HELSON SETS, SPECTRAL SYNTHESIS AND APPLICATIONS TO OPERATORS.

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Abstract. In this paper we highlight the role played by Helson sets and/or sets of spectral synthesis in some recent results in operator theory. We consider in particular the results on the cyclicity in $L^p$ spaces, the Katznelson-Tzafriri type theorems and polynomially bounded operators.

1. Introduction

In this survey we discuss some recent results in operator theory involving Helson sets and/or sets of spectral synthesis. Let $G$ be a locally compact abelian group and $\Gamma$ its dual group. A closed subset $E$ of $\Gamma$ is called a Helson set if every continuous function on $E$ vanishing at infinity is the restriction to $E$ of the Fourier transform of a function in $L^1(G)$; $E$ is said to be of spectral synthesis if every function in $L^1(G)$ whose Fourier transform vanishes on $E$, can be approximated in norm by functions in $L^1(G)$ whose Fourier transforms vanish in a neighborhood of $E$.

Helson proved in [11] that a Helson set in the unit circle $T$ does not support any non-zero measure with Fourier coefficients tending to 0. A closed subset $E$ of $T$ is called a set of multiplicity if $E$ supports a non-zero distribution with Fourier coefficients tending to zero. It was shown by Piatetski-Shapiro that there exists a multiplicity set in $T$ which does not support any non-zero measure with Fourier coefficients tending to 0. This result was extended by Körner who proved that there exists a Helson set which is also a multiplicity set ([21]). Notice that such a set is not a set of spectral synthesis. The proof of Körner is quite difficult and Kaufman presents a shorter one in [20]. Finally Lev and Olevskii obtain the following stronger result: for any real $q > 2$, there exists a Helson set which supports a non-zero distribution $S$ such that the sequence of Fourier coefficients $(\hat{S}(n))_{n \in \mathbb{Z}}$ is in $\ell^q(\mathbb{Z})$. This result allows them to refute the conjecture of Wiener about bicyclicity for translation of a function in $L^p(G)$ spaces. For $G = \mathbb{Z}$ and $G = \mathbb{R}$, Wiener
has characterized the functions which are bicyclic for translation in $L^1(G)$ and in $L^2(G)$. This characterization is actually given in terms of the zero set of the Fourier transform of the function in question (see [37]). However, Lev and Olevskii proved that this is not possible in $L^p(G)$ for $G = \mathbb{Z}$ or $\mathbb{R}$ and $1 < p < 2$ (see [24] and [23]).

We now turn to the quantitative version of Katznelson-Tzafriri type theorems. Let $S$ be a suitable subsemigroup of $G$ and let $T$ be a representation of $G$ by contractions on a Banach space $X$. For $f \in L^1(S)$, we set

$$\hat{f}(T) : x \mapsto \int_S f(t)T(t)x \, dm_G(t), \quad x \in X,$$

where $m_G$ is the Haar measure on $G$. The unitary spectrum of $T$ with respect to $S$ is defined by

$$\text{Sp}_u(T, S) = \{ \chi \in \Gamma; |\hat{f}(\chi)| \leq \|\hat{f}(T)\| \text{ for all } f \in L^1(S)\}.$$

We consider the following order on $S$: $s \preceq t \iff t - s \in S$. It was proved that if $f \in L^1(S)$ satisfies spectral synthesis for $\text{Sp}_u(T, S)$, then

$$\lim_{t \to \infty} \|T(t)\hat{f}(T)\| = 0.$$  

This result was first proved by Katznelson and Tzafriri ([19]) in the case $S = \mathbb{Z}$ and then extended to $S = \mathbb{R}$ in [9] and finally to the general case in [3] and [31]. Further results can be found in [6], [8] and [22].

It is shown in [41] that if $X$ is a Hilbert space and if $S = \mathbb{Z}_+^k$ or $S = \mathbb{R}_+$ then

$$\lim_{t \to \infty} \|T(t)\hat{f}(T)\| = \sup \{|f(\lambda)|, \lambda \in \text{Sp}_u(T, S)\}.$$  

(1.1)

This equality does not extend to representations by contractions defined on arbitrary Banach space. The second author obtained in [41] that if $E$ is a closed subset of $\Gamma$, then the following are equivalent:

(i) Equality (1.1) holds for every representation $T$ of $S$ by contractions such that $\text{Sp}_u(T, S) = E$.

