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On numerical semigroups with at most 12 left elements

S. Eliahou and D. Marín-Aragón

Abstract

For a numerical semigroup $S \subseteq \mathbb{N}$ with embedding dimension e , conductor c and left part $L = S \cap [0, c - 1]$, set $W(S) = e|L| - c$. In 1978 Wilf asked, in equivalent terms, whether $W(S) \geq 0$ always holds, a question known since as Wilf's conjecture. Using a closely related lower bound $W_0(S) \leq W(S)$, we show that if $|L| \leq 12$ then $W_0(S) \geq 0$, thereby settling Wilf's conjecture in this case. This is best possible, since cases are known where $|L| = 13$ and $W_0(S) = -1$. Wilf's conjecture remains open for $|L| \geq 13$.

1 Introduction

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ denote as usual the set of nonnegative integers. Given integers $a \leq b$, we denote by $[a, b[= [a, b - 1] = \{z \in \mathbb{Z} \mid a \leq z < b\}$, and by $[a, \infty[= \{z \in \mathbb{Z} \mid a \leq z\}$, the integer intervals they span. A *numerical semigroup* is a submonoid S of $(\mathbb{N}, +)$ such that $|\mathbb{N} \setminus S|$ is finite. Equivalently, it is a subset S of \mathbb{N} of the form $S = \langle a_1, \dots, a_n \rangle = \mathbb{N}a_1 + \dots + \mathbb{N}a_n$ where $\gcd(a_1, \dots, a_n) = 1$. The least such n is called the *embedding dimension* of S and is often denoted e . The *multiplicity* of S is $m = \min S^*$, where $S^* = S \setminus \{0\}$. The *conductor* of S is $c = \max(\mathbb{Z} \setminus S) + 1$, or equivalently, the least $c \in \mathbb{N}$ such $[c, \infty[\subseteq S$. The *genus* of S is $g = |\mathbb{N} \setminus S|$. The *left part* of S is

$$L = \{s \in S \mid s < c\} = S \cap [0, c[.$$

The *left elements* of S are the elements of L . Finally, as in [5], we denote

$$W(S) = e|L| - c. \tag{1}$$

In 1978 Wilf asked, in equivalent terms, whether the inequality

$$W(S) \geq 0$$

holds for every numerical semigroup S [16]. This open question is now known as Wilf's conjecture. Various particular cases have been settled, including the six independent cases $e \leq 3$, $|L| \leq 6$, $m \leq 18$, $g \leq 60$, $c \leq 3m$ and $e \geq m/3$. See e.g. [1, 2, 4, 5, 6, 8, 9, 10, 12, 11, 14, 15]. See also [3] for a recent extensive survey on this topic.

The authors of [4] settled Wilf's conjecture in case $|L| \leq 4$. This was later extended in [5], where a certain lower bound $W_0(S) \leq W(S)$ was introduced and shown to satisfy $W_0(S) \geq 0$ whenever $|L| \leq 6$. Here we further extend this result by showing that $W_0(S) \geq 0$ holds whenever $|L| \leq 12$. This is best possible since, as shown in [7], there are numerical semigroups S such that $|L| = 13$ and $W_0(S) < 0$. See also Section 2.2.

This paper is organized as follows. In Section 2 we recall some notation and background, including the definition of $W_0(S)$. In Section 3 we establish $W_0(S) \geq 0$ in some special circumstances. Our main result, namely $W_0(S) \geq 0$ if $|L| \leq 12$, and hence Wilf's conjecture in that case, is proved in Section 4.

For extensive information on numerical semigroups, see [13].

2 Background and notation

In this section, we recall some notation and terminology introduced in [5]. Let S be a numerical semigroup. We denote by $P \subset S^*$ the unique minimal generating set of S , so that $S = \langle P \rangle$ and $|P| = e$, the embedding dimension. It coincides with the set of *primitive elements* of S , i.e. those $x \in S^*$ which are not the sum of two smaller elements of S^* . Let m, c be the multiplicity and conductor of S , respectively. The *depth* of S is $q = \lceil c/m \rceil$ and its *offset* is $\rho = qm - c$. Thus

$$c = qm - \rho, \quad \rho \in [0, m[. \quad (2)$$

The set of *decomposable elements* of S is

$$D = S^* \setminus P = S^* + S^*.$$

Note that D contains $[c + m, \infty[$. Indeed, if $z \geq c + m$, then $z = m + (z - m)$, so that $z \in S^* + S^*$ since both $m \in S^*$ and $z - m \in S^*$ as $z - m \geq c$. It follows that

$$P \subseteq [m, c + m[. \quad (3)$$

Throughout Section 2, the symbols m, c, q and ρ will denote, often tacitly so, the multiplicity, conductor, depth and offset of the numerical semigroup S under consideration, respectively.

2.1 The level function λ

Let $S \subseteq \mathbb{N}$ be a numerical semigroup. We shall further use the following notation, as in [5].

Notation 2.1. For all $j \in \mathbb{Z}$, we denote

$$\begin{aligned} I_j &= [jm - \rho, (j+1)m - \rho[, \\ S_j &= S \cap I_j, P_j = P \cap I_j, D_j = D \cap I_j. \end{aligned}$$

In particular, we have

$$I_q = [qm - \rho, (q+1)m - \rho[= [c, c + m[.$$

The following set addition rules are shown in [5]. The proof is straightforward and left to the reader.

Proposition 2.2. Let S be a numerical semigroup. For all $i, j \geq 1$, we have

$$\begin{aligned} S_1 + S_j &\subseteq S_{1+j} \cup S_{2+j}, \\ S_i + S_j &\subseteq S_{i+j-1} \cup S_{i+j} \cup S_{i+j+1} \quad \text{if } i, j \geq 2. \quad \square \end{aligned}$$

Notation 2.3. The level function $\lambda_S: \mathbb{N} \rightarrow \mathbb{N}$ associated to S is defined by

$$\lambda_S(x) = j \iff x \in I_j \iff jm - \rho \leq x \leq (j+1)m - \rho - 1$$

for all $x \in \mathbb{N}$. In particular, if $x \in S$ then $\lambda_S(x) = j$ if and only if $x \in S_j$.

In the sequel, for simplicity, we shall write λ for λ_S . Using this function, the above proposition translates as follows.

Corollary 2.4. Let $x, y \in S^*$. If $\lambda(x), \lambda(y) \geq 2$ then

$$\lambda(x) + \lambda(y) - 1 \leq \lambda(x+y) \leq \lambda(x) + \lambda(y) + 1. \quad (4)$$

If $\lambda(x) = 1$ or $\lambda(y) = 1$, then $\lambda(x) + \lambda(y) \leq \lambda(x+y) \leq \lambda(x) + \lambda(y) + 1. \quad \square$

Here are some more consequences.

Corollary 2.5. *Let $a, x, y \in S^*$. Then $\lambda(x + y) > \max\{\lambda(x), \lambda(y)\}$. If $\lambda(a + x) = \lambda(a + y)$, then $|\lambda(y) - \lambda(x)| \leq 1$.*

Proof. The first statement directly follows from Corollary 2.4. As for the second one, let $i = \lambda(x), j = \lambda(y)$. We may assume $i \leq j$. Let $k = \lambda(a + x)$. Then $a + x, a + y \in S_k$. Hence $|y - x| = |(a + y) - (a + x)| \leq m - 1$. It follows that $j \leq i + 1$, since if $j \geq i + 2$ then $\min S_j - \max S_i \geq m + 1$. \square

2.2 The number $W_0(S)$

Notation 2.6. *For a numerical semigroup S , we denote*

$$W_0(S) = |P \cap L| |L| - q |D_q| + \rho. \quad (5)$$

Introduced in [5], this number bounds $W(S)$ from below and is sometimes easier to evaluate. See also [2], where $W_0(S)$ is denoted $E(S)$. The following result is Proposition 3.11 in [5]. For convenience, we recall the short proof.

Proposition 2.7. *Let S be a numerical semigroup. Then $W(S) \geq W_0(S)$.*

Proof. We have $W(S) = |P| |L| - c = |P| |L| - qm + \rho$. We have $m = |P_q| + |D_q|$, since $m = |S_q| = |P_q \sqcup D_q|$, and $|P| = |P \cap L| + |P_q|$. It follows that

$$W(S) = W_0(S) + |P_q| (|L| - q).$$

Now $|L| \geq q$, since L contains the q -subset $\{0, 1, \dots, q - 1\}m$. \square

Corollary 2.8. *Let S be a numerical semigroup such that $W_0(S) \geq 0$. Then S satisfies Wilf's conjecture.*

Proof. We have $W(S) \geq W_0(S) \geq 0$. \square

This corollary is the basis of our approach in this paper, whose main result is that $W_0(S) \geq 0$ whenever $|L| \leq 12$. Note that in contrast to Wilf's conjecture, the number $W_0(S)$ can be negative, but such cases are extremely rare. For instance, among the more than 10^{13} numerical semigroups of genus $g \leq 60$, only five of them satisfy $W_0(S) < 0$. See [2, 7, 9]. More specifically, these five exceptions all satisfy $W_0(S) = -1$, $|L| = 13$ and $c = 4m$, and they occur at genus 43, 51, 55, 55 and 59, respectively. The first one, of genus $g = 43$, is $S = \langle 14, 22, 23 \rangle \cup [56, \infty[$.

The following result has been established in [5].

Theorem 2.9. *Let S be a numerical semigroup of depth $q \leq 3$. Then $W_0(S) \geq 0$. In particular, S satisfies Wilf's conjecture.*

Consequently, in proving here that $W_0(S) \geq 0$ if $|L| \leq 12$, we only need to consider the case of depth $q \geq 4$. The next three sections focus on the Apéry set of S with respect to m and provide tools to evaluate $W_0(S)$ and prove our main result.

