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# On numerical semigroups with at most 12 left elements

S. Eliahou and D. Marín-Aragón

## Abstract

For a numerical semigroup  $S \subseteq \mathbb{N}$  with embedding dimension  $e$ , conductor  $c$  and left part  $L = S \cap [0, c - 1]$ , set  $W(S) = e|L| - c$ . In 1978 Wilf asked, in equivalent terms, whether  $W(S) \geq 0$  always holds, a question known since as Wilf's conjecture. Using a closely related lower bound  $W_0(S) \leq W(S)$ , we show that if  $|L| \leq 12$  then  $W_0(S) \geq 0$ , thereby settling Wilf's conjecture in this case. This is best possible, since cases are known where  $|L| = 13$  and  $W_0(S) = -1$ . Wilf's conjecture remains open for  $|L| \geq 13$ .

## 1 Introduction

Let  $\mathbb{N} = \{0, 1, 2, \dots\}$  denote as usual the set of nonnegative integers. Given integers  $a \leq b$ , we denote by  $[a, b[ = [a, b - 1] = \{z \in \mathbb{Z} \mid a \leq z < b\}$ , and by  $[a, \infty[ = \{z \in \mathbb{Z} \mid a \leq z\}$ , the integer intervals they span. A *numerical semigroup* is a submonoid  $S$  of  $(\mathbb{N}, +)$  such that  $|\mathbb{N} \setminus S|$  is finite. Equivalently, it is a subset  $S$  of  $\mathbb{N}$  of the form  $S = \langle a_1, \dots, a_n \rangle = \mathbb{N}a_1 + \dots + \mathbb{N}a_n$  where  $\gcd(a_1, \dots, a_n) = 1$ . The least such  $n$  is called the *embedding dimension* of  $S$  and is often denoted  $e$ . The *multiplicity* of  $S$  is  $m = \min S^*$ , where  $S^* = S \setminus \{0\}$ . The *conductor* of  $S$  is  $c = \max(\mathbb{Z} \setminus S) + 1$ , or equivalently, the least  $c \in \mathbb{N}$  such  $[c, \infty[ \subseteq S$ . The *genus* of  $S$  is  $g = |\mathbb{N} \setminus S|$ . The *left part* of  $S$  is

$$L = \{s \in S \mid s < c\} = S \cap [0, c[.$$

The *left elements* of  $S$  are the elements of  $L$ . Finally, as in [5], we denote

$$W(S) = e|L| - c. \tag{1}$$

In 1978 Wilf asked, in equivalent terms, whether the inequality

$$W(S) \geq 0$$

holds for every numerical semigroup  $S$  [16]. This open question is now known as Wilf's conjecture. Various particular cases have been settled, including the six independent cases  $e \leq 3$ ,  $|L| \leq 6$ ,  $m \leq 18$ ,  $g \leq 60$ ,  $c \leq 3m$  and  $e \geq m/3$ . See e.g. [1, 2, 4, 5, 6, 8, 9, 10, 12, 11, 14, 15]. See also [3] for a recent extensive survey on this topic.

The authors of [4] settled Wilf's conjecture in case  $|L| \leq 4$ . This was later extended in [5], where a certain lower bound  $W_0(S) \leq W(S)$  was introduced and shown to satisfy  $W_0(S) \geq 0$  whenever  $|L| \leq 6$ . Here we further extend this result by showing that  $W_0(S) \geq 0$  holds whenever  $|L| \leq 12$ . This is best possible since, as shown in [7], there are numerical semigroups  $S$  such that  $|L| = 13$  and  $W_0(S) < 0$ . See also Section 2.2.

This paper is organized as follows. In Section 2 we recall some notation and background, including the definition of  $W_0(S)$ . In Section 3 we establish  $W_0(S) \geq 0$  in some special circumstances. Our main result, namely  $W_0(S) \geq 0$  if  $|L| \leq 12$ , and hence Wilf's conjecture in that case, is proved in Section 4.

For extensive information on numerical semigroups, see [13].

## 2 Background and notation

In this section, we recall some notation and terminology introduced in [5]. Let  $S$  be a numerical semigroup. We denote by  $P \subset S^*$  the unique minimal generating set of  $S$ , so that  $S = \langle P \rangle$  and  $|P| = e$ , the embedding dimension. It coincides with the set of *primitive elements* of  $S$ , i.e. those  $x \in S^*$  which are not the sum of two smaller elements of  $S^*$ . Let  $m, c$  be the multiplicity and conductor of  $S$ , respectively. The *depth* of  $S$  is  $q = \lceil c/m \rceil$  and its *offset* is  $\rho = qm - c$ . Thus

$$c = qm - \rho, \quad \rho \in [0, m[. \quad (2)$$

The set of *decomposable elements* of  $S$  is

$$D = S^* \setminus P = S^* + S^*.$$

Note that  $D$  contains  $[c + m, \infty[$ . Indeed, if  $z \geq c + m$ , then  $z = m + (z - m)$ , so that  $z \in S^* + S^*$  since both  $m \in S^*$  and  $z - m \in S^*$  as  $z - m \geq c$ . It follows that

$$P \subseteq [m, c + m[. \quad (3)$$

Throughout Section 2, the symbols  $m, c, q$  and  $\rho$  will denote, often tacitly so, the multiplicity, conductor, depth and offset of the numerical semigroup  $S$  under consideration, respectively.

## 2.1 The level function $\lambda$

Let  $S \subseteq \mathbb{N}$  be a numerical semigroup. We shall further use the following notation, as in [5].

**Notation 2.1.** For all  $j \in \mathbb{Z}$ , we denote

$$\begin{aligned} I_j &= [jm - \rho, (j+1)m - \rho[, \\ S_j &= S \cap I_j, P_j = P \cap I_j, D_j = D \cap I_j. \end{aligned}$$

In particular, we have

$$I_q = [qm - \rho, (q+1)m - \rho[ = [c, c + m[.$$

The following set addition rules are shown in [5]. The proof is straightforward and left to the reader.

**Proposition 2.2.** Let  $S$  be a numerical semigroup. For all  $i, j \geq 1$ , we have

$$\begin{aligned} S_1 + S_j &\subseteq S_{1+j} \cup S_{2+j}, \\ S_i + S_j &\subseteq S_{i+j-1} \cup S_{i+j} \cup S_{i+j+1} \quad \text{if } i, j \geq 2. \quad \square \end{aligned}$$

**Notation 2.3.** The level function  $\lambda_S: \mathbb{N} \rightarrow \mathbb{N}$  associated to  $S$  is defined by

$$\lambda_S(x) = j \iff x \in I_j \iff jm - \rho \leq x \leq (j+1)m - \rho - 1$$

for all  $x \in \mathbb{N}$ . In particular, if  $x \in S$  then  $\lambda_S(x) = j$  if and only if  $x \in S_j$ .

In the sequel, for simplicity, we shall write  $\lambda$  for  $\lambda_S$ . Using this function, the above proposition translates as follows.

**Corollary 2.4.** Let  $x, y \in S^*$ . If  $\lambda(x), \lambda(y) \geq 2$  then

$$\lambda(x) + \lambda(y) - 1 \leq \lambda(x+y) \leq \lambda(x) + \lambda(y) + 1. \quad (4)$$

If  $\lambda(x) = 1$  or  $\lambda(y) = 1$ , then  $\lambda(x) + \lambda(y) \leq \lambda(x+y) \leq \lambda(x) + \lambda(y) + 1. \quad \square$

Here are some more consequences.

**Corollary 2.5.** *Let  $a, x, y \in S^*$ . Then  $\lambda(x+y) > \max\{\lambda(x), \lambda(y)\}$ . If  $\lambda(a+x) = \lambda(a+y)$ , then  $|\lambda(y) - \lambda(x)| \leq 1$ .*

*Proof.* The first statement directly follows from Corollary 2.4. As for the second one, let  $i = \lambda(x), j = \lambda(y)$ . We may assume  $i \leq j$ . Let  $k = \lambda(a+x)$ . Then  $a+x, a+y \in S_k$ . Hence  $|y-x| = |(a+y) - (a+x)| \leq m-1$ . It follows that  $j \leq i+1$ , since if  $j \geq i+2$  then  $\min S_j - \max S_i \geq m+1$ .  $\square$

## 2.2 The number $W_0(S)$

**Notation 2.6.** *For a numerical semigroup  $S$ , we denote*

$$W_0(S) = |P \cap L||L| - q|D_q| + \rho. \quad (5)$$

Introduced in [5], this number bounds  $W(S)$  from below and is sometimes easier to evaluate. See also [2], where  $W_0(S)$  is denoted  $E(S)$ . The following result is Proposition 3.11 in [5]. For convenience, we recall the short proof.

**Proposition 2.7.** *Let  $S$  be a numerical semigroup. Then  $W(S) \geq W_0(S)$ .*

*Proof.* We have  $W(S) = |P||L| - c = |P||L| - qm + \rho$ . We have  $m = |P_q| + |D_q|$ , since  $m = |S_q| = |P_q \sqcup D_q|$ , and  $|P| = |P \cap L| + |P_q|$ . It follows that

$$W(S) = W_0(S) + |P_q|(|L| - q).$$

Now  $|L| \geq q$ , since  $L$  contains the  $q$ -subset  $\{0, 1, \dots, q-1\}m$ .  $\square$

**Corollary 2.8.** *Let  $S$  be a numerical semigroup such that  $W_0(S) \geq 0$ . Then  $S$  satisfies Wilf's conjecture.*

*Proof.* We have  $W(S) \geq W_0(S) \geq 0$ .  $\square$

This corollary is the basis of our approach in this paper, whose main result is that  $W_0(S) \geq 0$  whenever  $|L| \leq 12$ . Note that in contrast to Wilf's conjecture, the number  $W_0(S)$  can be negative, but such cases are extremely rare. For instance, among the more than  $10^{13}$  numerical semigroups of genus  $g \leq 60$ , only five of them satisfy  $W_0(S) < 0$ . See [2, 7, 9]. More specifically, these five exceptions all satisfy  $W_0(S) = -1$ ,  $|L| = 13$  and  $c = 4m$ , and they occur at genus 43, 51, 55, 55 and 59, respectively. The first one, of genus  $g = 43$ , is  $S = \langle 14, 22, 23 \rangle \cup [56, \infty[$ .

The following result has been established in [5].