(ii) $E$ is of spectral synthesis and is a Helson set with $\alpha(E) = 1$, where $\alpha(E)$ is the Helson constant of $E$.

On the other hand, it follows from Corollary 4.3 of [35] that if $T$ is a representation of $S$ by contractions on a Hilbert space and if $\text{Sp}_u(T, S)$ is contained in a Helson set $E$ with $\alpha(E) = 1$, then equality (1.1) holds for $T$ (see Theorem 4.1).

Similar results are obtained in [40] about polynomially bounded operators. Thanks to the von Neumann inequality, a contraction on a Hilbert space is polynomially bounded. The situation is actually quite different for contractions on Banach spaces. It is shown in [40] that if $E$ is a closed subset of $\mathbb{T}$ then the following are equivalent:
(i) Every contraction on a Banach space whose spectrum is contained in $E$ is polynomially bounded.

(ii) $E$ is a Helson set and of spectral synthesis.

In section 5, we extend this result to all representations of $S$ by contractions on Banach spaces (see Theorem 5.1).

2. Definitions and classical results

Let $G$ be a locally compact abelian group equipped with the Haar measure $m_G$ and let $\Gamma$ be its dual group. For $f \in L^1(G)$ and $\chi \in \Gamma$, we set

$$\hat{f}(\chi) = \int_G f(t)\chi(t)dm_G(t).$$

If $I$ is a closed ideal of $L^1(G)$, we set

$$h(I) = \{\chi \in \Gamma : \hat{f}(\chi) = 0, \text{ for all } f \in I\}.$$

Let $E$ be a closed subset of $\Gamma$. We set

$$I(E) = \{f \in L^1(G), \hat{f} = 0 \text{ on } E\}$$

and

$$J(E) = \text{clos } \{f \in L^1(G), \hat{f} = 0 \text{ on a neighborhood of } E\}.$$

Clearly $I(E)$ and $J(E)$ are closed ideals of $L^1(G)$, $J(E) \subset I(E)$ and we have $h(I(E)) = h(J(E)) = E$ (see [32], p. 161). The set $E$ is said to be of spectral synthesis if $J(E) = I(E)$. Notice that $E$ is of spectral synthesis if there exists a unique closed ideal $I$ such that $h(I) = E$. A function $f \in L^1(G)$ (or $\hat{f}$) is said to satisfy spectral synthesis for $E$ if $f \in J(E)$. Here are some examples of sets of spectral synthesis:

(1) The compact countable sets ([18], Corollary, p. 240).

(2) The Cantor set in $\mathbb{R}$ or the unit circle $\mathbb{T} = \{z \in \mathbb{C}; |z| = 1\}$ ([13]).

(3) The sphere $S^1$ in $\mathbb{R}^2$ ([15]).

(4) All closed sets if $\Gamma$ is discrete ([4], p. 71).

Notice also that a function $f$ on the unit circle $\mathbb{T}$ satisfy the spectral synthesis for $Z(f) = \{z \in \mathbb{T}; f(z) = 0\}$ whenever it satifies the following condition: for every $z, z' \in \mathbb{T}$, $|f(z) - f(z')| \leq C_f |z - z'|^{1/2+\epsilon}$ for some constant $C_f$ and $\epsilon > 0$ ([17], Corollaire I, p. 123).

On the other hand it was shown by L. Schwartz that for $n \geq 3$ the sphere $S^{n-1} \subset \mathbb{R}^n$ is not of spectral synthesis ([36]). Later P. Malliavin proved that any non discrete $\Gamma$ contains a subset which is not of spectral synthesis (see [25],[26],[27] and [10], pp. 70-71).