2.3 The Apéry profile of S

Let $S \subseteq \mathbb{N}$ be a numerical semigroup. We denote by

$$A = \text{Ap}(S, m) = \{x \in S \mid x - m \notin S\}$$

the Apéry set of S with respect to m . Equivalently, $A = S \setminus (m + S)$. Each Apéry element $x \in A$ is the smallest element in S of its class mod m , since $x - m \notin S$. Hence $|A| = m$. We have

$$A \subseteq [0, c + m[.$$

Indeed, this follows from the inclusion $[c + m, \infty[= m + [c, \infty[\subseteq m + S$. We now introduce the Apéry profile of S .

Notation 2.10. *Let S be a numerical semigroup of depth $q \geq 1$. For all $0 \leq i \leq q$, we set $A_i = A \cap I_i$ and*

$$\alpha_i = |A_i|.$$

We have $A_0 = \{0\}$, so $\alpha_0 = 1$. Moreover, $P_1 = \{m\} \sqcup A_1$ and $P_i \subseteq A_i$ for all $i \geq 2$.

Definition 2.11. *We call Apéry profile of S the $(q - 1)$ -tuple*

$$\alpha(S) = (\alpha_1, \dots, \alpha_{q-1}) \in \mathbb{N}^{q-1}.$$

As noted above, we have

$$\alpha_1 = |P_1| - 1 \text{ and } \alpha_i \geq |P_i| \text{ for all } 2 \leq i < q. \quad (6)$$

Moreover, since $A \subseteq [0, c + m[= I_0 \sqcup I_1 \sqcup \dots \sqcup I_q$, we have

$$A = A_0 \sqcup A_1 \sqcup \dots \sqcup A_q. \quad (7)$$

Therefore

$$\sum_{i=0}^q \alpha_i = |A| = m. \quad (8)$$

This justifies why α_0, α_q are not included in the profile $\alpha(S)$, as $\alpha_0 = 1$ and α_q may be recovered from $\alpha(S)$ and m by the above formula.

2.4 Primitive and decomposable Apéry elements

Let S be a numerical semigroup of multiplicity m and Apéry set $A = S \setminus (m + S)$. A key point in the sequel is to distinguish, in $A^* = A \setminus \{0\}$, the primitive elements from the decomposable ones. Indeed, the partition

$$A^* = (A \cap P) \sqcup (A \cap D)$$

plays an important role and motivates the following notation.

Notation 2.12. *Let S be a numerical semigroup of depth $q \geq 1$. For all $1 \leq i \leq q$, we set*

$$\alpha'_i = |A_i \cap P|, \quad \alpha''_i = |A_i \cap D|.$$

Thus $\alpha_i = \alpha'_i + \alpha''_i$ for all i .

Since $P \subseteq [m, c + m[$ as seen above, and since $[m, c + m[\subseteq I_1 \cup \dots \cup I_q$, we have

$$|P| = 1 + \alpha'_1 + \dots + \alpha'_q. \quad (9)$$

In particular,

$$\begin{aligned} \alpha'_q &= |P \setminus L| = |P \cap [c, c + m[|, \\ |D_q| &= |D \cap [c, c + m[| = |[c, c + m[\setminus P|, \\ m &= |P_q| + |D_q|. \end{aligned}$$

The following properties of the Apéry set A will be widely used below, often tacitly so.

Lemma 2.13. *Let $z \in A \cap D$. If $z = x + y$ with $x, y \in S^*$, then $x, y \in A^*$.*

Proof. If $x \notin A$, then $x = m + s$ for some $s \in S$, implying $z = x + y = m + (s + y)$. Since $s + y \in S$, it follows that $z \notin A$, contrary to the hypothesis. \square

Proposition 2.14. *For all $k \geq 2$, we have $A_k \cap D \subseteq \cup_{i,j}(A_i + A_j)$ where $1 \leq i \leq j$ and $k - 1 \leq i + j \leq k + 1$.*

Proof. Directly follows from Proposition 2.2 and Lemma 2.13. \square

Corollary 2.15. *If $A_k \neq \emptyset$ and $A_i = \emptyset$ for all $1 \leq i \leq k - 1$ for some $k \geq 2$, then $A_k = P_k$.*

Proof. Directly follows from the above proposition. \square

2.5 Compressed Apéry elements

Throughout, let S denote a numerical semigroup with multiplicity m , conductor c and Apéry set $A = S \setminus (m + S)$.

Definition 2.16. *Let $s \in S^*$. We say that s is compressed if there exist $x, y \in S^*$ such that $s = x + y$ and $\lambda(s) < \lambda(x) + \lambda(y)$.*

By Corollary 2.4, we have $\lambda(x + y) \geq \lambda(x) + \lambda(y) - 1$ for all $x, y \in S^*$. Thus, the inequality $\lambda(x + y) < \lambda(x) + \lambda(y)$ is equivalent to $\lambda(x + y) = \lambda(x) + \lambda(y) - 1$. Estimating the number of compressed elements in A is important in the sequel. This motivates the following notation.

Notation 2.17. $C = C(S) = \{z \in A \cap D \mid z \text{ is compressed}\}$.

Recall from Lemma 2.13 that if $z \in A \cap D$ and $z = x + y$ with $x, y \in S^*$, then in fact $x, y \in A^*$. Consequently, for all $i, j \geq 2$, we have

$$(S_i + S_j) \cap A_{i+j-1} = (A_i + A_j) \cap A_{i+j-1} \subseteq C. \quad (10)$$

More generally, even if the description below will not be needed here, we have

$$C = \bigcup_{k=3}^q A_k \cap (\cup_{i=1}^{k-1} (A_i + A_{k+1-i})).$$

The next result provides a key lower bound on ρ , where ρ is the offset as defined in (2). See also Proposition 3.20 in [6].

Proposition 2.18. *Let S be a numerical semigroup. Then $\rho \geq |C|$.*

Proof. Let $z \in C$, and assume $z = x + y$ with $x, y \in A^*$ such that $\lambda(z) = \lambda(x) + \lambda(y) - 1$. Say $\lambda(x) = i$, $\lambda(y) = j$ and $\lambda(z) = i + j - 1$. By the definition of S_i , we have

$$(S_i + S_j) \cap S_{i+j-1} \subseteq [(i + j)m - 2\rho, (i + j)m - \rho[.$$

Thus $z \in [(i + j)m - 2\rho, (i + j)m - \rho[$. Now, the only classes mod m occurring in the latter interval are those in $[-2\rho, -\rho[$, a set of cardinality ρ . Since there is only one element in A per class mod m , and since $C \subset A$, the statement follows. \square

In particular, we shall invoke the following simplified version.

Corollary 2.19. *For all $i, j \geq 2$, we have $\rho \geq |(A_i + A_j) \cap A_{i+j-1}|$.*

Proof. Follows from (10) and Proposition 2.18. \square

2.6 Computing $W_0(S)$

The following formulas allow to evaluate $W_0(S)$ using the Apéry profile of S as defined in Definition 2.10 and the decomposition $\alpha_q = \alpha'_q + \alpha''_q$ given by Notation 2.12. Recall that both $|L|$ and $|D_q|$ are involved in the expression of $W_0(S)$.

Proposition 2.20. *Let $S \subseteq \mathbb{N}$ be a numerical semigroup with Apéry profile $\alpha(S) = (\alpha_1, \dots, \alpha_{q-1})$. Then*

$$\begin{aligned} |L| &= q + (q-1)\alpha_1 + \dots + \alpha_{q-1}, \\ |D_q| &= \alpha_0 + \alpha_1 + \dots + \alpha_{q-1} + \alpha''_q. \end{aligned}$$

Proof. Let $s \in L$ be minimal in its class mod m . Then $s \in L \cap A$. Let $i \geq 0$ be the unique integer such that $s \in A_i$. Then $0 \leq i \leq q-1$ since $s \in L$. Let $z \in L$ be such that $z \equiv s \pmod{m}$. Then $z = s + jm$ with $0 \leq j \leq q-1-i$. It follows that

$$|L \cap (s + m\mathbb{N})| = q - i.$$

Letting now s run through all elements of L which are minimal in their respective classes mod m , the above discussion implies

$$|L| = q|A_0| + (q-1)|A_1| + \dots + |A_{q-1}|,$$

yielding the first formula. Since $m = |P_q| + |D_q| = \alpha'_q + |D_q|$, it follows that

$$|D_q| = m - \alpha'_q = m - \alpha_q + \alpha''_q.$$

The second formula now follows from (8), i.e. $m = \sum_{i=0}^q \alpha_i$. □

2.7 Notation

We shall use the following notation throughout the remainder of the paper. Given a numerical semigroup S , we denote by m its multiplicity, by c its conductor, by q its depth, by ρ its offset, by L its left part, by P its set of primitive elements, by D its set of decomposable elements, by $A = S \setminus (m + S)$ its Apéry set with respect to m , and by

$$\alpha(S) = (\alpha_1, \dots, \alpha_{q-1})$$

its Apéry profile, where $\alpha_i = |A_i|$ for all $0 \leq i \leq q$. For $i \geq 1$, we have $\alpha_i = \alpha'_i + \alpha''_i$ where $\alpha'_i = |A_i \cap P|$ and $\alpha''_i = |A_i \cap D|$. We shall constantly use the formulas below to compute the ingredients involved in

$$W_0(S) = |P \cap L| |L| - q |D_q| + \rho,$$

namely

$$|P \cap L| = 1 + \sum_{i=1}^{q-1} \alpha'_i, \quad |L| = \sum_{i=0}^{q-1} (q-i)\alpha_i, \quad |D_q| = \sum_{i=0}^{q-1} \alpha_i + \alpha''_q. \quad (11)$$

Finally, as in the preceding section, we denote by C the set of compressed Apéry elements, i.e

$$C = \{z \in A \cap D \mid \exists x, y \in A^*, z = x + y, \lambda(z) < \lambda(x) + \lambda(y)\}.$$

In a few cases, the estimate $\rho \geq |C|$ provided by Proposition 2.18 will be crucially needed in order to be able to conclude $W_0(S) \geq 0$.

