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► **To cite this version:**

Brahim Bouya, Mohamed Zarrabi. HELSON SETS, SPECTRAL SYNTHESIS AND APPLICATIONS TO OPERATORS. Harmonic analysis, function theory, operator theory, and their applications, 71–80, Theta Ser. Adv. Math., 19, Theta, Bucharest, 2017., 2015, Bordeaux, France. hal-02657265

HAL Id: hal-02657265

<https://hal.science/hal-02657265>

Submitted on 30 May 2020

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HELSON SETS, SPECTRAL SYNTHESIS AND APPLICATIONS TO OPERATORS.

BRAHIM BOUYA AND MOHAMED ZARRABI

ABSTRACT. In this paper we highlight the role played by Helson sets and/or sets of spectral synthesis in some recent results in operator theory. We consider in particular the results on the cyclicity in L^p spaces, the Katznelson-Tzafriri type theorems and polynomially bounded operators.

1. INTRODUCTION

In this survey we discuss some recent results in operator theory involving Helson sets and/or sets of spectral synthesis. Let G be a locally compact abelian group and Γ its dual group. A closed subset E of Γ is called a Helson set if every continuous function on E vanishing at infinity is the restriction to E of the Fourier transform of a function in $L^1(G)$; E is said to be of spectral synthesis if every function in $L^1(G)$ whose Fourier transform vanishes on E , can be approximated in norm by functions in $L^1(G)$ whose Fourier transforms vanish in a neighborhood of E .

Helson proved in [11] that a Helson set in the unit circle \mathbb{T} does not support any non-zero measure with Fourier coefficients tending to 0. A closed subset E of \mathbb{T} is called a set of multiplicity if E supports a non-zero distribution with Fourier coefficients tending to zero. It was shown by Piatetski-Shapiro that there exists a multiplicity set in \mathbb{T} which does not support any non-zero measure with Fourier coefficients tending to 0. This result was extended by Körner who proved that there exists a Helson set which is also a multiplicity set ([21]). Notice that such a set is not a set of spectral synthesis. The proof of Körner is quite difficult and Kaufman presents a shorter one in [20]. Finally Lev and Olevskii obtain the following stronger result: for any real $q > 2$, there exists a Helson set which supports a non-zero distribution S such that the sequence of Fourier coefficients $(\widehat{S}(n))_{n \in \mathbb{Z}}$ is in $\ell^q(\mathbb{Z})$. This result allows them to refute the conjecture of Wiener about bicyclicity for translation of a function in $L^p(G)$ spaces. For $G = \mathbb{Z}$ and $G = \mathbb{R}$, Wiener

2000 *Mathematics Subject Classification.* 47D03; 43A46.

Key words and phrases. Helson sets, Spectral synthesis, Cyclicity, Representations, Locally compact abelian group, semigroup, contractions, polynomially bounded operators.

has characterized the functions which are bicyclic for translation in $L^1(G)$ and in $L^2(G)$. This characterization is actually given in terms of the zero set of the Fourier transform of the function in question (see [37]). However, Lev and Olevskii proved that this is not possible in $L^p(G)$ for $G = \mathbb{Z}$ or \mathbb{R} and $1 < p < 2$ (see [24] and [23]).

We now turn to the quantitative version of Katznelson-Tzafriri type theorems. Let S be a suitable subsemigroup of G and let T be a representation of G by contractions on a Banach space X . For $f \in L^1(S)$, we set

$$\widehat{f}(T) : x \longmapsto \int_S f(t)T(t)x \, dm_G(t), \quad x \in X,$$

where m_G is the Haar measure on G . The unitary spectrum of T with respect to S is defined by

$$\mathrm{Sp}_u(T, S) = \{\chi \in \Gamma; |\widehat{f}(\chi)| \leq \|\widehat{f}(T)\| \text{ for all } f \text{ in } L^1(S)\}.$$

We consider the following order on S : $s \preceq t \Leftrightarrow t - s \in S$. It was proved that if $f \in L^1(S)$ satisfies spectral synthesis for $\mathrm{Sp}_u(T, S)$, then $\lim_{t \rightarrow \infty} \|T(t)\widehat{f}(T)\| = 0$. This result was first proved by Katznelson and Tzafriri ([19]) in the case $S = \mathbb{Z}_+$ and then extended to $S = \mathbb{R}_+$ in [9] and finally to the general case in [3] and [31]. Further results can be found in [6], [8] and [22].