Let $A(\Gamma)$ denote the Banach algebra of functions $\hat{f}$, $f \in L^1(G)$ endowed with the norm $\|\hat{f}\|_{A(\Gamma)} = \|f\|_{L^1(G)}$. Let $A(E) = \{\hat{f}_{|E}, f \in L^1(G)\}$ be the Banach space equipped with the quotient norm of $L^1(G)/I(E)$. A closed
subset $E$ of $\Gamma$ is called a Helson set if the canonical injection $i: A(E) \to C_0(E)$ is onto, where $C_0(E)$ is the space of continuous functions vanishing at infinity, equipped with the supremum norm. This is equivalent to the invertibility of $i$ and in this case the norm of $i^{-1}$ is called the Helson constant of $E$ and is denoted by $\alpha(E)$. By duality $E$ is a Helson set if and only if there exists a positive constant $C$ such that for every $\mu \in M(E)$, $\|\mu\| \leq C\|\hat{\mu}\|_{\infty}$, where $M(E)$ is the set of Borel measures on $\Gamma$ supported by $E$, $\|\mu\|$ is the total variation of $\mu$ and $\|\hat{\mu}\|_{\infty} = \sup_{x \in G} |\hat{\mu}(x)|$. The smallest constant satisfying the last condition is equal to the Helson constant $\alpha(E)$.

The problem of whether a Helson set is of spectral synthesis was solved by Körner in [21] (see also [20]). More precisely, he proved that there exists a multiplicity set $E$ which is a Helson set with $\alpha(E) = 1$ and this set is not of spectral synthesis. For further discussion on sets from harmonic analysis, see the books [4], [10], [12], [16], [17], [32].

We now give some examples of Helson sets which are of spectral synthesis. Let $E$ be a closed subset of $\Gamma$; $E$ is called independent if for every $\chi_1, \ldots, \chi_k$ in $E$ and every set of integers $\{n_1, \ldots, n_k\}$,

$$\chi_1^{n_1} \cdots \chi_k^{n_k} = 1 \Rightarrow \chi_1^{n_1} = \cdots = \chi_k^{n_k} = 1.$$ 

A compact subset $E$ of $\Gamma$ is called a Kronecker set if the following property holds: for every continuous function $h$ on $E$ with $|h| = 1$ and for every $\epsilon > 0$ there exists $t \in G$ such that $\sup_{\chi \in E} |h(\chi) - \chi(t)| < \epsilon$. Every Kronecker set $E$ is of spectral synthesis and is a Helson set with $\alpha(E) = 1$ (see [33], [39] and [32], Theorem 5.5.2, p. 113). Notice that every Kronecker set is independent. If $E$ is a finite independent set and if every element of $E$ is of infinite order, then $E$ is a Kronecker set (see [32], p. 98).

Finally we note that any countable, compact, independent set is a Helson set and of spectral synthesis (see [32], Theorem 5.6.7, p. 117 and p.161).


Let $G$ be a locally compact abelian group, $\Gamma$ its dual group and $p \geq 1$. For a function $f$ on $G$ and $y \in G$, we set $\tau_y f(x) = f(x - y)$, $x \in G$. A subspace $V$ of $L^p(G)$ is called invariant by translation if for every $f \in V$ and $y \in G$, $\tau_y f \in V$. A function $f \in L^p(G)$ is called bicyclic in $L^p(G)$ if the subspace span$\{\tau_y f, y \in G\}$ is dense in $L^p(G)$. The Wiener theorem characterize the bicyclic function for translation in $L^1(G)$ and $L^2(G)$ ([37]).

**Theorem 3.1.** ([Wiener])

1. A function $f$ is bicyclic in $L^1(G)$ if and only if for every $\chi \in \Gamma$, $\hat{f}(\chi) \neq 0$.

2. A function $f$ is bicyclic in $L^2(G)$ if and only if $\hat{f}(\chi) \neq 0$ for almost all $\chi \in \Gamma$. 

The space $L^1(G)$ is a Banach algebra for the convolution product and $\Gamma$ can be identified with the space of maximal ideals by using the map $\chi \rightarrow I_\chi = \{ f \in L^1(G), \hat{f}(\chi) = 0 \}$. Moreover the closed translation invariant subspaces of $L^1(G)$ are just the closed ideals of $L^1(G)$ (see [32], Theorem 7.1.2, p. 158). For $f \in L^1(G)$ the closure of span$\{ \tau_y f, y \in G \}$ is invariant by translation and therefore is a closed ideal. It follows that span$\{ \tau_y f, y \in G \}$ is dense in $L^1(G)$ if and only if it is not contained in any maximal ideal, that is $\hat{f}(\chi) \neq 0$ for every $\chi \in \Gamma$. This proves part (1) of Theorem 3.1. Part (2) of Theorem 3.1 follows from the more general Wiener theorem on closed translation invariant subspaces:

A closed subspace $V$ of $L^2(G)$ is invariant by translation if and only if there exists a measurable subset $E$ of $\Gamma$ such that

$$V = \{ f \in L^2(G), \hat{f}(\chi) = 0 \text{ for almost all } \chi \in E \}.$$

This result and Theorem 3.1 were proved by Wiener in the case $G = \mathbb{Z}$ and $G = \mathbb{R}$ ([37]). For any group $G$, the proof of part (2) of Theorem 3.1 is given in ([12], Theorem (31.39)).