3 An occurrence of $W_0(S) \geq 0$

In this section, we establish $W_0(S) \geq 0$ for numerical semigroups S under suitable assumptions on $\alpha(S)$ but not on $|L|$. We use the notation of Section 2.7 throughout.

Theorem 3.1. *Let S be a numerical semigroup of depth $q \geq 4$. Let $h = \lceil q/2 \rceil$. Assume that $\alpha_i = 0$ for all $1 \leq i \leq h-1$. Then $W_0(S) \geq 0$.*

Proof. Since $P = \cup_{1 \leq i \leq q} P_i$ and $P_1 = \{m\} \sqcup A_1$, we have $|P| = 1 + \alpha_1 + \alpha'_2 + \cdots + \alpha'_q$ and

$$|P \cap L| = 1 + \alpha_1 + \alpha'_2 + \cdots + \alpha'_{q-1}. \quad (12)$$

• **Assume q odd.** Then $h = (q+1)/2$. Since $A_1 = \cdots = A_{h-1} = \emptyset$, we have

$$A \cap D \subseteq \bigcup_{i,j=h}^{q-1} (A_i + A_j).$$

By Proposition 2.2, we have $A_i + A_j \subseteq S_{i+j-1} \sqcup S_{i+j} \sqcup S_{i+j+1}$. Since $A_t = \emptyset$ for $t \geq q+1$, and since $2h = q-1$, it follows from the above that

$$A \cap D \subseteq 2A_h.$$

Hence $\alpha''_q \leq |2A_h|$. It also follows that $A_i = P_i$ for all $h \leq i \leq q-1$. Hence

$$\begin{aligned} |P \cap L| &= 1 + \alpha_h + \cdots + \alpha_{q-1} \\ &\geq 1 + \alpha_h, \\ |L| &= q + (h-1)\alpha_h + (h-2)\alpha_{h+1} + \cdots + \alpha_{q-1} \\ &\geq q + (h-1)\alpha_h, \\ |D_q| &= 1 + \alpha_h + \cdots + \alpha_{q-1} + \alpha''_q \\ &= |P \cap L| + \alpha''_q. \end{aligned}$$

Thus,

$$\begin{aligned}
W_0(S) &= |P \cap L||L| - q|D_q| + \rho \\
&= |P \cap L||L| - q(|P \cap L| + \alpha_q'') + \rho \\
&= |P \cap L|(|L| - q) - q\alpha_q'' + \rho \\
&= (1 + \alpha_h)(h - 1)\alpha_h - q\alpha_q'' + \rho.
\end{aligned}$$

Since $A \cap D_q \subseteq 2A_h$, and since $q = 2h - 1$, it follows from Lemma 2.18 that $A_q \cap D \subseteq C$, where C is the subset defined in that Lemma. Consequently, we have $|A_q \cap D| \leq |C| \leq \rho$, whence

$$\alpha_q'' \leq \min(|2A_h|, \rho).$$

Therefore

$$W_0(S) \geq (1 + \alpha_h)(h - 1)\alpha_h - q \min(|2A_h|, \rho) + \rho.$$

The following bound will take care of the last two summands.

Claim. For all $t \geq 0$, we have

$$-q \min(t, \rho) + \rho \geq -(q - 1)t. \quad (13)$$

Indeed, if $\rho \leq t$, then $-q \min(t, \rho) + \rho = -(q - 1)\rho \geq -(q - 1)t$. And if $\rho > t$, then $-q \min(t, \rho) + \rho = -qt + \rho > -(q - 1)t$. This proves the claim.

Moreover, as a very crude estimate, we have

$$|2A_h| \leq \alpha_h(\alpha_h + 1)/2.$$

Hence, using (13) and the above, we get

$$W_0(S) \geq (1 + \alpha_h)(h - 1)\alpha_h - (q - 1)\alpha_h(\alpha_h + 1)/2.$$

Since $(q - 1)/2 = h - 1$, it follows that $W_0(S) \geq 0$, as desired.

• **Assume q even.** Then $h = q/2$. Since $A_1 = \dots = A_{h-1} = \emptyset$, we have

$$A \cap D \subseteq \bigcup_{i,j=h}^{q-1} (A_i + A_j).$$

By Proposition 2.2, we have $A_i + A_j \subseteq S_{i+j-1} \sqcup S_{i+j} \sqcup S_{i+j+1}$. Since $A_t = \emptyset$ for $t \geq q + 1$, and since $2h = q$, it follows from the above that

$$A \cap D \subseteq (2A_h) \cup (A_h + A_{h+1}).$$

Moreover, we have

$$2A_h \cap A \subseteq A_{q-1} \cup A_q, \quad (A_h + A_{h+1}) \cap A \subseteq A_q.$$

Hence

$$A \cap D \subseteq (2A_h \cap A_{q-1}) \cup (2A_h \cap A_q) \cup (A_h + A_{h+1}) \cap A_q.$$

We have

$$\begin{aligned} \alpha''_{q-1} &= |2A_h \cap A_{q-1}|, \\ \alpha''_q &= |(2A_h \cap A_q) \cup (A_h + A_{h+1}) \cap A_q|. \end{aligned}$$

Hence

$$\alpha''_{q-1} + \alpha''_q \leq |2A_h| + |(A_h + A_{h+1}) \cap A_q|. \quad (14)$$

Moreover, since $q = 2h$, Corollary 2.19 yields

$$|(A_h + A_{h+1}) \cap A_q| \leq \rho.$$

Moreover, we have $|(A_h + A_{h+1}) \cap A_q| \leq \alpha_h \alpha_{h+1}$. Hence

$$|(A_h + A_{h+1}) \cap A_q| \leq \min(\rho, \alpha_h \alpha_{h+1}). \quad (15)$$

Combining (14) and (15), we get

$$\alpha''_{q-1} + \alpha''_q \leq |2A_h| + \min(\rho, \alpha_h \alpha_{h+1}). \quad (16)$$

It also follows that $A_i = P_i$ for all $h \leq i \leq q-2$. Hence

$$\begin{aligned} |P \cap L| &= 1 + \alpha_h + \cdots + \alpha_{q-2} + \alpha'_{q-1} \\ &\geq 1 + \alpha_h + \alpha_{h+1}, \\ |L| &= q + h\alpha_h + (h-1)\alpha_{h+1} + \cdots + \alpha_{q-1} \\ &\geq q + h\alpha_h + (h-1)\alpha_{h+1}, \\ |D_q| &= 1 + \alpha_h + \cdots + \alpha_{q-1} + \alpha''_q \\ &= |P \cap L| + \alpha''_{q-1} + \alpha''_q. \end{aligned}$$

Thus,

$$\begin{aligned} W_0(S) &= |P \cap L| |L| - q |D_q| + \rho \\ &= |P \cap L| |L| - q(|P \cap L| + \alpha''_{q-1} + \alpha''_q) + \rho \\ &= |P \cap L|(|L| - q) - q(\alpha''_{q-1} + \alpha''_q) + \rho \\ &\geq (1 + \alpha_h + \alpha_{h+1})(h\alpha_h + (h-1)\alpha_{h+1} - q(\alpha''_{q-1} + \alpha''_q)) + \rho. \end{aligned}$$

Therefore, using (16), we get

$$W_0(S) \geq (1 + \alpha_h + \alpha_{h+1})(h\alpha_h + (h-1)\alpha_{h+1}) - q(|2A_h| + \min(\rho, \alpha_h\alpha_{h+1})) + \rho.$$

Using (13) again for the last two summands, we have

$$-q \min(\rho, \alpha_h\alpha_{h+1}) + \rho \geq -(q-1)\alpha_h\alpha_{h+1}.$$

We also have the very crude estimate

$$|2A_h| \leq \alpha_h(\alpha_h + 1)/2.$$

Hence

$$W_0(S) \geq (1 + \alpha_h + \alpha_{h+1})(h\alpha_h + (h-1)\alpha_{h+1}) - q\alpha_h(\alpha_h + 1)/2 - (q-1)\alpha_h\alpha_{h+1}.$$

Using $q = 2h$, it follows that

$$W_0(S) = (h-1)\alpha_{h+1}(\alpha_{h+1} + 1).$$

Hence $W_0(S) \geq 0$, as desired. \square

We conclude this section with an easy particular case.

Proposition 3.2. *Let S be a numerical semigroup of depth $q \geq 4$ such that $|P \cap L| \geq \max(\alpha''_q, q)$. Then $W_0(S) \geq \rho$.*

Proof. $W_0(S) = |P \cap L||L| - q|D_q| + \rho$. Recall that $|L| = \sum_{i=0}^{q-1} (q-i)\alpha_i$ and that $|D_q| = \sum_{i=0}^{q-1} \alpha_i + \alpha''_q$. Hence

$$W_0(S) = q(|P \cap L| - \alpha''_q) + \sum_{i=1}^{q-1} ((q-i)|P \cap L| - q)\alpha_i + \rho.$$

Since $|P \cap L| \geq \alpha''_q$ and $|P \cap L| \geq q$ by hypothesis, the claimed inequality follows. \square

4 Main result

Let S be a numerical semigroup. We use the notation of Section 2.7 throughout. Wilf's conjecture has been successively settled for $|L| \leq 4$ and $|L| \leq 6$ in [4] and [5], respectively. Here we extend these results to the case $|L| \leq 12$. Even more so, we show that if $|L| \leq 12$ then $W_0(S) \geq 0$. As mentioned earlier, this is best possible, since there are numerical semigroups S satisfying $|L| = 13$ and $W_0(S) < 0$. At the time of writing, it remains an open problem to determine whether all numerical semigroups S with $|L| = 13$ satisfy Wilf's conjecture. In this section we prove the following result.