It is shown in [41] that if X is a Hilbert space and if $S = \mathbb{Z}_+^k$ or $S = \mathbb{R}_+$ then

$$\lim_{t \rightarrow \infty} \|T(t)\widehat{f}(T)\| = \sup \{|f(\lambda)|, \lambda \in \mathrm{Sp}_u(T, S)\}. \quad (1.1)$$

This equality does not extend to representations by contractions defined on arbitrary Banach space. The second author obtained in [41] that if E is a closed subset of Γ , then the following are equivalent:

(i) Equality (1.1) holds for every representation T of S by contractions such that $\mathrm{Sp}_u(T, S) = E$.

(ii) E is of spectral synthesis and is a Helson set with $\alpha(E) = 1$, where $\alpha(E)$ is the Helson constant of E .

On the other hand, it follows from Corollary 4.3 of [35] that if T is a representation of S by contractions on a Hilbert space and if $\mathrm{Sp}_u(T, S)$ is contained in a Helson set E with $\alpha(E) = 1$, then equality (1.1) holds for T (see Theorem 4.1).

Similar results are obtained in [40] about polynomially bounded operators. Thanks to the von Neumann inequality, a contraction on a Hilbert space is polynomially bounded. The situation is actually quite different for contractions on Banach spaces. It is shown in [40] that if E is a closed subset of \mathbb{T} then the following are equivalent:

(i) Every contraction on a Banach space whose spectrum is contained in E is polynomially bounded.

(ii) E is a Helson set and of spectral synthesis.

In section 5, we extend this result to all representations of S by contractions on Banach spaces (see Theorem 5.1).

2. DEFINITIONS AND CLASSICAL RESULTS

Let G be a locally compact abelian group equipped with the Haar measure m_G and let Γ be its dual group. For $f \in L^1(G)$ and $\chi \in \Gamma$, we set

$$\widehat{f}(\chi) = \int_G f(t)\chi(t)dm_G(t).$$

If I is a closed ideal of $L^1(G)$, we set

$$h(I) = \{\chi \in \Gamma : \widehat{f}(\chi) = 0, \text{ for all } f \in I\}.$$

Let E be a closed subset of Γ . We set

$$I(E) = \{f \in L^1(G), \widehat{f} = 0 \text{ on } E\}$$

and

$$J(E) = \text{clos } \{f \in L^1(G), \widehat{f} = 0 \text{ on a neighborhood of } E\}.$$

Clearly $I(E)$ and $J(E)$ are closed ideals of $L^1(G)$, $J(E) \subset I(E)$ and we have $h(I(E)) = h(J(E)) = E$ (see [32], p. 161). The set E is said to be of spectral synthesis if $J(E) = I(E)$. Notice that E is of spectral synthesis if there exists a unique closed ideal I such that $h(I) = E$. A function $f \in L^1(G)$ (or \widehat{f}) is said to satisfy spectral synthesis for E if $f \in J(E)$. Here are some examples of sets of spectral synthesis :

- (1) The compact countable sets ([18], Corollary, p. 240).
- (2) The Cantor set in \mathbb{R} or the unit circle $\mathbb{T} = \{z \in \mathbb{C}; |z| = 1\}$ ([13]).
- (3) The sphere S^1 in \mathbb{R}^2 ([15]).
- (4) All closed sets if Γ is discrete ([4], p. 71).

Notice also that a function f on the unit circle \mathbb{T} satisfy the spectral synthesis for $Z(f) = \{z \in \mathbb{T}; f(z) = 0\}$ whenever it satisfies the following condition : for every $z, z' \in \mathbb{T}$, $|f(z) - f(z')| \leq C_f |z - z'|^{1/2+\epsilon}$ for some constant C_f and $\epsilon > 0$ ([17], Corollaire I, p. 123).

On the other hand it was shown by L. Schwartz that for $n \geq 3$ the sphere $S^{n-1} \subset \mathbb{R}^n$ is not of spectral synthesis ([36]). Later P. Malliavin proved that any non discrete Γ contains a subset which is not of spectral synthesis (see [25],[26],[27] and [10], pp. 70-71).