It follows from Theorem 3.1 that the bicyclicity of a function $f$ in $L^1(G)$ for $p = 1$ and $p = 2$, is characterized in term the zero set of $\hat{f}$, that is $Z_{\hat{f}} = \{ \chi \in \Gamma, \hat{f} = 0 \}$. Wiener conjectured that for $1 < p < 2$, a similar characterization holds. In [24], N. Lev and A. Olevskii disproved this conjecture (see also [23]).

**Theorem 3.2.** ([Lev-Olevskii]) Let $G = \mathbb{Z}$ or $G = \mathbb{R}$. If $1 < p < 2$, then there exist two functions $f$ and $g$ in $L^1(G) \cap L^p(G)$ such that $Z_{\hat{f}} = Z_{\hat{g}}$, $f$ is cyclic in $L^p(G)$ while $g$ is not.

The proof of this theorem is deduced from the following result.

**Theorem 3.3.** ([Lev-Olevskii]) Let $q > 2$. Then there exists a Helson set $E \subset \mathbb{T}$ which supports a non-zero distribution $S$ such that the sequence of the Fourier coefficients $(\hat{S}(n))_{n \in \mathbb{Z}}$ is in $\ell^q(\mathbb{Z})$.

We now make some observations on how Theorem 3.3 implies Theorem 3.2 (see [24] and [23]). Let $q > 2$ and let $p$ be its conjugate number, that is $\frac{1}{p} + \frac{1}{q} = 1$. Let $E$ be a closed subset of $\mathbb{T}$ which support a non-zero distribution $S$ such that $\hat{S} = (\hat{S}(n))_{n \in \mathbb{Z}} \in \ell^q(\mathbb{Z})$. We take an infinitely differentiable function $h$ on the circle which vanishes exactly on $E$. We set $f = (\hat{h}(n))_{n \in \mathbb{Z}}$. Then $f \in \ell^1(\mathbb{Z})$, $Z_{\hat{f}} = E$ and, for every $k \in \mathbb{Z},$

$$\langle \hat{S} | (f(n-k))_{n \in \mathbb{Z}} \rangle_{\ell^1, \ell^p} := \sum_{n \in \mathbb{Z}} \hat{S}(-n) f(n-k) = (S | e^{ik(\cdot)} h(\cdot))_{D(\mathbb{T}), D'(\mathbb{T})},$$
where $(.,.)_{X',X}$ denote the duality between a topological space $X$ and its dual $X'$. Since $e^{ik(.)}h(.)$ is a smooth function, it satisfies the spectral synthesis for $E$ ([17], Corollaire I, p. 123). Therefore $(S|e^{ik(.)}h(.)|_{D'(T),D(T)}) = 0$. It follows that, for every $k$, $(\hat{S}((f(n-k))_{n\in\mathbb{Z}})_{\ell^1,\ell^p} = 0$, which implies that $f$ is not bicyclic in $\ell^p(\mathbb{Z})$.

On the other hand Lev and Olevskii showed by using Baire’s theorem that for every Helson set $E$ in $\mathbb{T}$, there exists $g \in \ell^1(\mathbb{Z})$ which is bicyclic in $\ell^p(\mathbb{Z})$ for every $p > 1$ and such that $Z_g = E$ (see [24], Lemma 9). Thus, we see that if $E$ is the set constructed in Theorem 3.3 then there exists $f, g \in \ell^1(\mathbb{Z})$ as in Theorem 3.2 and such that $Z_{\hat{f}} = Z_{\hat{g}} = E$.