**Theorem 2.9.** *Let  $S$  be a numerical semigroup of depth  $q \leq 3$ . Then  $W_0(S) \geq 0$ . In particular,  $S$  satisfies Wilf's conjecture.*

Consequently, in proving here that  $W_0(S) \geq 0$  if  $|L| \leq 12$ , we only need to consider the case of depth  $q \geq 4$ . The next three sections focus on the Apéry set of  $S$  with respect to  $m$  and provide tools to evaluate  $W_0(S)$  and prove our main result.

### 2.3 The Apéry profile of $S$

Let  $S \subseteq \mathbb{N}$  be a numerical semigroup. We denote by

$$A = \text{Ap}(S, m) = \{x \in S \mid x - m \notin S\}$$

the Apéry set of  $S$  with respect to  $m$ . Equivalently,  $A = S \setminus (m + S)$ . Each Apéry element  $x \in A$  is the smallest element in  $S$  of its class mod  $m$ , since  $x - m \notin S$ . Hence  $|A| = m$ . We have

$$A \subseteq [0, c + m[.$$

Indeed, this follows from the inclusion  $[c + m, \infty[ = m + [c, \infty[ \subseteq m + S$ . We now introduce the Apéry profile of  $S$ .

**Notation 2.10.** *Let  $S$  be a numerical semigroup of depth  $q \geq 1$ . For all  $0 \leq i \leq q$ , we set  $A_i = A \cap I_i$  and*

$$\alpha_i = |A_i|.$$

We have  $A_0 = \{0\}$ , so  $\alpha_0 = 1$ . Moreover,  $P_1 = \{m\} \sqcup A_1$  and  $P_i \subseteq A_i$  for all  $i \geq 2$ .

**Definition 2.11.** *We call Apéry profile of  $S$  the  $(q - 1)$ -tuple*

$$\alpha(S) = (\alpha_1, \dots, \alpha_{q-1}) \in \mathbb{N}^{q-1}.$$

As noted above, we have

$$\alpha_1 = |P_1| - 1 \text{ and } \alpha_i \geq |P_i| \text{ for all } 2 \leq i < q. \quad (6)$$

Moreover, since  $A \subseteq [0, c + m[ = I_0 \sqcup I_1 \sqcup \dots \sqcup I_q$ , we have

$$A = A_0 \sqcup A_1 \sqcup \dots \sqcup A_q. \quad (7)$$

Therefore

$$\sum_{i=0}^q \alpha_i = |A| = m. \quad (8)$$

This justifies why  $\alpha_0, \alpha_q$  are not included in the profile  $\alpha(S)$ , as  $\alpha_0 = 1$  and  $\alpha_q$  may be recovered from  $\alpha(S)$  and  $m$  by the above formula.

## 2.4 Primitive and decomposable Apéry elements

Let  $S$  be a numerical semigroup of multiplicity  $m$  and Apéry set  $A = S \setminus (m + S)$ . A key point in the sequel is to distinguish, in  $A^* = A \setminus \{0\}$ , the primitive elements from the decomposable ones. Indeed, the partition

$$A^* = (A \cap P) \sqcup (A \cap D)$$

plays an important role and motivates the following notation.

**Notation 2.12.** *Let  $S$  be a numerical semigroup of depth  $q \geq 1$ . For all  $1 \leq i \leq q$ , we set*

$$\alpha'_i = |A_i \cap P|, \quad \alpha''_i = |A_i \cap D|.$$

Thus  $\alpha_i = \alpha'_i + \alpha''_i$  for all  $i$ .

Since  $P \subseteq [m, c + m[$  as seen above, and since  $[m, c + m[ \subseteq I_1 \cup \dots \cup I_q$ , we have

$$|P| = 1 + \alpha'_1 + \dots + \alpha'_q. \quad (9)$$

In particular,

$$\begin{aligned} \alpha'_q &= |P \setminus L| = |P \cap [c, c + m[|, \\ |D_q| &= |D \cap [c, c + m[| = |[c, c + m[ \setminus P|, \\ m &= |P_q| + |D_q|. \end{aligned}$$

The following properties of the Apéry set  $A$  will be widely used below, often tacitly so.

**Lemma 2.13.** *Let  $z \in A \cap D$ . If  $z = x + y$  with  $x, y \in S^*$ , then  $x, y \in A^*$ .*

*Proof.* If  $x \notin A$ , then  $x = m + s$  for some  $s \in S$ , implying  $z = x + y = m + (s + y)$ . Since  $s + y \in S$ , it follows that  $z \notin A$ , contrary to the hypothesis.  $\square$

**Proposition 2.14.** *For all  $k \geq 2$ , we have  $A_k \cap D \subseteq \cup_{i,j}(A_i + A_j)$  where  $1 \leq i \leq j$  and  $k - 1 \leq i + j \leq k + 1$ .*

*Proof.* Directly follows from Proposition 2.2 and Lemma 2.13.  $\square$

**Corollary 2.15.** *If  $A_k \neq \emptyset$  and  $A_i = \emptyset$  for all  $1 \leq i \leq k - 1$  for some  $k \geq 2$ , then  $A_k = P_k$ .*

*Proof.* Directly follows from the above proposition.  $\square$

## 2.5 Compressed Apéry elements

Throughout, let  $S$  denote a numerical semigroup with multiplicity  $m$ , conductor  $c$  and Apéry set  $A = S \setminus (m + S)$ .

**Definition 2.16.** *Let  $s \in S^*$ . We say that  $s$  is compressed if there exist  $x, y \in S^*$  such that  $s = x + y$  and  $\lambda(s) < \lambda(x) + \lambda(y)$ .*

By Corollary 2.4, we have  $\lambda(x + y) \geq \lambda(x) + \lambda(y) - 1$  for all  $x, y \in S^*$ . Thus, the inequality  $\lambda(x + y) < \lambda(x) + \lambda(y)$  is equivalent to  $\lambda(x + y) = \lambda(x) + \lambda(y) - 1$ . Estimating the number of compressed elements in  $A$  is important in the sequel. This motivates the following notation.

**Notation 2.17.**  $C = C(S) = \{z \in A \cap D \mid z \text{ is compressed}\}$ .

Recall from Lemma 2.13 that if  $z \in A \cap D$  and  $z = x + y$  with  $x, y \in S^*$ , then in fact  $x, y \in A^*$ . Consequently, for all  $i, j \geq 2$ , we have

$$(S_i + S_j) \cap A_{i+j-1} = (A_i + A_j) \cap A_{i+j-1} \subseteq C. \quad (10)$$

More generally, even if the description below will not be needed here, we have

$$C = \bigcup_{k=3}^q A_k \cap (\cup_{i=1}^{k-1} (A_i + A_{k+1-i})).$$

The next result provides a key lower bound on  $\rho$ , where  $\rho$  is the offset as defined in (2). See also Proposition 3.20 in [6].

**Proposition 2.18.** *Let  $S$  be a numerical semigroup. Then  $\rho \geq |C|$ .*

*Proof.* Let  $z \in C$ , and assume  $z = x + y$  with  $x, y \in A^*$  such that  $\lambda(z) = \lambda(x) + \lambda(y) - 1$ . Say  $\lambda(x) = i$ ,  $\lambda(y) = j$  and  $\lambda(z) = i + j - 1$ . By the definition of  $S_i$ , we have

$$(S_i + S_j) \cap S_{i+j-1} \subseteq [(i + j)m - 2\rho, (i + j)m - \rho[.$$

Thus  $z \in [(i + j)m - 2\rho, (i + j)m - \rho[$ . Now, the only classes mod  $m$  occurring in the latter interval are those in  $[-2\rho, -\rho[$ , a set of cardinality  $\rho$ . Since there is only one element in  $A$  per class mod  $m$ , and since  $C \subset A$ , the statement follows.  $\square$

In particular, we shall invoke the following simplified version.

**Corollary 2.19.** *For all  $i, j \geq 2$ , we have  $\rho \geq |(A_i + A_j) \cap A_{i+j-1}|$ .*

*Proof.* Follows from (10) and Proposition 2.18.  $\square$

## 2.6 Computing $W_0(S)$

The following formulas allow to evaluate  $W_0(S)$  using the Apéry profile of  $S$  as defined in Definition 2.10 and the decomposition  $\alpha_q = \alpha'_q + \alpha''_q$  given by Notation 2.12. Recall that both  $|L|$  and  $|D_q|$  are involved in the expression of  $W_0(S)$ .

**Proposition 2.20.** *Let  $S \subseteq \mathbb{N}$  be a numerical semigroup with Apéry profile  $\alpha(S) = (\alpha_1, \dots, \alpha_{q-1})$ . Then*

$$\begin{aligned} |L| &= q + (q-1)\alpha_1 + \dots + \alpha_{q-1}, \\ |D_q| &= \alpha_0 + \alpha_1 + \dots + \alpha_{q-1} + \alpha''_q. \end{aligned}$$

*Proof.* Let  $s \in L$  be minimal in its class mod  $m$ . Then  $s \in L \cap A$ . Let  $i \geq 0$  be the unique integer such that  $s \in A_i$ . Then  $0 \leq i \leq q-1$  since  $s \in L$ . Let  $z \in L$  be such that  $z \equiv s \pmod{m}$ . Then  $z = s + jm$  with  $0 \leq j \leq q-1-i$ . It follows that

$$|L \cap (s + m\mathbb{N})| = q - i.$$

Letting now  $s$  run through all elements of  $L$  which are minimal in their respective classes mod  $m$ , the above discussion implies

$$|L| = q|A_0| + (q-1)|A_1| + \dots + |A_{q-1}|,$$

yielding the first formula. Since  $m = |P_q| + |D_q| = \alpha'_q + |D_q|$ , it follows that

$$|D_q| = m - \alpha'_q = m - \alpha_q + \alpha''_q.$$

The second formula now follows from (8), i.e.  $m = \sum_{i=0}^q \alpha_i$ . □

## 2.7 Notation

We shall use the following notation throughout the remainder of the paper. Given a numerical semigroup  $S$ , we denote by  $m$  its multiplicity, by  $c$  its conductor, by  $q$  its depth, by  $\rho$  its offset, by  $L$  its left part, by  $P$  its set of primitive elements, by  $D$  its set of decomposable elements, by  $A = S \setminus (m + S)$  its Apéry set with respect to  $m$ , and by

$$\alpha(S) = (\alpha_1, \dots, \alpha_{q-1})$$

its Apéry profile, where  $\alpha_i = |A_i|$  for all  $0 \leq i \leq q$ . For  $i \geq 1$ , we have  $\alpha_i = \alpha'_i + \alpha''_i$  where  $\alpha'_i = |A_i \cap P|$  and  $\alpha''_i = |A_i \cap D|$ . We shall constantly use the formulas below to compute the ingredients involved in

$$W_0(S) = |P \cap L| |L| - q |D_q| + \rho,$$

namely

$$|P \cap L| = 1 + \sum_{i=1}^{q-1} \alpha'_i, \quad |L| = \sum_{i=0}^{q-1} (q-i)\alpha_i, \quad |D_q| = \sum_{i=0}^{q-1} \alpha_i + \alpha''_q. \quad (11)$$

Finally, as in the preceding section, we denote by  $C$  the set of compressed Apéry elements, i.e

$$C = \{z \in A \cap D \mid \exists x, y \in A^*, z = x + y, \lambda(z) < \lambda(x) + \lambda(y)\}.$$

In a few cases, the estimate  $\rho \geq |C|$  provided by Proposition 2.18 will be crucially needed in order to be able to conclude  $W_0(S) \geq 0$ .

