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Iasson Karafyllis, Nicolás Espitia, Miroslav Krstic

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42 mations (which traces back to the seminal work [3]). In this line, some contributions
43 have rigorously handled the well-posedness of time-varying kernels solutions where the
44 reaction term is time and space dependent as in [27, 43] and some efficient algorithms
45 to better handle the solvability of kernels have been proposed e.g. in [15].

46 More recent contributions focus on coupled parabolic PDEs [29], with space vary-
47 ing reaction coefficients [42, 4] and time- and space- varying coefficients [19] all of
48 which able to handle more challenging issues related to the solvability, suitable choices
49 of target systems, coupling matrices structure and well-posedness issues in general.

50

51 In this paper, we aim at stabilizing scalar 1D reaction–diffusion PDEs with a time-
52 and space- varying reaction coefficient from a different perspective. Our approach
53 combines some ideas from hybrid systems, specifically from the framework of state-
54 dependent switching laws, sampled-data and event-triggered sampling/control strate-
55 gies. For an overview of the literature on sampled-data, event-triggered and switching
56 strategies, we refer to e.g. [14, 23, 40, 13, 11, 32, 16, 24] for finite-dimensional systems
57 and to [25, 10, 17, 35, 12],[41, 33, 22, 5, 6, 31] for some classes of infinite dimensional
58 systems.

59 Having said that, the main ideas in this paper state that instead of handling
60 a time-varying kernel capturing the time- and space- varying coefficient, we use a
61 simpler kernel capturing only the spatial variation of the reaction coefficient. This is
62 possible as long as the reaction coefficient is sampled in time, thus the kernel-PDE
63 reduces to a form involving space-varying coefficient only between two successive
64 sampling times. In order to determine the time instants, we introduce event-triggered
65 mechanisms that form an increasing sequence of triggering times (or time of events).
66 At those event times, kernels are computed/updated in aperiodic fashion and only
67 when needed. In other words, kernels for the control are scheduled according to some
68 event triggering condition (state-dependent law). Doing so, the approach constitutes a
69 kind of a gain scheduling strategy suggesting then the adopted name for our approach:
70 *event-triggered gain scheduling*.

71 Sampling in time the reaction coefficient introduces an error (called error when
72 sampling) that is reflected in the target system after transformation. This requires
73 the study of well-posedness issues, ISS properties ([18]) and the exponential stability
74 of the closed loop system when the control gains are scheduled according to the
75 event-triggered mechanisms. In this paper we propose two strategies: the first one
76 relies on a static triggering condition which takes into account the effect of the error
77 when sampling after transformation and the current state of the closed-loop system.
78 The second strategy relies on a dynamic triggering condition which makes uses of
79 a dynamic variable that can be seen as a filtered version of the static triggering
80 condition. Moreover, under the two proposed strategies, the avoidance of the so
81 called Zeno phenomenon is proved. Hence, we can guarantee the well-posedness as
82 well as the exponential stability of the closed-loop system provided that the reaction
83 coefficient is slowly time-varying.

84 The paper is organized as follows. In Section 2, we introduce the class of reaction-
85 diffusion parabolic systems, the control design which includes the introduction of the
86 event-triggered strategies for gain scheduling and the notion of existence and unique-
87 ness of solutions. Section 3 provides the main results which include the avoidance
88 of the Zeno phenomenon, the well-posedness of the closed-loop system and the expo-
89 nential stability result. Section 4 provides a numerical example to illustrate the main
90 results. Finally, conclusions and perspectives are given in Section 5. The Appendix
91 contains the proof of an auxiliary result.

92 *Notation.* \mathbb{R}_+ will denote the set of nonnegative real numbers. Let $S \subseteq \mathbb{R}^n$ be
93 an open set and let $A \subseteq \mathbb{R}^n$ be a set that satisfies $S \subseteq A \subseteq \bar{S}$ where \bar{S} denotes
94 the closure of S . By $C^0(A; \Omega)$, we denote the class of continuous functions on A ,
95 which take values in $\Omega \subseteq \mathbb{R}$. By $C^k(A; \Omega)$, where $k \geq 1$ is an integer, we denote the
96 class of functions on A , which takes values in Ω and has continuous derivatives of
97 order k . In other words, the functions of class $C^k(A; \Omega)$ are the functions which have
98 continuous derivatives of order k in $S = \text{int}(A)$ that can be continued continuously
99 to all points in $\partial S \cap A$. $L^2(0, 1)$ denotes the equivalence class of Lebesgue measurable
100 functions $f, g : [0, 1] \rightarrow \mathbb{R}$ for which $\|f\| = \left(\int_0^1 |f(x)|^2 dx \right)^{1/2} < \infty$ and with inner
101 product $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$. $L^\infty(0, 1)$ denotes the equivalence class of Lebesgue
102 measurable functions $f : [0, 1] \rightarrow \mathbb{R}$ for which $\|f\|_\infty = \text{ess sup}_{x \in (0, 1)} (|f(x)|) < +\infty$.
103 Let $u : \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}$ be given. $u[t]$ denotes the profile of u at certain $t \geq 0$, i.e.
104 $(u[t])(x) = u(t, x)$, for all $x \in [0, 1]$. For an interval $I \subseteq \mathbb{R}_+$, the space $C^0(I; L^2(0, 1))$
105 is the space of continuous mappings $I \ni t \mapsto u[t] \in L^2(0, 1)$. $H^2(0, 1)$ denotes the
106 Sobolev space of functions $f \in L^2(0, 1)$ with square integrable (weak) first and second-
107 order derivatives $f'(\cdot), f''(\cdot) \in L^2(0, 1)$.

108 **2. Problem description and control design.** Consider the following scalar
109 reaction-diffusion system with time- and space- varying reaction coefficient:

$$110 \quad (2.1) \quad u_t(t, x) = \varepsilon u_{xx}(t, x) + \lambda(t, x)u(t, x),$$

$$111 \quad (2.2) \quad u_x(t, 0) = qu(t, 0),$$

$$112 \quad (2.3) \quad u(t, 1) = U(t), \quad \text{or} \quad u_x(t, 1) = U(t),$$

113 and initial condition:

$$114 \quad (2.4) \quad u(0, x) = u_0(x),$$

115 where $\varepsilon > 0$, $q \in (-\infty, +\infty]$ (the case $q = +\infty$ is interpreted as the Dirichlet case
116 (see [36])) and $\lambda \in \mathcal{C}^0(\mathbb{R}_+ \times [0, 1])$ with $\lambda[t] \in \mathcal{C}^1([0, 1])$. $u : [0, \infty) \times [0, 1] \rightarrow \mathbb{R}$ is the
117 system state and $U(t) \in \mathbb{R}$ is the control input. The initial condition $u_0(x)$ in (2.4)
118 is assumed to belong to $L^2(0, 1)$. We assume that $\lambda \in \mathcal{C}^0(\mathbb{R}_+ \times [0, 1])$ is bounded, i.e.
119 there exists a constant $\bar{\lambda} > 0$ such that

$$120 \quad (2.5) \quad |\lambda(t, x)| \leq \bar{\lambda}, \quad \forall t \geq 0, \quad x \in [0, 1].$$

121 Moreover, we assume the following:

122 **ASSUMPTION 2.1.** *There exists a constant $\varphi > 0$ such that the following inequality*
123 *holds:*

$$124 \quad (2.6) \quad |\lambda(t, x) - \lambda(s, x)| \leq \varphi|t - s|, \quad \forall x \in [0, 1], \quad t, s \geq 0.$$

125 Assumption 2.1 means that the reaction coefficient is Lipschitz with respect to time
126 with Lipschitz constant φ . The constant φ is a quantity that depends on the rate
127 of change of the reaction coefficient: a high rate of change of the reaction coefficient
128 implies a large value for φ . All subsequent results are also valid (with some modifi-
129 cations) if Assumption 2.1 is replaced by the less demanding assumption of Hölder
130 continuity instead of Lipschitz continuity, i.e., if we replace the right hand side of (2.6)
131 by $\varphi|t - s|^a$, where $a \in (0, 1)$ is a constant. However, for simplicity we will restrict
132 the presentation of the results to the Lipschitz case.

133

134 In what follows, we do not consider system (2.1)-(2.3) as a time-varying system
135 but we consider system (2.1)-(2.3) as a time-invariant system with two inputs: the
136 control input $U(t)$ and the distributed disturbance input $\lambda[t]$ (see [38, Chapter 2]).
137 Furthermore, it should be noticed that the disturbance input appears in a multiplica-
138 tive way and system (2.1)-(2.3) is bilinear. Multiplicative control inputs in parabolic
139 PDEs have been studied in [20] and multiplicative disturbance inputs in abstract
140 infinite-dimensional systems have been studied in [28]. The fact of considering (2.1)-
141 (2.3) as a time-invariant system is very important for theoretical reasons: we can
142 always assume that the initial time is zero. Therefore, the proposed event-triggered
143 control scheme may be seen as a feedforward control scheme that compensates the
144 effect of the distributed disturbance input $\lambda[t]$. Feedforward control for infinite di-
145 mensional systems has been studied in [1].