We finish this section by recalling some known results on the bicylicity problem. In [5], Beurling showed that if $f \in \ell^1(\mathbb{Z})$ and if the Hausdorff dimension of $Z_{\hat{f}}$ is less or equal to $\frac{2(p-1)}{p}$ with $1 < p < 2$, then $f$ is bicyclic in $\ell^p(\mathbb{Z})$. Salem showed in [34] that this result is sharp. On the other hand Newman construct in [28] a set $E$ of Hausdorff dimension $1$ such that, if $f \in \ell^1(\mathbb{Z})$ and $Z_{\hat{f}} = E$ then $f$ is bicyclic in $\ell^p(\mathbb{Z})$, for every $p > 1$.

4. KATZNELSON-TZAFIRI TYPE THEOREMS

Let $G$ be a locally compact abelian group and $S$ be a measurable semigroup of $G$ with non empty interior in $G$ and such that $G = S - S$. Denote by $S^*$ the set of all the non-zero, continuous, bounded, homomorphisms of $S$ into the multiplicative semigroup $\mathbb{C}$. Let

$$S_u^* = \{ \chi \in S^*; |\chi(s)| = 1 \text{ for all } s \text{ in } S \}.$$  

We shall identify $S_u^*$ with the dual group $\Gamma$ of $G$, in the natural way. For $f \in L^1(S)$ and $\chi \in S^*$, we set

$$\hat{f}(\chi) = \int_S f(t)\chi(t)dt.$$ 

Finally we shall assume that $\{\hat{f}, f \in L^1(S)\}$ separates the points of $S^*$ from each other and from zero and that the interior $S^\circ$ is dense in $S$.

Let $X$ be a Banach space, $L(X)$ the Banach algebra of all bounded linear operators on $X$. Let $T$ be a representation of $S$ by contractions on $X$, that is a strongly continuous homomorphism from $S$ into $L(X)$ such that $T(s)$ is a contraction whenever $s \in S$. For $f \in L^1(S)$, let $\hat{f}(T) : X \mapsto X$ be the bounded operator defined by

$$\hat{f}(T) : x \mapsto \int_S f(t)T(t)xdm_G(t).$$ 

The spectrum of $T$ with respect to $S$ is defined by

$$\text{Sp}(T,S) = \{ \chi \in S^*; |\hat{f}(\chi)| \leq \|\hat{f}(T)\| \text{ for all } f \text{ in } L^1(S) \}.$$
The unitary spectrum of $T$ is defined by

$$\text{Sp}_u(T, S) = \text{Sp}(T, S) \cap S_u^* = \text{Sp}(T, S) \cap \Gamma.$$ 

In the sequel of the paper we will use the following notation: If $E$ is a closed subset of $\Gamma$, we denote by $C_0(E)$ the space of all continuous functions on $E$ vanishing at infinity. For a continuous bounded function $h$ on $E$ we set $\|h\|_{C(E)} = \sup \{|f(\chi)|, \chi \in E\}$.

**Examples.** (1) Let $G = \mathbb{R}$ and $S = \mathbb{R}_+$. The set $S^*$ will be identified with $\mathbb{C}_- = \{z \in \mathbb{C} : \text{Im}(z) < 0\}$ and $S_u^*$ with $\mathbb{R}$. For $f \in L^1(\mathbb{R}_+)$ and $z \in \mathbb{C}_-$, we have

$$\hat{f}(z) = \int_{\mathbb{R}_+} f(t)e^{-itz}dt.$$

A $C_0$-semigroup $T = (T_t)_{t \geq 0}$ of contractions on $X$ is a representation of $\mathbb{R}_+$. In this case we have $\text{Sp}_u(T, \mathbb{R}_+) = i\sigma(A) \cap \mathbb{R}$, where $A$ is the generator of $T$ and $\sigma(A)$ is the spectrum of $A$.