Let $A(\Gamma)$ denote the Banach algebra of functions \widehat{f} , $f \in L^1(G)$ endowed with the norm $\|\widehat{f}\|_{A(\Gamma)} = \|f\|_{L^1(G)}$. Let $A(E) = \{\widehat{f}|_E, f \in L^1(G)\}$ be the Banach space equipped with the quotient norm of $L^1(G)/I(E)$. A closed

subset E of Γ is called a Helson set if the canonical injection $i : A(E) \rightarrow \mathcal{C}_0(E)$ is onto, where $\mathcal{C}_0(E)$ is the space of continuous functions vanishing at infinity, equipped with the supremum norm. This is equivalent to the invertibility of i and in this case the norm of i^{-1} is called the Helson constant of E and is denoted by $\alpha(E)$. By duality E is a Helson set if and only if there exists a positive constant C such that for every $\mu \in M(E)$, $\|\mu\| \leq C\|\hat{\mu}\|_\infty$, where $M(E)$ is the set of Borel measures on Γ supported by E , $\|\mu\|$ is the total variation of μ and $\|\hat{\mu}\|_\infty = \sup_{x \in G} |\hat{\mu}(x)|$. The smallest constant satisfying the last condition is equal to the Helson constant $\alpha(E)$.

The problem of whether a Helson set is of spectral synthesis was solved by Körner in [21] (see also [20]). More precisely, he proved that there exists a multiplicity set E which is a Helson set with $\alpha(E) = 1$ and this set is not of spectral synthesis. For further discussion on sets from harmonic analysis, see the books [4], [10], [12], [16], [17], [32].

We now give some examples of Helson sets which are of spectral synthesis. Let E be a closed subset of Γ ; E is called independent if for every χ_1, \dots, χ_k in E and every set of integers $\{n_1, \dots, n_k\}$,

$$\chi_1^{n_1} \dots \chi_k^{n_k} = 1 \Rightarrow \chi_1^{n_1} = \dots = \chi_k^{n_k} = 1.$$

A compact subset E of Γ is called a Kronecker set if the following property holds: for every continuous function h on E with $|h| = 1$ and for every $\epsilon > 0$ there exists $t \in G$ such that $\sup_{\chi \in E} |h(\chi) - \chi(t)| < \epsilon$. Every Kronecker set E is of spectral synthesis and is a Helson set with $\alpha(E) = 1$ (see [33], [39] and [32], Theorem 5.5.2, p. 113). Notice that every Kronecker set is independent. If E is a finite independent set and if every element of E is of infinite order, then E is a Kronecker set (see [32], p. 98).

Finally we note that any countable, compact, independent set is a Helson set and of spectral synthesis (see [32], Theorem 5.6.7, p. 117 and p.161).

3. ON CYCLICITY IN L^p SPACES.

Let G be a locally compact abelian group, Γ its dual group and $p \geq 1$. For a function f on G and $y \in G$, we set $\tau_y f(x) = f(x - y)$, $x \in G$. A subspace V of $L^p(G)$ is called invariant by translation if for every $f \in V$ and $y \in G$, $\tau_y f \in V$. A function $f \in L^p(G)$ is called bicyclic in $L^p(G)$ if the subspace $\text{span}\{\tau_y f, y \in G\}$ is dense in $L^p(G)$. The Wiener theorem characterizes the bicyclic function for translation in $L^1(G)$ and $L^2(G)$ ([37]).

Theorem 3.1. (*[Wiener]*)

- (1) A function f is bicyclic in $L^1(G)$ if and only if for every $\chi \in \Gamma$, $\hat{f}(\chi) \neq 0$.
- (2) A function f is bicyclic in $L^2(G)$ if and only if $\hat{f}(\chi) \neq 0$ for almost all $\chi \in \Gamma$.