### 3 An occurrence of $W_0(S) \geq 0$

In this section, we establish  $W_0(S) \geq 0$  for numerical semigroups  $S$  under suitable assumptions on  $\alpha(S)$  but not on  $|L|$ . We use the notation of Section 2.7 throughout.

**Theorem 3.1.** *Let  $S$  be a numerical semigroup of depth  $q \geq 4$ . Let  $h = \lceil q/2 \rceil$ . Assume that  $\alpha_i = 0$  for all  $1 \leq i \leq h-1$ . Then  $W_0(S) \geq 0$ .*

*Proof.* Since  $P = \cup_{1 \leq i \leq q} P_i$  and  $P_1 = \{m\} \sqcup A_1$ , we have  $|P| = 1 + \alpha_1 + \alpha'_2 + \cdots + \alpha'_q$  and

$$|P \cap L| = 1 + \alpha_1 + \alpha'_2 + \cdots + \alpha'_{q-1}. \quad (12)$$

• **Assume  $q$  odd.** Then  $h = (q+1)/2$ . Since  $A_1 = \cdots = A_{h-1} = \emptyset$ , we have

$$A \cap D \subseteq \bigcup_{i,j=h}^{q-1} (A_i + A_j).$$

By Proposition 2.2, we have  $A_i + A_j \subseteq S_{i+j-1} \sqcup S_{i+j} \sqcup S_{i+j+1}$ . Since  $A_t = \emptyset$  for  $t \geq q+1$ , and since  $2h = q-1$ , it follows from the above that

$$A \cap D \subseteq 2A_h.$$

Hence  $\alpha''_q \leq |2A_h|$ . It also follows that  $A_i = P_i$  for all  $h \leq i \leq q-1$ . Hence

$$\begin{aligned} |P \cap L| &= 1 + \alpha_h + \cdots + \alpha_{q-1} \\ &\geq 1 + \alpha_h, \\ |L| &= q + (h-1)\alpha_h + (h-2)\alpha_{h+1} + \cdots + \alpha_{q-1} \\ &\geq q + (h-1)\alpha_h, \\ |D_q| &= 1 + \alpha_h + \cdots + \alpha_{q-1} + \alpha''_q \\ &= |P \cap L| + \alpha''_q. \end{aligned}$$

Thus,

$$\begin{aligned}
W_0(S) &= |P \cap L||L| - q|D_q| + \rho \\
&= |P \cap L||L| - q(|P \cap L| + \alpha_q'') + \rho \\
&= |P \cap L|(|L| - q) - q\alpha_q'' + \rho \\
&= (1 + \alpha_h)(h - 1)\alpha_h - q\alpha_q'' + \rho.
\end{aligned}$$

Since  $A \cap D_q \subseteq 2A_h$ , and since  $q = 2h - 1$ , it follows from Lemma 2.18 that  $A_q \cap D \subseteq C$ , where  $C$  is the subset defined in that Lemma. Consequently, we have  $|A_q \cap D| \leq |C| \leq \rho$ , whence

$$\alpha_q'' \leq \min(|2A_h|, \rho).$$

Therefore

$$W_0(S) \geq (1 + \alpha_h)(h - 1)\alpha_h - q \min(|2A_h|, \rho) + \rho.$$

The following bound will take care of the last two summands.

**Claim.** For all  $t \geq 0$ , we have

$$-q \min(t, \rho) + \rho \geq -(q - 1)t. \quad (13)$$

Indeed, if  $\rho \leq t$ , then  $-q \min(t, \rho) + \rho = -(q - 1)\rho \geq -(q - 1)t$ . And if  $\rho > t$ , then  $-q \min(t, \rho) + \rho = -qt + \rho > -(q - 1)t$ . This proves the claim.

Moreover, as a very crude estimate, we have

$$|2A_h| \leq \alpha_h(\alpha_h + 1)/2.$$

Hence, using (13) and the above, we get

$$W_0(S) \geq (1 + \alpha_h)(h - 1)\alpha_h - (q - 1)\alpha_h(\alpha_h + 1)/2.$$

Since  $(q - 1)/2 = h - 1$ , it follows that  $W_0(S) \geq 0$ , as desired.

• **Assume  $q$  even.** Then  $h = q/2$ . Since  $A_1 = \dots = A_{h-1} = \emptyset$ , we have

$$A \cap D \subseteq \bigcup_{i,j=h}^{q-1} (A_i + A_j).$$

By Proposition 2.2, we have  $A_i + A_j \subseteq S_{i+j-1} \sqcup S_{i+j} \sqcup S_{i+j+1}$ . Since  $A_t = \emptyset$  for  $t \geq q + 1$ , and since  $2h = q$ , it follows from the above that

$$A \cap D \subseteq (2A_h) \cup (A_h + A_{h+1}).$$

Moreover, we have

$$2A_h \cap A \subseteq A_{q-1} \cup A_q, \quad (A_h + A_{h+1}) \cap A \subseteq A_q.$$

Hence

$$A \cap D \subseteq (2A_h \cap A_{q-1}) \cup (2A_h \cap A_q) \cup (A_h + A_{h+1}) \cap A_q.$$

We have

$$\begin{aligned} \alpha''_{q-1} &= |2A_h \cap A_{q-1}|, \\ \alpha''_q &= |(2A_h \cap A_q) \cup (A_h + A_{h+1}) \cap A_q|. \end{aligned}$$

Hence

$$\alpha''_{q-1} + \alpha''_q \leq |2A_h| + |(A_h + A_{h+1}) \cap A_q|. \quad (14)$$

Moreover, since  $q = 2h$ , Corollary 2.19 yields

$$|(A_h + A_{h+1}) \cap A_q| \leq \rho.$$

Moreover, we have  $|(A_h + A_{h+1}) \cap A_q| \leq \alpha_h \alpha_{h+1}$ . Hence

$$|(A_h + A_{h+1}) \cap A_q| \leq \min(\rho, \alpha_h \alpha_{h+1}). \quad (15)$$

Combining (14) and (15), we get

$$\alpha''_{q-1} + \alpha''_q \leq |2A_h| + \min(\rho, \alpha_h \alpha_{h+1}). \quad (16)$$

It also follows that  $A_i = P_i$  for all  $h \leq i \leq q-2$ . Hence

$$\begin{aligned} |P \cap L| &= 1 + \alpha_h + \cdots + \alpha_{q-2} + \alpha'_{q-1} \\ &\geq 1 + \alpha_h + \alpha_{h+1}, \\ |L| &= q + h\alpha_h + (h-1)\alpha_{h+1} + \cdots + \alpha_{q-1} \\ &\geq q + h\alpha_h + (h-1)\alpha_{h+1}, \\ |D_q| &= 1 + \alpha_h + \cdots + \alpha_{q-1} + \alpha''_q \\ &= |P \cap L| + \alpha''_{q-1} + \alpha''_q. \end{aligned}$$

Thus,

$$\begin{aligned} W_0(S) &= |P \cap L| |L| - q |D_q| + \rho \\ &= |P \cap L| |L| - q(|P \cap L| + \alpha''_{q-1} + \alpha''_q) + \rho \\ &= |P \cap L| (|L| - q) - q(\alpha''_{q-1} + \alpha''_q) + \rho \\ &\geq (1 + \alpha_h + \alpha_{h+1})(h\alpha_h + (h-1)\alpha_{h+1} - q(\alpha''_{q-1} + \alpha''_q)) + \rho. \end{aligned}$$

Therefore, using (16), we get

$$W_0(S) \geq (1 + \alpha_h + \alpha_{h+1})(h\alpha_h + (h-1)\alpha_{h+1}) - q(|2A_h| + \min(\rho, \alpha_h\alpha_{h+1})) + \rho.$$

Using (13) again for the last two summands, we have

$$-q \min(\rho, \alpha_h\alpha_{h+1}) + \rho \geq -(q-1)\alpha_h\alpha_{h+1}.$$

We also have the very crude estimate

$$|2A_h| \leq \alpha_h(\alpha_h + 1)/2.$$

Hence

$$W_0(S) \geq (1 + \alpha_h + \alpha_{h+1})(h\alpha_h + (h-1)\alpha_{h+1}) - q\alpha_h(\alpha_h + 1)/2 - (q-1)\alpha_h\alpha_{h+1}.$$

Using  $q = 2h$ , it follows that

$$W_0(S) = (h-1)\alpha_{h+1}(\alpha_{h+1} + 1).$$

Hence  $W_0(S) \geq 0$ , as desired.  $\square$

We conclude this section with an easy particular case.

**Proposition 3.2.** *Let  $S$  be a numerical semigroup of depth  $q \geq 4$  such that  $|P \cap L| \geq \max(\alpha''_q, q)$ . Then  $W_0(S) \geq \rho$ .*

*Proof.*  $W_0(S) = |P \cap L||L| - q|D_q| + \rho$ . Recall that  $|L| = \sum_{i=0}^{q-1} (q-i)\alpha_i$  and that  $|D_q| = \sum_{i=0}^{q-1} \alpha_i + \alpha''_q$ . Hence

$$W_0(S) = q(|P \cap L| - \alpha''_q) + \sum_{i=1}^{q-1} ((q-i)|P \cap L| - q)\alpha_i + \rho.$$

Since  $|P \cap L| \geq \alpha''_q$  and  $|P \cap L| \geq q$  by hypothesis, the claimed inequality follows.  $\square$

## 4 Main result

Let  $S$  be a numerical semigroup. We use the notation of Section 2.7 throughout. Wilf's conjecture has been successively settled for  $|L| \leq 4$  and  $|L| \leq 6$  in [4] and [5], respectively. Here we extend these results to the case  $|L| \leq 12$ . Even more so, we show that if  $|L| \leq 12$  then  $W_0(S) \geq 0$ . As mentioned earlier, this is best possible, since there are numerical semigroups  $S$  satisfying  $|L| = 13$  and  $W_0(S) < 0$ . At the time of writing, it remains an open problem to determine whether all numerical semigroups  $S$  with  $|L| = 13$  satisfy Wilf's conjecture. In this section we prove the following result.