(2) Let $k \geq 1$ be an integer, $G = \mathbb{Z}^k$ and $S = \mathbb{Z}_+^k$, where $\mathbb{Z}_+$ is the set of all nonnegative integers; $S^*$ will be identified with $\mathbb{D}^k$ and $S_u^*$ with $\mathbb{T}^k$. For $f \in L^1(\mathbb{Z}_+^k)$ and $z = (z_1, \ldots, z_k) \mapsto \chi_z \in \mathbb{D}^k$, we have

$$\hat{f}(z) = \sum_{n \in \mathbb{Z}_+^k} f(n)z_1^{n_1} \ldots z_k^{n_k},$$

where $n = (n_1, \ldots, n_k)$. Let $T_1, \ldots, T_k$ be a finite many commuting contractions on $X$. Then $T = (T_1^{n_1} \ldots T_k^{n_k})_{n \in \mathbb{Z}_+^k}$ is a representation of $\mathbb{Z}_+^k$ by contractions on $X$. For $f = (f(n))_{n \in \mathbb{Z}_+^k}$ in $l^1(\mathbb{Z}_+^k)$, we have

$$\hat{f}(T) = \sum_{n \in \mathbb{Z}_+^k} f(n)T_1^{n_1} \ldots T_k^{n_k}.$$ 

Let $\mathcal{A}$ be a commutative Banach algebra that contains $T_1, \ldots, T_k$ and $I$ the identity map. We define the joint spectrum $\sigma_{\mathcal{A}}(T_1, \ldots, T_k)$, relative to $\mathcal{A}$, to be the set of all $k$-tuples of complex numbers $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that the ideal (and/or the closed ideal) of $\mathcal{A}$ generated by $\lambda_1 - T_1, \ldots, \lambda_k - T_k$ is a proper subset of $\mathcal{A}$.

We deduce from the above observations that $\text{Sp}_u(T, \mathbb{Z}_+^k) = \sigma_{\mathcal{A}}(T_1, \ldots, T_k) \cap \mathbb{T}^n$. We see that $\sigma_{\mathcal{A}}(T_1, \ldots, T_k) \cap \mathbb{T}^n$ is the same for any commutative Banach algebra that contain $T_1, \ldots, T_k$ and $I$. We note that for $k = 1$, we have $\text{Sp}_u(T, \mathbb{Z}_+) = \sigma(T_1) \cap \mathbb{T}$.

It is shown in [41] that if $G = \mathbb{R}$ and $S = \mathbb{R}_+$ or if $G = \mathbb{Z}^k$ and $S = \mathbb{Z}_+^k$ then

$$\lim_{t \to \infty} \|T(t)\hat{f}(T)\| = \|\hat{f}\|_{C(\text{Sp}_u(T, S))}, \quad (4.1)$$
whenever \( f \in L^1(S) \) and \( T \) is a representation by contractions on a Hilbert space. For any group \( G \) and a semigroup \( S \), Seifert get the following result.

**Theorem 4.1.** ([35]). Let \( T \) be a representation of \( S \) by contractions on a Hilbert space and \( f \in L^1(S) \). If \( E = \text{Sp}_u(T, S) \) is a Helson set then

\[
\lim_{t \to \infty} \| T(t) \hat{f}(T) \| \leq \alpha(E) \| \hat{f} \|_{C(E)}.
\]

In particular if \( \alpha(E) = 1 \), then \( \lim_{t \to \infty} \| T(t) \hat{f}(T) \| = \| \hat{f} \|_{C(E)} \).

These results are not true for representations by contractions acting on a Banach space. As it is shown in the following theorem it depends on the thinness of \( \text{Sp}_u(T, S) \).

**Theorem 4.2.** ([41]). Let \( c \geq 1 \) and \( E \) a closed subset of \( \Gamma \). Then the following are equivalent.

(i) \( E \) is of spectral synthesis and is a Helson set with \( \alpha(E) \leq c \).

(ii) For every representation \( T \) of \( S \) by contractions on a Banach space \( X \) such that \( \text{Sp}_u(T, S) = E \), we have

\[
\lim_{t \to \infty} \| T(t) \hat{f}(T) \| \leq c \| \hat{f} \|_{C(E)}, \quad f \in L^1(S).
\]

Notice that the inequality \( \| \hat{f} \|_{C(E)} \leq \lim_{t \to \infty} \| T(t) \hat{f}(T) \| \) always holds for \( E = \text{Sp}_u(T, S) \). From this observation and the above theorem we get (see [40]).

**Corollary 4.3.** ([41]). Let \( E \) be a closed subset of \( \Gamma \). The following are equivalent.

(i) \( E \) is of spectral synthesis and is a Helson set with \( \alpha(E) = 1 \).

