mathcal{C}^{k-1}} + \|\varphi^{-1}(t,\cdot)\|_{\mathcal{C}^{k-1}} \le C(M)e^{\lambda(M)|t|}.$$

Proof: Let $t_0 \in I$. Due to theorem 4.2, there is $\varphi(t_0)$ with the required space regularity satisfying (4.1) at $t = t_0$. Setting $\varphi(t) = \tilde{\varphi}(t) \circ \varphi(t_0)$, we replace (4.1) by the condition

$$\det(D_y\tilde{\varphi}(t,y)) = \frac{f(t,\varphi^{-1}(t_0,y))}{f(t_0,\varphi^{-1}(t_0,y))} \ .$$

Thus, we may assume without loss of generality that $f(t_0,y) \equiv 1$.

Let L^{-1} be the right-inverse of the divergence introduced in Appendix A.3. Notice that $\partial_t f(t,\cdot)$ is of class $\mathcal{C}^{k-1}$ and hence of class $\mathcal{C}^{k-2,\alpha}$. Since $\int_{\Omega_0} \partial_t f = \partial_t \int_{\Omega_0} f = 0$ we get that $f(t) \in Y_m^{k-2,\alpha}$ for all $t \in I$ and we can define

$$U(t,\cdot) \ = \ -\frac{1}{f(t,\cdot)}L^{-1}(\partial_t f(t,\cdot)) \ . \tag{4.2}$$

We get that U is well defined and it is of class $\mathcal{C}^{r-1}$ in time and $\mathcal{C}^{k-1}$ in space. For $x \in \overline{\Omega}_0$, we define $t \mapsto \psi(t,x)$ as the flow corresponding to the ODE

$$\psi(t_0,x) = x \quad \text{and} \quad \partial_t\psi(t,x) = U(t,\psi(t,x)) \quad t \in I \ . \tag{4.3}$$

Notice that ψ is locally well defined because $(t,y) \mapsto U(t,y)$ is at least lipschitzian in space and continuous in time. Moreover, the trajectories $t \in I \mapsto \psi(t,x) \in \overline{\Omega}_0$ are globally defined because $U(t,y) = 0$ on the boundary of Ω_0, providing a barrier of equilibrium points. The classical regularity results for ODEs show that ψ is of class $\mathcal{C}^r$ with respect to time and $\mathcal{C}^{k-1}$ in space (see Proposition A.4 in the appendix). Moreover, by reversing the time, the flow of a classical ODE as (4.3) is invertible and $\psi(t,\cdot)$ is a diffeomorphism for all $t \in I$. We set $\varphi(t,\cdot) = \psi^{-1}(t,\cdot)$.

Using Proposition A.2, we compute for $t \in I$

$$\partial_t\big[\det(D_y\psi(t,y))\, f(t,\psi(t,y))\big] = (\partial_t \det(D_y\psi(t,y)))\, f \circ \psi(t,y) + \det(D_y\psi(t,y))\, (\partial_t f) \circ \psi(t,y)$$
$$+ \det(D_y\psi(t,y))((D_y f) \circ \psi) \cdot (\partial_t\psi)(t,y)$$
$$= \det(D_y\psi)\Big[\text{Tr}\,\big((D_y\psi)^{-1}\partial_t(D_y\psi)\big) f \circ \psi$$
$$+ (\partial_t f) \circ \psi + \big((D_y f) \circ \psi\big) \cdot (\partial_t\psi)\Big](t,y) \ .$$

Since we have $D_y(\rho \circ \varphi) \circ \psi = (D_y\rho)(D_y\psi)^{-1}$ for any function ρ, we use the trick

$$\text{Tr}\,\big((D_y\psi)^{-1}\partial_t D_y(\psi)\big) = \text{Tr}\,\big(D_y(\partial_t\psi)(D_y\psi)^{-1}\big) = \text{Tr}\,\Big[D_y\big((\partial_t\psi) \circ \varphi\big)\Big] \circ \psi = \text{div}_y\,\big((\partial_t\psi) \circ \varphi\big) \circ \psi.$$

Since $(\partial_t\psi)(t,\varphi(t,y)) = U(t,\psi \circ \varphi(t,y)) = U(t,y)$ due to (4.3), we obtain

$$\partial_t\Big[\det\big(D_y\psi(t,y)\big)\, f\big(t,\psi(t,y)\big)\Big] = \det\big(D_y\psi(t,y)\big)\big[\text{div}_y(U)f + (\partial_t f) + (D_y f)U\big] \circ \psi \tag{4.4}$$
$$= \det\big(D_y\psi(t,y)\big)\big[\text{div}_y(Uf) + (\partial_t f)\big] \circ \psi \ . \tag{4.5}$$

It remains to notice that $\text{div}_y(Uf) = -\partial_t f$ by (4.2), that $f(t,y_0) = 1$ and that $\det(D_y\psi(t_0,x)) = \det(\text{id}) = 1$. Now, from (4.4), we have $\det\big(D_y\psi(t,y)\big) f\big(t,\psi(t,y)\big) = \det\big(D_y\psi(t_0,y)\big) f\big(t,\psi(t_0,y)\big) = 1$ and then

$$\forall t \in I, x \in \Omega_0 \ , \quad \det(D_y\psi(t,y)) = \frac{1}{f(t,\psi(t,y))} \ . \tag{4.6}$$

We conclude that, as required,

$$\forall t \in I, x \in \Omega_0 \ , \quad \det(D\varphi(t,y)) = \det((D\psi)^{-1}(t,\varphi(t,y))) = f(t,\psi \circ \varphi(t,y)) = f(t,y) \ .$$

Finally, it remains to notice that the bounds on f yield bounds on U due to (4.2). The classical bounds on the flow recalled in Proposition A.4 in the appendix provide the claimed bounds on φ and $\varphi^{-1} = \psi$. Indeed, they satisfy (4.3) and the dual ODE where the time is reversed. $\qquad\square$

The trick of the above proof is the use of the flow of the ODE (4.3), which has of course a geometrical background. Actually, this idea can be extended to other differential forms than the volume form considered here, see [15] and [42]. Also notice the explicit bounds stated in our version, which is not usual. We will need them for the proof of Theorem 4.1 because we slightly adapt the strategy of [16].

4.3 The fixed point method

Proposition 4.4 below is an improvement of Proposition 4.3 regarding regularity in both time and space, but it only deals with local perturbations of $f(t,y) \equiv 1$. It is proved following the fixed point method of [15, 16].