Theorem 4.1. *Let S be a numerical semigroup such that $|L| \leq 12$. Then $W_0(S) \geq 0$. In particular, S satisfies Wilf's conjecture.*

By Theorem 2.9, the bound $W_0(S) \geq 0$ holds for all numerical semigroups of depth $q \leq 3$. Consequently, in the sequel, we shall freely assume $q \geq 4$, since it suffices to prove Theorem 4.1 in that case. In fact, it also suffices to consider the case $q \leq 7$, as follows from the following proposition.

Proposition 4.2. *Let S be a numerical semigroup of depth $q \geq 8$ such that $|L| \leq 12$. Then $W_0(S) \geq 0$.*

Proof. Let $h = \lceil q/2 \rceil$. Then $12 \geq |L| \geq q + (q-1)\alpha_1 + \cdots + (q-h+1)\alpha_{h-1}$. This implies $\alpha_i = 0$ for all $1 \leq i \leq h-1$. For if not, then $|L| \geq q + (q-h+1)$, and since $h \leq (q+1)/2$, we would get $12 \geq |L| \geq 2q - (q+1)/2 + 1 = (3q+1)/2$ and hence $3q+1 \leq 24$, contrary to the hypothesis $q \geq 8$. It now follows from Theorem 3.1 that $W_0(S) \geq 0$. \square

Finally, the following result strongly restricts the values of α_1 to consider.

Lemma 4.3. *Let S be a numerical semigroup of depth $q \geq 4$ such that $|L| \leq 12$. Then $\alpha_1 \leq 2$.*

Proof. By Proposition 2.20, we have $|L| = q + (q-1)\alpha_1 + (q-2)\alpha_2 + \cdots + \alpha_{q-1}$. Hence $|L| \geq q + (q-1)\alpha_1$. We have $q \geq 4$. If $\alpha_1 \geq 3$ then $|L| \geq 4 + 9 = 13$, contrary to the hypothesis on $|L|$. \square

The cases $\alpha_1 = 2, 1$ and 0 will now be treated successively. We shall occasionally use the following notation.

Notation 4.4. *For all $i \geq 1$, we denote by $\mathbf{1}_i = \mathbf{1}_{A_i}$ the indicator function of A_i .*

4.1 When $\alpha_1 = 2$

Proposition 4.5. *Let S be a numerical semigroup of depth $q \geq 4$ such that $|L| \leq 12$. If $\alpha_1 = 2$, then $q = 4$ and $W_0(S) \geq 0$.*

Proof. Since $\alpha_1 = 2$, we have $12 \geq |L| \geq q + 2(q - 1) = 3q - 2$. It follows that $q < 5$, whence $q = 4$ since $q \geq 4$ by hypothesis. We have

$$|P \cap L| \geq 3, \quad |L| = 10 + 2\alpha_2 + \alpha_3, \quad |D_4| = 3 + \alpha_2 + \alpha_3 + \alpha_4''.$$

Hence $2\alpha_2 + \alpha_3 \leq 2$ and so $\alpha_2 \leq 1$.

• If $\alpha_2 = 0$, then $|L| = 10 + \alpha_3$ and so $\alpha_3 \leq 2$. Since $2A_1 \subset S_2 \cup S_3$, and since $A_2 = \emptyset$ and $2S_3 \cap S_4 = \emptyset$, it follows that

$$A_4 \cap D \subseteq (A_1 + A_3).$$

Hence $\alpha_4'' \leq 2\alpha_3$. Therefore

$$\begin{aligned} W_0(S) &\geq 3(10 + \alpha_3) - 4(3 + \alpha_3 + 2\alpha_3) + \rho \\ &= 18 - 9\alpha_3 + \rho \\ &\geq \rho. \end{aligned}$$

• If $\alpha_2 = 1$, then $|L| = 12 + \alpha_3$ and so $|L| = 12$ and $\alpha_3 = 0$. Since $2A_1 \subset S_2 \cup S_3$ and $A_3 = \emptyset$, it follows that

$$A_4 \cap D \subseteq (A_1 + A_2) \cup (2A_2).$$

Hence $\alpha_4'' \leq 2 + 1 = 3$. Therefore

$$\begin{aligned} W_0(S) &\geq 3 \cdot 12 - 4(3 + 1 + 3) + \rho \\ &= 8 + \rho. \quad \square \end{aligned}$$

Remark 4.6. *A better lower bound on $W_0(S)$ may be obtained by splitting α_i as $\alpha_i' + \alpha_i''$ for $i = 2, 3$ in the above proof. For instance, we have only used $|P \cap L| \geq 3$. But we could have used $|P \cap L| \geq 4$ if either α_2' or α_3' were assumed positive, while if $\alpha_2' = \alpha_3' = 0$, a sharper estimate on α_4'' can be derived.*

4.2 When $\alpha_1 = 1$

Since $12 \geq |L| \geq q + (q - 1) = 2q - 1$, it follows that $q \leq 6$. We shall successively treat the cases $q = 4, 5$ and 6 . Throughout Section 4.2, we set

$$A_1 = \{x\}.$$

4.2.1 Case $q = 4$

Then $\alpha(S) = (1, \alpha_2, \alpha_3)$. We have $|L| = 4 + 3 + 2\alpha_2 + \alpha_3$, whence $2\alpha_2 + \alpha_3 \leq 5$, implying $\alpha_2 \leq 2$. We successively examine the cases $\alpha_2 = 2, 1, 0$. To start with, we have

$$A_4 \cap D \subseteq (A_1 + A_2) \cup (A_1 + A_3) \cup (A_2 + A_2) \cup (A_2 + A_3). \quad (17)$$

Subcase $\alpha_2 = 2$. Then $\alpha(S) = (1, 2, \alpha_3)$. We have $|L| = 11 + \alpha_3$, whence $\alpha_3 \leq 1$. Denote

$$A_2 = \{y_1, y_2\}.$$

Since $A_2 \cap D \subseteq 2A_1$, and since $|2A_1| = 1$, we have $\alpha_2'' \leq 1$ whence $\alpha_2' \in \{1, 2\}$.

• Assume first $\alpha_2' = 1$. Say $y_1 \in P, y_2 \in D$. Then $y_2 = 2x$, and $|P \cap L| = 3 + \alpha_3'$. We have $|D_4| = 4 + \alpha_3 + \alpha_4''$, and

$$(A_1 + A_2) \cup (2A_2) = \{4x, 3x, 2x + y_1, x + y_1, 2y_1\}.$$

Since $4x \notin S_3$ and since $\alpha_3 \leq 1$, there are at most four possibilities for $A_3 \cap D$, listed below together with their consequences on $A_4 \cap D$. Note that Lemma 2.13 plays a key role to deduce these consequences. For example, if $4x \in A_4$ or $2x + y_1 \in A_4$, then necessarily $3x \in A_3$ or $x + y_1 \in A_3$, respectively. Note also that the level function $\lambda = \lambda_S$ is nondecreasing. Consequently, in the last case $A_3 = \{2y_1\}$ below, it follows that $x + y_1 \notin A_4$, for $\lambda(x + y_1) \leq \lambda(2y_1) = 3$ since $x < y_1$. Here then are the possibilities for $A_3 \cap D$:

1. If $A_3 \cap D = \emptyset$ then $A_4 \cap D \subseteq \{3x, x + y_1, 2y_1\}$.
2. If $A_3 \cap D = \{3x\}$ then $A_4 \cap D \subseteq \{4x, x + y_1, 2y_1\}$.
3. If $A_3 \cap D = \{x + y_1\}$ then $A_4 \cap D \subseteq \{3x, 2x + y_1, 2y_1\}$.
4. If $A_3 \cap D = \{2y_1\}$ then $A_4 \cap D \subseteq \{3x, 2y_1\}$.

In either case, we have $\alpha_4'' \leq 3$. Recall also that $\alpha_3 \leq 1$ here. Hence

$$\begin{aligned} W_0(S) &= (3 + \alpha_3')(11 + \alpha_3) - 4(4 + \alpha_3 + \alpha_4'') + \rho \\ &\geq (3 + \alpha_3')(11 + \alpha_3) - 4(7 + \alpha_3) + \rho \\ &\geq 4 + 11\alpha_3' + \alpha_3'\alpha_3 + \rho. \end{aligned}$$

• Assume now $\alpha_2' = 2$, so that $y_1, y_2 \in P$. Then $|P \cap L| = 4 + \alpha_3'$. We have $|D_4| = 4 + \alpha_3 + \alpha_4''$, and

$$(2A_1) \cup (A_1 + A_2) \cup (2A_2) = \{2x, x + y_1, x + y_2, 2y_1, y_1 + y_2, 2y_2\}.$$

Up to permutation of y_1, y_2 , and using $\alpha_3 \leq 1$ and $2x \notin S_4$, here are the possibilities for $A_3 \cap D$, together with their consequences for $A_4 \cap D$:

1. If $A_3 \cap D = \emptyset$ then $A_4 \cap D \subseteq \{x + y_1, x + y_2, 2y_1, y_1 + y_2, 2y_2\}$.
2. If $A_3 \cap D = \{2x\}$ then $A_4 \cap D \subseteq \{3x, x + y_1, x + y_2, 2y_1, y_1 + y_2, 2y_2\}$.
3. If $A_3 \cap D = \{x + y_1\}$ then $A_4 \cap D \subseteq \{x + y_2, 2y_1, y_1 + y_2, 2y_2\}$.
4. If $A_3 \cap D = \{2y_1\}$ then $A_4 \cap D \subseteq \{x + y_1, x + y_2, y_1 + y_2, 2y_2\}$.
5. If $A_3 \cap D = \{y_1 + y_2\}$ then $A_4 \cap D \subseteq \{x + y_1, x + y_2, 2y_1, 2y_2\}$.