The space $L^1(G)$ is a Banach algebra for the convolution product and Γ can be identified with the space of maximal ideals by using the map $\chi \rightarrow I_\chi = \{f \in L^1(G), \widehat{f}(\chi) = 0\}$. Moreover the closed translation invariant subspaces of $L^1(G)$ are just the closed ideals of $L^1(G)$ (see [32], Theorem 7.1.2, p. 158). For $f \in L^1(G)$ the closure of $\text{span}\{\tau_y f, y \in G\}$ is invariant by translation and therefore is a closed ideal. It follows that $\text{span}\{\tau_y f, y \in G\}$ is dense in $L^1(G)$ if and only if it is not contained in any maximal ideal, that is $\widehat{f}(\chi) \neq 0$ for every $\chi \in \Gamma$. This proves part (1) of Theorem 3.1. Part (2) of Theorem 3.1 follows from the more general Wiener theorem on closed translation invariant subspaces:

A closed subspace V of $L^2(G)$ is invariant by translation if and only if there exists a measurable subset E of Γ such that

$$V = \{f \in L^2(G), \widehat{f}(\chi) = 0 \text{ for almost all } \chi \in E\}.$$

This result and Theorem 3.1 were proved by Wiener in the case $G = \mathbb{Z}$ and $G = \mathbb{R}$ ([37]). For any group G , the proof of part (2) of Theorem 3.1 is given in ([12], Theorem (31.39)).

It follows from Theorem 3.1 that the bicyclicity of a function f in $L^1(G)$ for $p = 1$ and $p = 2$, is characterized in term the zero set of \widehat{f} , that is $Z_{\widehat{f}} = \{\chi \in \Gamma, \widehat{f}(\chi) = 0\}$. Wiener conjectured that for $1 < p < 2$, a similar characterization holds. In [24], N. Lev and A. Olevskii disproved this conjecture (see also [23]).

Theorem 3.2. *([Lev-Olevskii]) Let $G = \mathbb{Z}$ or $G = \mathbb{R}$. If $1 < p < 2$, then there exist two functions f and g in $L^1(G) \cap L^p(G)$ such that $Z_{\widehat{f}} = Z_{\widehat{g}}$, f is cyclic in $L^p(G)$ while g is not.*

The proof of this theorem is deduced from the following result.

Theorem 3.3. *([Lev-Olevskii]) Let $q > 2$. Then there exists a Helson set $E \subset \mathbb{T}$ which supports a non-zero distribution S such that the sequence of the Fourier coefficients $(\widehat{S}(n))_{n \in \mathbb{Z}}$ is in $\ell^q(\mathbb{Z})$.*

We now make some observations on how Theorem 3.3 implies Theorem 3.2 (see [24] and [23]). Let $q > 2$ and let p be its conjugate number, that is $\frac{1}{p} + \frac{1}{q} = 1$. Let E be a closed subset of \mathbb{T} which support a non-zero distribution S such that $\widehat{S} = (\widehat{S}(n))_{n \in \mathbb{Z}} \in \ell^q(\mathbb{Z})$. We take an infinitely differentiable function h on the circle which vanishes exactly on E . We set $f = (\widehat{h}(n))_{n \in \mathbb{Z}}$. Then $f \in \ell^1(\mathbb{Z})$, $Z_{\widehat{f}} = E$ and, for every $k \in \mathbb{Z}$,

$$(\widehat{S} | (f(n-k))_{n \in \mathbb{Z}})_{\ell^q, \ell^p} := \sum_{n \in \mathbb{Z}} \widehat{S}(-n) f(n-k) = (S | e^{ik(\cdot)} h(\cdot))_{\mathcal{D}'(\mathbb{T}), \mathcal{D}(\mathbb{T})}.$$

where $(\cdot|\cdot)_{X',X}$ denote the duality between a topological space X and its dual X' . Since $e^{ik(\cdot)}h(\cdot)$ is a smooth function, it satisfies the spectral synthesis for E ([17], Corollaire I, p. 123). Therefore $(S|e^{ik(\cdot)}h(\cdot))_{\mathcal{D}'(\mathbb{T}),\mathcal{D}(\mathbb{T})} = 0$. It follows that, for every k , $(\widehat{S}|(f(n-k))_{n \in \mathbb{Z}})_{\ell^q, \ell^p} = 0$, which implies that f is not bicyclic in $\ell^p(\mathbb{Z})$.

On the other hand Lev and Olevskii showed by using Baire's theorem that for every Helson set E in \mathbb{T} , there exists $g \in \ell^1(\mathbb{Z})$ which is bicyclic in $\ell^p(\mathbb{Z})$ for every $p > 1$ and such that $Z_{\widehat{g}} = E$ (see [24], Lemma 9). Thus, we see that if E is the set constructed in Theorem 3.3 then there exists $f, g \in \ell^1(\mathbb{Z})$ as in Theorem 3.2 and such that $Z_{\widehat{f}} = Z_{\widehat{g}} = E$.