146 **2.1. Backstepping control design.** Our aim is the global exponential stabi-
147 lization of the system (2.1)-(2.3) at zero using boundary control. To that end, we
148 follow the backstepping approach which makes uses of an invertible Volterra trans-
149 formation to map the system into a target system simpler to handle and with desired
150 stability properties. Since the reaction coefficient is both time and space varying,
151 typically, the kernels of the transformation have to be chosen to depend on time.
152 This brings an additional source of complexity since the resulting kernel PDE equa-
153 tion contains a time-derivative of the kernel and involves the time and space varying
154 coefficient. Overall, the problem is much harder to solve but has been the object of
155 extensive investigation since the seminal work [3]. Numerical strategies such as the
156 method of successive approximation have been widely employed (see e.g. [27] and
157 [43]).

158 Our approach takes a different direction. We avoid solving a kernel-PDE hyper-
159 bolic spatial operator and first order derivative with respect to time capturing the
160 reaction coefficient (dependent on both time and space). We use simpler kernels for
161 the control under which we are still able to stabilize exponentially the closed-loop
162 system. This brings some degree of robustness to the controller. Inspired by event-
163 triggered control strategies (in both finite and infinite-dimensional settings), the key
164 idea of our approach is to schedule the kernel gain at a certain increasing sequence of
165 times. More precisely the computation and updating of the kernel are on events and
166 only when needed. The time instants are determined by event-triggered mechanisms
167 that form an increasing sequence $\{t_j\}_{j \in \mathbb{N}}$ with $t_0 = 0$ which will be characterized later
168 on.

169 Let $\{t_j\}_{j \in \mathbb{N}}$ be an increasing sequence of times with $t_0 = 0$ and define:

$$170 \quad (2.7) \quad b_j(x) := \lambda(t_j, x), \quad \text{for } x \in [0, 1].$$

171 which is the sampled version of the reaction coefficient $\lambda(t, x)$. We define also the
172 error when sampling:

$$173 \quad (2.8) \quad e_j(t, x) := \lambda(t, x) - b_j(x), \quad \text{for } t \in [t_j, t_{j+1}), \quad x \in [0, 1].$$

174 Therefore, we can rewrite (2.1)-(2.3), for $t \in [t_j, t_{j+1})$ as follows:

$$175 \quad (2.9) \quad u_t(t, x) = \varepsilon u_{xx}(t, x) + b_j(x)u(t, x) + e_j(t, x)u(t, x),$$

$$176 \quad (2.10) \quad u_x(t, 0) = qu(t, 0),$$

$$177 \quad (2.11) \quad u(t, 1) = U(t), \quad \text{or } u_x(t, 1) = U(t).$$

178 The backstepping boundary control design is performed by transforming (2.9)-(2.11)
 179 into a target system which will reflect of the error when sampling $e_j(t, x)$ (2.8). There-
 180 fore, consider the following invertible Volterra transformation, for $j \geq 0$,

$$181 \quad (2.12) \quad w_j(t, x) = u(t, x) - \int_0^x K_j(x, y)u(t, y)dy := (\mathcal{K}_j u[t])(x)$$

182 whose inverse ¹ is given by

$$183 \quad (2.13) \quad u(t, x) = w_j(t, x) + \int_0^x L_j(x, y)w(t, y)dy := (\mathcal{L}_j w_j[t])(x)$$

184 with kernels $K_j, L_j \in \mathcal{C}^2(\mathcal{T})$ evolving in a triangular domain given by $\mathcal{T} = \{(x, y) :$
 185 $0 \leq y \leq x \leq 1\}$ and satisfying [36, Theorem 2]:

$$186 \quad (2.14) \quad K_{j,xx}(x, y) - K_{j,yy}(x, y) = \frac{(b_j(y) + c)}{\varepsilon} K_j(x, y),$$

187

$$188 \quad (2.15) \quad K_{jy}(x, 0) = qK_j(x, 0),$$

189

$$190 \quad (2.16) \quad K_j(x, x) = -\frac{1}{2\varepsilon} \int_0^x (b_j(s) + c)ds,$$

191

$$192 \quad (2.17) \quad L_{j,xx}(x, y) - L_{j,yy}(x, y) = -\frac{(b_j(x) + c)}{\varepsilon} L_j(x, y),$$

193

$$194 \quad (2.18) \quad L_{jy}(x, 0) = qL_j(x, 0),$$

195

$$196 \quad (2.19) \quad L_j(x, x) = -\frac{1}{2\varepsilon} \int_0^x (b_j(s) + c)ds.$$

197 Under (2.12), (2.14)-(2.16) and selecting the control $U(t)$ to satisfy

$$198 \quad (2.20) \quad U(t) = \int_0^1 K_j(1, y)u(t, y)dy, \quad t \in (t_j, t_{j+1})$$

199 for Dirichlet actuation or by

$$200 \quad (2.21) \quad U(t) = K_j(1, 1)u(t, 1) + \int_0^1 K_{j,x}(1, y)u(t, y)dy, \quad t \in (t_j, t_{j+1})$$

201 for Neumann actuation, the transformed system, for all $j \geq 0$, $t \in (t_j, t_{j+1})$, is as
 202 follows:

$$203 \quad (2.22) \quad w_{j,t}(t, x) = \varepsilon w_{j,xx}(t, x) - cw_j(t, x) + (\mathcal{K}_j f_j[t])(x),$$

$$204 \quad (2.23) \quad w_{jx}(t, 0) = qw_j(t, 0),$$

$$205 \quad (2.24) \quad w_j(t, 1) = 0, \quad \text{or} \quad w_{jx}(t, 1) = 0,$$

¹see [34, Chapter 4] for details on the invertibility of the Volterra transformation.

206 where

$$207 \quad (2.25) \quad f_j(t, x) := e_j(t, x)u(t, x)$$

208 and c is a design parameter which is chosen as $c \geq \varepsilon \bar{q}^2$ (for Dirichlet actuation) or
 209 $c \geq \varepsilon \bar{q}^2 + \varepsilon/2$ (for Neumann actuation) where $\bar{q} = \max\{0, -q\}$ (see [36]).

210 Moreover, the following estimate holds, for all $j \geq 0$

$$211 \quad (2.26) \quad \max\{|K_j(x, y)|, |L_j(x, y)|\} \leq M \exp(2Mx), \quad \text{for } (x, y) \in \mathcal{T},$$

212 where $M := \frac{\bar{\lambda} + c}{\varepsilon}$ (see [36]).

213 Definitions (2.12), (2.13) imply the following estimates, for all $j \geq 0$

$$214 \quad (2.27) \quad \|(\mathcal{K}_j u[t])\| \leq \tilde{K}_j \|u[t]\|,$$

215

$$216 \quad (2.28) \quad \|(\mathcal{L}_j w_j[t])\| \leq \tilde{L}_j \|w_j[t]\|,$$

217 where \tilde{K}_j and \tilde{L}_j are defined, respectively by $\tilde{K}_j := 1 + \left(\int_0^1 \left(\int_0^x |K_j(x, y)|^2 dy\right) dx\right)^{1/2}$

218 and $\tilde{L}_j := 1 + \left(\int_0^1 \left(\int_0^x |L_j(x, y)|^2 dy\right) dx\right)^{1/2}$.

219 In addition, inequality (2.26) implies the existence of a constant $G > 0$ such that:

$$220 \quad (2.29) \quad \max\{\tilde{K}_j, \tilde{L}_j\} \leq G, \quad \text{for } j \geq 0,$$

221 where

$$222 \quad (2.30) \quad G := 1 + \sqrt{\frac{\bar{\lambda} + c}{4\varepsilon} \left(\exp\left(\frac{4(\bar{\lambda} + c)}{\varepsilon}\right) - 1 \right)}$$

223 which is independent of j .