In either case, we have $\alpha_4'' \leq 6$. Hence

$$\begin{aligned} W_0(S) &= (4 + \alpha_3')(11 + \alpha_3) - 4(4 + \alpha_3 + \alpha_4'') + \rho \\ &\geq (4 + \alpha_3')(11 + \alpha_3) - 4(10 + \alpha_3) + \rho \\ &= 4 + 11\alpha_3' + \alpha_3'\alpha_3 + \rho. \end{aligned}$$

Subcase $\alpha_2 = 1$. Then $\alpha(S) = (1, 1, \alpha_3)$. We have $|L| = 9 + \alpha_3$, whence $\alpha_3 \leq 3$. We also have $|P \cap L| = 2 + \alpha_2' + \alpha_3'$ and $|D_4| = 3 + \alpha_3 + \alpha_4''$. Denote

$$A_2 = \{y\}.$$

• Assume first $y \in A_2 \cap D$. Then $y = 2x$, $|P \cap L| = 2 + \alpha_3'$ and $A_3 \cap D \subseteq \{3x\}$. Thus $\alpha_3'' \leq 1$, and either

$$A_4 \cap D \subseteq \{3x\} \cup (x + P_3) \quad \text{or} \quad A_4 \cap D \subseteq \{4x\} \cup (2x + P_3) \cup (x + P_3).$$

We claim that $\alpha_4'' \leq 1 + \alpha_3'$ in both cases. This is clear in the first one. In the second one, for all $z \in P_3$ we have

$$|A_4 \cap \{2x + z, x + z\}| \leq 1.$$

Therefore $|A_4 \cap ((2x + P_3) \cup (x + P_3))| \leq |P_3|$, implying $\alpha_4'' \leq 1 + \alpha_3'$ here as well. Using $\alpha_3 = \alpha_3' + \alpha_3'' \leq \alpha_3' + 1$, we have

$$\begin{aligned} W_0(S) &= (2 + \alpha_3')(9 + \alpha_3) - 4(3 + \alpha_3 + \alpha_4'') + \rho \\ &\geq (2 + \alpha_3')(9 + \alpha_3) - 4(4 + \alpha_3' + \alpha_3) + \rho \\ &= 2 + 5\alpha_3' - 2\alpha_3 + \alpha_3\alpha_3' + \rho \\ &\geq 2 + 5\alpha_3' - 2(\alpha_3' + 1) + \alpha_3\alpha_3' + \rho \\ &\geq 3\alpha_3' + \alpha_3\alpha_3' + \rho. \end{aligned}$$

- Assume now $y \in A_2 \cap P$. Hence $|P \cap L| = 3 + \alpha'_3$. Here we have

$$A_3 \cap D \subseteq \{2x, x+y, 2y\}$$

and so $\alpha''_3 \leq 3$. Let us examine in turn the possibilities for $A_3 \cap D$ and their consequences for $A_4 \cap D$:

1. If $A_3 \cap D = \emptyset$ then $A_4 \cap D \subseteq \{x+y, 2y\} \cup (\{x, y\} + P_3)$.
2. If $A_3 \cap D = \{2x\}$ then $A_4 \cap D \subseteq \{3x, x+y, 2y\} \cup (\{x, y\} + P_3)$.
3. If $A_3 \cap D = \{x+y\}$ then $A_4 \cap D \subseteq \{2y\} \cup (\{x, y\} + P_3)$.
4. If $A_3 \cap D = \{2y\}$ then $A_4 \cap D \subseteq \{x+y, 3y\} \cup (\{x, y\} + P_3)$.
5. If $A_3 \cap D = \{2x, x+y\}$ then $A_4 \cap D \subseteq \{3x, 2x+y, 2y\} \cup (\{x, y\} + P_3)$.
6. If $A_3 \cap D = \{2x, 2y\}$ then $A_4 \cap D \subseteq \{3x, x+y, 2y\} \cup (\{x, y\} + P_3)$.
7. If $A_3 \cap D = \{x+y, 2y\}$ then $A_4 \cap D \subseteq \{x+2y, 3y\} \cup (\{x, y\} + P_3)$.
8. If $A_3 \cap D = \{2x, x+y, 2y\}$ then $A_4 \cap D \subseteq \{3x, 2x+y, x+2y, 3y\} \cup (\{x, y\} + P_3)$.

Note that $|\{x, y\} + P_3| \leq 2|P_3| = 2\alpha'_3$. Consequently, distinguishing between the first seven cases and the last one, we have

$$\begin{aligned} \alpha''_3 \leq 2 &\Rightarrow \alpha''_4 \leq 3 + 2\alpha'_3 \\ \alpha''_3 = 3 &\Rightarrow \alpha''_4 \leq 4 + 2\alpha'_3. \end{aligned}$$

- Assume first $\alpha''_3 \leq 2$, so that $\alpha_3 \leq \alpha'_3 + 2$ and $\alpha''_4 \leq 3 + 2\alpha'_3$. We have

$$\begin{aligned} W_0(S) &= (3 + \alpha'_3)(9 + \alpha_3) - 4(3 + \alpha_3 + \alpha''_4) + \rho \\ &\geq (3 + \alpha'_3)(9 + \alpha_3) - 4(6 + \alpha_3 + 2\alpha'_3) + \rho \\ &= 3 + \alpha'_3 - \alpha_3 + \alpha'_3\alpha_3 + \rho \\ &\geq 1 + \alpha'_3\alpha_3 + \rho. \end{aligned}$$

- Assume now $\alpha''_3 = 3$, so that $\alpha''_4 \leq 4 + 2\alpha'_3$. Since $\alpha_3 \leq 3$, it follows that $\alpha_3 = \alpha''_3 = 3$, whence $\alpha'_3 = 0$ and $\alpha''_4 \leq 4$. We have

$$\begin{aligned} W_0(S) &= (3 + \alpha'_3)(9 + \alpha_3) - 4(3 + \alpha_3 + \alpha''_4) + \rho \\ &\geq 3 \cdot 12 - 4 \cdot 10 + \rho \\ &= -4 + \rho. \end{aligned}$$

We now show that $\rho \geq 6$ here. Indeed, since $y \in A_2$ and $3y \in A_4$, we have

$$2m - \rho \leq y, \quad 3y \leq 5m - \rho - 1.$$

Therefore $3(2m - \rho) \leq 3y \leq 5m - \rho - 1$, implying $m \leq 2\rho - 1$. Now, since $m = 1 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \geq 10$ by (8), it follows that $\rho \geq 6$, whence $W_0(S) \geq 2$ and we are done.

Subcase $\alpha_2 = 0$. Then $\alpha(S) = (1, 0, \alpha_3)$ here, and incidentally $\alpha_3 \leq 5$ since $|L| \leq 12$. We have

$$|P \cap L| = 2 + \alpha'_3, \quad |L| = 7 + \alpha_3, \quad |D_4| = 2 + \alpha_3 + \alpha''_4.$$

Recalling that $A_1 = \{x\}$, we distinguish the cases where $2x \in A$ or not.

• Assume first $2x \in A$. Since $\lambda(2x) \in \{2, 3\}$ and since $A_2 = \emptyset$, it follows that $2x \in A_3$. Hence $\alpha''_3 = 1$ and so $\alpha_3 = 1 + \alpha'_3$. Thus

$$|L| = 8 + \alpha'_3, \quad |D_4| = 3 + \alpha'_3 + \alpha''_4.$$

Since $A_4 \cap D \subseteq \{3x\} \cup P_3$, it follows that $\alpha''_4 \leq 1 + \alpha'_3$ and hence $|D_4| \leq 4 + 2\alpha'_3$. A straightforward computation then yields

$$W_0(S) \geq \rho.$$

• Assume now $2x \notin A$. It follows that $A_3 \subseteq P$, i.e. $\alpha_3 = \alpha'_3$, and $A_4 \cap D \subseteq x + A_3$. Therefore $\alpha''_4 \leq \alpha_3$ and so $|D_4| = 2 + \alpha_3 + \alpha''_4 \leq 2 + 2\alpha_3$. This implies here

$$W_0(S) \geq 6 + \rho.$$

This concludes the case $\alpha_1 = 1$ and $q = 4$.

4.2.2 Case $q = 5$

We now tackle the case $\alpha_1 = 1$ and $q = 5$, i.e. $\alpha(S) = (1, \alpha_2, \alpha_3, \alpha_4)$. We have $|L| = 9 + 3\alpha_2 + 2\alpha_3 + \alpha_4$, whence $3\alpha_2 + 2\alpha_3 + \alpha_4 \leq 3$, implying $\alpha_2 \leq 1$.

Subcase $\alpha_2 = 1$. Then $\alpha(S) = (1, 1, \alpha_3, \alpha_4)$. Then $|L| = 12$ and $\alpha_3 = \alpha_4 = 0$. Moreover, $|D_5| = \alpha_0 + \alpha_1 + \alpha_2 + \alpha'_5 = 3 + \alpha'_5$. Set

$$A_2 = \{y\}.$$

• Assume first $y \in D$. Then $y = 2x$. Since $A_3 = A_4 = \emptyset$ by hypothesis, it follows that $|P \cap L| = 2$ and $A_5 \cap D \subseteq \{3x\}$. Therefore $\alpha_5'' \leq 1$. We conclude that

$$\begin{aligned} W_0(S) &= 2 \cdot 12 - 5(3 + \alpha_5'') + \rho \\ &\geq 4 + \rho. \end{aligned}$$

• Assume now $y \in P$. Then $|P \cap L| = 3$ and $A_5 \cap D \subseteq \{2y\}$ since $\lambda(2x) \leq 3$ and $\lambda(x+y) \leq 4$. Hence $\alpha_5'' \leq 1$ and

$$\begin{aligned} W_0(S) &= 3 \cdot 12 - 5(3 + \alpha_5'') + \rho \\ &\geq 16 + \rho. \end{aligned}$$

Subcase $\alpha_2 = 0$. Then $\alpha(S) = (1, 0, \alpha_3, \alpha_4)$ and $|L| = 9 + 2\alpha_3 + \alpha_4$. Hence $\alpha_3 \leq 1$. Moreover, $|D_5| = \alpha_0 + \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5'' = 2 + \alpha_3 + \alpha_4 + \alpha_5''$. We have $A_2 = \emptyset$.