**Theorem 4.1.** *Let  $S$  be a numerical semigroup such that  $|L| \leq 12$ . Then  $W_0(S) \geq 0$ . In particular,  $S$  satisfies Wilf's conjecture.*

By Theorem 2.9, the bound  $W_0(S) \geq 0$  holds for all numerical semigroups of depth  $q \leq 3$ . Consequently, in the sequel, we shall freely assume  $q \geq 4$ , since it suffices to prove Theorem 4.1 in that case. In fact, it also suffices to consider the case  $q \leq 7$ , as follows from the following proposition.

**Proposition 4.2.** *Let  $S$  be a numerical semigroup of depth  $q \geq 8$  such that  $|L| \leq 12$ . Then  $W_0(S) \geq 0$ .*

*Proof.* Let  $h = \lceil q/2 \rceil$ . Then  $12 \geq |L| \geq q + (q-1)\alpha_1 + \cdots + (q-h+1)\alpha_{h-1}$ . This implies  $\alpha_i = 0$  for all  $1 \leq i \leq h-1$ . For if not, then  $|L| \geq q + (q-h+1)$ , and since  $h \leq (q+1)/2$ , we would get  $12 \geq |L| \geq 2q - (q+1)/2 + 1 = (3q+1)/2$  and hence  $3q+1 \leq 24$ , contrary to the hypothesis  $q \geq 8$ . It now follows from Theorem 3.1 that  $W_0(S) \geq 0$ .  $\square$

Finally, the following result strongly restricts the values of  $\alpha_1$  to consider.

**Lemma 4.3.** *Let  $S$  be a numerical semigroup of depth  $q \geq 4$  such that  $|L| \leq 12$ . Then  $\alpha_1 \leq 2$ .*

*Proof.* By Proposition 2.20, we have  $|L| = q + (q-1)\alpha_1 + (q-2)\alpha_2 + \cdots + \alpha_{q-1}$ . Hence  $|L| \geq q + (q-1)\alpha_1$ . We have  $q \geq 4$ . If  $\alpha_1 \geq 3$  then  $|L| \geq 4 + 9 = 13$ , contrary to the hypothesis on  $|L|$ .  $\square$

The cases  $\alpha_1 = 2, 1$  and  $0$  will now be treated successively. We shall occasionally use the following notation.

**Notation 4.4.** *For all  $i \geq 1$ , we denote by  $\mathbf{1}_i = \mathbf{1}_{A_i}$  the indicator function of  $A_i$ .*

## 4.1 When $\alpha_1 = 2$

**Proposition 4.5.** *Let  $S$  be a numerical semigroup of depth  $q \geq 4$  such that  $|L| \leq 12$ . If  $\alpha_1 = 2$ , then  $q = 4$  and  $W_0(S) \geq 0$ .*

*Proof.* Since  $\alpha_1 = 2$ , we have  $12 \geq |L| \geq q + 2(q - 1) = 3q - 2$ . It follows that  $q < 5$ , whence  $q = 4$  since  $q \geq 4$  by hypothesis. We have

$$|P \cap L| \geq 3, \quad |L| = 10 + 2\alpha_2 + \alpha_3, \quad |D_4| = 3 + \alpha_2 + \alpha_3 + \alpha_4''.$$

Hence  $2\alpha_2 + \alpha_3 \leq 2$  and so  $\alpha_2 \leq 1$ .

• If  $\alpha_2 = 0$ , then  $|L| = 10 + \alpha_3$  and so  $\alpha_3 \leq 2$ . Since  $2A_1 \subset S_2 \cup S_3$ , and since  $A_2 = \emptyset$  and  $2S_3 \cap S_4 = \emptyset$ , it follows that

$$A_4 \cap D \subseteq (A_1 + A_3).$$

Hence  $\alpha_4'' \leq 2\alpha_3$ . Therefore

$$\begin{aligned} W_0(S) &\geq 3(10 + \alpha_3) - 4(3 + \alpha_3 + 2\alpha_3) + \rho \\ &= 18 - 9\alpha_3 + \rho \\ &\geq \rho. \end{aligned}$$

• If  $\alpha_2 = 1$ , then  $|L| = 12 + \alpha_3$  and so  $|L| = 12$  and  $\alpha_3 = 0$ . Since  $2A_1 \subset S_2 \cup S_3$  and  $A_3 = \emptyset$ , it follows that

$$A_4 \cap D \subseteq (A_1 + A_2) \cup (2A_2).$$

Hence  $\alpha_4'' \leq 2 + 1 = 3$ . Therefore

$$\begin{aligned} W_0(S) &\geq 3 \cdot 12 - 4(3 + 1 + 3) + \rho \\ &= 8 + \rho. \quad \square \end{aligned}$$

**Remark 4.6.** *A better lower bound on  $W_0(S)$  may be obtained by splitting  $\alpha_i$  as  $\alpha_i' + \alpha_i''$  for  $i = 2, 3$  in the above proof. For instance, we have only used  $|P \cap L| \geq 3$ . But we could have used  $|P \cap L| \geq 4$  if either  $\alpha_2'$  or  $\alpha_3'$  were assumed positive, while if  $\alpha_2' = \alpha_3' = 0$ , a sharper estimate on  $\alpha_4''$  can be derived.*

## 4.2 When $\alpha_1 = 1$

Since  $12 \geq |L| \geq q + (q - 1) = 2q - 1$ , it follows that  $q \leq 6$ . We shall successively treat the cases  $q = 4, 5$  and  $6$ . Throughout Section 4.2, we set

$$A_1 = \{x\}.$$

### 4.2.1 Case $q = 4$

Then  $\alpha(S) = (1, \alpha_2, \alpha_3)$ . We have  $|L| = 4 + 3 + 2\alpha_2 + \alpha_3$ , whence  $2\alpha_2 + \alpha_3 \leq 5$ , implying  $\alpha_2 \leq 2$ . We successively examine the cases  $\alpha_2 = 2, 1, 0$ . To start with, we have

$$A_4 \cap D \subseteq (A_1 + A_2) \cup (A_1 + A_3) \cup (A_2 + A_2) \cup (A_2 + A_3). \quad (17)$$

**Subcase  $\alpha_2 = 2$ .** Then  $\alpha(S) = (1, 2, \alpha_3)$ . We have  $|L| = 11 + \alpha_3$ , whence  $\alpha_3 \leq 1$ . Denote

$$A_2 = \{y_1, y_2\}.$$

Since  $A_2 \cap D \subseteq 2A_1$ , and since  $|2A_1| = 1$ , we have  $\alpha_2'' \leq 1$  whence  $\alpha_2' \in \{1, 2\}$ .

• Assume first  $\alpha_2' = 1$ . Say  $y_1 \in P, y_2 \in D$ . Then  $y_2 = 2x$ , and  $|P \cap L| = 3 + \alpha_3'$ . We have  $|D_4| = 4 + \alpha_3 + \alpha_4''$ , and

$$(A_1 + A_2) \cup (2A_2) = \{4x, 3x, 2x + y_1, x + y_1, 2y_1\}.$$

Since  $4x \notin S_3$  and since  $\alpha_3 \leq 1$ , there are at most four possibilities for  $A_3 \cap D$ , listed below together with their consequences on  $A_4 \cap D$ . Note that Lemma 2.13 plays a key role to deduce these consequences. For example, if  $4x \in A_4$  or  $2x + y_1 \in A_4$ , then necessarily  $3x \in A_3$  or  $x + y_1 \in A_3$ , respectively. Note also that the level function  $\lambda = \lambda_S$  is nondecreasing. Consequently, in the last case  $A_3 = \{2y_1\}$  below, it follows that  $x + y_1 \notin A_4$ , for  $\lambda(x + y_1) \leq \lambda(2y_1) = 3$  since  $x < y_1$ . Here then are the possibilities for  $A_3 \cap D$ :

1. If  $A_3 \cap D = \emptyset$  then  $A_4 \cap D \subseteq \{3x, x + y_1, 2y_1\}$ .
2. If  $A_3 \cap D = \{3x\}$  then  $A_4 \cap D \subseteq \{4x, x + y_1, 2y_1\}$ .
3. If  $A_3 \cap D = \{x + y_1\}$  then  $A_4 \cap D \subseteq \{3x, 2x + y_1, 2y_1\}$ .
4. If  $A_3 \cap D = \{2y_1\}$  then  $A_4 \cap D \subseteq \{3x, 2y_1\}$ .

In either case, we have  $\alpha_4'' \leq 3$ . Recall also that  $\alpha_3 \leq 1$  here. Hence

$$\begin{aligned} W_0(S) &= (3 + \alpha_3')(11 + \alpha_3) - 4(4 + \alpha_3 + \alpha_4'') + \rho \\ &\geq (3 + \alpha_3')(11 + \alpha_3) - 4(7 + \alpha_3) + \rho \\ &\geq 4 + 11\alpha_3' + \alpha_3'\alpha_3 + \rho. \end{aligned}$$

• Assume now  $\alpha_2' = 2$ , so that  $y_1, y_2 \in P$ . Then  $|P \cap L| = 4 + \alpha_3'$ . We have  $|D_4| = 4 + \alpha_3 + \alpha_4''$ , and

$$(2A_1) \cup (A_1 + A_2) \cup (2A_2) = \{2x, x + y_1, x + y_2, 2y_1, y_1 + y_2, 2y_2\}.$$

Up to permutation of  $y_1, y_2$ , and using  $\alpha_3 \leq 1$  and  $2x \notin S_4$ , here are the possibilities for  $A_3 \cap D$ , together with their consequences for  $A_4 \cap D$ :

1. If  $A_3 \cap D = \emptyset$  then  $A_4 \cap D \subseteq \{x + y_1, x + y_2, 2y_1, y_1 + y_2, 2y_2\}$ .
2. If  $A_3 \cap D = \{2x\}$  then  $A_4 \cap D \subseteq \{3x, x + y_1, x + y_2, 2y_1, y_1 + y_2, 2y_2\}$ .
3. If  $A_3 \cap D = \{x + y_1\}$  then  $A_4 \cap D \subseteq \{x + y_2, 2y_1, y_1 + y_2, 2y_2\}$ .
4. If  $A_3 \cap D = \{2y_1\}$  then  $A_4 \cap D \subseteq \{x + y_1, x + y_2, y_1 + y_2, 2y_2\}$ .
5. If  $A_3 \cap D = \{y_1 + y_2\}$  then  $A_4 \cap D \subseteq \{x + y_1, x + y_2, 2y_1, 2y_2\}$ .