224 **2.2. Well-posedness analysis.** In order to study well-posedness issues for
 225 (2.22)-(2.24) and in turn for (2.9)-(2.11) (by virtue of the bounded invertibility the
 226 backstepping transformations (2.12)-(2.13)) under any event-triggered gain scheduling
 227 implementation, we need first a more general result (see Theorem 2.2 below) which
 228 establishes the notion of solution for the following reaction-diffusion system:

$$229 \quad (2.31) \quad \begin{aligned} w_t(t, x) &= \varepsilon w_{xx}(t, x) - cw(t, x) + (\mathcal{F}(t)w[t])(x), \quad x \in (0, 1), \\ a_0 w(t, 0) + b_0 w_x(t, 0) &= a_1 w(t, 1) + b_1 w_x(t, 1) = 0, \end{aligned}$$

230 where $\varepsilon > 0, c, a_0, b_0, a_1, b_1$ are constants with $a_0^2 + b_0^2 > 0, a_1^2 + b_1^2 > 0$ and for each
 231 $t \geq 0$ the operator $\mathcal{F}(t) : L^2(0, 1) \rightarrow L^2(0, 1)$ is a linear bounded operator for which
 232 there exist constants $\Omega_1, \Omega_2 > 0$ such that

$$233 \quad (2.32) \quad \|\mathcal{F}[t]\| \leq \Omega_1, \quad \text{for all } t \geq 0,$$

234

$$235 \quad (2.33) \quad \|\mathcal{F}(t) - \mathcal{F}(s)\| \leq \Omega_2 |t - s|, \quad \text{for all } t, s \geq 0.$$

236

237 THEOREM 2.2. For every initial condition $w_0 \in L^2(0, 1)$ and $T > 0$, the initial-
 238 boundary value problem (2.31) with

$$239 \quad (2.34) \quad w[0] = w_0,$$

240 has a unique solution $w \in C^0([0, T]; L^2(0, 1)) \cap C^1((0, T); L^2(0, 1))$ with $w[t] \in D$ for
 241 all $t \in (0, T)$, where

$$242 \quad D := \{f \in H^2(0, 1) : a_0 f(0) + b_0 f'(0) = a_1 f(1) + b_1 f'(1) = 0\},$$

243 that satisfies (2.34) and (2.31) for all $t \in (0, T)$.

244 *Proof.* See Appendix A. □

245 We are in position to specialize the well-posedness result to the system (2.22)-
 246 (2.24) so as we can construct the solution by the step method. To do so, let us take
 247 in (2.34) $a_0 = q$, $b_0 = -1$ for $q < +\infty$; $a_0 = 1$, $b_0 = 0$ for $q = +\infty$, $a_1 = 1$, $b_1 = 0$ for
 248 Dirichlet actuation and $a_1 = 0$, $b_1 = 1$ for Neumann actuation. In addition, it suffices
 249 to observe that the operator $(\mathcal{K}_j f_j[t])(x)$ in (2.22) has the form of $(\mathcal{F}(t)w_j[t])(x)$, i.e.,

$$250 \quad (2.35) \quad (\mathcal{F}(t)w_j[t])(x) = (\mathcal{K}_j(e_j[t]\mathcal{L}_j w_j[t]))(x), \quad t \in [t_j, t_{j+1})$$

251 which is indeed the case by virtue of (2.8), (2.12), (2.13) and (2.25). Moreover,
 252 due to (2.6) in Assumption 2.1, the operator $\mathcal{F}(t)$ satisfies (2.32)-(2.33). Extending
 253 continuously the operator $\mathcal{F}(t)$ defined by (2.35) for $t \geq t_{j+1}$ and using Theorem 2.2
 254 we obtain the following proposition:

255 PROPOSITION 2.3. For every initial data $w_j[t_j] = (\mathcal{K}_j u[t_j])(x) \in L^2(0, 1)$, there
 256 exists a unique function $w_j \in C^0([t_j, t_{j+1}]; L^2(0, 1)) \cap C^1((t_j, t_{j+1}); L^2(0, 1))$ with
 257 $w_j[t] \in D$ for $t \in (t_j, t_{j+1}]$ that satisfies (2.22)-(2.24) for all $t \in (t_j, t_{j+1})$ where
 258 $D \subset H^2([0, 1])$ is the set of functions $f : [0, 1] \rightarrow \mathbb{R}$ for which one has $f'(0) = qf(0)$
 259 and $f(1) = 0$ for the case of Dirichlet actuation or $f'(1) = 0$ for the case of Neumann
 260 actuation.

261 Consequently, by the bounded invertibility of the backstepping transformation, for
 262 every $u_0 \in L^2(0, 1)$, we can construct a solution $u \in C^0([0, \lim_{j \rightarrow \infty}(t_j)]; L^2(0, 1))$
 263 with $u[t] \in H^2(0, 1)$ for $t \in (0, \lim_{j \rightarrow \infty}(t_j))$ and $u \in C^1(\tilde{I}; L^2(0, 1))$ where $\tilde{I} =$
 264 $[0, \lim_{j \rightarrow \infty}(t_j)) \setminus \{t_j : j = 0, 1, 2, \dots\}$ which also satisfies (2.9)-(2.11) for all $t \in \tilde{I}$.

265 **2.3. Event-triggered gain scheduling.** Let us consider the following Sturm-
 266 Liouville operator $B : D \rightarrow L^2(0, 1)$ defined by

$$267 \quad (2.36) \quad (Bh)(x) = -\varepsilon \frac{d^2 h}{dx^2}(x),$$

268 for all $h \in D$ and $x \in (0, 1)$ where $D \subset H^2([0, 1])$ is the set of functions $h : [0, 1] \rightarrow \mathbb{R}$
 269 for which one has $h'(0) = qh(0)$ and $h(1) = 0$ for the case of Dirichlet actuation or
 270 $h'(1) = 0$ for the case of Neumann actuation.

271 We denote $\mu_1 < \mu_2 < \dots < \mu_n < \dots$ with $\lim_{n \rightarrow \infty}(\mu_n) = +\infty$ and $\phi_n(x) \in$
 272 $\mathcal{C}^2([0, 1], \mathbb{R})$, ($n = 1, 2, \dots$) the eigenvalues and the eigenfunctions, respectively, of the
 273 operator B .

274 **2.3.1. Event-triggered gain scheduling with a static triggering condi-**
 275 **tion.** We introduce the first event-triggering strategy (or mechanism) for gain sched-
 276 uling considered in this paper. The triggering condition is a state-dependent law and

277 determines the time instants at which the reaction coefficient has to be sampled and
 278 thereby when the kernel computation/updating has to be done.

279 **DEFINITION 2.4** (Definition of the static event-triggered mechanism for gain sched-
 280 uling). Let K_j be the kernel satisfying (2.14)-(2.16) and let $f_j(t, x)$ be given by (2.25),
 281 $j \geq 0$. Let μ_1 be the principal eigenvalue of the Sturm-Liouville operator B (2.36).
 282 Let $R \in (0, 1)$ be a design parameter and define

$$283 \quad (2.37) \quad \mu := c + \varepsilon\mu_1.$$

284 The static event-triggered gain scheduler is defined as follows:

285 The times of events $t_j \geq 0$ with $t_0 = 0$ form a finite or countable set of times
 286 which is determined by the following rules for some $j \geq 0$:

287
 288 a) if $\{t > t_j : \langle (\mathcal{K}_j u[t]), (\mathcal{K}_j f_j[t]) \rangle > \mu R \|(\mathcal{K}_j u[t])\|^2\} = \emptyset$ then the set of the
 289 times of the events is $\{t_0, \dots, t_j\}$.

290
 291 b) if $\{t > t_j : \langle (\mathcal{K}_j u[t]), (\mathcal{K}_j f_j[t]) \rangle > \mu R \|(\mathcal{K}_j u[t])\|^2\} \neq \emptyset$, then the next event
 292 time is given by:

$$293 \quad (2.38) \quad t_{j+1} := \inf\{t > t_j : \langle (\mathcal{K}_j u[t]), (\mathcal{K}_j f_j[t]) \rangle > \mu R \|(\mathcal{K}_j u[t])\|^2\},$$

294 where $u[t]$ denotes the solution of (2.1)-(2.3) with (2.20) or (2.21) for $t \geq t_j$.

295 **2.3.2. Event-triggered gain scheduling with a dynamic triggering con-**
 296 **dition.** Inspired by [11] and [6], we introduce the second event-triggering mechanism
 297 for gain scheduling in this paper. It involves a dynamic variable which can be viewed
 298 as a filtered value of the static triggering condition in (2.38). With this strategy we
 299 expect to reduce updating times for the kernel scheduling and obtain larger inter-
 300 execution times.

301 **DEFINITION 2.5** (Definition of the dynamic event-triggered mechanism for gain
 302 scheduling). Let K_j be the kernel satisfying (2.14)-(2.16), $f_j(t, x)$ be given by (2.25),
 303 $j \geq 0$ and μ be given by (2.37). Let $R \in (0, 1)$, $\eta \geq 2\mu(1 - R)$ and $\theta > 0$ be design
 304 parameters.

305 The dynamic event-triggered gain scheduler is defined as follows:

306 The times of events $t_j \geq 0$ with $t_0 = 0$ form a finite or countable set of times
 307 which is determined by the following rules for some $j \geq 0$:

308
 309 a) if $\{t > t_j : \langle (\mathcal{K}_j u[t]), (\mathcal{K}_j f_j[t]) \rangle - \mu R \|(\mathcal{K}_j u[t])\|^2 > \frac{1}{\theta} m(t)\} = \emptyset$ then the set
 310 of the times of the events is $\{t_0, \dots, t_j\}$.

