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A new regularity criterion for weak solutions to the 3D micropolar fluid flows in terms of the pressure

Sadek Gala · Maria Alessandra Ragusa · Michel Théra

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Abstract This note aims to giving a new regularity criterion for weak solutions to the three-dimensional micropolar fluid flows by imposing a critical growth condition on the field of pressure.

Keywords Micropolar fluid flows; · weak solutions; · Pressure criterion · Besov spaces..

Mathematics Subject Classification (2000) 35Q35 · 35B65 · 76D05

In this note we consider the following Cauchy problem (1) for the incompressible micropolar fluid equations in \mathbb{R}^3 :

$$\partial_t u - \Delta u + (u \cdot \nabla) u + \nabla \pi - \nabla \times \omega = 0, \quad (1a)$$

$$\partial_t \omega - \Delta \omega - \nabla \nabla \cdot \omega + 2\omega + (u \cdot \nabla) \omega - \nabla \times u = 0, \quad (1b)$$

$$\nabla \cdot u = 0, \quad (1c)$$

$$u(x, 0) = u_0(x), \quad \omega(x, 0) = \omega_0(x), \quad (1d)$$

where $u = u(x, t) \in \mathbb{R}^3$, $\omega = \omega(x, t) \in \mathbb{R}^3$ and $\pi = \pi(x, t)$ denote the unknown velocity of the fluid, the micro-rotational velocity of the fluid particles and the unknown scalar pressure of the fluid at the point $(x, t) \in \mathbb{R}^3 \times (0, T)$, respectively, while u_0, ω_0 are given initial data satisfying $\nabla \cdot u = 0$ in the sense of distributions.

This model for micropolar fluid flows proposed by Eringen [6] enables to consider some physical phenomena that cannot be treated by the classical Navier-Stokes equations for the viscous incompressible fluids, such as for example, the motion of animal blood, muddy fluids, liquid crystals and dilute aqueous polymer solutions, colloidal suspensions, etc.

When the micro-rotation effects are neglected or $\omega = 0$, (1) reduces to the incompressible Navier-Stokes equations, and it is well known that regularity criteria

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for weak solution to the fluid dynamical models attracts more and more attention. Velocity or vorticity or pressure blow-up criteria for Navier-Stokes equations, micropolar fluid equations and magnetohydrodynamics (MHD) equations and so on (see e.g., [1–4, 7–10, 15–17] and the references therein) attracted during the last years the attention of many researchers.

As for the pressure criterion, let us first recall some results on pressure regularity of Navier-Stokes equations. In [14], He and Gala proved regularity of weak solutions under the condition

$$\int_0^T \|\pi(\cdot, t)\|_{\dot{B}_{\infty, \infty}^{-1}}^2 dt < \infty. \quad (2)$$

Here and thereafter, $\dot{B}_{\infty, \infty}^{-1}$ stands for the homogeneous Besov space, (for the definition see e.g. [14] and [13]). Later on, Guo and Gala [13] refined the condition (2) to

$$\int_0^T \frac{\|\pi(\cdot, t)\|_{\dot{B}_{\infty, \infty}^{-1}}^2}{1 + \log\left(e + \|\pi(\cdot, t)\|_{\dot{B}_{\infty, \infty}^{-1}}\right)} dt < \infty. \quad (3)$$

Motivated by the paper of Guo and Gala [13], the aim of this paper is to give a new regularity criterion for weak solutions to the 3D micropolar fluid flows in terms of the pressure in critical Besov spaces.

1 Main result

Let us start by stating the main result of this note.

Theorem 1.1 *Let $T > 0$ and $(u_0, \omega_0) \in L^2(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$ in the sense of distributions. Assume that (u, ω) is a weak solution to the 3D micropolar fluid flows (1) on $(0, T)$. If the pressure π satisfies the following condition :*

$$\int_0^T \frac{\|\pi(\cdot, t)\|_{\dot{B}_{\infty, \infty}^{-1}}^2}{\left(e + \log\left(e + \|\pi(\cdot, t)\|_{\dot{B}_{\infty, \infty}^{-1}}\right)\right) \log\left(e + \log\left(e + \|\pi(\cdot, t)\|_{\dot{B}_{\infty, \infty}^{-1}}\right)\right)} dt < \infty, \quad (4)$$

then (u, ω) is regular on $(0, T]$, i.e., $(u, \omega) \in C^\infty(\mathbb{R}^3 \times (0, T])$.