Proposition 4.4. *Let $k \geq 1$ and $r \geq 0$ with $k \geq r$ and let $\alpha \in (0,1)$. Let Ω_0 be a bounded connected open $\mathcal{C}^{k+1,\alpha}$ domain of $\mathbb{R}^N$. There exists $\varepsilon > 0$ such that, for any interval $I \subset \mathbb{R}$ and any $f \in \mathcal{C}^r(I, \mathcal{C}^{k-1,\alpha}(\overline{\Omega}_0, \mathbb{R}_+^*))$ being such that*

$$\forall t \in I , \quad \int_{\Omega_0} f(t,y)\,\mathrm{d}y = \mathrm{meas}(\Omega_0) \quad and \quad \|f(t,\cdot) - 1\|_{\mathcal{C}^{k-1,\alpha}} \leq \varepsilon .$$

Then, there exists a family $(\varphi(t))_{t \in I}$ of diffeomorphisms of $\overline{\Omega}_0$ of class $\mathcal{C}^r(I, \mathcal{C}^{k,\alpha}(\overline{\Omega}_0, \overline{\Omega}_0))$ satisfying (4.1). Moreover, there exists a constant K independent of f such that

$$\sup_{t \in I} \|\varphi(t)\|_{\mathcal{C}^{k,\alpha}} + \sup_{t \in I} \|\varphi^{-1}(t)\|_{\mathcal{C}^{k,\alpha}} \leq K \sup_{t \in I} \|f(t,\cdot) - 1\|_{\mathcal{C}^{k-1,\alpha}} .$$

Proof: We follow Section 5.6.2 of [15]. We reproduce the proof for sake of completeness and to explain why we can add for free the time-dependence. We set

$$Q : M \in \mathcal{M}_d(\mathbb{R}) \longmapsto \det(\mathrm{id} + M) - 1 - \mathrm{Tr}(M) \in \mathbb{R} .$$

Notice that Q is the sum of monomials of degree between 2 and d with respect to the coefficients because 1 and $\mathrm{Tr}(M)$ are the first order terms of the development of $\det(\mathrm{id}+M)$. Using the fact that $\mathcal{C}^{k,\alpha}$ is a Banach algebra (see for example [14]), there exists a constant K_2 such that, for any functions $u, v \in \mathcal{C}^{k,\alpha}(\overline{\Omega}, \mathbb{R}^N)$

$$\|Q(D_y u) - Q(D_y v)\|_{\mathcal{C}^{k-1,\alpha}} \leq K_1 (1 + \|u\|_{\mathcal{C}^{k,\alpha}}^{d-2} + \|v\|_{\mathcal{C}^{k,\alpha}}^{d-2})$$
$$\max(\|u\|_{\mathcal{C}^{k,\alpha}}, \|v\|_{\mathcal{C}^{k,\alpha}}) \|u - v\|_{\mathcal{C}^{k,\alpha}} . \tag{4.7}$$

We seek for a function $\varphi(t,y)$ solving (4.1) for times t close to t_0 by setting

$$\varphi(t,y) = \mathrm{id} + \eta(t,y) .$$

Since we would like that $\det(\mathrm{id} + D_y \eta) = f$ and as $\mathrm{div}\,\eta = \mathrm{Tr}(D_y \eta)$, the identity (4.1) is equivalent to

$$\begin{cases} \mathrm{div}\,\eta(y,t) = f(t,y) - 1 - Q(D_y \eta(t,y)) & y \in \Omega_0 \\ \eta(t,y) = 0 & y \in \partial\Omega_0 \end{cases} \tag{4.8}$$

Let $X_0^{k,\alpha}$ and $Y_m^{k-1,\alpha}$ the spaces introduced in Appendix A.3. First notice that, by assumption,

$$\int_{\Omega_0} f(t,y)\,\mathrm{d}y = \mathrm{meas}(\Omega_0) = \int_{\Omega_0} 1 \,\mathrm{d}y$$

and thus $(f(t)-1)$ belongs to $Y_m^{k-1,\alpha}$. Since $f(t,\cdot)$ is close to 1, we seek for η small in $\mathcal{C}^r(I, \mathcal{C}^{k,\alpha}(\overline{\Omega}_0, \mathbb{R}^N))$. By Corollary A.8, there exists $R > 0$ small such that if $\|\eta(t,\cdot)\|_{\mathcal{C}^{k,\alpha}} \leq R$ for all $t \in I$, then $\varphi(t,y) = \mathrm{id} + \eta(t,y)$ is a diffeomorphism from Ω_0 onto itself. If this is the case, we have that

$$\int_{\Omega_0} Q(D_y \eta(t,y))\,\mathrm{d}y = \int_{\Omega_0} \big(\det(\mathrm{id} + D_y \eta)(t,y) - 1 + \mathrm{Tr}(D_y \eta(t,y)) \big)\,\mathrm{d}y$$
$$= \int_{\Omega_0} \det(D_y \varphi(t,y))\,\mathrm{d}y - \mathrm{meas}(\Omega_0) + \int_{\Omega_0} \mathrm{div}(\eta(t,y))\,\mathrm{d}y$$
$$= \int_{\Omega_0} \mathbb{1}(\varphi(y))\,\mathrm{d}y - \mathrm{meas}(\Omega_0) + \int_{\partial\Omega_0} \eta(y)\,\mathrm{d}y$$
$$= \mathrm{meas}(\Omega_0) - \mathrm{meas}(\Omega_0) + 0 = 0 .$$

This means that we can look for η small as a solution in $\mathcal{C}^r(I, \mathcal{C}^{k,\alpha}(\overline{\Omega}_0, \mathbb{R}^N))$ of

$$\eta(t) = L^{-1} \big(f(t) - 1 - Q(D_y \eta(t)) \big) \tag{4.9}$$

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since we expect $Q(D_y\eta(t))$ to belong to $Y_m^{k-1,\alpha}$. We construct $\eta(t)$ by a fixed point argument by applying Theorem A.3 to the function

$$\Phi \ : \ (\eta, t) \in \mathcal{B}_R \times I \ \longmapsto \ L^{-1}\big(f(t) - 1 - Q(D_y\eta)\big) \in X_0^{k,\alpha}$$

where L^{-1} is the right-inverse of the divergence (see Appendix A.3) and $\mathcal{B}_R = \{\eta \in X_0^{k,\alpha}, \|\eta\|_{X^{k,\alpha}} \leq R\}$ with R as above. As a consequence, $\mathrm{id} + \eta$ is a diffeomorphism and the above computations are valid. Due to (4.7), we have

$$\|\Phi(\eta_1, t) - \Phi(\eta_2, t)\|_{\mathcal{C}^{k,\alpha}} \ \leq \ R K_1 K_2 (1 + 2R^{d-2}) \|\eta_1 - \eta_2\|_{\mathcal{C}^{k,\alpha}}$$

where $K_2 = |||L^{-1}|||$. Using that $Q(0) = 0$, we also have

$$\|\Phi(\eta, t)\|_{\mathcal{C}^{k,\alpha}} \ \leq \ K_2\big(\|f(t) - 1\|_{\mathcal{C}^{k-1,\alpha}} + K_1(R^2 + R^d)\big) \ .$$