(ii) For every representation \( T \) of \( S \) by contractions on a Banach space such that \( \text{Sp}_u(T, S) = E \), we have

\[
\lim_{t \to \infty} \| T(t) \hat{f}(T) \| = \| \hat{f} \|_{C(E)}, \quad f \in L^1(S).
\]

5. **Polynomially bounded operators**

Let \( T_1, T_2, \ldots, T_k \) be finitely many commuting contractions on a Hilbert space. We say that the multidimensional von Neumann inequality holds for \( (T_1, T_2, \ldots, T_k) \) if

\[
\| P(T_1, T_2, \ldots, T_k) \| \leq \sup_{(z_1, \ldots, z_k) \in \mathbb{D}^k} | P(z_1, \ldots, z_k) |,
\]

for every polynomial \( P \in \mathbb{C}[z_1, \ldots, z_k] \). By the classical von Neumann inequality and [1], the multidimensional von Neumann inequality holds for
Theorem 4.2. For every \( P_k \in \mathbb{T} \), if \( E^{(ii)} \) is the Banach algebra of holomorphic functions in \( D \) to the boundary and equipped with the supremum norm. Then inequality (5.1) is equivalent to

\[
\|P(T_1, T_2, \ldots, T_k)\| \leq C \sup_{(z_1, \ldots, z_k) \in \mathbb{D}^k} |P(z_1, \ldots, z_k)|.
\]

for every \( P \in \mathbb{C}[z_1, \ldots, z_k] \) ([7], [2]). Notice that \( T^k \) is the Shilov boundary of the Banach algebra of holomorphic functions in \( \mathbb{D}^n \) which are continuous up to the boundary and equipped with the supremum norm. Then inequality (5.1) is equivalent to

\[
\|P(T_1, T_2, \ldots, T_k)\| \leq C \sup_{(z_1, \ldots, z_k) \in \mathbb{T}^k} |P(z_1, \ldots, z_k)|.
\]

We will now be interested in the last inequality for representations of semigroups. Let \( G, \Gamma \) and \( S \) be as in section 4. In [40], Theorem 4.1, sharp conditions are given on the thinness of the spectrum so that a contraction on a Banach space is polynomially bounded. This result is extended to representations in the following theorem.

**Theorem 5.1.** Let \( E \) be a closed subset of \( \Gamma \). Then the following are equivalent.

(i) For any representation \( T \) of \( S \) by invertible isometries on a Banach space with \( \text{Sp}(T, S) \subset E \), there exists a constant \( C \) such that \( \|\hat{f}(T)\| \leq C \sup_{\chi \in \Gamma} |\hat{f}(\chi)| \), for every \( f \in L^1(S) \).

(ii) \( E \) is a Helson set and of spectral synthesis.

**Proof.** If \( T \) is a representation of \( S \) by invertible isometries then, for every \( t \in S \), \( \|\hat{f}(T)\| = \|T(t)\hat{f}(T)\| \). Thus the implication (ii) \( \Rightarrow \) (i) follows from Theorem 4.2.

Now we prove the implication (i) \( \Rightarrow \) (ii). Let \( I \) be a closed ideal of \( L^1(G) \) with \( h(I) = E \). Let \( \pi : L^1(G) \to L^1(G)/I \) denote the canonical surjection and, for \( t \in G \), \( \tau_t \) the translation operator defined on \( L^1(G) \) by: \( \tau_t f(x) = f(x - t), x \in G \). We set \( T(t)(\pi(f)) = \pi(\tau_t f), f \in L^1(G) \); \( T \) is a representation of \( G \) by invertible isometries satisfying the following properties (see the proof of Theorem 5.4 in [41]):

1. For \( f \in L^1(G) \), \( \hat{f}(T) \) is the operator defined by: \( \pi(g) \to \pi(f * g) \).
2. For \( f \in L^1(G) \), \( f(T) = 0 \) holds if and only if \( f \in I \).
3. For \( f \in L^1(G) \), \( \|\pi(f)\| = \|\hat{f}(T)\| \).
4. \( \text{Sp}_u(T, S) = \text{Sp}(T, G) = E \).