Remark 1.1 This result provides a new information concerning the question of the regularity of weak solutions to the micropolar fluid equations and extends those of [14] and [13]. In particular, the double-logarithm estimate (4) is sharper than any other results [13, 14].

Before stating our result, let us recall what we mean by a weak solution.

Definition 1.1 ([16]) Let $(u_0, \omega_0) \in L^2(\mathbb{R}^3)$ and suppose that $\nabla \cdot u_0 = 0$. A measurable function $(u(x, t), \omega(x, t))$ is called a weak solution to the 3D micropolar flows equations (1) on $(0, T)$ if (u, ω) satisfies three properties :

- (1) $(u, \omega) \in L^\infty((0, T); L^2(\mathbb{R}^3)) \cap L^2((0, T); H^1(\mathbb{R}^3))$ for all $T > 0$;
- (2) $(u(x, t), \omega(x, t))$ verifies (1) in the sense of distribution;
- (3) For all $0 \leq t \leq T$ it holds :

$$\begin{aligned} & \|u(\cdot, t)\|_{L^2}^2 + \|\omega(\cdot, t)\|_{L^2}^2 + 2 \int_0^t (\|\nabla u(\cdot, \tau)\|_{L^2}^2 + \|\nabla \omega(\cdot, \tau)\|_{L^2}^2 + \|\nabla \cdot \omega(\cdot, \tau)\|_{L^2}^2) d\tau \\ & \leq \|u_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2, \end{aligned}$$

By a strong solution we mean a weak solution (u, ω) such that

$$(u, \omega) \in L^\infty((0, T); H^1(\mathbb{R}^3)) \cap L^2((0, T); H^2(\mathbb{R}^3)).$$

It is well known that strong solutions are regular (say, classical) and unique in the class of weak solutions.

In order to prove Theorem 1.1, we first establish some estimates between pressure and velocity. Taking div and $\nabla \operatorname{div}$ to both sides of the micropolar fluid flows for smooth solution (u, π) , separately, we get the well-known pressure-velocity relation in \mathbb{R}^3 , given by

$$\pi = (-\Delta)^{-1} \sum_{i,j=1}^3 \frac{\partial^2}{\partial x_i \partial x_j} (u_i u_j) \quad \text{and} \quad \nabla \pi = (-\Delta)^{-1} \sum_{i,j=1}^3 \frac{\partial^2}{\partial x_i \partial x_j} (\nabla(u_i u_j)).$$

Then, the Calderón-Zygmund inequality implies that for any $1 < \alpha < +\infty$:

$$\|\pi\|_{L^\alpha} \leq C \|u\|_{L^{2\alpha}}^2 \quad \text{and} \quad \|\nabla \pi\|_{L^\alpha} \leq C \| |u| \nabla u \|_{L^\alpha}. \quad (5)$$

2 Proof of Theorem 1.1

Now we are in the position to prove Theorem 1.1. Firstly, by means of the local existence result, which is similar to the one used in the theory of Navier-Stokes equations (refer to Giga [12], see also Dong et al. [5]), and the standard local solution extension technique, equation (1) with $(u_0, \omega_0) \in L^2(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)$ admits a unique L^4 -strong solution (u, ω) on a maximal time interval. For the simplicity notation, we may suppose that the maximal time interval is $[0, T)$. Thus, in order to prove Theorem 1.1, it remains to show that

$$\lim_{t \rightarrow T} (\|u(t)\|_4 + \|\omega(t)\|_4) < \infty.$$

This will lead to a contradiction to the estimates to be derived below.