Subsubcase $\alpha_3 = 1$. Then $\alpha(S) = (1, 0, 1, \alpha_4)$. We have $|L| = 11 + \alpha_4$, hence $\alpha_4 \leq 1$. Since $A_2 = \emptyset$, we have $2x \in A$ if and only if $2x \in A_3$.

• Assume first $2x \in A$. Then $A_3 = \{2x\}$ since $A_2 = \emptyset$ and $\alpha_3 = 1$. Hence $A_4 \cap D \subseteq \{3x\}$ and $A_5 \cap D \subseteq \{3x, 4x\} \cup (x + A_4 \cap P)$. Therefore

$$\alpha_4'' = \mathbf{1}_4(3x), \quad \alpha_5'' \leq \mathbf{1}_5(3x) + \mathbf{1}_5(4x) + \alpha_4'. \quad (18)$$

Note also that if $4x \in A_5$, then $A_4 = \{3x\}$ by Lemma 2.13 and the bound $\alpha_4 \leq 1$. We have

$$|P \cap L| = 2 + \alpha_4', \quad |L| = 11 + \alpha_4, \quad |D_5| = 3 + \alpha_4 + \alpha_5'' = 3 + \alpha_4' + \mathbf{1}_4(3x) + \alpha_5''.$$

A straightforward computation, using (18), then yields

$$\begin{aligned} W_0(S) &= (2 + \alpha_4')(11 + \alpha_4) - 5(3 + \alpha_4' + \mathbf{1}_4(3x) + \alpha_5'') + \rho \\ &\geq (2 + \alpha_4')(11 + \alpha_4) - 5(3 + 2\alpha_4' + \mathbf{1}_4(3x) + \mathbf{1}_5(3x) + \mathbf{1}_5(4x)) + \rho \\ &\geq 7 - 3 \cdot \mathbf{1}_4(3x) - 5 \cdot \mathbf{1}_5(3x) - 5 \cdot \mathbf{1}_5(4x) + \rho. \end{aligned}$$

If $4x \notin A_5$ then $W_0(S) \geq 2 + \rho$ since $\mathbf{1}_4(3x) + \mathbf{1}_5(3x) \leq 1$ and we are done. If $4x \in A_5$, then $3x \in A_4$ as noted above, whence $W_0(S) \geq -1 + \rho$. But then $4x \in C$, since $\lambda(4x) = 5$ whereas $\lambda(2x) = 3$. Thus $\rho \geq |C| \geq 1$, implying $W_0(S) \geq 0$, as desired.

• Assume now $2x \notin A$. Then $\alpha'_3 = \alpha_3 = 1$ and $|P \cap L| = 3 + \alpha'_4$. We have $A_5 \cap D \subseteq (A_1 + A_3) \cup (A_1 + A_4)$, whence $\alpha''_5 \leq 2$ since $\alpha_1 = \alpha_3 = 1$ and $\alpha_4 \leq 1$. Therefore $|D_5| = 3 + \alpha_4 + \alpha''_5 \leq 6$, so that

$$\begin{aligned} W_0(S) &= (3 + \alpha'_4)(11 + \alpha_4) - 5|D_5| + \rho \\ &\geq 3 \cdot 11 - 30 + \rho \\ &\geq 3 + \rho. \end{aligned}$$

Subsubcase $\alpha_3 = 0$. Then $\alpha(S) = (1, 0, 0, \alpha_4)$. We have $|L| = 9 + \alpha_4$, hence $\alpha_4 \leq 3$ and $|D_5| = 2 + \alpha_4 + \alpha''_5$. Since $A_2 = A_3 = \emptyset$, and since $2 \leq \lambda(2x) \leq 3$, it follows that $2x \notin A$ and $A_4 = P_4$. Thus $A_5 \cap D \subseteq x + P_4$, so that $\alpha''_5 \leq \alpha_4$ and $|P \cap L| = 2 + \alpha_4$. It follows that

$$\begin{aligned} W_0(S) &\geq (2 + \alpha'_4)(9 + \alpha_4) - 5(2 + \alpha_4 + \alpha''_5) + \rho \\ &\geq (2 + \alpha'_4)(9 + \alpha_4) - 5(2 + 2\alpha_4) + \rho \\ &\geq 8 + \alpha_4 + \alpha_4^2 + \rho. \end{aligned}$$

This concludes the case $\alpha_1 = 1$ and $q = 5$.

4.2.3 Case $q = 6$

Still for $\alpha_1 = 1$, we now tackle the last case $q = 6$. Then $\alpha(S) = (1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$. We have $|L| = 11 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5$, whence $\alpha_2 = \alpha_3 = \alpha_4 = 0$ and $\alpha_5 \leq 1$.

Subcase $\alpha_5 = 1$. Then $\alpha(S) = (1, 0, 0, 0, 1)$. We have $|L| = 12$ and $|D_6| = \alpha_0 + \alpha_1 + \alpha_5 + \alpha''_6 = 3 + \alpha''_6$. Recall that $A_1 = \{x\}$. Since $\lambda(2x) \leq 3$ and $A_2 = A_3 = A_4 = \emptyset$, it follows that $2x \notin A$. Whence $A_5 = P_5$, i.e. $\alpha'_5 = \alpha_5 = 1$, and $A_6 \cap D \subseteq x + A_5$. Hence $|P \cap L| = 4$ and $\alpha''_6 \leq 1$. Consequently $|D_6| \leq 4$, and

$$\begin{aligned} W_0(S) &\geq 3 \cdot 12 - 6 \cdot 4 + \rho \\ &= 12 + \rho. \end{aligned}$$

Subcase $\alpha_5 = 0$. Then $\alpha(S) = (1, 0, 0, 0, 0)$. We have $|L| = 11$ and $|D_6| = 2 + \alpha''_6$. Since $A_1 = \{x\}$ and A_2, A_3, A_4, A_5 are all empty, it follows that $|P \cap L| = 2$ and $A_6 \cap D = \emptyset$, i.e. $\alpha''_6 = 0$. Thus

$$W_0(S) = 2 \cdot 11 - 6 \cdot 2 + \rho = 10 + \rho.$$

Summarizing, we have shown that if $|L| \leq 12$, $\alpha_1 = 1$ and $q = 6$, then $W_0(S) \geq 10 + \rho$. This concludes the case $|L| \leq 12$ and $\alpha_1 = 1$.

4.3 When $\alpha_1 = 0$

As noted at the beginning of Section 4, it suffices to consider the cases $4 \leq q \leq 7$.

4.3.1 Case $q = 4$

Then $\alpha(S) = (0, \alpha_2, \alpha_3)$. Since $\lceil q/2 \rceil = 2$ here, Theorem 3.1 yields $W_0(S) \geq 0$.

4.3.2 Case $q = 5$

Then $\alpha(S) = (0, \alpha_2, \alpha_3, \alpha_4)$. We have $12 \geq |L| = 5 + 3\alpha_2 + 2\alpha_3 + \alpha_4$. Hence $\alpha_2 \leq 2$. We now examine successively the cases $\alpha_2 = 2, 1, 0$. Since $A_1 = \emptyset$, it follows that $A_2 \subset P$.

Subcase $\alpha_2 = 2$. Then $\alpha(S) = (0, 2, \alpha_3, \alpha_4)$. We have $|L| = 11 + 2\alpha_3 + \alpha_4$. Hence $\alpha_3 = 0$ and $\alpha_4 \leq 1$. Since $A_2 \subset P$, we have $\alpha'_2 = \alpha_2 = 2$ here. Thus $|P \cap L| \geq 3$. Set

$$A_2 = \{x_1, x_2\}.$$

Assume first $\alpha_4 = 1$. Then $\alpha(S) = (0, 2, 0, 1)$, so that $|L| = 12$ and $|D_5| = 4 + \alpha''_5$. Denote $A_4 = \{z\}$.

- If $z \in P$, then $|P \cap L| = 4$ and $2A_2 \cap (A_3 \cup A_4) = \emptyset$. Now

$$A_5 \cap D \subseteq 2A_2 \cup (A_2 + A_4).$$

Since $|2A_2| \leq 3$ and $|A_2 + A_4| \leq |A_2|$, it follows that $\alpha''_5 \leq 5$. Hence

$$\begin{aligned} W_0(S) &= |P \cap L||L| - 5|D_5| + \rho \\ &\geq 4 \cdot 12 - 5 \cdot 9 + \rho \\ &= 3 + \rho \end{aligned}$$

and we are done if $z \in P$.

- If $z \in D$, then $|P \cap L| = 3$ and $z \in A_2$ since $A_3 = \emptyset$. Hence, up to renumbering, either $z = 2x_1$ or $z = x_1 + x_2$. If $z = 2x_1$ then $A_5 \cap D \subseteq \{3x_1, x_1 + x_2, 2x_2\}$, whereas if $z = x_1 + x_2$, then $A_5 \cap D \subseteq \{2x_1, 2x_2\}$. In either case, we have $\alpha''_5 \leq 3$. Therefore $|D_5| \leq 7$. It follows that

$$\begin{aligned} W_0(S) &= |P \cap L||L| - 5|D_5| + \rho \\ &\geq 3 \cdot 12 - 5 \cdot 7 + \rho \\ &= 1 + \rho \end{aligned}$$

and we are done as well if $z \in D$.