In either case, we have  $\alpha_4'' \leq 6$ . Hence

$$\begin{aligned} W_0(S) &= (4 + \alpha_3')(11 + \alpha_3) - 4(4 + \alpha_3 + \alpha_4'') + \rho \\ &\geq (4 + \alpha_3')(11 + \alpha_3) - 4(10 + \alpha_3) + \rho \\ &= 4 + 11\alpha_3' + \alpha_3'\alpha_3 + \rho. \end{aligned}$$

**Subcase  $\alpha_2 = 1$ .** Then  $\alpha(S) = (1, 1, \alpha_3)$ . We have  $|L| = 9 + \alpha_3$ , whence  $\alpha_3 \leq 3$ . We also have  $|P \cap L| = 2 + \alpha_2' + \alpha_3'$  and  $|D_4| = 3 + \alpha_3 + \alpha_4''$ . Denote

$$A_2 = \{y\}.$$

• Assume first  $y \in A_2 \cap D$ . Then  $y = 2x$ ,  $|P \cap L| = 2 + \alpha_3'$  and  $A_3 \cap D \subseteq \{3x\}$ . Thus  $\alpha_3'' \leq 1$ , and either

$$A_4 \cap D \subseteq \{3x\} \cup (x + P_3) \quad \text{or} \quad A_4 \cap D \subseteq \{4x\} \cup (2x + P_3) \cup (x + P_3).$$

We claim that  $\alpha_4'' \leq 1 + \alpha_3'$  in both cases. This is clear in the first one. In the second one, for all  $z \in P_3$  we have

$$|A_4 \cap \{2x + z, x + z\}| \leq 1.$$

Therefore  $|A_4 \cap ((2x + P_3) \cup (x + P_3))| \leq |P_3|$ , implying  $\alpha_4'' \leq 1 + \alpha_3'$  here as well. Using  $\alpha_3 = \alpha_3' + \alpha_3'' \leq \alpha_3' + 1$ , we have

$$\begin{aligned} W_0(S) &= (2 + \alpha_3')(9 + \alpha_3) - 4(3 + \alpha_3 + \alpha_4'') + \rho \\ &\geq (2 + \alpha_3')(9 + \alpha_3) - 4(4 + \alpha_3' + \alpha_3) + \rho \\ &= 2 + 5\alpha_3' - 2\alpha_3 + \alpha_3\alpha_3' + \rho \\ &\geq 2 + 5\alpha_3' - 2(\alpha_3' + 1) + \alpha_3\alpha_3' + \rho \\ &\geq 3\alpha_3' + \alpha_3\alpha_3' + \rho. \end{aligned}$$

- Assume now  $y \in A_2 \cap P$ . Hence  $|P \cap L| = 3 + \alpha'_3$ . Here we have

$$A_3 \cap D \subseteq \{2x, x+y, 2y\}$$

and so  $\alpha''_3 \leq 3$ . Let us examine in turn the possibilities for  $A_3 \cap D$  and their consequences for  $A_4 \cap D$ :

1. If  $A_3 \cap D = \emptyset$  then  $A_4 \cap D \subseteq \{x+y, 2y\} \cup (\{x, y\} + P_3)$ .
2. If  $A_3 \cap D = \{2x\}$  then  $A_4 \cap D \subseteq \{3x, x+y, 2y\} \cup (\{x, y\} + P_3)$ .
3. If  $A_3 \cap D = \{x+y\}$  then  $A_4 \cap D \subseteq \{2y\} \cup (\{x, y\} + P_3)$ .
4. If  $A_3 \cap D = \{2y\}$  then  $A_4 \cap D \subseteq \{x+y, 3y\} \cup (\{x, y\} + P_3)$ .
5. If  $A_3 \cap D = \{2x, x+y\}$  then  $A_4 \cap D \subseteq \{3x, 2x+y, 2y\} \cup (\{x, y\} + P_3)$ .
6. If  $A_3 \cap D = \{2x, 2y\}$  then  $A_4 \cap D \subseteq \{3x, x+y, 2y\} \cup (\{x, y\} + P_3)$ .
7. If  $A_3 \cap D = \{x+y, 2y\}$  then  $A_4 \cap D \subseteq \{x+2y, 3y\} \cup (\{x, y\} + P_3)$ .
8. If  $A_3 \cap D = \{2x, x+y, 2y\}$  then  $A_4 \cap D \subseteq \{3x, 2x+y, x+2y, 3y\} \cup (\{x, y\} + P_3)$ .

Note that  $|\{x, y\} + P_3| \leq 2|P_3| = 2\alpha'_3$ . Consequently, distinguishing between the first seven cases and the last one, we have

$$\begin{aligned} \alpha''_3 \leq 2 &\Rightarrow \alpha''_4 \leq 3 + 2\alpha'_3 \\ \alpha''_3 = 3 &\Rightarrow \alpha''_4 \leq 4 + 2\alpha'_3. \end{aligned}$$

- Assume first  $\alpha''_3 \leq 2$ , so that  $\alpha_3 \leq \alpha'_3 + 2$  and  $\alpha''_4 \leq 3 + 2\alpha'_3$ . We have

$$\begin{aligned} W_0(S) &= (3 + \alpha'_3)(9 + \alpha_3) - 4(3 + \alpha_3 + \alpha''_4) + \rho \\ &\geq (3 + \alpha'_3)(9 + \alpha_3) - 4(6 + \alpha_3 + 2\alpha'_3) + \rho \\ &= 3 + \alpha'_3 - \alpha_3 + \alpha'_3\alpha_3 + \rho \\ &\geq 1 + \alpha'_3\alpha_3 + \rho. \end{aligned}$$

- Assume now  $\alpha''_3 = 3$ , so that  $\alpha''_4 \leq 4 + 2\alpha'_3$ . Since  $\alpha_3 \leq 3$ , it follows that  $\alpha_3 = \alpha''_3 = 3$ , whence  $\alpha'_3 = 0$  and  $\alpha''_4 \leq 4$ . We have

$$\begin{aligned} W_0(S) &= (3 + \alpha'_3)(9 + \alpha_3) - 4(3 + \alpha_3 + \alpha''_4) + \rho \\ &\geq 3 \cdot 12 - 4 \cdot 10 + \rho \\ &= -4 + \rho. \end{aligned}$$

We now show that  $\rho \geq 6$  here. Indeed, since  $y \in A_2$  and  $3y \in A_4$ , we have

$$2m - \rho \leq y, \quad 3y \leq 5m - \rho - 1.$$

Therefore  $3(2m - \rho) \leq 3y \leq 5m - \rho - 1$ , implying  $m \leq 2\rho - 1$ . Now, since  $m = 1 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \geq 10$  by (8), it follows that  $\rho \geq 6$ , whence  $W_0(S) \geq 2$  and we are done.

**Subcase  $\alpha_2 = 0$ .** Then  $\alpha(S) = (1, 0, \alpha_3)$  here, and incidentally  $\alpha_3 \leq 5$  since  $|L| \leq 12$ . We have

$$|P \cap L| = 2 + \alpha'_3, \quad |L| = 7 + \alpha_3, \quad |D_4| = 2 + \alpha_3 + \alpha''_4.$$

Recalling that  $A_1 = \{x\}$ , we distinguish the cases where  $2x \in A$  or not.

• Assume first  $2x \in A$ . Since  $\lambda(2x) \in \{2, 3\}$  and since  $A_2 = \emptyset$ , it follows that  $2x \in A_3$ . Hence  $\alpha''_3 = 1$  and so  $\alpha_3 = 1 + \alpha'_3$ . Thus

$$|L| = 8 + \alpha'_3, \quad |D_4| = 3 + \alpha'_3 + \alpha''_4.$$

Since  $A_4 \cap D \subseteq \{3x\} \cup P_3$ , it follows that  $\alpha''_4 \leq 1 + \alpha'_3$  and hence  $|D_4| \leq 4 + 2\alpha'_3$ . A straightforward computation then yields

$$W_0(S) \geq \rho.$$

• Assume now  $2x \notin A$ . It follows that  $A_3 \subseteq P$ , i.e.  $\alpha_3 = \alpha'_3$ , and  $A_4 \cap D \subseteq x + A_3$ . Therefore  $\alpha''_4 \leq \alpha_3$  and so  $|D_4| = 2 + \alpha_3 + \alpha''_4 \leq 2 + 2\alpha_3$ . This implies here

$$W_0(S) \geq 6 + \rho.$$

This concludes the case  $\alpha_1 = 1$  and  $q = 4$ .

#### 4.2.2 Case $q = 5$

We now tackle the case  $\alpha_1 = 1$  and  $q = 5$ , i.e.  $\alpha(S) = (1, \alpha_2, \alpha_3, \alpha_4)$ . We have  $|L| = 9 + 3\alpha_2 + 2\alpha_3 + \alpha_4$ , whence  $3\alpha_2 + 2\alpha_3 + \alpha_4 \leq 3$ , implying  $\alpha_2 \leq 1$ .

**Subcase  $\alpha_2 = 1$ .** Then  $\alpha(S) = (1, 1, \alpha_3, \alpha_4)$ . Then  $|L| = 12$  and  $\alpha_3 = \alpha_4 = 0$ . Moreover,  $|D_5| = \alpha_0 + \alpha_1 + \alpha_2 + \alpha'_5 = 3 + \alpha'_5$ . Set

$$A_2 = \{y\}.$$

• Assume first  $y \in D$ . Then  $y = 2x$ . Since  $A_3 = A_4 = \emptyset$  by hypothesis, it follows that  $|P \cap L| = 2$  and  $A_5 \cap D \subseteq \{3x\}$ . Therefore  $\alpha_5'' \leq 1$ . We conclude that

$$\begin{aligned} W_0(S) &= 2 \cdot 12 - 5(3 + \alpha_5'') + \rho \\ &\geq 4 + \rho. \end{aligned}$$

• Assume now  $y \in P$ . Then  $|P \cap L| = 3$  and  $A_5 \cap D \subseteq \{2y\}$  since  $\lambda(2x) \leq 3$  and  $\lambda(x+y) \leq 4$ . Hence  $\alpha_5'' \leq 1$  and

$$\begin{aligned} W_0(S) &= 3 \cdot 12 - 5(3 + \alpha_5'') + \rho \\ &\geq 16 + \rho. \end{aligned}$$

**Subcase  $\alpha_2 = 0$ .** Then  $\alpha(S) = (1, 0, \alpha_3, \alpha_4)$  and  $|L| = 9 + 2\alpha_3 + \alpha_4$ . Hence  $\alpha_3 \leq 1$ . Moreover,  $|D_5| = \alpha_0 + \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5'' = 2 + \alpha_3 + \alpha_4 + \alpha_5''$ . We have  $A_2 = \emptyset$ .