Proof Before going into the proof, we recall the following inequality established in [11] (see also [13]):

$$\|f\|_{L^4}^2 \leq C \|f\|_{B_{\infty,\infty}^{-1}} \|\nabla f\|_{L^2}. \quad (6)$$

Testing (1a) by $u|u|^2$ and using (1c), we get

$$\begin{aligned} & \frac{1}{4} \frac{d}{dt} \|u\|_{L^4}^4 + \| |u| \nabla u \|_{L^2}^2 + \frac{1}{2} \left\| \nabla |u|^2 \right\|_{L^2}^2 \\ &= \int_{\mathbb{R}^3} (\nabla \times \omega) \cdot u |u|^2 dx - \int_{\mathbb{R}^3} (u \cdot \nabla \pi) |u|^2 dx \\ &= \int_{\mathbb{R}^3} \omega [\nabla \times (u |u|^2)] dx + \int_{\mathbb{R}^3} \pi u \cdot \nabla |u|^2 dx \end{aligned} \quad (7)$$

Testing (1b) by $\omega |\omega|^2$, and using (1c) infers that

$$\begin{aligned} & \frac{1}{4} \frac{d}{dt} \|\omega\|_{L^4}^4 + \| |\omega| \nabla \cdot \omega \|_{L^2}^2 + \frac{1}{2} \left\| \nabla |\omega|^2 \right\|_{L^2}^2 \\ &= \int_{\mathbb{R}^3} (\nabla \times u) \cdot \omega |\omega|^2 dx - 2 \|\omega\|_{L^4}^4 \\ &= \int_{\mathbb{R}^3} u [\nabla \times (\omega |\omega|^2)] dx - 2 \|\omega\|_{L^4}^4, \end{aligned} \quad (8)$$

where we have used the following identities due to the divergence free property of the velocity field u :

$$\int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot u |u|^2 dx = 0 = \int_{\mathbb{R}^3} (u \cdot \nabla) \omega \cdot \omega |\omega|^2 dx.$$

Summing up (7) and (8), it follows that

$$\begin{aligned} & \frac{1}{4} \frac{d}{dt} (\|u\|_{L^4}^4 + \|\omega\|_{L^4}^4) + \|u\| \nabla u\|_{L^2}^2 + \frac{1}{2} \left\| \nabla |u|^2 \right\|_{L^2}^2 + \|\omega\| \nabla \cdot \omega\|_{L^2}^2 + \frac{1}{2} \left\| \nabla |\omega|^2 \right\|_{L^2}^2 \\ &= \int_{\mathbb{R}^3} \omega [\nabla \times (u |u|^2)] dx + \int_{\mathbb{R}^3} u [\nabla \times (\omega |\omega|^2)] dx - 2 \|\omega\|_{L^4}^4 - \int_{\mathbb{R}^3} (u \cdot \nabla \pi) |u|^2 dx. \end{aligned} \quad (9)$$

Using the Hölder inequality and the Young inequality and integrating by parts, we derive the estimate of the first three terms on the right-hand side of (9) as follows:

$$\begin{aligned} & \int_{\mathbb{R}^3} \omega [\nabla \times (u |u|^2)] dx + \int_{\mathbb{R}^3} u [\nabla \times (\omega |\omega|^2)] dx - 2 \|\omega\|_{L^4}^4 \\ & \leq \|u\|_{L^4} \|\omega\|_{L^4} (\|u\| \nabla u\|_{L^2} + \|\omega\| \nabla \omega\|_{L^2}) - 2 \|\omega\|_{L^4}^4 \\ & \leq \|u\| \nabla u\|_{L^2}^2 + \|\omega\| \nabla \omega\|_{L^2}^2 + C \|u\|_{L^4}^4. \end{aligned} \quad (10)$$

To estimate the last term of the right-hand side of (9), we have after integrating by parts and employing the Hölder inequality and the Young inequality,

$$\begin{aligned} \left| \int_{\mathbb{R}^3} \pi u \cdot \nabla |u|^2 dx \right| & \leq \int_{\mathbb{R}^3} |\pi| |u| |\nabla |u|^2| dx \\ & \leq C \|\pi\|_{L^4}^2 \|u\|_{L^4}^2 + \frac{1}{4} \left\| \nabla |u|^2 \right\|_{L^2}^2 \\ & \leq C \|\pi\|_{\dot{B}_{\infty,\infty}^{-1}} \|\nabla \pi\|_{L^2} \|u\|_{L^4}^2 + \frac{1}{4} \left\| \nabla |u|^2 \right\|_{L^2}^2 \\ & \leq C \|\pi\|_{\dot{B}_{\infty,\infty}^{-1}} \|u\|_{L^4}^2 \|u\| \nabla u\|_{L^2} + \frac{1}{4} \left\| \nabla |u|^2 \right\|_{L^2}^2 \\ & \leq C \|\pi\|_{\dot{B}_{\infty,\infty}^{-1}}^2 \|u\|_{L^4}^4 + \frac{1}{2} \|u\| \nabla u\|_{L^2}^2 + \frac{1}{4} \left\| \nabla |u|^2 \right\|_{L^2}^2, \end{aligned} \quad (11)$$