311
 312 b) if $\{t > t_j : \langle (\mathcal{K}_j u[t]), (\mathcal{K}_j f_j[t]) \rangle - \mu R \|(\mathcal{K}_j u[t])\|^2 > \frac{1}{\theta} m(t)\} \neq \emptyset$, then the next
 313 event time is given by:

$$314 \quad (2.39) \quad t_{j+1} := \inf\{t > t_j : \langle (\mathcal{K}_j u[t]), (\mathcal{K}_j f_j[t]) \rangle - \mu R \|(\mathcal{K}_j u[t])\|^2 > \frac{1}{\theta} m(t)\},$$

315 where $u[t]$ denotes the solution of (2.1)-(2.3) with (2.20) or (2.21) for $t > t_j$
 316 and m satisfies the ordinary differential equation

$$317 \quad (2.40) \quad \dot{m}(t) = -\eta m(t) + \left(\mu R \|(\mathcal{K}_j u[t])\|^2 - \langle (\mathcal{K}_j u[t]), (\mathcal{K}_j f_j[t]) \rangle \right), \quad \text{for } t \geq t_j,$$

318 and we set $m(t_j) = 0$.

319 *Remark 2.6.* Let us remark that the static event-triggered strategy has only one
320 design parameter (i.e. $R \in (0, 1)$) whereas the dynamic event-triggered strategy
321 has three additional design parameters, namely, R (as in the static case), η and θ .
322 Essentially, η adjusts the convergence rate of the filter (2.40) that can be characterized
323 as $\eta \geq 2\mu(1 - R)$. The parameter θ , on the other hand, can be selected to contribute
324 to sample less frequent than with the static event-triggered strategy. As a matter of
325 fact, one can see the static event-triggering condition (2.38) as the limiting case of
326 the dynamic event-triggering condition (2.39)-(2.40) when θ goes to $+\infty$.

327 The following result guarantees that the dynamic variable $m(t)$ remains always posi-
328 tive between two successive triggering times. This fact is going to be helpful in the
329 stability analysis of the closed-loop system.

330 **LEMMA 2.7.** *Under the definition of the event-triggered gain scheduling with dy-*
331 *namic trigger condition (2.39)-(2.40), it holds, for $t \in [t_j, t_{j+1})$, $j \geq 0$, that $\frac{1}{\theta}m(t) +$
332 $\mu R\|(\mathcal{K}_j u[t])\|^2 - \langle (\mathcal{K}_j u[t]), (\mathcal{K}_j f_j[t]) \rangle \geq 0$ and $m(t) \geq 0$.*

333 *Proof.* From definition of the the event-triggered gain scheduling with dynamic
334 triggering condition (2.39)-(2.40), events are triggered to guarantee, for $t \in [t_j, t_{j+1})$,
335 $j \geq 0$ that $\frac{1}{\theta}m(t) + \mu R\|(\mathcal{K}_j u[t])\|^2 - \langle (\mathcal{K}_j u[t]), (\mathcal{K}_j f_j[t]) \rangle \geq 0$. This inequality in
336 conjunction with (2.40) yields:

$$337 \quad (2.41) \quad \dot{m}(t) \geq -(\eta + \frac{1}{\theta})m(t)$$

338 for which the Comparison principle can be used to guarantee $m(t) \geq 0$, for all $t \in$
339 $[t_j, t_{j+1})$, $j \geq 0$ and provided that $m(t_j) \geq 0$. \square

340 Lemma 2.7 guarantees that $\frac{1}{\theta}m(t_{j+1}) + \mu R\|(\mathcal{K}_j u[t_{j+1}])\|^2 - \langle (\mathcal{K}_j u[t_{j+1}]), (\mathcal{K}_j f_j[t_{j+1}]) \rangle$
341 ≥ 0 and that $m(t_{j+1}) \geq 0$ when $t_{j+1} < +\infty$ (by continuity).

342 **PROPOSITION 2.8.** *If the time of the next event generated by (2.38) is finite, then*
343 *the time of the next event generated by the dynamic event triggered mechanism (2.39)-*
344 *(2.40) is strictly larger than the time of the next event generated by the static event*
345 *triggered mechanism (2.38).*

346 *Proof.* Without loss of generality we may assume that $j = 0$ (and consequently
347 $t_0 = 0$). Notice that if $u[0] = 0$ then both the static and dynamic event triggering
348 conditions give $t_1 = +\infty$. By assumption, the time of the next event generated by the
349 static strategy is finite; therefore it holds that $u[0]$ is not zero. Consequently, $\mathcal{K}_0 u[0]$
350 is not zero.

351 Let t_1 be the time of the next event generated by the static event triggered mechanism
352 and let T be the time of the next event generated by the dynamic one. We show next
353 that $T > t_1$ by contradiction. Assume that $T \leq t_1$. Define

$$354 \quad (2.42) \quad q(t) := \mu R\|(\mathcal{K}_0 u[t])\|^2 - \langle (\mathcal{K}_0 u[t]), (\mathcal{K}_0 f_0[t]) \rangle.$$

355 Then we have by virtue of (2.38), (2.42) $q(t) \geq 0$ for all $t \in [0, t_1]$ and by virtue
356 of (2.39), (2.42) $q(T) = -\frac{1}{\theta}m(T)$, implying that $m(T) \leq 0$. Since $m(0) = 0$ and
357 $\dot{m}(t) = -\eta m(t) + q(t)$ for all $t \in [0, T]$, we have

$$358 \quad (2.43) \quad m(t) = \int_0^t \exp(-\eta(t-s))q(s)ds, \quad \text{for all } t \in [0, T].$$

359 Since $q(t) \geq 0$ for all $t \in [0, T]$ we get $m(T) \geq 0$ and thus we conclude that $m(T) = 0$.
360 By continuity of $q(t)$ (which follows by virtue of Proposition 2.3, the continuity of

361 all mappings involved with respect to time, and since the scalar product and the
362 norm preserve continuity) and the fact that $q(t) \geq 0$ for all $t \in [0, T]$, the integral
363 $\int_0^T \exp(-\eta(T-s))q(s)ds$ is zero only if $q(t)$ is identically zero on $[0, T]$. However,
364 that is not possible since $f_0[0] = 0$ (recall (2.7),(2.8) and (2.25)) and since $q(0) =$
365 $\mu R \|\mathcal{K}_0 u[0]\|^2 - \langle \mathcal{K}_0 u[0], K_0 f_0[0] \rangle = \mu R \|\mathcal{K}_0 u[0]\|^2 > 0$. Thus, it must hold that $T >$
366 t_1 . \square

367 **3. Analysis of the closed-loop system and main results.** In this section we
368 present our main results: the avoidance for the Zeno behavior ², the well-posedness
369 and the exponential stability of the closed-loop system under boundary controller
370 whose gains are scheduled according to the two event-triggered strategies.

371 **3.1. Avoidance of the Zeno phenomenon.**

372 **3.1.1. Event-triggered gain scheduling with a static triggering condi-**
373 **tion.**

374 **PROPOSITION 3.1.** *Under (2.38), there exists a minimal dwell-time between two*
375 *triggering times, i.e. there exists a constant $\tau > 0$ (independent of the initial condition*
376 *u_0) such that $t_{j+1} - t_j \geq \tau$, for all $j \geq 0$. More specifically, τ satisfies:*

$$377 \quad (3.1) \quad \tau = \frac{1}{\varphi} \frac{\mu R}{G^2},$$

378 with $\mu = c + \varepsilon \mu_1$ (recall (2.37) with μ_1 being the principal eigenvalue of the Sturm-
379 Liouville operator B (2.36)), $R \in (0, 1)$ being the design parameter involved in the
380 event-triggering condition (2.38), φ as in Assumption 2.1 and G given by (2.30).

381 *Proof.* Assume that an event occurs at $t = t_{j+1}$. Then, from (2.38) and using
382 (2.12), continuity of all mappings involved with respect to time and the Cauchy-
383 Schwarz inequality, the following more conservative estimate holds:

$$384 \quad (3.2) \quad \|w_j[t_{j+1}]\| \|(\mathcal{K}_j f_j[t_{j+1}])\| \geq \langle w_j[t_{j+1}], (\mathcal{K}_j f_j[t_{j+1}]) \rangle \geq \mu R \|w_j[t_{j+1}]\|^2.$$

385 Using (2.13), (2.25), (2.27)-(2.30) we get from (3.2):

$$386 \quad (3.3) \quad G^2 \|w_j[t_{j+1}]\|^2 \|e_j[t_{j+1}]\|_\infty \geq \mu R \|w_j[t_{j+1}]\|^2.$$

387 Therefore,

$$388 \quad (3.4) \quad G^2 \|e_j[t_{j+1}]\|_\infty \geq \mu R.$$

389 By virtue of Assumption 2.1 in conjunction with (2.7) and (2.8), we obtain for all
390 $j \geq 0$

$$391 \quad (3.5) \quad G^2 \varphi (t_{j+1} - t_j) \geq \mu R,$$

392 from which we can deduce (using definition (3.1))

$$393 \quad (3.6) \quad t_{j+1} - t_j \geq \tau,$$

394 being τ the minimal dwell-time (independent on initial conditions). \square

²We recall, the Zeno phenomenon means infinite triggering times in a finite-time interval. In practice, Zeno phenomenon would represent infeasible implementation into digital platforms since one would require to sample infinitely fast.