We finish this section by recalling some known results on the bicyclicity problem. In [5], Beurling showed that if $f \in \ell^1(\mathbb{Z})$ and if the Hausdorff dimension of $Z_{\widehat{f}}$ is less or equal to $\frac{2(p-1)}{p}$ with $1 < p < 2$, then f is bicyclic in $\ell^p(\mathbb{Z})$. Salem showed in [34] that this result is sharp. On the other hand Newman construct in [28] a set E of Hausdorff dimension 1 such that, if $f \in \ell^1(\mathbb{Z})$ and $Z_{\widehat{f}} = E$ then f is bicyclic in $\ell^p(\mathbb{Z})$, for every $p > 1$.

4. KATZNELSON-TZAFRIRI TYPE THEOREMS

Let G be a locally compact abelian group and S be a measurable semi-group of G with non empty interior in G and such that $G = S - S$. Denote by S^* the set of all the non-zero, continuous, bounded, homomorphisms of S into the multiplicative semigroup \mathbb{C} . Let

$$S_u^* = \{\chi \in S^*; |\chi(s)| = 1 \text{ for all } s \text{ in } S\}.$$

We shall identify S_u^* with the dual group Γ of G , in the natural way. For $f \in L^1(S)$ and $\chi \in S^*$, we set

$$\widehat{f}(\chi) = \int_S f(t)\chi(t)dt.$$

Finally we shall assume that $\{\widehat{f}, f \in L^1(S)\}$ separates the points of S^* from each other and from zero and that the interior S° is dense in S .

Let X be a Banach space, $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators on X . Let T be a representation of S by contractions on X , that is a strongly continuous homomorphism from S into $\mathcal{L}(X)$ such that $T(s)$ is a contraction whenever $s \in S$. For $f \in L^1(S)$, let $\widehat{f}(T) : X \rightarrow X$ be the bounded operator defined by

$$\widehat{f}(T) : x \mapsto \int_S f(t)T(t)x dm_G(t).$$

The spectrum of T with respect to S is defined by

$$\text{Sp}(T, S) = \{\chi \in S^*; |\widehat{f}(\chi)| \leq \|\widehat{f}(T)\| \text{ for all } f \text{ in } L^1(S)\}.$$

The unitary spectrum of T is defined by

$$\mathrm{Sp}_u(T, S) = \mathrm{Sp}(T, S) \cap S_u^* = \mathrm{Sp}(T, S) \cap \Gamma.$$

In the sequel of the paper we will use the following notation: If E is a closed subset of Γ , we denote by $\mathcal{C}_0(E)$ the space of all continuous functions on E vanishing at infinity. For a continuous bounded function h on E we set $\|h\|_{\mathcal{C}(E)} = \sup \{|f(\chi)|, \chi \in E\}$.

Examples. (1) Let $G = \mathbb{R}$ and $S = \mathbb{R}_+$. The set S^* will be identified with $\mathbb{C}_- = \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}$ and S_u^* with \mathbb{R} . For $f \in L^1(\mathbb{R}_+)$ and $z \in \overline{\mathbb{C}_-}$, we have

$$\widehat{f}(z) = \int_{\mathbb{R}_+} f(t)e^{-itz} dt.$$

A C_0 -semigroup $T = (T_t)_{t \geq 0}$ of contractions on X is a representation of \mathbb{R}_+ . In this case we have $\mathrm{Sp}_u(T, \mathbb{R}_+) = i\sigma(A) \cap \mathbb{R}$, where A is the generator of T and $\sigma(A)$ is the spectrum of A .