**Subsubcase  $\alpha_3 = 1$ .** Then  $\alpha(S) = (1, 0, 1, \alpha_4)$ . We have  $|L| = 11 + \alpha_4$ , hence  $\alpha_4 \leq 1$ . Since  $A_2 = \emptyset$ , we have  $2x \in A$  if and only if  $2x \in A_3$ .

• Assume first  $2x \in A$ . Then  $A_3 = \{2x\}$  since  $A_2 = \emptyset$  and  $\alpha_3 = 1$ . Hence  $A_4 \cap D \subseteq \{3x\}$  and  $A_5 \cap D \subseteq \{3x, 4x\} \cup (x + A_4 \cap P)$ . Therefore

$$\alpha_4'' = \mathbf{1}_4(3x), \quad \alpha_5'' \leq \mathbf{1}_5(3x) + \mathbf{1}_5(4x) + \alpha_4'. \quad (18)$$

Note also that if  $4x \in A_5$ , then  $A_4 = \{3x\}$  by Lemma 2.13 and the bound  $\alpha_4 \leq 1$ . We have

$$|P \cap L| = 2 + \alpha_4', \quad |L| = 11 + \alpha_4, \quad |D_5| = 3 + \alpha_4 + \alpha_5'' = 3 + \alpha_4' + \mathbf{1}_4(3x) + \alpha_5''.$$

A straightforward computation, using (18), then yields

$$\begin{aligned} W_0(S) &= (2 + \alpha_4')(11 + \alpha_4) - 5(3 + \alpha_4' + \mathbf{1}_4(3x) + \alpha_5'') + \rho \\ &\geq (2 + \alpha_4')(11 + \alpha_4) - 5(3 + 2\alpha_4' + \mathbf{1}_4(3x) + \mathbf{1}_5(3x) + \mathbf{1}_5(4x)) + \rho \\ &\geq 7 - 3 \cdot \mathbf{1}_4(3x) - 5 \cdot \mathbf{1}_5(3x) - 5 \cdot \mathbf{1}_5(4x) + \rho. \end{aligned}$$

If  $4x \notin A_5$  then  $W_0(S) \geq 2 + \rho$  since  $\mathbf{1}_4(3x) + \mathbf{1}_5(3x) \leq 1$  and we are done. If  $4x \in A_5$ , then  $3x \in A_4$  as noted above, whence  $W_0(S) \geq -1 + \rho$ . But then  $4x \in C$ , since  $\lambda(4x) = 5$  whereas  $\lambda(2x) = 3$ . Thus  $\rho \geq |C| \geq 1$ , implying  $W_0(S) \geq 0$ , as desired.

• Assume now  $2x \notin A$ . Then  $\alpha'_3 = \alpha_3 = 1$  and  $|P \cap L| = 3 + \alpha'_4$ . We have  $A_5 \cap D \subseteq (A_1 + A_3) \cup (A_1 + A_4)$ , whence  $\alpha''_5 \leq 2$  since  $\alpha_1 = \alpha_3 = 1$  and  $\alpha_4 \leq 1$ . Therefore  $|D_5| = 3 + \alpha_4 + \alpha''_5 \leq 6$ , so that

$$\begin{aligned} W_0(S) &= (3 + \alpha'_4)(11 + \alpha_4) - 5|D_5| + \rho \\ &\geq 3 \cdot 11 - 30 + \rho \\ &\geq 3 + \rho. \end{aligned}$$

**Subsubcase  $\alpha_3 = 0$ .** Then  $\alpha(S) = (1, 0, 0, \alpha_4)$ . We have  $|L| = 9 + \alpha_4$ , hence  $\alpha_4 \leq 3$  and  $|D_5| = 2 + \alpha_4 + \alpha''_5$ . Since  $A_2 = A_3 = \emptyset$ , and since  $2 \leq \lambda(2x) \leq 3$ , it follows that  $2x \notin A$  and  $A_4 = P_4$ . Thus  $A_5 \cap D \subseteq x + P_4$ , so that  $\alpha''_5 \leq \alpha_4$  and  $|P \cap L| = 2 + \alpha_4$ . It follows that

$$\begin{aligned} W_0(S) &\geq (2 + \alpha'_4)(9 + \alpha_4) - 5(2 + \alpha_4 + \alpha''_5) + \rho \\ &\geq (2 + \alpha'_4)(9 + \alpha_4) - 5(2 + 2\alpha_4) + \rho \\ &\geq 8 + \alpha_4 + \alpha_4^2 + \rho. \end{aligned}$$

This concludes the case  $\alpha_1 = 1$  and  $q = 5$ .

### 4.2.3 Case $q = 6$

Still for  $\alpha_1 = 1$ , we now tackle the last case  $q = 6$ . Then  $\alpha(S) = (1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ . We have  $|L| = 11 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5$ , whence  $\alpha_2 = \alpha_3 = \alpha_4 = 0$  and  $\alpha_5 \leq 1$ .

**Subcase  $\alpha_5 = 1$ .** Then  $\alpha(S) = (1, 0, 0, 0, 1)$ . We have  $|L| = 12$  and  $|D_6| = \alpha_0 + \alpha_1 + \alpha_5 + \alpha''_6 = 3 + \alpha''_6$ . Recall that  $A_1 = \{x\}$ . Since  $\lambda(2x) \leq 3$  and  $A_2 = A_3 = A_4 = \emptyset$ , it follows that  $2x \notin A$ . Whence  $A_5 = P_5$ , i.e.  $\alpha'_5 = \alpha_5 = 1$ , and  $A_6 \cap D \subseteq x + A_5$ . Hence  $|P \cap L| = 4$  and  $\alpha''_6 \leq 1$ . Consequently  $|D_6| \leq 4$ , and

$$\begin{aligned} W_0(S) &\geq 3 \cdot 12 - 6 \cdot 4 + \rho \\ &= 12 + \rho. \end{aligned}$$

**Subcase  $\alpha_5 = 0$ .** Then  $\alpha(S) = (1, 0, 0, 0, 0)$ . We have  $|L| = 11$  and  $|D_6| = 2 + \alpha''_6$ . Since  $A_1 = \{x\}$  and  $A_2, A_3, A_4, A_5$  are all empty, it follows that  $|P \cap L| = 2$  and  $A_6 \cap D = \emptyset$ , i.e.  $\alpha''_6 = 0$ . Thus

$$W_0(S) = 2 \cdot 11 - 6 \cdot 2 + \rho = 10 + \rho.$$

Summarizing, we have shown that if  $|L| \leq 12$ ,  $\alpha_1 = 1$  and  $q = 6$ , then  $W_0(S) \geq 10 + \rho$ . This concludes the case  $|L| \leq 12$  and  $\alpha_1 = 1$ .

### 4.3 When $\alpha_1 = 0$

As noted at the beginning of Section 4, it suffices to consider the cases  $4 \leq q \leq 7$ .

#### 4.3.1 Case $q = 4$

Then  $\alpha(S) = (0, \alpha_2, \alpha_3)$ . Since  $\lceil q/2 \rceil = 2$  here, Theorem 3.1 yields  $W_0(S) \geq 0$ .

#### 4.3.2 Case $q = 5$

Then  $\alpha(S) = (0, \alpha_2, \alpha_3, \alpha_4)$ . We have  $12 \geq |L| = 5 + 3\alpha_2 + 2\alpha_3 + \alpha_4$ . Hence  $\alpha_2 \leq 2$ . We now examine successively the cases  $\alpha_2 = 2, 1, 0$ . Since  $A_1 = \emptyset$ , it follows that  $A_2 \subset P$ .

Subcase  $\alpha_2 = 2$ . Then  $\alpha(S) = (0, 2, \alpha_3, \alpha_4)$ . We have  $|L| = 11 + 2\alpha_3 + \alpha_4$ . Hence  $\alpha_3 = 0$  and  $\alpha_4 \leq 1$ . Since  $A_2 \subset P$ , we have  $\alpha'_2 = \alpha_2 = 2$  here. Thus  $|P \cap L| \geq 3$ . Set

$$A_2 = \{x_1, x_2\}.$$

Assume first  $\alpha_4 = 1$ . Then  $\alpha(S) = (0, 2, 0, 1)$ , so that  $|L| = 12$  and  $|D_5| = 4 + \alpha''_5$ . Denote  $A_4 = \{z\}$ .

- If  $z \in P$ , then  $|P \cap L| = 4$  and  $2A_2 \cap (A_3 \cup A_4) = \emptyset$ . Now

$$A_5 \cap D \subseteq 2A_2 \cup (A_2 + A_4).$$

Since  $|2A_2| \leq 3$  and  $|A_2 + A_4| \leq |A_2|$ , it follows that  $\alpha''_5 \leq 5$ . Hence

$$\begin{aligned} W_0(S) &= |P \cap L||L| - 5|D_5| + \rho \\ &\geq 4 \cdot 12 - 5 \cdot 9 + \rho \\ &= 3 + \rho \end{aligned}$$

and we are done if  $z \in P$ .

- If  $z \in D$ , then  $|P \cap L| = 3$  and  $z \in A_2$  since  $A_3 = \emptyset$ . Hence, up to renumbering, either  $z = 2x_1$  or  $z = x_1 + x_2$ . If  $z = 2x_1$  then  $A_5 \cap D \subseteq \{3x_1, x_1 + x_2, 2x_2\}$ , whereas if  $z = x_1 + x_2$ , then  $A_5 \cap D \subseteq \{2x_1, 2x_2\}$ . In either case, we have  $\alpha''_5 \leq 3$ . Therefore  $|D_5| \leq 7$ . It follows that

$$\begin{aligned} W_0(S) &= |P \cap L||L| - 5|D_5| + \rho \\ &\geq 3 \cdot 12 - 5 \cdot 7 + \rho \\ &= 1 + \rho \end{aligned}$$

and we are done as well if  $z \in D$ .