395 Proposition 3.1 allows us to conclude that $\lim(t_j) = +\infty$ and thereby we can ap-
 396 ply Proposition 2.3 to get the following result on the existence of solutions of the
 397 closed-loop system (2.1)-(2.4) with control (2.20), under the static event-triggered
 398 gain scheduler (2.38).

399 **COROLLARY 3.2.** *For every initial condition $u_0 \in L^2(0, 1)$, there exists a unique*
 400 *mapping $u \in C^0(\mathbb{R}_+; L^2(0, 1)) \cap C^1(I; L^2(0, 1))$, $u[t] \in H^2(0, 1)$ for $t > 0$ that satisfies*
 401 *(2.1)-(2.4) with control (2.20) or (2.21), under the static event-triggered gain scheduler*
 402 *(2.38) for all $t \in I$ and $I = \mathbb{R}_+ \setminus \{t_j \geq 0, j = 0, 1, 2, \dots\}$.*

403 *Proof.* It is an immediate consequence of Proposition 2.3 and Proposition 3.1.
 404 Indeed, the solution is constructed (by the step method) iteratively between successive
 405 triggering times. \square

406 **Remark 3.3.** The minimal dwell-time depends on the rate of change of the reac-
 407 tion coefficient. A higher rate of change of the reaction coefficient (i.e., large φ) would
 408 give a smaller minimal dwell-time (i.e., more frequent event triggering). This is ex-
 409 pected since a high rate of change of the reaction coefficient requires a more frequent
 410 update of the control law.

411 **3.1.2. Event-triggered gain scheduling with a dynamic triggering con-**
 412 **dition.** By virtue of Proposition 2.8, the inter-execution time for the dynamic event
 413 triggered mechanism (2.39)-(2.40) always exceeds the inter-execution time for the
 414 static event triggered mechanism (2.38). Due to Proposition 3.1, the Zeno phenome-
 415 non is immediately excluded.

416 Therefore, as in the static case, we can also conclude that $\lim(t_j) = +\infty$ and
 417 thereby we can apply Proposition 2.3 to get the following result on the existence of
 418 solutions of the closed-loop system (2.1)-(2.4) with control (2.20) or (2.21), under the
 419 dynamic event-triggered gain scheduler (2.39)-(2.40).

420 **COROLLARY 3.4.** *For every initial condition $u_0 \in L^2(0, 1)$, there exists a unique*
 421 *mapping $u \in C^0(\mathbb{R}_+; L^2(0, 1)) \cap C^1(I; L^2(0, 1))$, $u[t] \in H^2(0, 1)$ for $t > 0$ that satis-*
 422 *fies (2.1)-(2.4) with control (2.20) or (2.21), under the dynamic event-triggered gain*
 423 *scheduler (2.39)-(2.40) for all $t \in I$ and $I = \mathbb{R}_+ \setminus \{t_j \geq 0, j = 0, 1, 2, \dots\}$.*

424 **3.2. Exponential stability analysis.** We present next the stability results un-
 425 der our two event-triggered gain scheduling strategies.

426 **3.2.1. Event-triggered gain scheduling with a static triggering condi-**
 427 **tion.**

428 **THEOREM 3.5.** *Under Assumption 2.1, if the following condition is fulfilled,*

$$429 \quad (3.7) \quad \varphi < \frac{\mu^2 R(1 - R)}{G^2 \ln(G)},$$

430 *where φ , G , μ , R are defined, respectively, in (2.6), (2.30), (2.37), (2.38); then, the*
 431 *closed-loop system (2.1)-(2.4) with control (2.20) or (2.21), under the static event-*
 432 *triggered gain scheduler (2.38), is globally exponentially stable. More specifically, there*
 433 *exists a constant $\sigma > 0$ such that*

$$434 \quad (3.8) \quad \|u[t]\| \leq G \exp(-\sigma t) \|u[0]\|, \quad \text{for all } t \geq 0.$$

435 *Proof.* By using the variational characterization of eigenvalues (see [39, Section
436 11.4]) in conjunction with (2.22)-(2.24),(2.25), the following estimate holds for all
437 $t \in (t_j, t_{j+1})$:

$$438 \quad (3.9) \quad \frac{d}{dt} \left(\frac{1}{2} \|w_j[t]\|^2 \right) \leq -\mu \|w_j[t]\|^2 + \langle w_j[t], (\mathcal{K}_j f_j[t]) \rangle,$$

439 where $\mu = c + \varepsilon\mu_1$ (recall (2.37) μ_1 being the principal eigenvalue of the Sturm-
440 Liouville operator B (2.36)). We can rewrite (3.9) as follows:

$$441 \quad (3.10) \quad \frac{d}{dt} \left(\frac{1}{2} \|w_j[t]\|^2 \right) \leq -\mu(1-R) \|w_j[t]\|^2 - \mu R \|w_j[t]\|^2 + \langle w_j[t], (\mathcal{K}_j f_j[t]) \rangle,$$

442 where $R \in (0, 1)$ is the parameter involved in (2.38).

443 Therefore, from the definition of the static event-triggered gain scheduler, events
444 are triggered to guarantee, $\langle w_j[t], (\mathcal{K}_j f_j[t]) \rangle \leq \mu R \|w_j[t]\|^2$, for all $t \in (t_j, t_{j+1})$.
445 Then, we obtain for all $t \in [t_j, t_{j+1})$:

$$446 \quad (3.11) \quad \|w_j[t]\|^2 \leq \exp(-2\mu(1-R)(t-t_j)) \|w_j[t_j]\|^2.$$

447 Using (2.12), (2.13), (2.27)-(2.30) and (3.11), we get:

$$448 \quad (3.12) \quad \|u[t]\|^2 \leq G^2 \exp(-2\mu(1-R)(t-t_j)) \|u[t_j]\|^2,$$

449 for all $t \in [t_j, t_{j+1})$. Since $u \in C^0(\mathbb{R}_+; L^2(0, 1))$, it follows that (3.12) holds for $t = t_{j+1}$
450 as well, i.e.

$$451 \quad (3.13) \quad \|u[t_{j+1}]\|^2 \leq G^2 \exp(-2\mu(1-R)(t_{j+1}-t_j)) \|u[t_j]\|^2.$$

452 Now, for all $t \geq 0$, an estimate of $\|u[t]\|$ in terms of $\|u[0]\|$ can be derived recursively,
453 by using (3.6) and the fact that there have been j events and that $j\tau$ units of time
454 have (at least) been passed until t . To that end, we can apply induction on j and
455 prove that, for all $j \geq 0$,

$$456 \quad (3.14) \quad \|u[t_j]\|^2 \leq (G^{2j}) \exp(-2\mu(1-R)t_j) \|u[0]\|^2,$$

457 and that $t_j \geq j\tau$. Let $j \geq 0$ be given (arbitrary) and $t \in [t_j, t_{j+1})$ (arbitrary). We
458 obtain from (3.12),(3.13) and (3.14):

$$459 \quad (3.15) \quad \|u[t]\|^2 \leq (G^2)^{j+1} \exp(-2\mu(1-R)t) \|u[0]\|^2.$$

460 Moreover, since $j \leq \frac{t}{\tau}$, it holds:

$$461 \quad (3.16) \quad \|u[t]\|^2 \leq G^2 \exp\left(-\left(2\mu(1-R) - \frac{2\ln(G)}{\tau}\right)t\right) \|u[0]\|^2.$$

462 In light of condition (3.7) in conjunction with (3.6) where $\tau = \frac{1}{\varphi} \frac{\mu R}{G^2}$, we finally obtain:

$$463 \quad (3.17) \quad \|u[t]\| \leq G \exp(-\sigma t) \|u[0]\|, \quad \text{for all } t \geq 0,$$

464 with $\sigma = \frac{\mu^2 R(1-R) - \varphi G^2 \ln(G)}{\mu R} > 0$. This concludes the proof. \square

465 *Remark 3.6.* Notice that $\mu^2 R(1-R) - \varphi G^2 \ln(G) > 0$ holds true provided that φ
466 is sufficiently small (this corresponds to the case where $\lambda(t, x)$ is slowly time-varying
467 coefficient). In addition, it is worth remarking that we can select $R = \frac{1}{2}$ in order to
468 maximize the allowable upper bound φ . Nevertheless, different values of R may be
469 used in practice since the obtained estimates are conservative. The proof of Theorem
470 2 provides a (conservative) explicit estimate of the convergence rate $\sigma > 0$. The
471 obtained estimate shows that the smaller φ is (i.e., the slower the change of the
472 reaction coefficient), the higher the convergence rate is.