and hence,

$$\begin{aligned} & \frac{d}{dt} (\|u(t)\|_{L^4}^4 + \|\omega(t)\|_{L^4}^4) + \|u\| \nabla u\|_{L^2}^2 + \left\| \nabla |u|^2 \right\|_{L^2}^2 + \|\omega\| \nabla \cdot \omega\|_{L^2}^2 + \|\omega\| \nabla \omega\|_{L^2}^2 \\ & \leq C(1 + \|\pi\|_{\dot{B}_{\infty,\infty}^{-1}}^2) \|u\|_{L^4}^4. \end{aligned}$$

Applying the Gronwall inequality, yields

$$\begin{aligned} & \|u(\cdot, t)\|_{L^4}^4 + \|\omega(\cdot, t)\|_{L^4}^4 \\ & \leq (\|u_0\|_{L^4}^4 + \|\omega_0\|_{L^4}^4) \exp \left(C \int_0^t (1 + \|\pi(\cdot, \tau)\|_{\dot{B}_{\infty,\infty}^{-1}}^2) d\tau \right). \end{aligned}$$

Taking the inner product of (1a) with $-\Delta u$, (1b) with $-\Delta \omega$ in $L^2(\mathbb{R}^3)$, adding the resulting equations together, and using the Gagliardo-Nirenberg inequalities:

$$\|\nabla u\|_{L^4} \leq C \|u\|_{L^4}^{\frac{1}{5}} \|\Delta u\|_{L^2}^{\frac{4}{5}} \quad \text{and} \quad \|\nabla \omega\|_{L^2} \leq C \|u\|_{L^2}^{\frac{1}{2}} \|\Delta u\|_{L^2}^{\frac{1}{2}}.$$

we obtain,

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} (\|\nabla u(\cdot, t)\|_{L^2}^2 + \|\nabla \omega(\cdot, t)\|_{L^2}^2) + \|\Delta u\|_{L^2}^2 + \|\Delta \omega\|_{L^2}^2 + \|\nabla \nabla \cdot \omega\|_{L^2}^2 + 2\|\nabla \omega\|_{L^2}^2 \\
& \int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot \Delta u dx - \int_{\mathbb{R}^3} (\nabla \times \omega) \cdot \Delta u dx + \int_{\mathbb{R}^3} (u \cdot \nabla) \omega \cdot \Delta \omega dx - \int_{\mathbb{R}^3} (\nabla \times u) \cdot \Delta \omega dx \\
& \leq \|u\|_{L^4} \|\nabla u\|_{L^4} \|\Delta u\|_{L^2} + \|\nabla \omega\|_{L^2} \|\Delta u\|_{L^2} + \|u\|_{L^4} \|\nabla \omega\|_{L^4} \|\Delta \omega\|_{L^2} + \|\nabla u\|_{L^2} \|\Delta \omega\|_{L^2} \\
& \leq C \|u\|_{L^4}^{\frac{6}{5}} \|\Delta u\|_{L^2}^{\frac{9}{5}} + \|\omega\|_{L^2}^{\frac{1}{2}} \|\Delta \omega\|_{L^2}^{\frac{1}{2}} \|\Delta u\|_{L^2} + \|u\|_{L^4} \|\omega\|_{L^4}^{\frac{1}{5}} \|\Delta \omega\|_{L^2}^{\frac{9}{5}} + \|u\|_{L^2}^{\frac{1}{2}} \|\Delta u\|_{L^2}^{\frac{1}{2}} \|\Delta \omega\|_{L^2} \\
& \leq C \|u\|_{L^4}^{12} + C \|\omega\|_{L^2}^2 + C(\|u\|_{L^4}^{12} + \|\omega\|_{L^4}^{12}) + C \|u\|_{L^2}^2 + \frac{1}{2} (\|\Delta u\|_{L^2}^2 + \|\Delta \omega\|_{L^2}^2) \\
& \leq C(1 + \|u\|_{L^4}^{12} + \|\omega\|_{L^4}^{12}) + \frac{1}{2} (\|\Delta u\|_{L^2}^2 + \|\Delta \omega\|_{L^2}^2).
\end{aligned}$$