Assume now $\alpha_4 = 0$. Then $\alpha(S) = (0, 2, 0, 0)$, so that $|L| = 11$ and $|D_5| = 3 + \alpha_5''$. In that case, we have $A_5 \subseteq 2A_2$, whence $\alpha_5'' \leq 3$ and $|D_5| \leq 6$. It follows that

$$W_0(S) \geq 3 \cdot 11 - 5 \cdot 6 + \rho \geq 3 + \rho.$$

That concludes the subcase $\alpha_2 = 2$, i.e. $\alpha(S) = (0, 2, \alpha_3, \alpha_4)$ here.

Subcase $\alpha_2 = 1$. Then $\alpha(S) = (0, 1, \alpha_3, \alpha_4)$. We have $|L| = 8 + 2\alpha_3 + \alpha_4$. Hence $\alpha_3 \leq 2$. Since $A_2 \subset P$, we have $\alpha_2' = \alpha_2 = 1$ here. Set

$$A_2 = \{x\}.$$

Subsubcase $\alpha_3 = 2$. Then $\alpha(S) = (0, 1, 2, \alpha_4)$. We have $|L| = 12 + \alpha_4$, whence $|L| = 12$ and $\alpha_4 = 0$. Thus $\alpha(S) = (0, 1, 2, 0)$. Hence $|D_5| = 4 + \alpha_5''$. Since $A_3 \cap D \subseteq 2A_2$, it follows that $\alpha_3'' \leq 1$ and hence $\alpha_3' \in \{1, 2\}$. Therefore $|P \cap L| \geq 3$. Set

$$A_3 = \{y_1, y_2\}.$$

• Assume first $\alpha_3' = 1$. Hence $\alpha_3'' = 1$, and up to renumbering, we may assume $y_1 = 2x, y_2 \in P$. Hence $|P \cap L| = 3$ and

$$A_5 \cap D \subseteq \{3x, x + y_2, 2y_2\}.$$

Therefore $\alpha_5'' \leq 3$ and so $|D_5| \leq 7$. We then have

$$\begin{aligned} W_0(S) &= |P \cap L| |L| - 5 |D_5| + \rho \\ &\geq 3 \cdot 12 - 5 \cdot 7 + \rho \\ &= 1 + \rho \end{aligned}$$

and we are done here.

• Assume now $\alpha_3' = \alpha_3 = 2$. Thus $y_1, y_2 \in P$ and so $|P \cap L| = 4$. We have

$$A_5 \cap D \subseteq \{2x\} \cup (x + A_3) \cup (2A_3 \cap A_5).$$

Thus $\alpha_5'' \leq 3 + |2A_3 \cap A_5|$, and of course $|2A_3 \cap A_5| \leq |2A_3| \leq 3$ since $|A_3| = 2$. At this point we have

$$|D_5| \leq 7 + |2A_3 \cap A_5|.$$

Therefore

$$\begin{aligned}
W_0(S) &= |P \cap L| |L| - 5|D_5| + \rho \\
&\geq 4 \cdot 12 - 5 \cdot (7 + |2A_3 \cap A_5|) + \rho \\
&= 13 - 5|2A_3 \cap A_5| + \rho.
\end{aligned}$$

Now $\rho \geq |2A_3 \cap A_5|$, since $2A_3 \cap A_5 \subseteq C$ and $|C| \leq \rho$. Hence

$$W_0(S) \geq 13 - 4|2A_3 \cap A_5|.$$

But $|2A_3 \cap A_5| \leq 3$ as seen above. It follows that $W_0(S) \geq 1$ and we are done here.

Subsubcase $\alpha_3 = 1$. Then $\alpha(S) = (0, 1, 1, \alpha_4)$. We have

$$|P \cap L| = 2 + \alpha'_3 + \alpha'_4, \quad |L| = 10 + \alpha_4, \quad |D_5| = 3 + \alpha_4 + \alpha''_5. \quad (19)$$

We examine four subcases, depending on which multiples of x belong to A .

Case 1. $2x \in A_3, 3x \in A_4$. Then $A_3 = \{2x\}$ since $\alpha_3 = 1$, and $A_4 \cap D = \{3x\}$. Hence $\alpha_4 = \alpha'_4 + 1$. Moreover,

$$A_5 \cap D \subseteq \{4x\} \cup (x + A_4 \cap P).$$

Whether $4x$ belongs to A_5 or not will be measured by $\mathbf{1}_5(4x) \in \{0, 1\}$, where $\mathbf{1}_i = \mathbf{1}_{A_i}$ denotes the indicator function of A_i for all i . Thus,

$$\alpha''_5 \leq \mathbf{1}_5(4x) + \alpha'_4,$$

and so

$$|P \cap L| = 2 + \alpha'_4, \quad |L| = 11 + \alpha'_4, \quad |D_5| = 4 + \alpha'_4 + \alpha''_5 \leq 4 + 2\alpha'_4 + \mathbf{1}_5(4x). \quad (20)$$

Plugging this data into (5), we get

$$W_0(S) \geq 2 + 3\alpha'_4 + \alpha_4^2 - 5 \cdot \mathbf{1}_5(4x) + \rho.$$

If either $\alpha'_4 \neq 0$ or $\mathbf{1}_5(4x) = 0$, then $W_0(S) \geq \rho$ and we are done. If $\alpha'_4 = 0$ and $\mathbf{1}_5(4x) = 1$, then $W_0(S) \geq -3 + \rho$. But in this case, we claim that $\rho \geq 3$. Indeed, both $2x, 3x$ are *compressed* Apéry elements, since $\lambda(x) = 2, \lambda(2x) = 3, \lambda(3x) = 4$ and $2x = x + x, 3x = x + 2x$. Moreover, since $\mathbf{1}_5(4x) = 1$ here, then $4x$ is also a compressed Apéry element, using $\lambda(4x) = 5$ and $4x = 2x + 2x$. Hence $\{2x, 3x, 4x\} \subseteq C$. Since $\rho \geq |C|$, this proves $\rho \geq 3$ as claimed and yields $W_0(S) \geq 0$, as desired.

Case 2. $2x \in A_3, 3x \notin A_4$. Then $A_3 = \{2x\}$ and $A_4 \subset P$, so that $\alpha_4 = \alpha'_4$. We have

$$A_5 \cap D \subseteq \{3x\} \cup (x + A_4).$$

Hence $\alpha''_5 \leq 1 + \alpha_4$, and so

$$|P \cap L| = 2 + \alpha_4, \quad |L| = 10 + \alpha_4, \quad |D_5| = 3 + \alpha_4 + \alpha''_5 \leq 4 + 2\alpha_4.$$

Hence $W_0(S) \geq 2\alpha_4 + \alpha_4^2 + \rho$ and we are done in the present case.

Case 3. $2x \in A_4$. Then $A_3 = A_3 \cap P$, so that $\alpha'_3 = \alpha_3 = 1$. We set $A_3 = \{y\}$. We have $\{2x\} \subseteq A_4 \cap D \subseteq \{2x, x + y\}$ and $A_5 \cap D \subseteq \{3x, 2y, x + y\}$. Since $x + y$ cannot belong to both A_4, A_5 , it follows that $\alpha''_4 + \alpha''_5 \leq 4$.

Assume first $\alpha_4 = 2$. We then have

$$|P \cap L| = 3 + \alpha'_4, \quad |L| = 12, \quad |D_5| = 3 + \alpha'_4 + \alpha''_4 + \alpha''_5 \leq 7 + \alpha'_4. \quad (21)$$

It follows that $W_0(S) \geq 12(3 + \alpha'_4) - 5(7 + \alpha'_4) + \rho \geq 1 + \rho$ and we are done.

Assume now $\alpha_4 = 1$. Then $A_4 = \{2x\}$ and $|L| = 11$. If $\alpha''_5 \leq 2$, then

$$|P \cap L| = 3, \quad |L| = 11, \quad |D_5| = 4 + \alpha''_5 \leq 6.$$

It follows that $W_0(S) \geq 3 \cdot 11 - 5 \cdot 6 + \rho \geq 3 + \rho$ and we are done. But if $\alpha''_5 = 3$, i.e. if $A_5 \cap D = \{3x, 2y, x + y\}$, then we only get $W_0(S) \geq -2 + \rho$. But in this case, since $3x, 2y \in C$, it follows that $\rho \geq 2$ whence $W_0(S) \geq 0$ as desired.

Case 4. $2x \notin A$. Then as above, $A_3 = A_3 \cap P$, so that $\alpha'_3 = \alpha_3 = 1$, and we set $A_3 = \{y\}$. If $x + y \in A$, then it belongs to either A_4 or A_5 . But in any case, $x + 2y \notin A$ since $\lambda(x + 2y) \geq 6$, as easily seen. It follows that $A_4 \cap D \subseteq \{x + y\}$ and $A_5 \cap D \subseteq \{x + y, 2y\}$. Hence

$$\alpha_4 = \alpha'_4 + \mathbf{1}_4(x + y), \quad \alpha''_5 \leq 1 + \mathbf{1}_5(x + y).$$

It follows that

$$|P \cap L| = 3 + \alpha'_4, \quad |L| = 10 + \alpha'_4 + \mathbf{1}_4(x + y), \quad |D_5| \leq 4 + \alpha'_4 + \mathbf{1}_4(x + y) + \mathbf{1}_5(x + y).$$

A straightforward computation, using $\mathbf{1}_4(x + y) + \mathbf{1}_5(x + y) \leq 1$, then yields

$$\begin{aligned} W_0(S) &\geq 10 + 8\alpha'_4 + \alpha_4'^2 + (\alpha'_4 - 2) \cdot \mathbf{1}_4(x + y) - 5 \cdot \mathbf{1}_5(x + y) + \rho \\ &\geq 5 + \rho \end{aligned}$$

and we are done.