473 **3.2.2. Event-triggered gain scheduling with a dynamic triggering con-**
474 **dition.**

475 **THEOREM 3.7.** *If condition (3.7) of Theorem 3.5 holds, then, the closed-loop sys-*
476 *tem (2.1)-(2.4) with control (2.20) or (2.21), under dynamic event-triggered gain*
477 *scheduler (2.39)-(2.40) is globally exponentially stable. More specifically, there exists*
478 *a constant $\sigma > 0$ such that*

479 (3.18)
$$\|u[t]\| \leq G \exp(-\sigma t) \|u[0]\|, \quad \text{for all } t \geq 0.$$

480 *Proof.* An estimate of the time-derivative of the following function $W(t) :=$
481 $\frac{1}{2}\|w_j[t]\|^2 + m(t)$ along the solutions (2.22)-(2.24),(2.25),(2.40) is given by:
(3.19)

482
$$\dot{W}(t) \leq -\mu\|w_j[t]\|^2 + \langle w_j[t], (\mathcal{K}_j f_j[t]) \rangle - \eta m(t) - \langle w_j[t], (\mathcal{K}_j f_j[t]) \rangle + \mu R \|w_j[t]\|^2$$

483 which can be rewritten as follows:

484 (3.20)
$$\dot{W}(t) \leq -\mu(1-R) (\|w_j[t]\|^2 + 2m(t)) - m(t)(\eta - 2\mu(1-R)).$$

485 By Lemma 2.7, we guarantee that $m(t) \geq 0$ and since $\eta \geq 2\mu(1-R)$ (recall Definition
486 2.5), thus we get:

487 (3.21)
$$\dot{W}(t) \leq -2\mu(1-R)W(t)$$

488 Therefore, we obtain for $t \in [t_j, t_{j+1})$:

489 (3.22)
$$\frac{1}{2}\|w_j[t]\|^2 + m(t) \leq \exp(-2\mu(1-R)(t-t_j)) \left(\frac{1}{2}\|w_j[t_j]\|^2 + m(t_j) \right).$$

490 Notice that $\frac{1}{2}\|w_j[t]\|^2 \leq \frac{1}{2}\|w_j[t_j]\|^2 + m(t)$ and that by Definition 2.5, $m(t_j) = 0$.
491 Therefore, from (3.22) we have, for all $t \in [t_j, t_{j+1})$:

492 (3.23)
$$\|w_j[t]\|^2 \leq \exp(-2\mu(1-R)(t-t_j))\|w_j[t_j]\|^2.$$

493 The remaining part of the proof follows the same reasoning as the proof of Theorem
494 3.5 (see from (3.12)). This concludes the proof. \square

495 *Remark 3.8.* The function $W(t)$ is monotonically decreasing (see (3.21)), for all
496 $t \in [t_j, t_{j+1})$. However, the function $\frac{1}{2}\|w_j[t]\|^2$ may not be monotonically decreasing
497 on that interval. The design parameter θ involved in the dynamic event-triggering
498 condition (2.39)-(2.40) and also discussed in Remark 2.6, allows to limit the poten-
499 tial increase of $\frac{1}{2}\|w_j[t]\|^2$. Indeed, since events are triggered to guarantee $\frac{1}{\theta}m(t) +$
500 $\mu R \|(\mathcal{K}_j u[t])\|^2 - \langle (\mathcal{K}_j u[t]), (\mathcal{K}_j f_j[t]) \rangle \geq 0$, it holds that

501
$$\frac{d}{dt} \left(\frac{1}{2}\|w_j[t]\|^2 \right) \leq -\mu(1-R)\|w_j[t]\|^2 + \frac{1}{\theta}m(t).$$

502 Notice that the larger the value of θ , the more limited the increase. We approach
503 then to the case as we were dealing with the static event-triggered gain scheduler.

504 **4. Numerical simulations.** We illustrate the results by considering (2.1)-(2.4)
505 with $\varepsilon = 1$, $q = +\infty$ and initial condition $u_0(x) = 2(x - x^2)$. For numerical simula-
506 tions, the state of the system has been discretized by finite differences on a uniform
507 grid with the step $h = 0.02$ for the space variable. The discretization with respect to

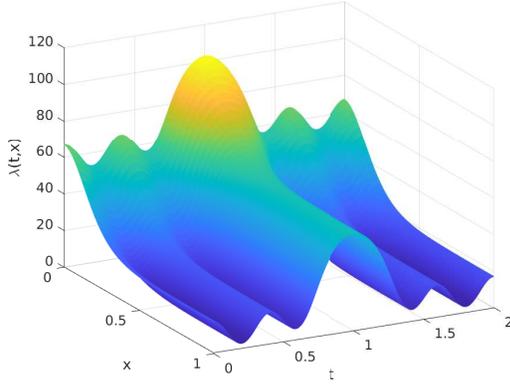


FIG. 1. Profile of the time- and space- varying reaction coefficient $\lambda(t, x) = 10 + \frac{50}{\cosh^2(5(t-1))} + 7 \cos(5\pi t) + \frac{50}{\cosh^2(5x)}$ for the reaction-diffusion system (2.1)-(2.4).

508 time was done using the implicit Euler scheme with step size $\Delta t = h^2$. We run sim-
 509 ulations on a frame of 2s. We choose the time- and space- varying coefficient $\lambda(t, x)$
 510 to have a simple form as $\lambda(t, x) = \lambda_c + \lambda_t(t) + \lambda_x(x)$. More specifically:

$$511 \quad (4.1) \quad \lambda(t, x) = 10 + \frac{50}{\cosh^2(5(t-1))} + 7 \cos(5\pi t) + \frac{50}{\cosh^2(5x)}, \quad t > 0, \quad x \in [0, 1],$$

512 which has a profile depicted in Figure 1. We stabilize the closed-loop system (2.1)-
 513 (2.4) under Dirichlet actuation with boundary control (2.20) whose kernel gains satisfy
 514 (2.14)-(2.16) and are scheduled according to the two event-triggered mechanisms we
 515 introduced in Definition 2.4 (static-based triggering condition) and Definition 2.5
 516 (dynamic-based triggering condition).

517 The parameters of the triggering conditions are set $R = 0.15$, $\mu = c + \varepsilon\pi^2 = \pi^2$
 518 with $c = 0$. In addition, $\eta = 16.7$ and $\theta = 0.15$.

519 From (2.7), $b_j(x) = \lambda(t_j, x) = \tilde{\lambda}_j + \frac{50}{\cosh^2(5x)}$, where $\tilde{\lambda}_j := 10 + \frac{50}{\cosh^2(5(t_j-1))} +$
 520 $7 \cos(5\pi t_j)$ with $\{t_j\}_{j \in \mathbb{N}}$ according to (2.38) (static) or (2.39)-(2.40) (dynamic). In ei-
 521 ther cases, kernels K_j satisfying (2.14)-(2.16), for all $t \in [t_j, t_{j+1})$, admit a closed-form
 522 solution which is given as follows [36, Section VIII.E]:

$$523 \quad (4.2) \quad K_j(x, y) = -\tilde{\lambda}_j y \frac{I_1 \left(\sqrt{\tilde{\lambda}_j} (x^2 - y^2) \right)}{\sqrt{\tilde{\lambda}_j} (x^2 - y^2)} - 5 \tanh(5y) I_0 \left(\sqrt{\tilde{\lambda}_j} (x^2 - y^2) \right),$$

524 where $I_m(\cdot)$, $m \in \mathbb{Z}$ is a modified Bessel function of the first kind of order m . Figure 2
 525 shows the event-triggered sampled version of the profile of the time- and space- vary-
 526 ing reaction coefficient (4.1) for all $t \in [t_j, t_{j+1})$, $j \geq 0$ according to the static event-
 527 triggered gain scheduler (2.38) (depicted on the left) and the dynamic event-triggered
 528 gain scheduler (2.39)-(2.40) (depicted on the right). Hence, the kernel updating is
 529 done on events and aperiodically. One of the main features of this approach is that
 530 the kernel of the control does not need to be computed using the method of succes-
 531 sive approximations to solve a PDE kernel which involves a time- and space- varying
 532 coefficient (see e.g. [15, 19] which do deal with successive approximations method
 533 and efficient numerical schemes for the computation of kernels and further consider-
 534 ations for the stabilization of reaction-diffusion PDEs with time- and space- varying

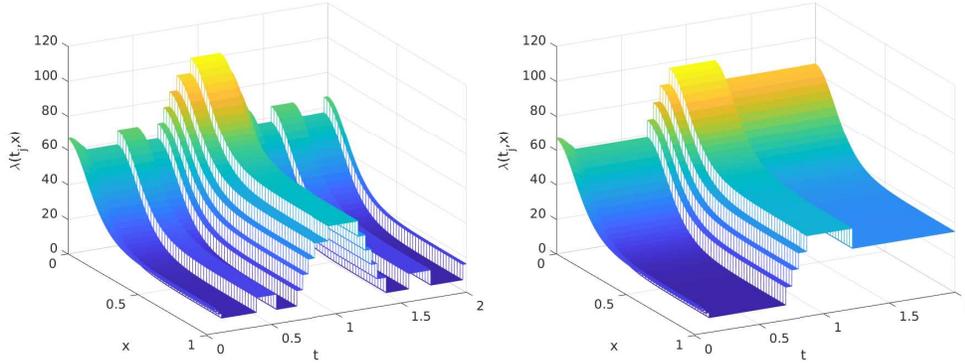


FIG. 2. Sampled version of the profile of the time- and space- varying reaction coefficient (4.1), i.e. $\lambda(t_j, x)$ for all $\{t_j\}_{j \in \mathbb{N}}$ according to the static event-triggered gain scheduler (2.38) (depicted on the left) and the dynamic event-triggered gain scheduler (2.39)-(2.40) (depicted on the right).