So by assumption (i) and property (4), there exists a constant \( C > 0 \) such that for every \( f \in L^1(S) \), \( \|\hat{f}(T)\| \leq C \sup_{\chi \in \Gamma} |\hat{f}(\chi)| \). It follows then from Lemma 2.2 in [41], that for every \( f \in L^1(S) \) and for every \( y \in L^1(G)/I \), \( \|\hat{f}(T)y\| \leq C\|\hat{f}\|_{L^1(G)/I} \|y\| \). Then we deduce from property (3) that

\[
\|\pi(f)\| \leq C\|\hat{f}\|_{L^1(E)}, \quad f \in L^1(S).
\]

(5.2)
Now we want to extend this inequality to all functions in $L^1(G)$. Let $f \in L^1(G)$ and $s \in S$, we set

$$f_s(t) = \begin{cases} f(t - s) & \text{if } t \in S \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\|f\| \leq \|f_s\| + \int_{G \setminus (S - s)} |f(t)|dm_G(t), \quad (5.3)$$

and

$$\|\hat{f}_s\|_{C(E)} \leq \|\hat{f}\|_{C(E)} + \int_{G \setminus (S - s)} |f(t)|dm_G(t). \quad (5.4)$$

Let $\epsilon > 0$. There exists $s \in S$ such that

$$\int_{G \setminus (S - s)} |f(t)|dm_G(t) < \epsilon.$$ 

Since $I$ is invariant under translation (([32], Theorem 7.1.2, p. 157), for every $g \in I$, we have $\tau_{-s}g \in I$ and

$$\|\pi(f)\| \leq \|f - \tau_{-s}g\| = \|\tau_s(f - \tau_{-s}g)\| = \|\tau_sf - g\| \leq \|f_s - g\| + \int_{G \setminus (S - s)} |f(t)|dm_G(t) \leq \|f_s - g\| + \epsilon.$$ 

Taking now the infimum over $g \in I$, we get $\|\pi(f)\| \leq \|\pi(f_s)\| + \epsilon$. Now applying (5.2) to $f_s$, we obtain

$$\|\pi(f)\| \leq \|\pi(f_s)\| + \epsilon \leq C\|\hat{f}_s\|_{C(E)} + \epsilon$$

By (5.4), we have $\|\pi(f)\| \leq C(\|\hat{f}\|_{C(E)} + \epsilon) + \epsilon$. Letting $\epsilon \to 0$, we get

$$\|\pi(f)\| \leq C\|\hat{f}\|_{C(E)}, \quad f \in L^1(G). \quad (5.5)$$

Let $I = J(E)$. It follows from (5.5) that if $f \in L^1(G)$ with $\hat{f} = 0$ on $E$, then $\pi(f) = 0$ and thus $f \in J(E)$. This shows that $E$ is of spectral synthesis.

Let now $I = I(E)$. Let $i : A(E) \to C_0(E)$ be the canonical injection. Inequality (5.5) shows that the range of $i$ is closed. On the other hand, since $A(\Gamma)$ is dense in $C_0(\Gamma)$ ([32], Theorem 1.2.4, p. 9), the range of $i$ is also dense in $C_0(E)$. So $i$ is surjective, which implies that $E$ is a Helson set with $\alpha(E) = \|i^{-1}\| \leq C$.

Let $G = Z$, $S = Z_+$ and $E$ a closed subset of $T$. For $T = (T^n_1)_{n \in Z_+}$ where $T_1$ is a bounded operator on a Banach space, Theorem 5.1 can be formulated in the following way: The two conditions below are equivalent:

(i) Any invertible operator $T_1$ on a Banach space such that $\sup_{n \in Z} \|T^n_1\| < +\infty$ and $\sigma(T_1) \subset E$, is polynomially bounded.
(ii) \( E \) is a Helson set and of spectral synthesis.

See ([40], Theorem 4.1 and Remark 5.5). The condition \( \sup_{n \in \mathbb{Z}} \| T_n^n \| < +\infty \) is sharp in property (i)'. Indeed, we have the following result from ([40], Theorem 5.4):

Let \( (\beta_n)_{n \geq 1} \) be a sequence of real numbers such that \( \lim_{n \to +\infty} \beta_n = +\infty \) and \( \beta_n > 1 \) for every \( n \geq 1 \). Then there exists a contraction such that \( \| T_1^n - T \| \leq \beta_n \), \( n \geq 1 \), \( \text{Sp}(T_1) \) is a Helson set and of spectral synthesis, and \( T_1 \) is not polynomially bounded.

References