(2) Let $k \geq 1$ be an integer, $G = \mathbb{Z}^k$ and $S = \mathbb{Z}_+^k$, where \mathbb{Z}_+ is the set of all nonnegative integers; S^* will be identified with $\overline{\mathbb{D}^k}$ and S_u^* with \mathbb{T}^k . For $f \in L^1(\mathbb{Z}_+^k)$ and $z = (z_1, \dots, z_k) \rightarrow \chi_z \in \overline{\mathbb{D}^k}$, we have

$$\widehat{f}(z) = \sum_{n \in \mathbb{Z}_+^k} f(n) z_1^{n_1} \dots z_k^{n_k},$$

where $n = (n_1, \dots, n_k)$. Let T_1, \dots, T_k be a finite many commuting contractions on X . Then $T = (T_1^{n_1} \dots T_k^{n_k})_{n \in \mathbb{Z}_+^k}$ is a representation of \mathbb{Z}_+^k by contractions on X . For $f = (f(n))_{n \in \mathbb{Z}_+^k}$ in $\ell^1(\mathbb{Z}_+^k)$, we have

$$\widehat{f}(T) = \sum_{n \in \mathbb{Z}_+^k} f(n) T_1^{n_1} \dots T_k^{n_k}.$$

Let \mathcal{A} be a commutative Banach algebra that contains T_1, \dots, T_k and I the identity map. We define the joint spectrum $\sigma_{\mathcal{A}}(T_1, \dots, T_k)$, relative to \mathcal{A} , to be the set of all k -tuples of complex numbers $\lambda = (\lambda_1, \dots, \lambda_k)$ such that the ideal (and/or the closed ideal) of \mathcal{A} generated by $\lambda_1 - T_1, \dots, \lambda_k - T_k$ is a proper subset of \mathcal{A} .

We deduce from the above observations that $\mathrm{Sp}_u(T, \mathbb{Z}_+^k) = \sigma_{\mathcal{A}}(T_1, \dots, T_k) \cap \mathbb{T}^n$. We see that $\sigma_{\mathcal{A}}(T_1, \dots, T_k) \cap \mathbb{T}^n$ is the same for any commutative Banach algebra that contain T_1, \dots, T_k and I . We note that for $k = 1$, we have $\mathrm{Sp}_u(T, \mathbb{Z}_+) = \sigma(T_1) \cap \mathbb{T}$.

It is shown in [41] that if $G = \mathbb{R}$ and $S = \mathbb{R}_+$ or if $G = \mathbb{Z}^k$ and $S = \mathbb{Z}_+^k$ then

$$\lim_{t \rightarrow \infty} \|T(t)\widehat{f}(T)\| = \|\widehat{f}\|_{\mathcal{C}(\mathrm{Sp}_u(T, S))}, \quad (4.1)$$

whenever $f \in L^1(S)$ and T is a representation by contractions on a Hilbert space. For any group G and a semigroup S , Seifert get the following result.

Theorem 4.1. ([35]). *Let T be a representation of S by contractions on a Hilbert space and $f \in L^1(S)$. If $E = Sp_u(T, S)$ is a Helson set then*

$$\lim_{t \rightarrow \infty} \|T(t)\widehat{f}(T)\| \leq \alpha(E)\|f\|_{\mathcal{C}(E)}.$$

In particular if $\alpha(E) = 1$, then $\lim_{t \rightarrow \infty} \|T(t)\widehat{f}(T)\| = \|f\|_{\mathcal{C}(E)}$.

These results are not true for representations by contractions acting on a Banach space. As it is shown in the following theorem it depends on the thinness of $Sp_u(T, S)$.

Theorem 4.2. ([41]). *Let $c \geq 1$ and E a closed subset of Γ . Then the following are equivalent.*

- (i) *E is of spectral synthesis and is a Helson set with $\alpha(E) \leq c$.*
- (ii) *For every representation T of S by contractions on a Banach space X such that $Sp_u(T, S) = E$, we have*

$$\lim_{t \rightarrow \infty} \|T(t)\widehat{f}(T)\| \leq c\|f\|_{\mathcal{C}(E)}, \quad f \in L^1(S).$$

Notice that the inequality $\|f\|_{\mathcal{C}(E)} \leq \lim_{t \rightarrow \infty} \|T(t)\widehat{f}(T)\|$ always holds for $E = Sp_u(T, S)$. From this observation and the above theorem we get (see [40]).