This yields,

$$\begin{aligned}
& \frac{d}{dt} (\|\nabla u(\cdot, t)\|_{L^2}^2 + \|\nabla \omega(\cdot, t)\|_{L^2}^2) + \|\Delta u\|_{L^2}^2 + \|\Delta \omega\|_{L^2}^2 + \|\nabla \nabla \cdot \omega\|_{L^2}^2 + 2\|\nabla \omega\|_{L^2}^2 \\
& \leq C(1 + \|u\|_{L^4}^{12} + \|\omega\|_{L^4}^{12}).
\end{aligned}$$

Integrating the above inequality over $(0, t)$, we have

$$\begin{aligned}
& \|\nabla u(t)\|_{L^2}^2 + \|\nabla \omega(t)\|_{L^2}^2 + \int_0^t (\|\Delta u(\tau)\|_{L^2}^2 + \|\Delta \omega(\tau)\|_{L^2}^2 + \|\nabla \nabla \cdot \omega(\tau)\|_{L^2}^2 \\
& + 2\|\nabla \omega(\tau)\|_{L^2}^2) d\tau \\
& \leq \|\nabla u_0\|_{L^2}^2 + \|\nabla \omega_0\|_{L^2}^2 + C \int_0^t (1 + \|u(\tau)\|_{L^4}^{12} + \|\omega(\tau)\|_{L^4}^{12}) d\tau. \tag{12}
\end{aligned}$$

On the other hand, combining a Sobolev embedding theorem $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, (12) and (5), we obtain that

$$\begin{aligned}
e + \|\pi(\cdot, t)\|_{L^3} & \leq e + C \|u(\cdot, t)\|_{L^6}^2 \leq e + C \|\nabla u(\cdot, t)\|_{L^2}^2 \\
& \leq e + C(\|\nabla u_0\|_{L^2}^2 + \|\nabla \omega_0\|_{L^2}^2) + C \int_0^t (1 + \|u(\cdot, \tau)\|_{L^4}^{12}) d\tau \\
& \leq e + C(\|\nabla u_0\|_{L^2}^2 + \|\nabla \omega_0\|_{L^2}^2) + C(e+t) \sup_{0 \leq \tau \leq t} (1 + \|u(\cdot, \tau)\|_{L^4}^{12}) \\
& \leq C \left(e + \|\nabla u_0\|_{L^2}^2 + \|\nabla \omega_0\|_{L^2}^2 \right) (e+t) \sup_{0 \leq \tau \leq t} (1 + \|u(\cdot, \tau)\|_{L^4}^{12}) \\
& \leq C_0(e+t) \exp \left(C \int_0^t (1 + \|\pi(\cdot, \tau)\|_{\dot{B}_{\infty, \infty}^{-1}}^2) d\tau \right), \tag{13}
\end{aligned}$$

where the constant $C_0 = C(e, \|\nabla u_0\|_{L^2}, \|\nabla \omega_0\|_{L^2}, \|u_0\|_{L^4}, \|\omega_0\|_{L^4})$. Using the fact that $L^3(\mathbb{R}^3) \subset \dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3)$, it follows that

$$e + \|\pi(\cdot, t)\|_{L^3} \leq C(e+t) \exp \left(C \int_0^t (1 + \|\pi(\cdot, \tau)\|_{\dot{B}_{\infty, \infty}^{-1}}^2) d\tau \right) \tag{14}$$

Now, taking the logarithm on both sides of (14), we can conclude that

$$\log(e + \|\pi(\cdot, t)\|_{L^3}) \leq \log(C(e+t)) + C \int_0^t (1 + \|\pi(\cdot, \tau)\|_{\dot{B}_{\infty, \infty}^{-1}}^2) d\tau. \tag{15}$$