We choose $R \in (0, 1/2]$ small enough such that $\mathrm{id} + \eta$ is a diffeomorphism and

$$RK_1 K_2 (1 + 2R^{d-2}) \ \leq \ \frac{1}{2}$$

which also implies that $K_1 K_2 (R^2 + R^d) \leq R/2$. Then, we choose $\varepsilon = R/2K_2$ and assume that

$$\forall t \in I \ , \quad \|f(t) - 1\|_{\mathcal{C}^{k-1,\alpha}} \ \leq \ \varepsilon = \frac{R}{2K_1} \ . \tag{4.10}$$

By construction, Φ is $1/2-$lipschitzian from $\mathcal{B}_R$ into $\mathcal{B}_R$. The classical fixed point theorem for contraction maps shows the existence of $\eta(t)$ solving (4.9) for each t. The regularity of η with respect to the time t is directly given by Theorem A.3. By construction $\eta(t) = L^{-1}(f(t) - 1 - Q(D_y\eta(t)))$, and since $LL^{-1} = \mathrm{id}$, this implies that $\mathrm{div}\,\eta(t) = f(t) - 1 - Q(D_y\eta(t))$ and thus $f(t) = \det(\mathrm{id} + \eta(t)) = \det(\varphi(t))$. Also remember that, by construction, φ is $\mathcal{C}^k$ close enough to the identity in Ω_0 and is the identity on $\partial\Omega_0$ so that $\varphi(t)$ is a $\mathcal{C}^k-$diffeomorphism (see Appendix A.6 and the topological arguments of [41]). $\qquad\square$

4.4 Proof of Theorem 4.1

As in the work of Dacorogna and Moser (see [15] and [16]), we obtain Theorem 4.1 by combining the propositions 4.3 and 4.4. In the original method, the authors used first the fixed point method and then the flow method. However, this approach does not provide the optimal time regularity. Therefore, we couple the two methods in the other sense. In this case, we need the $\mathcal{C}^{k-1}$-bounds on the diffeomorphism provided by Proposition 4.3. This is the reason why we made them explicit, which is not common for this type of results.

Let $f \in \mathcal{C}^r(I, \mathcal{C}^{k-1,\alpha}(\overline{\Omega}_0, \mathbb{R}_+^*))$ with $\int_\Omega f(t, y)\,\mathrm{d}y = \mathrm{meas}(\Omega_0)$ for all $t \in I$. Let $\varepsilon > 0$. We consider a regularization f_1 of f, which is of class $\mathcal{C}^{r'}(I, \mathcal{C}^{k'-1}(\overline{\Omega}_0, \mathbb{R}_+^*))$ with $r' = \max(1, r)$ and $k' = k+2$, satisfying

$$\forall t \in I \ , \quad \left\|\frac{f(t, \cdot)}{f_1(t, \cdot)} - 1\right\|_{\mathcal{C}^{k-1,\alpha}} \leq \varepsilon(t) \ . \tag{4.11}$$

Assume that we first apply the flow method of Proposition 4.3 to obtain a smooth family of diffeomorphisms $\varphi_1 \in \mathcal{C}^{r'}(I, \mathcal{C}^{k'-1}(\overline{\Omega}_0, \overline{\Omega}_0)) \subset \mathcal{C}^r(I, \mathcal{C}^{k+1}\overline{\Omega}_0, \overline{\Omega}_0))$ such that

$$\det(D_y\varphi_1(t, y)) = f_1(t, y) \quad \text{and} \quad \varphi_{1|\partial\Omega_0}(t, \cdot) = \mathrm{id}_{|\partial\Omega_0} \ .$$

Now, we seek for φ of the form $\varphi = \varphi_2 \circ \varphi_1$ satisfying the statement of Theorem 4.1. Thus, we need to find $\varphi_2 \in \mathcal{C}^r(I, \mathcal{C}^{k,\alpha}(\overline{\Omega}_0, \mathbb{R}_+^*))$ such that

$$\det(D_y\varphi_2)(t, \varphi_1(t, y))\det(D_y\varphi_1(t, y)) = f(t, y)$$

i.e.

$$\det(D_y\varphi_2)(t, y) = \frac{f(t, \varphi_1^{-1}(t, y))}{f_1(t, \varphi_1^{-1}(t, y))} \ . \tag{4.12}$$

We would like to apply the fixed point method of Proposition 4.4 by choosing $\varepsilon(t)$ small enough such that

$$\forall t \in I \ , \quad \left\|\frac{f(t, \varphi_1^{-1}(t, y))}{f_1(t, \varphi_1^{-1}(t, y))} - 1\right\|_{\mathcal{C}^{k-1,\alpha}} \leq \varepsilon \tag{4.13}$$

with ε as in Proposition 4.4. However, we notice that in (4.13) appears the composition with φ_1^{-1}. Fix ε as in Proposition 4.4 and smaller than $1/2$. When we consider Proposition 4.3 with $M = 2$, we have the upper bound for $C(2)e^{\lambda(2)t}$ for the derivatives of any φ_1 constructed as above when $\varepsilon(t) \leq \varepsilon$ in (4.11). In particular, these bounds indicate how much the composition by φ^{-1} increases the $\mathcal{C}^{k-1,\alpha}$−norm. Having this in mind, we may choose $\varepsilon(t)$ small (and exponentially decreasing) such that, for any f_1 satisfying (4.11), the associated diffeomorphisms φ_1 are such that (4.13) holds.

We construct the diffeomorphism as follows. We choose a regularization f_1 of the function f, which is of class $\mathcal{C}^{r'}(I, \mathcal{C}^{k'-1}(\overline{\Omega}_0, \mathbb{R}_+^*))$ with $r' = \max(1, r)$ and $k' = k + 2$, satisfying (4.11) for the suitable $\varepsilon(t)$. We will also need that $\int_{\Omega_0}(f/f_1)(t, y)\,\mathrm{d}y = \mathrm{meas}(\Omega_0)$, which is provided by first choosing $\tilde{f}$ satisfying (4.11) with a smaller $\varepsilon(t)$ and then set $f_1 = \tilde{f} \cdot \int(f/\tilde{f})/\mathrm{meas}(\Omega_0)$. By Proposition 4.3, there exists a family of diffeomorphisms $t \mapsto \varphi_1(t, \cdot)$ satisfying (4.11) and of class $\mathcal{C}^{r'}(I, \mathcal{C}^{k'-1}(\overline{\Omega}_0, \overline{\Omega}_0)) \subset \mathcal{C}^r(I, \mathcal{C}^{k+1}(\overline{\Omega}_0, \overline{\Omega}_0))$.