535 coefficients). As motivated throughout the paper, it suffices to schedule the kernel
 536 in a suitable way and only when needed while using a simpler kernel (in some cases
 537 admitting closed-form solution; or in some cases when it is approximated via a simpler
 538 successive approximation as one is not taking into account the time-dependence).

539 Figures 3 and 4 show the time-evolution of the L^2 norm of the closed-loop sys-
 540 tem (2.1)-(2.4), (4.1) and the time-evolution of the boundary control (2.20), respec-
 541 tively, under static event-triggered gain scheduler (2.38) (red dashed line) and dy-
 542 namic event-triggered gain scheduler (2.39)-(2.40) (blue line). For both figures, on
 543 the right, there are zooms of the two curves to illustrate the difference. It can be
 544 observed that under the two strategies, the behavior is similar with same theoretical
 guarantees. Finally, we run simulations for 100 different initial conditions given by

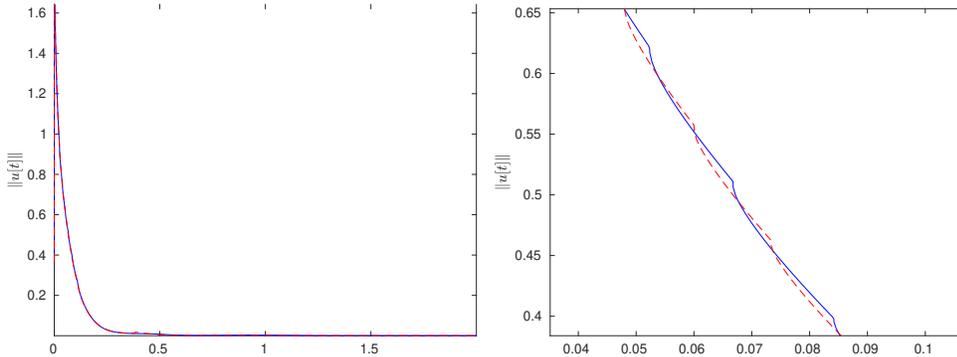


FIG. 3. Time-evolution of the L^2 norm of the closed-loop system (2.1)-(2.4), (4.1) with boundary control (2.20) under static event-triggered gain scheduler (2.38) (red dashed line) and dynamic event-triggered gain scheduler (2.39)-(2.40) (blue line). On the right, there is a zoom of the two curves to illustrate the difference.

545
 546 $u_0(x) = \sqrt{2/n} \sin(\sqrt{n}\pi x) + \sqrt{n}(x - x^2)$, for $n = 1, \dots, 100$ on a frame of 2s. We
 547 compare the static event triggered mechanism with respect to the dynamic one while
 548 computing the inter-execution times between two triggering times. We compare sev-
 549 eral cases by tuning different parameters. For all cases, η is selected as $\eta = 2\mu(1 - R)$.
 550 The mean value of the numbers of events generated under the two strategies is re-

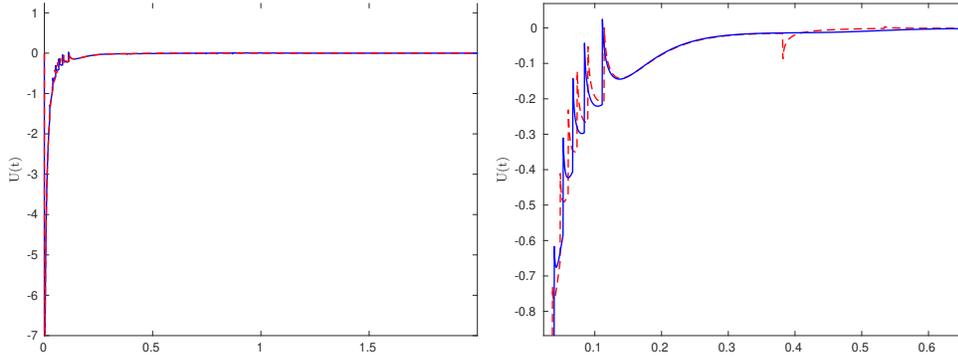


FIG. 4. Time-evolution of the boundary control under static event-triggered gain scheduler (2.38) (red dashed line) and under dynamic event-triggered gain scheduler (2.39)-(2.40) (blue line). On the right, there is a zoom of the two curves to illustrate the difference.

551 ported in Table 1. The mean value and coefficient of variation (ratio between the
 552 standard deviation and the mean value) of inter-execution times for both approaches
 553 are reported in Tables 2 and 3, respectively. In addition, Figure 5 shows the density of
 554 the inter-execution times (axis in logarithmic scale). The red bars correspond to the
 555 inter-execution times under the static event triggered mechanism (2.38); whereas the
 556 blue bars correspond to the dynamic event triggered mechanism (2.39)-(2.40) result-
 557 ing in larger inter-execution times. Therefore, it can be asserted that, as expected,
 558 with the dynamic triggering condition one obtains larger inter-execution times and
 559 we can reduce the number of events rendering the strategy slightly less conserva-
 560 tive. In general, dynamic event-triggered strategies may offer benefits with respect to
 561 static strategies as in the framework of even-triggered control (in finite and infinite
 562 dimensional settings).

TABLE 1

Mean value of the number of events generated under the static event-triggered gain scheduler (2.38) and under dynamic event-triggered gains scheduler (2.39)-(2.40).

	$R = 0.15, \eta = 16.7$	$R = 0.5 \eta = 9.86$
Static ET	39.93	18.58
Dynamic ET ($\theta = 100$)	37.08	17.6
Dynamic ET ($\theta = 1$)	29.8	16.02
Dynamic ET ($\theta = 0.015$)	17.01	8.99

TABLE 2

Mean value of inter-execution times for static event-triggered gain scheduler (2.38) and for dynamic event-triggered gains scheduler (2.39)-(2.40).

	$R = 0.15, \eta = 16.7$	$R = 0.5 \eta = 9.86$
Static ET	0.0460	0.0738
Dynamic ET ($\theta = 100$)	0.0354	0.0521
Dynamic ET ($\theta = 1$)	0.0374	0.0582
Dynamic ET ($\theta = 0.015$)	0.0546	0.1112

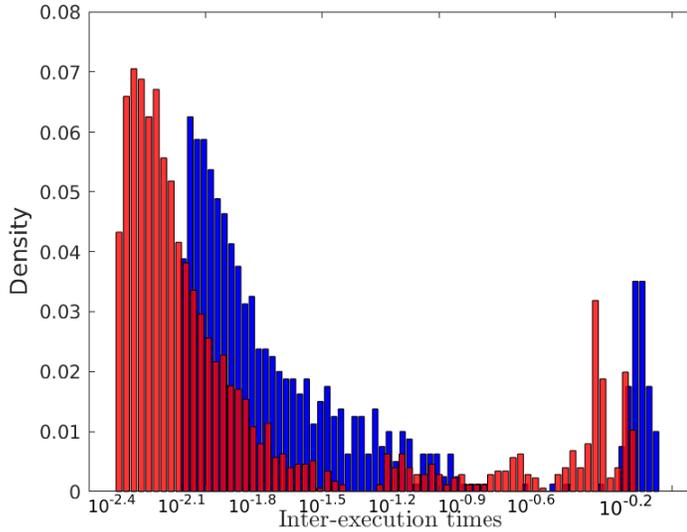


FIG. 5. Density of the inter-execution times (axis in logarithmic scale) computed for 100 different initial conditions given by $u_0(x) = \sqrt{2/n} \sin(\sqrt{n}\pi x) + \sqrt{n}(x-x^2)$, for $n = 1, \dots, 100$ on a frame of 2s. The parameters of the event-triggered strategies are: $R = 0.5$, $\eta = 9.86$ and $\theta = 0.015$. The red bars correspond to the density of inter-execution times under the static event triggered mechanism (2.38); whereas the blue bars correspond to the dynamic event triggered mechanism (2.39)-(2.40) resulting in larger inter-execution times.