Corollary 4.3. ([41]). *Let E be a closed subset of Γ . The following are equivalent.*

- (i) *E is of spectral synthesis and is a Helson set with $\alpha(E) = 1$.*
- (ii) *For every representation T of S by contractions on a Banach space such that $Sp_u(T, S) = E$, we have*

$$\lim_{t \rightarrow \infty} \|T(t)\widehat{f}(T)\| = \|f\|_{\mathcal{C}(E)}, \quad f \in L^1(S).$$

5. POLYNOMIALLY BOUNDED OPERATORS

Let T_1, T_2, \dots, T_k be finitely many commuting contractions on a Hilbert space. We say that the multidimensional von Neumann inequality holds for (T_1, T_2, \dots, T_k) if

$$\|P(T_1, T_2, \dots, T_k)\| \leq \sup_{(z_1, \dots, z_k) \in \overline{\mathbb{D}^k}} |P(z_1, \dots, z_k)|,$$

for every polynomial $P \in \mathbb{C}[z_1, \dots, z_k]$. By the classical von Neumann inequality and [1], the multidimensional von Neumann inequality holds for

$k = 1$ and $k = 2$, while it fails in general for $k \geq 3$. On the other hand, for $k \geq 3$, it is not known if there exists a finite constant C such that

$$\|P(T_1, T_2, \dots, T_k)\| \leq C \sup_{(z_1, \dots, z_k) \in \overline{\mathbb{D}^k}} |P(z_1, \dots, z_k)|, \quad (5.1)$$

for every $P \in \mathbb{C}[z_1, \dots, z_k]$ ([7], [2]). Notice that \mathbb{T}^k is the Shilov boundary of the Banach algebra of holomorphic functions in \mathbb{D}^n which are continuous up to the boundary and equipped with the supremum norm. Then inequality (5.1) is equivalent to

$$\|P(T_1, T_2, \dots, T_k)\| \leq C \sup_{(z_1, \dots, z_k) \in \mathbb{T}^k} |P(z_1, \dots, z_k)|.$$

We will now be interested in the last inequality for representations of semi-groups. Let G , Γ and S be as in section 4. In [40], Theorem 4.1, sharp conditions are given on the thinness of the spectrum so that a contraction on a Banach space is polynomially bounded. This result is extended to representations in the following theorem.

Theorem 5.1. *Let E be a closed subset of Γ . Then the following are equivalent.*

- (i) *For any representation T of S by invertible isometries on a Banach space with $\text{Sp}_u(T, S) \subset E$, there exists a constant C such that $\|\widehat{f}(T)\| \leq C \sup_{\chi \in \Gamma} |\widehat{f}(\chi)|$, for every $f \in L^1(S)$.*
- (ii) *E is a Helson set and of spectral synthesis.*

Proof. If T is a representation of S by invertible isometries then, for every $t \in S$, $\|\widehat{f}(T)\| = \|T(t)\widehat{f}(T)\|$. Thus the implication (ii) \Rightarrow (i) follows from Theorem 4.2.

Now we prove the implication (i) \Rightarrow (ii). Let I be a closed ideal of $L^1(G)$ with $h(I) = E$. Let $\pi : L^1(G) \rightarrow L^1(G)/I$ denote the canonical surjection and, for $t \in G$, τ_t the translation operator defined on $L^1(G)$ by: $\tau_t f(x) = f(x - t)$, $x \in G$. We set $T(t)(\pi(f)) = \pi(\tau_t f)$, $f \in L^1(G)$; T is a representation of G by invertible isometries satisfying the following properties (see the proof of Theorem 5.4 in [41]):

- (1) For $f \in L^1(G)$, $\widehat{f}(T)$ is the operator defined by: $\pi(g) \rightarrow \pi(f * g)$.
- (2) For $f \in L^1(G)$, $\widehat{f}(T) = 0$ holds if and only if $f \in I$.
- (3) For $f \in L^1(G)$, $\|\pi(f)\| = \|\widehat{f}(T)\|$.
- (4) $\text{Sp}_u(T, S) = \text{Sp}(T, G) = E$.