Now we consider $f_2 = f/f_1$ which satisfies by construction (4.13) with ε as in Proposition 4.4. Notice that f_2 is of class $\mathcal{C}^r(I, \mathcal{C}^{k-1,\alpha}(\overline{\Omega}_0, \mathbb{R}_+^*))$ and

$$\int_{\Omega_0} f_2(t, y)\,\mathrm{d}y = \int_{\Omega_0} \frac{f(t, \varphi_1^{-1}(t, y))}{\det(D_y\varphi_1(t, \varphi_1^{-1}(t, y)))}\,\mathrm{d}y$$
$$= \int_{\Omega_0} f(t, \varphi_1^{-1}(t, y))\det(D_y(\varphi_1^{-1}(t, y)))\,\mathrm{d}y$$
$$= \int_{\Omega_0} f(t, y)\,\mathrm{d}y = \mathrm{meas}(\Omega_0)\ .$$

Thus, we can apply Proposition 4.4 to obtain a family of diffeomorphisms φ_2 of class $\mathcal{C}^r(I, \mathcal{C}^{k,\alpha}(\overline{\Omega}_0, \overline{\Omega}_0))$ such that

$$\det(D_y\varphi_2(t, y)) = f_2(t, y) \quad \text{and} \quad \varphi_{2|\partial\Omega_0}(t, \cdot) = \mathrm{id}_{|\partial\Omega_0}\ .$$

By construction, $\varphi = \varphi_2 \circ \varphi_1$ satisfies the conclusion of Theorem 4.1.

5 Some applications of our results

5.1 Adiabatic dynamics for quantum states on moving domains

In this subsection, we show how to ensure an adiabatic result for the Schrödinger equation on a moving domain as (1.9). We consider the framework of Corollary 1.5. We denote by $H(\tau) = -\Delta$ the Dirichlet Laplacian in $L^2(\Omega(\tau), \mathbb{C})$, i.e. $D(H(\tau)) = H^2(\Omega(\tau)) \cap H_0^1(\Omega(\tau))$ and $\tau \in [0, 1] \mapsto \lambda(\tau)$, a continuous curve such that $\lambda(\tau)$ for every $\tau \in [0, 1]$ is in the discrete spectrum of $H(\tau)$. We also assume that $\lambda(\tau)$ is a simple isolated eigenvalue for every $\tau \in [0, 1]$, associated with the spectral projectors $P(\tau)$.

We consider on the time interval $[0, 1/\epsilon]$ the equation (3.7) in $L^2(\Omega_0, \mathbb{C})$ when $H(t)$ is a Dirichlet Laplacian. Fixed $\epsilon > 0$, we substitute t by $\tau = \epsilon t$ and set $\tilde{v}_\epsilon(\tau) = v(\tau/\epsilon) = h^\sharp u_\epsilon(\tau/\epsilon)$ to obtain

$$\begin{cases} i\epsilon\partial_\tau\tilde{v}_\epsilon(\tau) = \left(h^\sharp H(\tau)h_\sharp + \epsilon\mathcal{H}(\tau)\right)\tilde{v}_\epsilon(\tau), & \tau \in [0, 1] \\ \tilde{v}_\epsilon(\tau)_{|\partial\Omega_0} \equiv 0, \\ \tilde{v}_\epsilon(\tau = 0) = h^\sharp u_0 \end{cases} \tag{5.1}$$

where

$$\mathcal{H}(\tau) = -h^\sharp\left[i\,\mathrm{div}_x\left(A_h(\tau, x)\,\cdot\,\right)\ +\ i\big\langle A_h(\tau, x)\big|\nabla_x\,\cdot\,\big\rangle\right]h_\sharp\ , \qquad A_h(\tau, x) := -\frac{1}{2}(h_*\partial_\tau h)(\tau, x).$$

The problem (5.1) generates a unitary flow thanks to Section 3. Even though it is well known that the classical adiabatic theorem is valid for the dynamics $i\epsilon\partial_\tau u_\epsilon(\tau) = h^\sharp H(\tau)h_\sharp u_\epsilon(t)$, (see [11, Chapter 4] or [52]), we may wonder if it is the same for the equation (5.1) because

$$\tilde{H}_\epsilon(\tau) = h^\sharp H(\tau)h_\sharp + \epsilon\mathcal{H}(\tau)$$

depends on ϵ and no spectral assumptions have been made on this family. First, we notice that, by conjugation, $\lambda(\tau)$ also belongs to the discrete spectrum of the operator $h^\sharp H(\tau)h_\sharp$ in $L^2(\Omega_0, \mathbb{C})$ and is associated with the spectral projection $(h^\sharp Ph_\sharp)(\tau)$. Then, for each τ, $\tilde{H}_\epsilon(\tau)$ is a small relatively compact self-adjoint perturbation of $h^\sharp H(\tau)h_\sharp$. Thus, for all $\epsilon > 0$ small enough, there exists a curve $\tilde{\lambda}_\epsilon(\tau)$ of simple

isolated eigenvalues of $\tilde{H}_\epsilon(\tau)$, associated with spectral projectors $\tilde{P}_\epsilon(\tau)$, such that $\tilde{\lambda}_\epsilon$ and $\tilde{P}_\epsilon$ converge uniformly when $\epsilon \to 0$ to λ and $h^\sharp P h_\sharp$ respectively (see [33] for further details). In this framework, even if $\tilde{H}_\epsilon(\tau)$ depends on ϵ, it is known that the classical adiabatic arguments can still be applied (see for example Nenciu [45, Remarks; p. 16; (4)], Teufel [52, Theorem 4.15] or the works [2, 24, 32]). Thus, we obtain the following convergence for the solution of (5.1)

$$\langle \tilde{P}_\epsilon(1)\tilde{v}_\epsilon(1)|\tilde{v}_\epsilon(1)\rangle \xrightarrow[\epsilon \to 0]{} \langle \tilde{P}_0(0)v(0)|v(0)\rangle = \langle (h^\sharp P h_\sharp)(0).h^\sharp u_0|h^\sharp u_0\rangle = \langle P(0)u_0|u_0\rangle .$$

Finally, we notice that, since $\tilde{P}_\epsilon(1)$ converges to $h^\sharp P(1)h_\sharp$ when ϵ goes to zero,

$$\langle \tilde{P}_\epsilon(1)\tilde{v}_\epsilon(1)|\tilde{v}_\epsilon(1)\rangle = \langle (h^\sharp P(1)h_\sharp)\tilde{v}_\epsilon(1)|\tilde{v}_\epsilon(1)\rangle + o(1) = \langle P(1)u_\epsilon(1/\epsilon)|u_\epsilon(1/\epsilon)\rangle + o(1) .$$

This concludes the proof of Corollary 1.5.