TABLE 3

Coefficient of variation of inter-execution times for static event-triggered gain scheduler (2.38) and for dynamic event-triggered gains scheduler (2.39)-(2.40).

	$R = 0.15, \eta = 16.7$	$R = 0.5 \eta = 9.86$
Static ET	1.9814	2.203
Dynamic ET ($\theta = 100$)	2.110	2.693
Dynamic ET ($\theta = 1$)	2.6251	2.8905
Dynamic ET ($\theta = 0.015$)	2.3457	2.0965

563 **5. Conclusion.** In this paper, we have addressed the problem of exponential
564 stabilization of a reaction-diffusion PDE with time- and space- varying reaction coef-
565 ficient. The boundary control design relies on the backstepping method and the gains
566 are computed/updated on events according to two event-triggered gain scheduling
567 schemes. Two event-triggered strategies are proposed for gain scheduling: static and
568 dynamic. The latter involves a dynamic variable that can be viewed as the filtered
569 value of the static one. It has been observed that under this strategy it is possible
570 to reduce the number of events for the gain scheduling. We show that under the two
571 proposed event-triggered gain scheduling schemes Zeno behavior is avoided, which
572 allows to prove well-posedness as well as the exponential stability of the closed-loop
573 system.

574 Our approach can be seen as an efficient way of kernel computation as it is sched-
575 uled aperiodically, when needed and relying on the current state information of the

576 closed-loop system and the time- and space- varying reaction coefficient which is con-
 577 sidered as a distributed input disturbance. Furthermore, the boundary controller is
 578 seen as a feedforward one. This work constitutes an effort towards the “robustifica-
 579 tion” of boundary controllers designed under backstepping method.

580 In future work, we expect to combine these results with event-triggered control
 581 strategies for boundary controlled reaction-diffusion PDEs systems recently intro-
 582 duced in [7] (which deals with constant reaction coefficient only). The results in this
 583 paper may suggest that the triggering times for gain scheduling may be synchronized
 584 with the time instants for control updating. The control is going to be piecewise
 585 constant and not piecewise continuous as in the present work. This would represent
 586 a more realistic way of actuation on the PDE system towards digital realizations. Fi-
 587 nally, we expect to study observers to come up with an observer-based event-triggered
 588 gain scheduling scheme for this class of reaction diffusion PDEs. This would require
 589 to handle new arising challenging issues (e.g. event-triggered gain scheduling of the
 590 output injection gains, point-wise estimates whenever ones samples the output mea-
 591 surement, the avoidance of the Zeno phenomena, among many others).

592 Appendix A. Proof of Theorem 2.2.

593 *Proof.* It suffices to show that there exists $k > 0$ such that for each $w_0 \in L^2(0, 1)$
 594 and $T > 0$ the initial value problem

$$595 \quad (\text{A.1}) \quad \begin{aligned} \dot{y}[t] + (A + kI)y[t] &= \mathcal{F}(t)y[t], \\ y[0] &= w_0, \end{aligned}$$

596 has a unique classical solution on $[0, T]$ in the sense described in [30] where $A : D \rightarrow$
 597 $L^2(0, 1)$ is the Sturm-Liouville operator defined by the following formula for every
 598 $f \in D$:

$$599 \quad (\text{A.2}) \quad (Af)(x) = -\varepsilon f''(x) + cf(x), \text{ for } x \in (0, 1).$$

600 Notice that any solution $y[t]$ of (A.1) provides a solution of the initial-boundary value
 601 problem (2.31) with (2.34) by means of the formula $w[t] = \exp(kt)y[t]$ and any solution
 602 $w[t]$ of the initial boundary value problem (2.31) with (2.34) provides a solution of
 603 the initial value problem (A.1) by means of the formula $y[t] = \exp(-kt)w[t]$.

604 The fact that A is a (non-singular) Sturm-Liouville operator guarantees that
 605 the eigenvalues $\lambda_n, n = 1, 2, \dots$ of A are all real with $\lim (\lambda_n) = +\infty$ and that the
 606 eigenfunctions $\phi_n \in C^2([0, 1]), n = 1, 2, \dots$ of A with $A\phi_n = \lambda_n\phi_n$ and $\|\phi_n\| = 1$ form
 607 an orthonormal basis of $L^2(0, 1)$. It follows that the semigroup

$$608 \quad S(t)u = \sum_{n=1}^{\infty} \exp(-\lambda_n t) \langle \phi_n, u \rangle \phi_n, \text{ for } u \in L^2(0, 1) \text{ and } t > 0,$$

609 is a C_0 semigroup on $L^2(0, 1)$ (see [44], page 178). Consequently, $-A$ is the infinites-
 610 imal generator of a C_0 semigroup on $L^2(0, 1)$.

611 The fact that $-A$ is the infinitesimal generator of an analytic semigroup follows
 612 from the fact that the eigenvalues $\lambda_n, n = 1, 2, \dots$ of A are all real with $\lim (\lambda_n) = +\infty$
 613 Indeed, we can directly apply Theorem 5.2 on page 61 in [30] when $\lambda_1 > 0$. When
 614 $\lambda_1 < 0$ then we can apply Theorem 5.2 on page 61 in [30] to the operator $-A - rI$
 615 with $r > -\lambda_1$ (which generates the C_0 semigroup $\exp(-rt)S(t)$; see explanations on
 616 page 61 in [30]).

617 Since $-A$ is the infinitesimal generator of a C_0 semigroup $S(t), t \geq 0$ on $L^2(0, 1)$
618 and since for each $t \geq 0$ the operator $\mathcal{F}(t) : L^2(0, 1) \rightarrow L^2(0, 1)$ is a linear bounded
619 operator for which there exist constants $\Omega_1, \Omega_2 > 0$ such that (2.32) and (2.33) hold, it
620 follows from Theorem 1.2 on page 184 in [30] that there exists a unique mild solution
621 $y \in C^0([0, T]; L^2(0, 1))$ of the initial value problem (A.1), i.e.,

$$622 \quad (\text{A.3}) \quad y[t] = \exp(-kt)S(t)w_0 + \int_0^t \exp(-k(t-s))S(t-s)\mathcal{F}(s)y[s]ds, \text{ for all } t \in [0, T].$$

623 Theorem 2.2 on page 4 in [30] implies the existence of constants $M, \omega > 0$ such that
624 the estimate $\|S(t)\| \leq M \exp(\omega t)$ holds for all $t \geq 0$. Exploiting the previous estimate
625 in conjunction with (A.3) and (2.32), we get for all $t \in [0, T]$:

$$626 \quad (\text{A.4}) \quad \|y[t]\| \leq M \exp(-(k - \omega)t) \|w_0\| + \frac{Mc}{k - \omega} \max_{0 \leq s \leq T} (\|y[s]\|), \text{ for all } t \in [0, T].$$

627 Selecting $k > 0$ so that $\frac{Mc}{k - \omega} < 1$, estimate (A.4) implies the following estimate:

$$628 \quad (\text{A.5}) \quad \max_{0 \leq s \leq T} (\|y[s]\|) \leq \left(1 - \frac{Mc}{k - \omega}\right)^{-1} M \|w_0\|.$$

629 We next define

$$630 \quad (\text{A.6}) \quad g(t) = \mathcal{F}(t)y[t], \quad t \in [0, T].$$

631 Inequalities (A.5), (2.32) and definition (A.6) imply that $g \in L^p([0, T]; L^2(0, 1))$ for
632 every $p \in [1, +\infty)$. Definition (A.6) allows us to conclude that the mild solution
633 $y \in C^0([0, T]; L^2(0, 1))$ of the initial value problem (A.1) is also a mild solution of
634 the inhomogeneous initial value problem:

$$635 \quad (\text{A.7}) \quad \begin{aligned} \dot{z}[t] + (A + kI)z[t] &= g(t), \\ z[0] &= w_0, \end{aligned} \quad \square$$

636 i.e., $z[t] = y[t]$ for $t \in [0, T]$. Since $-A$ is the infinitesimal generator of an analytic
637 semigroup $S(t)$ on $L^2(0, 1)$, it follows from Theorem 3.1 on page 110 in [30] that for
638 every $p \in (1, +\infty)$, the mapping $t \rightarrow y[t]$ is locally Hölder continuous on $(0, T]$ with
639 exponent $\frac{p-1}{p}$. Using (2.32), (2.33) and the fact that $g(t) = \mathcal{F}(t)y[t]$ for $t \in [0, T]$, it
640 follows that for every $p \in (1, +\infty)$, the mapping $t \rightarrow g[t]$ is locally Hölder continuous
641 on $(0, T]$ with exponent $\frac{p-1}{p}$. By virtue of Corollary 3.3 on page 113 in [30] we conclude
642 that $y[t]$ is the unique classical solution of (A.1).

643

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