So by assumption (i) and property (4), there exists a constant $C > 0$ such that for every $f \in L^1(S)$, $\|\widehat{f}(T)\| \leq C \sup_{\chi \in \Gamma} |\widehat{f}(\chi)|$. It follows then from Lemma 2.2 in [41], that for every $f \in L^1(S)$ and for every $y \in L^1(G)/I$, $\|\widehat{f}(T)y\| \leq C \|\widehat{f}\|_{\mathcal{C}(\text{Sp}_u(T, S))} \|y\|$. Then we deduce from property (3) that

$$\|\pi(f)\| \leq C \|\widehat{f}\|_{\mathcal{C}(E)}, \quad f \in L^1(S). \quad (5.2)$$

Now we want to extend this inequality to all functions in $L^1(G)$. Let $f \in L^1(G)$ and $s \in S$, we set

$$f_s(t) = \begin{cases} f(t-s) & \text{if } t \in S \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\|f\| \leq \|f_s\| + \int_{G \setminus (S-s)} |f(t)| dm_G(t), \quad (5.3)$$

and

$$\|\widehat{f}_s\|_{\mathcal{C}(E)} \leq \|\widehat{f}\|_{\mathcal{C}(E)} + \int_{G \setminus (S-s)} |f(t)| dm_G(t). \quad (5.4)$$

Let $\epsilon > 0$. There exists $s \in S$ such that $\int_{G \setminus (S-s)} |f(t)| dm_G(t) < \epsilon$. Since I is invariant under translation ([32], Theorem 7.1.2, p. 157), for every $g \in I$, we have $\tau_{-s}g \in I$ and

$$\begin{aligned} \|\pi(f)\| &\leq \|f - \tau_{-s}g\| = \|\tau_s(f - \tau_{-s}g)\| = \|\tau_s f - g\| \\ &\leq \|f_s - g\| + \int_{G \setminus (S-s)} |f(t)| dm_G(t) \\ &\leq \|f_s - g\| + \epsilon. \end{aligned}$$

Taking now the infimum over $g \in I$, we get $\|\pi(f)\| \leq \|\pi(f_s)\| + \epsilon$. Now applying (5.2) to f_s , we obtain

$$\|\pi(f)\| \leq \|\pi(f_s)\| + \epsilon \leq C\|\widehat{f}_s\|_{\mathcal{C}(E)} + \epsilon$$

By (5.4), we have $\|\pi(f)\| \leq C(\|\widehat{f}\|_{\mathcal{C}(E)} + \epsilon) + \epsilon$. Letting $\epsilon \rightarrow 0$, we get

$$\|\pi(f)\| \leq C\|\widehat{f}\|_{\mathcal{C}(E)}, \quad f \in L^1(G). \quad (5.5)$$

Let $I = J(E)$. It follows from (5.5) that if $f \in L^1(G)$ with $\widehat{f} = 0$ on E , then $\pi(f) = 0$ and thus $f \in J(E)$. This shows that E is of spectral synthesis.

Let now $I = I(E)$. Let $i : A(E) \rightarrow \mathcal{C}_0(E)$ be the canonical injection. Inequality (5.5) shows that the range of i is closed. On the other hand, since $A(\Gamma)$ is dense in $\mathcal{C}_0(\Gamma)$ ([32], Theorem 1.2.4, p. 9), the range of i is also dense in $\mathcal{C}_0(E)$. So i is surjective, which implies that E is a Helson set with $\alpha(E) = \|i^{-1}\| \leq C$. □

Let $G = \mathbb{Z}$, $S = \mathbb{Z}_+$ and E a closed subset of \mathbb{T} . For $T = (T_1^n)_{n \in \mathbb{Z}_+}$ where T_1 is a bounded operator on a Banach space, Theorem 5.1 can be formulated in the following way: The two conditions below are equivalent:

(i)' Any invertible operator T_1 on a Banach space such that $\sup_{n \in \mathbb{Z}} \|T_1^n\| < +\infty$ and $\sigma(T_1) \subset E$, is polynomially bounded.

(ii) E is a Helson set and of spectral synthesis.

See ([40], Theorem 4.1 and Remark 5.5). The condition $\sup_{n \in \mathbb{Z}} \|T_1^n\| < +\infty$ is sharp in property (i)'. Indeed, we have the following result from ([40], Theorem 5.4):

Let $(\beta_n)_{n \geq 1}$ be a sequence of real numbers such that $\lim_{n \rightarrow +\infty} \beta_n = +\infty$ and $\beta_n > 1$ for every $n \geq 1$. Then there exists a contraction such that $\|T_1^{-n}\| \leq \beta_n$, $n \geq 1$, $\text{Sp}(T_1)$ is a Helson set and of spectral synthesis, and T_1 is not polynomially bounded.

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