5.2 Explicit examples of time-varying domains

Translation of a potential well

Let us consider any domain $\Omega_0 \subset \mathbb{R}^N$ and any smooth family of vectors $D(t) \in \mathcal{C}^2([0,T],\mathbb{R}^N)$. The family of translated domains is $\Omega(t) = h(t,\Omega_0)$ where $h(t,y) = y + D(t)$. By explicit computations, we obtain that $h^\sharp \Delta_x h_\sharp = \Delta_y$ and $(h_*\partial_t h)(t,x) = D'(t)$. Since $|J|$ does not depend on y, we do not need Moser's trick to get (1.17). We can apply the gauge transformation since $\phi(t,x) = \frac{1}{2}\langle D(t)|x\rangle$ satisfies $h_*\partial_t h = 2\nabla_x \phi$. Then, $w = h^\sharp e^{-i\phi}u$ satisfies Equation (1.15), which becomes in our framework

$$i\partial_t w = -\Delta_y w + \frac{1}{4}\Big(2\langle D''(t)|(y+D(t))\rangle + |D(t)|^2\Big)w. \tag{5.2}$$

In this very particular case, we can further simplify the expression thanks to an interesting fact. Two terms of the electric potential in (5.2) do not depend on the space variable. We can thus apply another transformation to the system by adding a phase which is an antiderivative of $2\langle D''(t)|D(t)\rangle + |D(t)|^2$. For example, we consider

$$\tilde{w} = e^{\frac{i}{4}(2\langle D'(t)|D(t)\rangle - \int_0^t |D(s)|^2 \, ds)}w$$

where w satisfies Equation (5.2). Then, $\tilde{w}$ is solution of the equation

$$i\partial_t \tilde{w} = -\Delta_y \tilde{w} + \frac{1}{2}\langle D''(t)|y\rangle \tilde{w}. \tag{5.3}$$

These explicit computations are not new and appear for example in [6] for the one dimensional case of a moving interval.

Rotating domains

Let us consider a family of rotating domains in $\mathbb{R}^2$. Clearly, the same results can be extended by considering rotations in $\mathbb{R}^N$ with $N \geq 3$. Let $\Omega_0 \subset \mathbb{R}^2$ and let $\Omega(t) = h(t,\Omega_0)$ with

$$h(t): \ y = (y_1,y_2) \in \Omega_0 \longmapsto \big(\cos(\omega t)y_1 - \sin(\omega t)y_2, \cos(\omega t)y_1 + \sin(\omega t)y_2\big)$$

and $\omega \in \mathbb{R}$. Using the classical notation $(y_1,y_2)^\perp = (-y_2,y_1)$, it is straightforward to check that $|J| = 1$, $J^{-1}(J^{-1})^t = Id$,

$$(h_*\partial_t h)(t,x) = \omega x^\perp \quad \text{and} \quad J^{-1}\partial_t h(t,y) = \omega y^\perp .$$

We obtain by direct computation or by the first line of (1.17) that $v = h^* u$ satisfies the following Schrödinger equation in the rotating frame

$$i\partial_t v = -\Delta_y v + i\omega\langle y^\perp, \nabla_y v\rangle . \tag{5.4}$$

This is an obvious and well-known computation (used for studying quantum systems in rotating potentials frames). The general Hamiltonian structure highlighted in this paper simply writes here as

$$i\partial_t v = -\Big(\nabla_y - i\frac{\omega}{2}y^\perp\Big)^2 v - \frac{\omega^2}{4}|y|^2 v .$$

which is given by the second line of (1.17) or obtained directly from (5.4) by using the fact that $y^\perp$ is divergence free. We recover a repulsive potential $\omega^2|y|^2/4$ corresponding to the centrifugal force.

Moving domains with diagonal diffeomorphisms

Let $\Omega_0 \subset \mathbb{R}^N$ and let $\Omega(t) = h(t, \Omega_0)$ with

$$h(t): \; y = (y_i)_{i=1...N} \in \Omega_0 \; \longmapsto \; \big(f_i(t)\,y_i\big)_{i=1...N}$$

and $f_i \in \mathcal{C}^2([0,T], \mathbb{R})$. As above, we obtain

$$h^\sharp \Delta_x h_\sharp = \sum_{i=1}^N \frac{1}{f_i(t)^2} \partial_{y_i y_i}^2, \qquad (h_* \partial_t h)(t,x) = \left(\frac{f_i'(t)}{f_i(t)} x_i\right)_{i=1...N}. \tag{5.5}$$

We apply again the gauge transformation and then,

$$\phi(t,x) = \frac{1}{4} \sum_{j=1}^N \left(\frac{f_i'(t)}{f_i(t)} x_i^2\right) \qquad \text{satisfies} \qquad h_* \partial_t h = 2\nabla_x \phi. \tag{5.6}$$

Finally, $w = h^\sharp e^{-i\phi} u$ satisfies the equation

$$i\partial_t w = -\sum_{i=1}^N \frac{1}{f_i(t)^2} \partial_{y_i y_i}^2 w + \frac{1}{4}\left(\sum_{i=1}^N f_i''(t) f_i(t) y_i^2\right) w. \tag{5.7}$$

In the homothetical case where $f_1 = f_2 = \ldots =: f(t)$ (see for instance [3, 5, 7, 38, 40, 43, 47, 50]). Equation (5.7) becomes

$$i\partial_t w(t) = -\frac{1}{f(t)^2} \Delta w(t) + \frac{1}{4} f''(t) f(t) |y|^2 w(t), \quad t \in I.$$

In this case, it is usual to make a further simplification to eliminate the time-dependence of the main operator by changing the time variable for

$$\tau = \int_0^t \frac{1}{f(s)^2} ds$$

and introducing the implicitly defined function

$$U(\tau) = \frac{f'(t) f(t)}{4}.$$

We obtain

$$i\partial_\tau w(\tau) = -\Delta w(\tau) + \Big(U'(\tau) - 4U(\tau)\Big)|y|^2 w(\tau), \qquad \tau \in \left[0, \int_0^T 1/f(s)^2 ds\right]. \tag{5.8}$$

In this simple case, we see that the general framework of this paper coincides with the previous computations introduced in dimension $d = 1$ by Beauchard in [5]. Indeed, the transformations described in [5, Section 1.3] corresponds to application $u \longmapsto w = h^\sharp e^{-i\phi} u$ as the multiplication for the square root of the exponential [5, relation (1.4)] is nothing else than the multiplication for the square root of the Jacobian appearing in the definition of $h^\sharp$. Our paper put the change of variable of [5] in a more general geometric framework.