Assume now  $\alpha_4 = 0$ . Then  $\alpha(S) = (0, 2, 0, 0)$ , so that  $|L| = 11$  and  $|D_5| = 3 + \alpha_5''$ . In that case, we have  $A_5 \subseteq 2A_2$ , whence  $\alpha_5'' \leq 3$  and  $|D_5| \leq 6$ . It follows that

$$W_0(S) \geq 3 \cdot 11 - 5 \cdot 6 + \rho \geq 3 + \rho.$$

That concludes the subcase  $\alpha_2 = 2$ , i.e.  $\alpha(S) = (0, 2, \alpha_3, \alpha_4)$  here.

Subcase  $\alpha_2 = 1$ . Then  $\alpha(S) = (0, 1, \alpha_3, \alpha_4)$ . We have  $|L| = 8 + 2\alpha_3 + \alpha_4$ . Hence  $\alpha_3 \leq 2$ . Since  $A_2 \subset P$ , we have  $\alpha_2' = \alpha_2 = 1$  here. Set

$$A_2 = \{x\}.$$

Subsubcase  $\alpha_3 = 2$ . Then  $\alpha(S) = (0, 1, 2, \alpha_4)$ . We have  $|L| = 12 + \alpha_4$ , whence  $|L| = 12$  and  $\alpha_4 = 0$ . Thus  $\alpha(S) = (0, 1, 2, 0)$ . Hence  $|D_5| = 4 + \alpha_5''$ . Since  $A_3 \cap D \subseteq 2A_2$ , it follows that  $\alpha_3'' \leq 1$  and hence  $\alpha_3' \in \{1, 2\}$ . Therefore  $|P \cap L| \geq 3$ . Set

$$A_3 = \{y_1, y_2\}.$$

• Assume first  $\alpha_3' = 1$ . Hence  $\alpha_3'' = 1$ , and up to renumbering, we may assume  $y_1 = 2x, y_2 \in P$ . Hence  $|P \cap L| = 3$  and

$$A_5 \cap D \subseteq \{3x, x + y_2, 2y_2\}.$$

Therefore  $\alpha_5'' \leq 3$  and so  $|D_5| \leq 7$ . We then have

$$\begin{aligned} W_0(S) &= |P \cap L||L| - 5|D_5| + \rho \\ &\geq 3 \cdot 12 - 5 \cdot 7 + \rho \\ &= 1 + \rho \end{aligned}$$

and we are done here.

• Assume now  $\alpha_3' = \alpha_3 = 2$ . Thus  $y_1, y_2 \in P$  and so  $|P \cap L| = 4$ . We have

$$A_5 \cap D \subseteq \{2x\} \cup (x + A_3) \cup (2A_3 \cap A_5).$$

Thus  $\alpha_5'' \leq 3 + |2A_3 \cap A_5|$ , and of course  $|2A_3 \cap A_5| \leq |2A_3| \leq 3$  since  $|A_3| = 2$ . At this point we have

$$|D_5| \leq 7 + |2A_3 \cap A_5|.$$

Therefore

$$\begin{aligned}
W_0(S) &= |P \cap L| |L| - 5|D_5| + \rho \\
&\geq 4 \cdot 12 - 5 \cdot (7 + |2A_3 \cap A_5|) + \rho \\
&= 13 - 5|2A_3 \cap A_5| + \rho.
\end{aligned}$$

Now  $\rho \geq |2A_3 \cap A_5|$ , since  $2A_3 \cap A_5 \subseteq C$  and  $|C| \leq \rho$ . Hence

$$W_0(S) \geq 13 - 4|2A_3 \cap A_5|.$$

But  $|2A_3 \cap A_5| \leq 3$  as seen above. It follows that  $W_0(S) \geq 1$  and we are done here.

**Subsubcase  $\alpha_3 = 1$ .** Then  $\alpha(S) = (0, 1, 1, \alpha_4)$ . We have

$$|P \cap L| = 2 + \alpha'_3 + \alpha'_4, \quad |L| = 10 + \alpha_4, \quad |D_5| = 3 + \alpha_4 + \alpha''_5. \quad (19)$$

We examine four subcases, depending on which multiples of  $x$  belong to  $A$ .

**Case 1.**  $2x \in A_3, 3x \in A_4$ . Then  $A_3 = \{2x\}$  since  $\alpha_3 = 1$ , and  $A_4 \cap D = \{3x\}$ . Hence  $\alpha_4 = \alpha'_4 + 1$ . Moreover,

$$A_5 \cap D \subseteq \{4x\} \cup (x + A_4 \cap P).$$

Whether  $4x$  belongs to  $A_5$  or not will be measured by  $\mathbf{1}_5(4x) \in \{0, 1\}$ , where  $\mathbf{1}_i = \mathbf{1}_{A_i}$  denotes the indicator function of  $A_i$  for all  $i$ . Thus,

$$\alpha''_5 \leq \mathbf{1}_5(4x) + \alpha'_4,$$

and so

$$|P \cap L| = 2 + \alpha'_4, \quad |L| = 11 + \alpha'_4, \quad |D_5| = 4 + \alpha'_4 + \alpha''_5 \leq 4 + 2\alpha'_4 + \mathbf{1}_5(4x). \quad (20)$$

Plugging this data into (5), we get

$$W_0(S) \geq 2 + 3\alpha'_4 + \alpha_4^2 - 5 \cdot \mathbf{1}_5(4x) + \rho.$$

If either  $\alpha'_4 \neq 0$  or  $\mathbf{1}_5(4x) = 0$ , then  $W_0(S) \geq \rho$  and we are done. If  $\alpha'_4 = 0$  and  $\mathbf{1}_5(4x) = 1$ , then  $W_0(S) \geq -3 + \rho$ . But in this case, we claim that  $\rho \geq 3$ . Indeed, both  $2x, 3x$  are *compressed* Apéry elements, since  $\lambda(x) = 2, \lambda(2x) = 3, \lambda(3x) = 4$  and  $2x = x + x, 3x = x + 2x$ . Moreover, since  $\mathbf{1}_5(4x) = 1$  here, then  $4x$  is also a compressed Apéry element, using  $\lambda(4x) = 5$  and  $4x = 2x + 2x$ . Hence  $\{2x, 3x, 4x\} \subseteq C$ . Since  $\rho \geq |C|$ , this proves  $\rho \geq 3$  as claimed and yields  $W_0(S) \geq 0$ , as desired.

**Case 2.**  $2x \in A_3, 3x \notin A_4$ . Then  $A_3 = \{2x\}$  and  $A_4 \subset P$ , so that  $\alpha_4 = \alpha'_4$ . We have

$$A_5 \cap D \subseteq \{3x\} \cup (x + A_4).$$

Hence  $\alpha''_5 \leq 1 + \alpha_4$ , and so

$$|P \cap L| = 2 + \alpha_4, \quad |L| = 10 + \alpha_4, \quad |D_5| = 3 + \alpha_4 + \alpha''_5 \leq 4 + 2\alpha_4.$$

Hence  $W_0(S) \geq 2\alpha_4 + \alpha_4^2 + \rho$  and we are done in the present case.

**Case 3.**  $2x \in A_4$ . Then  $A_3 = A_3 \cap P$ , so that  $\alpha'_3 = \alpha_3 = 1$ . We set  $A_3 = \{y\}$ . We have  $\{2x\} \subseteq A_4 \cap D \subseteq \{2x, x + y\}$  and  $A_5 \cap D \subseteq \{3x, 2y, x + y\}$ . Since  $x + y$  cannot belong to both  $A_4, A_5$ , it follows that  $\alpha''_4 + \alpha''_5 \leq 4$ .

Assume first  $\alpha_4 = 2$ . We then have

$$|P \cap L| = 3 + \alpha'_4, \quad |L| = 12, \quad |D_5| = 3 + \alpha'_4 + \alpha''_4 + \alpha''_5 \leq 7 + \alpha'_4. \quad (21)$$

It follows that  $W_0(S) \geq 12(3 + \alpha'_4) - 5(7 + \alpha'_4) + \rho \geq 1 + \rho$  and we are done.

Assume now  $\alpha_4 = 1$ . Then  $A_4 = \{2x\}$  and  $|L| = 11$ . If  $\alpha''_5 \leq 2$ , then

$$|P \cap L| = 3, \quad |L| = 11, \quad |D_5| = 4 + \alpha''_5 \leq 6.$$

It follows that  $W_0(S) \geq 3 \cdot 11 - 5 \cdot 6 + \rho \geq 3 + \rho$  and we are done. But if  $\alpha''_5 = 3$ , i.e. if  $A_5 \cap D = \{3x, 2y, x + y\}$ , then we only get  $W_0(S) \geq -2 + \rho$ . But in this case, since  $3x, 2y \in C$ , it follows that  $\rho \geq 2$  whence  $W_0(S) \geq 0$  as desired.

**Case 4.**  $2x \notin A$ . Then as above,  $A_3 = A_3 \cap P$ , so that  $\alpha'_3 = \alpha_3 = 1$ , and we set  $A_3 = \{y\}$ . If  $x + y \in A$ , then it belongs to either  $A_4$  or  $A_5$ . But in any case,  $x + 2y \notin A$  since  $\lambda(x + 2y) \geq 6$ , as easily seen. It follows that  $A_4 \cap D \subseteq \{x + y\}$  and  $A_5 \cap D \subseteq \{x + y, 2y\}$ . Hence

$$\alpha_4 = \alpha'_4 + \mathbf{1}_4(x + y), \quad \alpha''_5 \leq 1 + \mathbf{1}_5(x + y).$$

It follows that

$$|P \cap L| = 3 + \alpha'_4, \quad |L| = 10 + \alpha'_4 + \mathbf{1}_4(x + y), \quad |D_5| \leq 4 + \alpha'_4 + \mathbf{1}_4(x + y) + \mathbf{1}_5(x + y).$$

A straightforward computation, using  $\mathbf{1}_4(x + y) + \mathbf{1}_5(x + y) \leq 1$ , then yields

$$\begin{aligned} W_0(S) &\geq 10 + 8\alpha'_4 + \alpha_4'^2 + (\alpha'_4 - 2) \cdot \mathbf{1}_4(x + y) - 5 \cdot \mathbf{1}_5(x + y) + \rho \\ &\geq 5 + \rho \end{aligned}$$

and we are done.

**Subsubcase  $\alpha_3 = 0$ .** Then  $\alpha(S) = (0, 1, 0, \alpha_4)$ . We have

$$|P \cap L| = 2 + \alpha'_4, \quad |L| = 8 + \alpha_4, \quad |D_5| = 2 + \alpha_4 + \alpha''_5.$$

Here  $A_3 = \emptyset$ , and  $\alpha_4 \leq 4$  since  $|L| \leq 12$  by assumption.