For simplicity, set

$$\begin{aligned}
\mathcal{L}(t) & = \log(e + \|\pi(\cdot, t)\|_{L^3}), \\
\mathcal{E}(t) & = \log(C(e+t)) + C \int_0^t (1 + \|\nabla \pi(\cdot, \tau)\|_{\dot{B}_{\infty, \infty}^{-1}}^2) d\tau, \tag{16}
\end{aligned}$$

with $\mathcal{E}(0) = \log(Ce)$. Then, the above inequality (15) implies that

$$0 < \mathcal{Z}(t) \leq \mathcal{E}(t)$$

from which we easily get

$$(e + \mathcal{Z}(t)) \log(e + \mathcal{Z}(t)) \leq (e + \mathcal{E}(t)) \log(e + \mathcal{E}(t)).$$

On the other hand, we have

$$\begin{aligned} \frac{d}{dt} \log(e + \mathcal{E}(t)) &= \frac{1}{e + \mathcal{E}(t)} \left(\frac{1}{e + t} + C(1 + \|\nabla \pi(\cdot, t)\|_{\dot{B}_{\infty, \infty}^{-1}}^2) \right) \\ &\leq \frac{1}{e^2} + C \frac{1 + \|\nabla \pi(\cdot, t)\|_{\dot{B}_{\infty, \infty}^{-1}}^2}{e + \mathcal{E}(t)} \\ &= \frac{1}{e^2} + C \frac{1 + \|\nabla \pi(\cdot, t)\|_{\dot{B}_{\infty, \infty}^{-1}}^2}{(e + \mathcal{E}(t)) \ln(e + \mathcal{E}(t))} \log(e + \mathcal{E}(t)) \\ &\leq \frac{1}{e^2} + C \frac{1 + \|\nabla \pi(\cdot, t)\|_{\dot{B}_{\infty, \infty}^{-1}}^2}{(e + \mathcal{Z}(t)) \ln(e + \mathcal{Z}(t))} \log(e + \mathcal{E}(t)) \end{aligned}$$

Applying the Gronwall inequality to $\log(e + \mathcal{E}(t))$, we find

$$\begin{aligned} &\log(e + \mathcal{E}(t)) \\ &\leq \log(e + \mathcal{E}(0)) \exp \left(\frac{T}{e^2} + C \int_0^t \frac{1 + \|\nabla \pi(\cdot, \tau)\|_{\dot{B}_{\infty, \infty}^{-1}}^2}{(e + \mathcal{Z}(\tau)) \log(e + \mathcal{Z}(\tau))} d\tau \right), \end{aligned}$$

and equivalently

$$e + \mathcal{E}(t) \leq (e + \mathcal{E}(0)) \exp \left(\frac{T}{e^2} + C \int_0^t \frac{1 + \|\nabla \pi(\cdot, \tau)\|_{\dot{B}_{\infty, \infty}^{-1}}^2}{(e + \mathcal{Z}(\tau)) \log(e + \mathcal{Z}(\tau))} d\tau \right)$$

Using the fact that $L^3(\mathbb{R}^3) \subset \dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3)$, it follows from (16) that

$$\begin{aligned} \int_0^t \|\pi(\cdot, \tau)\|_{\dot{B}_{\infty, \infty}^{-1}} d\tau &\leq C \int_0^t \|\pi(\cdot, \tau)\|_{L^3} d\tau \\ &\leq (e + \mathcal{E}(0)) \exp \left(\frac{T}{e^2} + C \int_0^t \frac{1 + \|\pi(\cdot, \tau)\|_{\dot{B}_{\infty, \infty}^{-1}}^2}{(e + \mathcal{Z}(\tau)) \log(e + \mathcal{Z}(\tau))} d\tau \right) < \infty. \end{aligned} \quad (17)$$

Hence by virtue of (12) and (17), we conclude that

$$(u, \omega) \in L^\infty((0, T); H^1(\mathbb{R}^3)) \cap L^2((0, T); H^2(\mathbb{R}^3)),$$

which completes the proof of Theorem 1.1. \square

There is no conflict of interest.

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