Subsubcase $\alpha_3 = 0$. Then $\alpha(S) = (0, 1, 0, \alpha_4)$. We have

$$|P \cap L| = 2 + \alpha'_4, \quad |L| = 8 + \alpha_4, \quad |D_5| = 2 + \alpha_4 + \alpha''_5.$$

Here $A_3 = \emptyset$, and $\alpha_4 \leq 4$ since $|L| \leq 12$ by assumption.

Case 1. $2x \in A$. Then $2x \in A_4 \sqcup A_5$. Thus

$$A_4 \cap D \subseteq \{2x\}, \quad A_5 \cap D \subseteq \{2x, 3x\} \cup (x + A_4 \cap P).$$

Hence $\alpha_4 = \alpha'_4 + \alpha''_4 = \alpha'_4 + \mathbf{1}_4(2x)$ and $\alpha''_5 \leq \mathbf{1}_5(2x) + \mathbf{1}_5(3x) + \alpha'_4$. We have

$$|P \cap L| = 2 + \alpha'_4, \quad |L| = 8 + \alpha'_4 + \mathbf{1}_4(2x), \quad |D_5| \leq 2 + 2\alpha'_4 + \mathbf{1}_4(2x) + \mathbf{1}_5(2x) + \mathbf{1}_5(3x).$$

A straightforward computation then yields

$$W_0(S) \geq 6 - 3 \cdot \mathbf{1}_4(2x) - 5 \cdot \mathbf{1}_5(2x) - 5 \cdot \mathbf{1}_5(3x) + \rho.$$

If $3x \notin A$, then $\mathbf{1}_5(3x) = 0$ and $W_0(S) \geq 1 + \rho$. The case $3x \in A$ is more delicate. It implies $3x \in A_5$, i.e. $\mathbf{1}_5(3x) = 1$, and then of course $2x \in A_4$, i.e. $\mathbf{1}_4(2x) = 1$ and $\mathbf{1}_5(2x) = 0$. Thus $W_0(S) \geq -2 + \rho$. It remains to show $\rho \geq 2$. Since $(\lambda(x), \lambda(2x), \lambda(3x)) = (2, 4, 5)$ here, it follows that $3x \in C$, whence $\rho \geq 1$. This is not strong enough yet. However, since $(\lambda(2x), \lambda(3x)) = (4, 5)$, we have

$$4m - \rho \leq 2x, \quad 3x \leq 6m - \rho - 1.$$

Therefore $3(4m - \rho)/2 \leq 3x \leq 6m - \rho - 1$, implying $\rho \geq 2$ as desired, and hence $W_0(S) \geq 0$.

Case 2. $2x \notin A$. Then $A_3 = A_4 \cap D = \emptyset$. Hence $A_4 = A_4 \cap P$ and $A_5 \cap D \subseteq x + A_4$. That is, we have $\alpha_4 = \alpha'_4$ and $\alpha''_5 \leq \alpha_4$. Thus $|D_5| \leq 2 + 2\alpha_4$, and a straightforward computation yields

$$W_0(S) \geq 6 + \rho.$$

4.3.3 Case $q = 6$

Then $\alpha(S) = (0, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$. We have $12 \geq |L| = 6 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5$, whence $\alpha_2 \leq 1$.

Subcase $\alpha_2 = 1$. Then $|L| = 10 + 3\alpha_3 + 2\alpha_4 + \alpha_5$, implying $\alpha_3 = 0$ and $\alpha_4 \leq 1$. Thus $\alpha(S) = (0, 1, 0, \alpha_4, \alpha_5)$. We set

$$A_2 = \{x\},$$

with $x \in P$ since $A_2 \cap D = \emptyset$. We have $A_3 = \emptyset$.

Subsubcase $\alpha_4 = 1$. Then $|L| = 12$, whence $\alpha_5 = 0$ and so $\alpha(S) = (0, 1, 0, 1, 0)$. Set $A_4 = \{z\}$. Since $A_3 = A_5 = \emptyset$, it follows that $2x \in A$ if and only if $2x = z$. That is, either $z \in A \cap P$ and $2x \notin A$, or else $z = 2x$. In either case, we have

$$A_6 \cap D \subseteq x + A_4,$$

whence $\alpha_6'' \leq 1$. Summarizing, we have

$$|P \cap L| = 2 + \alpha_4', \quad |L| = 12, \quad |D_6| = 3 + \alpha_6'' \leq 4.$$

It follows that

$$W_0(S) \geq (2 + \alpha_4') \cdot 12 - 6 \cdot 4 + \rho \geq \rho$$

and we are done here.

Subsubcase $\alpha_4 = 0$. Then $\alpha(S) = (0, 1, 0, 0, \alpha_5)$. Since $\lambda(2x) \in [3, 5]$ and $A_3 = A_4 = \emptyset$, we have $2x \in A$ if and only if $2x \in A_5$. Therefore $A_5 \cap D \subseteq \{2x\}$, i.e. $\alpha_5'' \leq 1$. However $3x \notin A_6$. For if $3x \in A_6$, then $2x \in A_5$ by Lemma 2.13, whence

$$5m - \rho \leq 2x, \quad 3x \leq 7m - \rho - 1.$$

Thus $3(5m - \rho)/2 \leq 3x \leq 7m - \rho - 1$, whence $\rho \geq m + 2$, in contradiction with $\rho \in [0, m[$. Therefore $A_6 \cap D \subseteq x + (A_5 \cap P)$, i.e. $\alpha_6'' \leq \alpha_5'$. Summarizing, we have

$$|P \cap L| = 2 + \alpha_5', \quad |L| = 10 + \alpha_5, \quad |D_6| = 2 + \alpha_5 + \alpha_6'' \leq 2 + \alpha_5 + \alpha_5'.$$

A straightforward computation then yields

$$W_0(S) \geq 8 - 4\alpha_5'' + \rho,$$

whence $W_0(S) \geq 4 + \rho$ since $\alpha_5'' \leq 1$.

Subcase $\alpha_2 = 0$. Then $\alpha(S) = (0, 0, \alpha_3, \alpha_4, \alpha_5)$ and Theorem 3.1 yields $W_0(S) \geq 0$.

4.3.4 Case $q = 7$

Here $\alpha(S) = (0, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$. Then $12 \geq |L| \geq 7 + 5\alpha_2$, whence $\alpha_2 \leq 1$.

• If $\alpha_2 = 1$, then $|L| = 12$ and $\alpha_3 = \dots = \alpha_6 = 0$. It follows that $A \cap D = \emptyset$, whence $\alpha_7'' = 0$. Thus $|P \cap L| = 2$ and $|D_7| = 2$. Therefore

$$W_0(S) = 2 \cdot 12 - 7 \cdot 2 + \rho = 10 + \rho$$

and we are done in this case.

- If $\alpha_2 = 0$, then $|L| = 7 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$, whence $\alpha_3 \leq 1$.

- Assume $\alpha_3 = 1$. Then $\alpha(S) = (0, 0, 1, \alpha_4, \alpha_5, \alpha_6)$ and $|L| = 11 + 2\alpha_5 + \alpha_6$, whence $\alpha_4 = \alpha_5 = 0$ and $\alpha_6 \leq 1$. Thus in fact, $\alpha(S) = (0, 0, 1, 0, 0, \alpha_6)$ and $|L| = 11 + \alpha_6$. We set

$$A_3 = \{x\}.$$

Of course $x \in P$ since $A_1 = A_2 = \emptyset$. Since $A_4 = A_5 = \emptyset$, it follows that $A_6 \cap D \subseteq \{2x\}$. Moreover, since $(S_3 + S_6) \cap S_7 = \emptyset$, we also have $A_7 \cap D \subseteq \{2x\}$. Thus

$$\alpha''_6 = \mathbf{1}_6(2x), \quad \alpha''_7 = \mathbf{1}_7(2x).$$

Summarizing, we have

$$|P \cap L| = 2 + \alpha'_6, \quad |L| = 11 + \alpha_6, \quad |D_7| = 2 + \alpha_6 + \alpha''_7 = 2 + \alpha'_6 + \mathbf{1}_6(2x) + \mathbf{1}_7(2x).$$

A straightforward computation, using $\mathbf{1}_6(2x) + \mathbf{1}_7(2x) \leq 1$, then yields

$$W_0(S) \geq 1 + \rho$$

and we are done.

- Finally, assume $\alpha_3 = 0$. Then $\alpha(S) = (0, 0, 0, \alpha_4, \alpha_5, \alpha_6)$. In that case, since $\alpha_1 = \dots = \alpha_{h-1} = 0$ where $h = \lceil q/2 \rceil = 4$, Theorem 3.1 implies $W_0(S) \geq 0$.

4.3.5 Case $q \geq 8$

For depth $q \geq 8$, Proposition 4.2 yields $W_0(S) \geq 0$.

4.4 Concluding remarks

Having examined above all possible Apéry profiles $\alpha(S)$ compatible with $|L| \leq 12$, the proof of Theorem 4 is now complete.

As mentioned earlier, among the more than 10^{13} numerical semigroups of genus $g \leq 60$, exactly five of them satisfy $W_0(S) < 0$. These five exceptions have depth $q = 4$ and Apéry profile $\alpha(S) = (2, 0, 3)$, yielding $|L| = 4 + 2 \cdot 3 + 3 = 13$. Moreover, they satisfy $c = 4m$, $|P \cap L| = |P_1| = 3$ and $\alpha''_4 = 4$, whence $|D_4| = 10$ and $W_0(S) = 3 \cdot 13 - 4 \cdot 10 = -1$.

Infinite families of numerical semigroups satisfying $W_0(S) < 0$ have been constructed in [2, 7]. However, they all turn out to satisfy $W(S) \geq 0$. It would be very interesting to characterize those S satisfying $W_0(S) < 0$, but that will likely be hard to achieve. Less ambitiously, can one determine how many such cases occur in, say, genus $g \leq 100$?

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