**Case 1.**  $2x \in A$ . Then  $2x \in A_4 \sqcup A_5$ . Thus

$$A_4 \cap D \subseteq \{2x\}, \quad A_5 \cap D \subseteq \{2x, 3x\} \cup (x + A_4 \cap P).$$

Hence  $\alpha_4 = \alpha'_4 + \alpha''_4 = \alpha'_4 + \mathbf{1}_4(2x)$  and  $\alpha''_5 \leq \mathbf{1}_5(2x) + \mathbf{1}_5(3x) + \alpha'_4$ . We have

$$|P \cap L| = 2 + \alpha'_4, \quad |L| = 8 + \alpha'_4 + \mathbf{1}_4(2x), \quad |D_5| \leq 2 + 2\alpha'_4 + \mathbf{1}_4(2x) + \mathbf{1}_5(2x) + \mathbf{1}_5(3x).$$

A straightforward computation then yields

$$W_0(S) \geq 6 - 3 \cdot \mathbf{1}_4(2x) - 5 \cdot \mathbf{1}_5(2x) - 5 \cdot \mathbf{1}_5(3x) + \rho.$$

If  $3x \notin A$ , then  $\mathbf{1}_5(3x) = 0$  and  $W_0(S) \geq 1 + \rho$ . The case  $3x \in A$  is more delicate. It implies  $3x \in A_5$ , i.e.  $\mathbf{1}_5(3x) = 1$ , and then of course  $2x \in A_4$ , i.e.  $\mathbf{1}_4(2x) = 1$  and  $\mathbf{1}_5(2x) = 0$ . Thus  $W_0(S) \geq -2 + \rho$ . It remains to show  $\rho \geq 2$ . Since  $(\lambda(x), \lambda(2x), \lambda(3x)) = (2, 4, 5)$  here, it follows that  $3x \in C$ , whence  $\rho \geq 1$ . This is not strong enough yet. However, since  $(\lambda(2x), \lambda(3x)) = (4, 5)$ , we have

$$4m - \rho \leq 2x, \quad 3x \leq 6m - \rho - 1.$$

Therefore  $3(4m - \rho)/2 \leq 3x \leq 6m - \rho - 1$ , implying  $\rho \geq 2$  as desired, and hence  $W_0(S) \geq 0$ .

**Case 2.**  $2x \notin A$ . Then  $A_3 = A_4 \cap D = \emptyset$ . Hence  $A_4 = A_4 \cap P$  and  $A_5 \cap D \subseteq x + A_4$ . That is, we have  $\alpha_4 = \alpha'_4$  and  $\alpha''_5 \leq \alpha_4$ . Thus  $|D_5| \leq 2 + 2\alpha_4$ , and a straightforward computation yields

$$W_0(S) \geq 6 + \rho.$$

### 4.3.3 Case $q = 6$

Then  $\alpha(S) = (0, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ . We have  $12 \geq |L| = 6 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5$ , whence  $\alpha_2 \leq 1$ .

**Subcase  $\alpha_2 = 1$ .** Then  $|L| = 10 + 3\alpha_3 + 2\alpha_4 + \alpha_5$ , implying  $\alpha_3 = 0$  and  $\alpha_4 \leq 1$ . Thus  $\alpha(S) = (0, 1, 0, \alpha_4, \alpha_5)$ . We set

$$A_2 = \{x\},$$

with  $x \in P$  since  $A_2 \cap D = \emptyset$ . We have  $A_3 = \emptyset$ .

**Subsubcase  $\alpha_4 = 1$ .** Then  $|L| = 12$ , whence  $\alpha_5 = 0$  and so  $\alpha(S) = (0, 1, 0, 1, 0)$ . Set  $A_4 = \{z\}$ . Since  $A_3 = A_5 = \emptyset$ , it follows that  $2x \in A$  if and only if  $2x = z$ . That is, either  $z \in A \cap P$  and  $2x \notin A$ , or else  $z = 2x$ . In either case, we have

$$A_6 \cap D \subseteq x + A_4,$$

whence  $\alpha_6'' \leq 1$ . Summarizing, we have

$$|P \cap L| = 2 + \alpha_4', \quad |L| = 12, \quad |D_6| = 3 + \alpha_6'' \leq 4.$$

It follows that

$$W_0(S) \geq (2 + \alpha_4') \cdot 12 - 6 \cdot 4 + \rho \geq \rho$$

and we are done here.

**Subsubcase  $\alpha_4 = 0$ .** Then  $\alpha(S) = (0, 1, 0, 0, \alpha_5)$ . Since  $\lambda(2x) \in [3, 5]$  and  $A_3 = A_4 = \emptyset$ , we have  $2x \in A$  if and only if  $2x \in A_5$ . Therefore  $A_5 \cap D \subseteq \{2x\}$ , i.e.  $\alpha_5'' \leq 1$ . However  $3x \notin A_6$ . For if  $3x \in A_6$ , then  $2x \in A_5$  by Lemma 2.13, whence

$$5m - \rho \leq 2x, \quad 3x \leq 7m - \rho - 1.$$

Thus  $3(5m - \rho)/2 \leq 3x \leq 7m - \rho - 1$ , whence  $\rho \geq m + 2$ , in contradiction with  $\rho \in [0, m[$ . Therefore  $A_6 \cap D \subseteq x + (A_5 \cap P)$ , i.e.  $\alpha_6'' \leq \alpha_5'$ . Summarizing, we have

$$|P \cap L| = 2 + \alpha_5', \quad |L| = 10 + \alpha_5, \quad |D_6| = 2 + \alpha_5 + \alpha_6'' \leq 2 + \alpha_5 + \alpha_5'.$$

A straightforward computation then yields

$$W_0(S) \geq 8 - 4\alpha_5'' + \rho,$$

whence  $W_0(S) \geq 4 + \rho$  since  $\alpha_5'' \leq 1$ .

**Subcase  $\alpha_2 = 0$ .** Then  $\alpha(S) = (0, 0, \alpha_3, \alpha_4, \alpha_5)$  and Theorem 3.1 yields  $W_0(S) \geq 0$ .

#### 4.3.4 Case $q = 7$

Here  $\alpha(S) = (0, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$ . Then  $12 \geq |L| \geq 7 + 5\alpha_2$ , whence  $\alpha_2 \leq 1$ .

• If  $\alpha_2 = 1$ , then  $|L| = 12$  and  $\alpha_3 = \dots = \alpha_6 = 0$ . It follows that  $A \cap D = \emptyset$ , whence  $\alpha_7'' = 0$ . Thus  $|P \cap L| = 2$  and  $|D_7| = 2$ . Therefore

$$W_0(S) = 2 \cdot 12 - 7 \cdot 2 + \rho = 10 + \rho$$

and we are done in this case.

- If  $\alpha_2 = 0$ , then  $|L| = 7 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ , whence  $\alpha_3 \leq 1$ .

- Assume  $\alpha_3 = 1$ . Then  $\alpha(S) = (0, 0, 1, \alpha_4, \alpha_5, \alpha_6)$  and  $|L| = 11 + 2\alpha_5 + \alpha_6$ , whence  $\alpha_4 = \alpha_5 = 0$  and  $\alpha_6 \leq 1$ . Thus in fact,  $\alpha(S) = (0, 0, 1, 0, 0, \alpha_6)$  and  $|L| = 11 + \alpha_6$ . We set

$$A_3 = \{x\}.$$

Of course  $x \in P$  since  $A_1 = A_2 = \emptyset$ . Since  $A_4 = A_5 = \emptyset$ , it follows that  $A_6 \cap D \subseteq \{2x\}$ . Moreover, since  $(S_3 + S_6) \cap S_7 = \emptyset$ , we also have  $A_7 \cap D \subseteq \{2x\}$ . Thus

$$\alpha''_6 = \mathbf{1}_6(2x), \quad \alpha''_7 = \mathbf{1}_7(2x).$$

Summarizing, we have

$$|P \cap L| = 2 + \alpha'_6, \quad |L| = 11 + \alpha_6, \quad |D_7| = 2 + \alpha_6 + \alpha''_7 = 2 + \alpha'_6 + \mathbf{1}_6(2x) + \mathbf{1}_7(2x).$$

A straightforward computation, using  $\mathbf{1}_6(2x) + \mathbf{1}_7(2x) \leq 1$ , then yields

$$W_0(S) \geq 1 + \rho$$

and we are done.

- Finally, assume  $\alpha_3 = 0$ . Then  $\alpha(S) = (0, 0, 0, \alpha_4, \alpha_5, \alpha_6)$ . In that case, since  $\alpha_1 = \dots = \alpha_{h-1} = 0$  where  $h = \lceil q/2 \rceil = 4$ , Theorem 3.1 implies  $W_0(S) \geq 0$ .

#### 4.3.5 Case $q \geq 8$

For depth  $q \geq 8$ , Proposition 4.2 yields  $W_0(S) \geq 0$ .

## 4.4 Concluding remarks

Having examined above all possible Apéry profiles  $\alpha(S)$  compatible with  $|L| \leq 12$ , the proof of Theorem 4 is now complete.

As mentioned earlier, among the more than  $10^{13}$  numerical semigroups of genus  $g \leq 60$ , exactly five of them satisfy  $W_0(S) < 0$ . These five exceptions have depth  $q = 4$  and Apéry profile  $\alpha(S) = (2, 0, 3)$ , yielding  $|L| = 4 + 2 \cdot 3 + 3 = 13$ . Moreover, they satisfy  $c = 4m$ ,  $|P \cap L| = |P_1| = 3$  and  $\alpha''_4 = 4$ , whence  $|D_4| = 10$  and  $W_0(S) = 3 \cdot 13 - 4 \cdot 10 = -1$ .

Infinite families of numerical semigroups satisfying  $W_0(S) < 0$  have been constructed in [2, 7]. However, they all turn out to satisfy  $W(S) \geq 0$ . It would be very interesting to characterize those  $S$  satisfying  $W_0(S) < 0$ , but that will likely be hard to achieve. Less ambitiously, can one determine how many such cases occur in, say, genus  $g \leq 100$ ?

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Authors' addresses:

- Shalom Eliahou, Univ. Littoral Côte d'Opale, UR 2597 - LMPA - Laboratoire de Mathématiques Pures et Appliquées Joseph Liouville, F-62228 Calais, France and CNRS, FR2037, France.  
eliahou@univ-littoral.fr
- D. Marín-Aragón, Univ. Cádiz, Facultad de Ciencias Campus Universitario Río San Pedro s/n. 11510 Puerto Real, Cádiz, España  
daniel.marin@uca.es