A similar expression to (5.8) is also obtained by Moyano in [43] for the case of the two-dimensional disk, nevertheless the transformations adopted in [43] are different from the ones considered in our work. In particular, they are not unitary with respect to the classical L^2-norm.

For a second application of this simple case, we consider the case of a family of cylinders

$$\Omega(t) = \Big\{(x_1, x_2, x_3) \in \mathbb{R}_+^3 \quad x_1 \in (0, \ell(t)), \; x_2^2 + x_3^2 < r^2, \Big\}$$

for ℓ a $\mathcal{C}^2-$varying length. We would like to consider the Schrödinger equation $i\partial_t u = -\Delta_x u$ in $\Omega(t)$ with boundary conditions of the Neumann type. This example corresponds to the situation of Figure 2 in Section 1. As shown in this paper, the conditions at the boundaries cannot be pure homogeneous Neumann ones everywhere if $\ell(t)$ is not constant. Theorem 1.2 shows us the correct ones. We can choose

$$h(t, \cdot): (y_1, y_2, y_3) \in \Omega_0 \longmapsto \big(\ell(t) y_1, y_2, y_3\big) \in \Omega(t)$$

with Ω_0 the cylinder of length 1. Then, we get by (5.5) the term $h_*\partial_t h$ and we can compute that the suitable boundary conditions are

$$\partial_\nu u - \frac{i}{2}\ell'(t)u = 0 \quad \text{at } x \in \partial\Omega(t) \text{ with } x_1 = \ell(t) \tag{5.9}$$

and $\partial_\nu u = 0$ on the other parts of the boundary (see Figure 2). It is important to notice that, contrary to the above changes of variables, this computation is independent of the choice of $h(t)$. Equations as (5.8) can be seen as auxiliary equations and they depend on several choices, while (5.9) is stated for the original variable u and have physical meaning.

A Appendix

A.1 Unitary semigroups

Defining solutions of an evolution equation with a time-dependent family of operators is nowadays a classical result (see [51]). In the present article, we are interested in the Hamiltonian structure and we use the following result of [37] (see also [52]).

Theorem A.1. (Kisyński, 1963)
Let $\mathcal{X}$ be a Hilbert space and let $(\mathcal{H}(t))_{t\in[0,T]}$ be a family of self-adjoint positive operators on $\mathcal{X}$ such that $\mathcal{X}^{1/2} = D(\mathcal{H}(t)^{1/2})$ is independent of time t. Also set $\mathcal{X}^{-1/2} = D(\mathcal{H}(t)^{-1/2}) = (\mathcal{X}^{1/2})^*$ and assume that $\mathcal{H}(t) : \mathcal{X}^{1/2} \to \mathcal{X}^{-1/2}$ is of class $\mathcal{C}^1$ with respect to $t \in [0,T]$. In other words, we assume that the sesquilinear form $\phi_t(u,v) = \langle \mathcal{H}(t)u|v\rangle$ associated with $\mathcal{H}(t)$ has a domain $\mathcal{X}^{1/2}$ independent of the time and is of class $\mathcal{C}^1$ with respect to t. Also assume that there exist $\gamma > 0$ and $\kappa \in \mathbb{R}$ such that,

$$\forall t \in [0,T] \;,\; \forall u \in \mathcal{X}^{1/2} \;,\quad \phi_t(u,u) = \langle \mathcal{H}(t)u|u\rangle_{\mathcal{X}} \geq \gamma\|u\|^2_{\mathcal{X}^{1/2}} - \kappa\|u\|^2_{\mathcal{X}} \;. \tag{A.1}$$

Then, for any $u_0 \in \mathcal{X}^{1/2}$, there is a unique solution u belonging to $\mathcal{C}^0([0,T],\mathcal{X}^{1/2}) \cap \mathcal{C}^1([0,T],\mathcal{X}^{-1/2})$ of the equation

$$i\partial_t u(t) = \mathcal{H}(t)u(t) \quad u(0) = u_0 \;. \tag{A.2}$$

Moreover, $\|u_0\|_{\mathcal{X}} = \|u(t)\|_{\mathcal{X}}$ for all $t \in [0,t]$ and we may extend by density the flow of (A.2) on $\mathcal{X}$ as a unitary flow $U(t,s)$ such that $U(t,s)u(s) = u(t)$ for all solutions u of (A.2).

If in addition $\mathcal{H}(t) : \mathcal{X}^{1/2} \to \mathcal{X}^{-1/2}$ is of class $\mathcal{C}^2$ and $u_0 \in D(\mathcal{H}(0))$, then $u(t)$ belongs to $D(\mathcal{H}(t))$ for all $t \in [0,T]$ and u is of class $\mathcal{C}^1([0,T],\mathcal{X})$.

A.2 The derivative of the determinant

We recall the following standard result.

Proposition A.2. *Let $I \subset \mathbb{R}$ be an interval of times and let $N \geq 1$. Let $A(t)$ be a family of $N \times N$ complex matrices which is differentiable with respect to the parameter $t \in I$. If $A(t)$ is invertible for every $t \in I$, then*

$$\partial_t \det(A(t)) = \det(A(t))\operatorname{Tr}\left(A(t)^{-1}\partial_t A(t)\right).$$

More generally, we have $\partial_t \det(A(t)) = \operatorname{Tr}(\operatorname{com}(A(t))^t \partial_t A(t))$ where $\operatorname{com}(A)$ is the comatrix of A.

Proof: Without loss of generality, let us consider the derivative at time $t = 0$ and assume $N \geq 2$ (since $N = 1$ is a trivial case). First assume that $A(0) = I$, where I is the identity matrix. Then,

$$\partial_{t=0}\det(A(t)) = \lim_{t\to 0} \frac{\det\left[I + tA'(0) + o(t)\right] - \det(I)}{t}$$
$$= \lim_{t\to 0} \frac{\operatorname{Tr}\left[I + tA'(0)\right] + o(t) - \operatorname{Tr}(I)}{t} = \operatorname{Tr}(A'(0)).$$

In the case where $A(t)$ is invertible at $t = 0$, we write

$$\det(A(t)) = \det(A(0))\det(A(0)^{-1}A(t))$$

and apply the above computation to $\tilde{A}_h(t) = A(0)^{-1}A(t)$.

For any invertible A, we have $\operatorname{com}(A)^t = \det(A)A^{-1}$. Thus, we obtain the last statement by extending the formula $\partial_t \det(A(t)) = \operatorname{Tr}(\operatorname{com}(A(t))^t \partial_t A(t))$ by a density argument. $\qquad\square$

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A.3 Right-inverse of the divergence

Let $k \geq 1$, let $\alpha \in (0,1)$ and let Ω_0 be a $\mathcal{C}^{k+1,\alpha}$–domain of $\mathbb{R}^N$. We define

$$X_0^{k,\alpha} = \{u \in \mathcal{C}^{k,\alpha}(\overline{\Omega}_0, \mathbb{R}^N)\, , \, u = 0 \text{ on } \partial\Omega_0\},$$

$$Y_m^{k-1,\alpha} = \left\{v \in \mathcal{C}^{k-1,\alpha}(\overline{\Omega}_0, \mathbb{R}^N)\, , \, \int_{\Omega_0} v = 0\right\}$$

and

$$L \; : \; u \in X_0^{k,\alpha} \; \longmapsto \; \operatorname{div} u \in Y_m^{k-1,\alpha} \; .$$

Notice that L is well defined since $\int_{\Omega_0} \operatorname{div} u = 0$ if $u_{\partial\Omega_0} \equiv 0$. It is shown in [16, Theorem 30] that the operator L admits a bounded linear right-inverse

$$L^{-1} : Y_m^{k-1,\alpha} \to X_0^{k,\alpha}$$

that is $LL^{-1} = \operatorname{id}$ and there exists $K > 0$ such that $\|L^{-1}v\|_{\mathcal{C}^{k,\alpha}} \leq K\|v\|_{\mathcal{C}^{k-1,\alpha}}$.

This kind of result is classical, in particular in the framework of Sobolev spaces. In addition to [16] see also [12, 13, 15].

A.4 Fixed point theorem with parameter

Even though the Banach fixed point theorem is long-established, in this paper we need its extension to the case where the contraction depends on a parameter. Of course, this extension is also very classical. We briefly recall it for sake of completeness in order to detail the problem of the regularity.

Theorem A.3. *Let U be an open subset of a Banach space X and let V be an open subset of a Banach space Λ. Let $F \subset U$ be a closed subset of X and let*

$$\Phi \; : \; (x,\lambda) \in U \times V \; \longmapsto \; \Phi(x,\lambda) \in X \; .$$

Assume that

(i) For all λ, $\Phi(\cdot,\lambda)$ maps F into F.

(ii) The maps $\Phi(\cdot,\lambda)$ are uniformly contracting in the sense that there exists $k \in [0,1)$ such that

$$\forall (x,y,\lambda) \in U \times U \times V \; , \quad \|\Phi(x,\lambda) - \Phi(y,\lambda)\|_X \; \leq \; k\|x-y\|_X \; .$$

Then, for all $\lambda \in V$, there exists a unique solution $x(\lambda)$ of $x = \Phi(x,\lambda)$ in F. Moreover, if Φ is of class $\mathcal{C}^k(U \times V, X)$ with $k \in \mathbb{N}$, then $x(\lambda)$ is also of class $\mathcal{C}^k(V, F)$ (the derivatives being understood in the Fréchet sense).

Proof: The existence and uniqueness of $x(\lambda)$ correspond of course to the classical Banach fixed point theorem. Assume that Φ is continuous. Then, we write

$$\begin{aligned}
\|x(\lambda) - x(\lambda')\|_X &= \|\Phi(x(\lambda),\lambda) - \Phi(x(\lambda'),\lambda')\|_X \\
&\leq \|\Phi(x(\lambda),\lambda) - \Phi(x(\lambda),\lambda')\|_X + \|\Phi(x(\lambda),\lambda') - \Phi(x(\lambda'),\lambda')\|_X \\
&\leq \|\Phi(x(\lambda),\lambda) - \Phi(x(\lambda),\lambda')\|_X + k\|x(\lambda) - x(\lambda')\|_X.
\end{aligned}$$

Since $k < 1$ and $\lambda' \mapsto \Phi(x(\lambda),\lambda')$ is continuous, we obtain the continuity of $\lambda \mapsto x(\lambda)$. If Φ is of class $\mathcal{C}^k$ with $k \geq 1$, then we apply the implicit function theorem to the equation $\Psi(x,\lambda) = 0$ with $\Psi(x,\lambda) = x - \Phi(x,\lambda)$. Notice that, due to the contraction property, $\|D_x\Phi(x,\lambda)\|_{\mathcal{L}(X)} \leq k$ and thus $D_x\Psi$ is invertible everywhere. $\qquad\square$

A.5 The flow of a vector field on a compact domain

Let $d \geq 1$, $r \geq 0$ and $p \geq 1$. Let $\overline{\Omega}_0$ be a compact smooth domain of $\mathbb{R}^N$ and I be a compact interval of times. Let $(t, y) \in I \times \overline{\Omega}_0 \longmapsto U(t, y) \in \mathbb{R}^N$ a vector field which is of class $\mathcal{C}^r$ in time and $\mathcal{C}^p$ in space, meaning that all derivatives $\partial_t^{r'} \partial_y^{p'} U$ exist and they are continuous for all $r' \leq r$ and $p' \leq p$. We also assume that $U(t, y) = 0$ on $\partial\Omega_0$.

We define $t \mapsto \psi(t, x)$ as the flow corresponding to the ODE

$$\psi(t_0, x) = x \quad \text{and} \quad \partial_t \psi(t, x) = U(t, \psi(t, x)) \quad t \in I \ . \tag{A.3}$$

Notice that ψ is locally well defined because $(t, y) \mapsto U(t, y)$ is at least lipschitzian in space and continuous in time. Moreover, the trajectories $t \in I \mapsto \psi(t, x) \in \overline{\Omega}_0$ are globally defined because $U(t, y) = 0$ on the boundary of Ω_0, providing a barrier of equilibrium points.

The purpose of this appendix is to show the following regularity result. It is of course a well known property. However, the uniform bounds are often not stated explicitly and that is why we quickly recall here the arguments to obtain them.

Proposition A.4. *Let $r \geq 0$ and $p \geq 1$. If U is of class $\mathcal{C}^r$ in time and $\mathcal{C}^p$ in space, then the flow ψ defined by (A.3) is of class $\mathcal{C}^{r+1}$ in time and $\mathcal{C}^p$ in space. Moreover, for all $M > 0$, there exist $C(M) > 0$ and $\lambda(M) > 0$ such that, if U satisfies*

$$\forall t \in I \ , \quad \|U(t, \cdot)\|_{\mathcal{C}^p} \ \leq \ M,$$

then

$$\forall t \in I \ , \quad \|\psi(t, \cdot)\|_{\mathcal{C}^p} \ \leq \ C(M)e^{\lambda(M)|t - t_0|} \ .$$

Proof: The fact that the $\mathcal{C}^0$-bound on U yields a $\mathcal{C}^0-$bound on ψ simply comes from (A.3). If $U(t, y)$ is of class $\mathcal{C}^1$ with respect to y, it is well know that $\psi(t, y)$ is a of class $\mathcal{C}^1$ with respect to y and the derivatives solve the ODE

$$\partial_t(\partial_{y_i}\psi(t, y)) = D_y U(t, \psi(t, y)).\partial_{y_i}\psi(t, y) \ , \tag{A.4}$$

see for example [27]. We have $\partial_{y_i}\psi(t = 0) \equiv 0$, thus (A.4) and Grönwall's inequality ensures the bound on $\partial_{y_i}\psi$.

If U is of class $\mathcal{C}^2$ in y, the above arguments show that $y \mapsto D_y U(t, \psi(t, y))$ is of class $\mathcal{C}^1$ and we apply again the procedure to (A.4). We obtain that $\psi(t, y)$ is a of class $\mathcal{C}^2$ with respect to y and the derivatives solve the ODE

$$\partial_t(\partial_{y_i y_j}^2 \psi(t, y)) = D_y U(t, \psi(t, y)).\partial_{y_i y_j}^2 \psi(t, y) + D_{yy}^2 U(t, \psi(t, y)).(\partial_{y_i}\psi(t, y), \partial_{y_j}\psi(t, y)) \ ,$$

where $\partial_{y_i y_j}^2 \psi(t, y)$ is the unknown. Since we already have bounds on ψ and its first derivatives, again, Grönwall's inequality yields the bound on $\partial_{y_i y_j}^2 \psi$.

By applying the argument as many times as needed, we obtain the uniform bounds for all the wanted derivatives. We also proceed in the same way to obtain the regularity with respect to the time t. $\qquad\square$

A.6 Globalization of local diffeomorphisms

In this appendix, we consider a $\mathcal{C}^1-$function φ for a domain Ω_0 into itself such that φ is a local diffeomorphism and $\varphi_{|\partial\Omega} = \mathrm{id}$. We would like to obtain that φ is in fact a global diffeomorphism from Ω_0 into itself. This extension needs topological arguments contained in the article [41] of Meisters and Olech.

Theorem 1 of [41] applied to the ball of $\mathbb{R}^N$ writes as follows.

Theorem A.5. (Meisters-Olech, 1963)
Let B_R be the open ball of center 0 and radius $R > 0$ of $\mathbb{R}^N$ and let $\overline{B}_R$ the closed ball. Let f be a continuous mapping of $\overline{B}_R$ into itself which is locally one-to-one on $\overline{B}_R \setminus Z$, where $Z \cap B_R$ is discrete and Z does not cover the whole boundary ∂B_R. If f is one-to-one from ∂B_R into itself, then f is an homeomorphism of $\overline{B}_R$ onto itself.

In fact, the original result of [41] includes different domains than the balls. Nevertheless, if we consider any smooth domain, then it has to be diffeomorphic to a ball (typically, annulus are excluded). To consider more general domains, we assume that f is the identity at the boundary.

Theorem A.6. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open domain. Let f be a continuous mapping of $\overline{\Omega}$ into itself which is locally one-to-one on $\overline{\Omega} \setminus Z$, where Z is a finite set. Assume moreover that f is the identity on $\partial\Omega$. Then, f is an homeomorphism of $\overline{\Omega}$ onto itself.*

Proof: For R large enough, $\overline{\Omega}$ is included inside the ball B_R. We extend continuously f to a function $\tilde{f}$ by setting $\tilde{f} = $ id on $\overline{B}_R \setminus \Omega$. Notice that f maps $\overline{\Omega}$ into itself and is locally one-to-one at all the points of the boundary, except maybe at a finite number of them. This yields that $\tilde{f}$ is locally one-to-one at all these points since the extension maps the outside of $\overline{\Omega}$ into itself. We apply Theorem A.5 to $\tilde{f}$ and obtain that $\tilde{f}$ is an homeomorphism of B_R. Since it is the identity outside Ω, $f = \tilde{f}_{|\overline{\Omega}}$ must be an homeomorphism of $\overline{\Omega}$. $\square$

If we consider f of class $\mathcal{C}^k$ and Df its jacobian matrix, then we may check the local one-to-one property by assuming that $\det(Df)$ only vanishes at a finite number of points. More importantly, if $\det(Df)$ never vanishes, then f is a diffeomorphism.

Corollary A.7. *Let $k \geq 1$, let $\Omega \subset \mathbb{R}^N$ be a bounded open domain of class $\mathcal{C}^k$ and let $f \in \mathcal{C}^k(\overline{\Omega}, \overline{\Omega})$. Assume that $\det(Df)$ does not vanish on $\overline{\Omega}$ and that f is the identity on $\partial\Omega$. Then, f is a $\mathcal{C}^k-$diffeomorphism of $\overline{\Omega}$ onto itself.*

We could also be interested in the following other consequence.

Corollary A.8. *Let $k \geq 1$, let $\Omega \subset \mathbb{R}^N$ be a bounded open domain of class $\mathcal{C}^k$ and let f be a $\mathcal{C}^k-$diffeomorphism from $\overline{\Omega}$ onto itself. Assume that f is the identity on $\partial\Omega$. Then, for all $\varepsilon > 0$, there exists $\eta > 0$ such that, for all functions $g \in \mathcal{C}^k(\overline{\Omega}, \overline{\Omega})$ with $g_{\partial\Omega} = $ id and $\|f - g\|_{\mathcal{C}^k} \leq \eta$, g is also a $\mathcal{C}^k-$diffeomorphism of $\overline{\Omega}$ onto itself and $\|f^{-1} - g^{-1}\|_{\mathcal{C}^k} \leq \varepsilon$.*

Proof: We simply notice that $\overline{\Omega}$ is compact and so $|\det(Df)| \geq \alpha > 0$ for some uniform positive α. Thus, for g which is $\mathcal{C}^1-$close to f, Dg is still invertible everywhere and g is a $\mathcal{C}^k-$diffeomorphism due to Corollary A.7. $\square$

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