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## **An existence result for quasi-equilibrium problems via Ekeland's variational principle**

**John Cotrina · Michel Théra · Javier Zúñiga**

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**Abstract** This paper deals with the existence of solutions to equilibrium and quasi-equilibrium problems without any convexity assumption. Coverage includes some equivalences to the Ekeland variational principle for bifunctions and basic facts about transfer lower continuity. An application is given to systems of quasi-equilibrium problems

**Keywords** Keywords: Quasi-equilibrium problem, System of quasi-equilibrium problem, Ekeland variational principle, Transfer lower continuity

**Mathematics Subject Classification (2000)** 58E30 · 54E50 · 49J40 · 49J27

### **1 Introduction**

To the best of our knowledge, the first attempts to investigate equilibrium problems started with the paper of Blum and Oettli [9]. Equilibrium problems provide a framework to handle broad classes of problems arising in different areas such as optimization, variational inequalities, fixed point theory, Nash equilibria and some others. Motivated by these applications, many authors have been increasingly interested

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in studying conditions for the existence of solutions for equilibrium problems. Also, there exists a large literature about existence involving convexity assumptions on the constraint set and the bifunction, see for instance [7, 11, 12, 16, 25–27, 29] and their references therein. The main concepts used generally in these problems involve the famous minimax inequality due to Ky Fan [22] or the Brézis-Browder-Stampacchia result [10], while other authors used an important lemma, strongly related to generalized monotonicity concepts, due again to Ky Fan [28].

Most of the results concerning existence of equilibrium problems without any convexity assumptions, use Ekeland's variational principle, see [1, 6]. Indeed, this principle or its equivalents (see [30]) is a key tool in the theory of variational analysis. This principle uses the completeness of the space as well as the lower semicontinuity of the function under consideration. It permits to establish the existence of approximate solutions of minimization problems.

Our main interest concerns quasi-equilibrium problems, that is equilibrium problems with a constraint set depending on the current point. This problem has gained more and more attention, perhaps because it models generalized Nash equilibrium problems, which in turn models a large number of real life problems, see e.g. [21] and its references therein. Recent works on the existence of solutions for this kind of problem involving convexity assumptions are given in [4, 14, 17–19]. In [13] an existence result was provided for quasi-equilibrium problems, without any convexity condition, via Ekeland's variational principle.

It is our aim in this contribution to move away from lower semicontinuity by using a weaker notion called *transfer lower continuity* which was introduced in the framework of mathematical analysis for generalizations of the Weierstrass and maximum theorems and briefly studied by Tian and Zhou [32].

The rest of the paper is organized in five main sections. The next section is devoted to the concept of transfer lower continuity, while in section 3 we establish, for non necessarily lower semicontinuous functions, an Ekeland-type theorem which involves the lower semicontinuous regularization of the given functional. We show that this theorem is equivalent to a theorem by Bianchi et al [6] established for equilibrium problems.

Section 4 is concerned with new existence results of equilibria and quasi-equilibria, while the main result of section 5 is Theorem 13 that guarantees the existence of a solution to a system of quasi-equilibrium problems in complete metric spaces.

## 2 Definitions, notation and preliminaries results

In this section we introduce and remind tools that will be useful throughout the paper and we will use standard notations and terminology from real analysis.

Given a nonempty subset  $C$  of a topological space  $X$ , a function  $h : C \rightarrow \mathbb{R}$  is said to be *lower semicontinuous*<sup>1</sup> (lsc for short) if, for each  $x \in C$  and each  $\lambda \in \mathbb{R}$  such that  $h(x) > \lambda$ , there exists a neighbourhood  $V_x$  of  $x$  such that  $h(x') > \lambda$ , for all  $x' \in V_x \cap C$ .

Tian and Zhou ([32]) introduced a weaker notion of lower semicontinuity that they called *transfer lower continuity*. More precisely,  $h$  is transfer lower continuous (tlc, for short) if, for each  $x, y \in C$  such that  $h(x) > h(y)$ , there exist  $y' \in C$  and  $V_x$  a neighbourhood of  $x$  such that  $h(x') > h(y')$ , for all  $x' \in V_x \cap C$ .

Given  $h$  and  $\lambda \in \mathbb{R}$ , we denote by  $\text{Epi } h$  and  $S_h(\lambda)$  the *epigraph* and the *lower sub-level set* at level  $\lambda$  of  $h$ , respectively, i.e.,

$$\text{Epi } h := \{(x, \lambda) \in C \times \mathbb{R} : h(x) \leq \lambda\} \text{ and } S_h(\lambda) := \{x \in C : h(x) \leq \lambda\}.$$

It is well known that a function is lsc if and only if  $\text{Epi } h$  is closed in  $C \times \mathbb{R}$  or equivalently, if and only if  $S_h(\lambda)$  is closed in  $C$ , for all  $\lambda \in \mathbb{R}$ , see for instance [15].

We will write  $S_h(x)$  instead of  $S_h(h(x))$  in order to simplify the notation. Thanks to [32, Lemma 1 and Remark 7] a function  $h$  is tlc if and only if

$$\bigcap_{x \in C} S_h(x) = \bigcap_{x \in C} \overline{S_h(x)},$$

where  $\overline{S_h(x)}$  is the closure of  $S_h(x)$  in  $C$ .

Note that contrary to lower semicontinuity, transfer lower continuity is not closed under addition as the following simple example shows.

*Example 2.1* Let  $h, g : \mathbb{R} \rightarrow \mathbb{R}$  be two functions defined as

$$h(x) := \begin{cases} x+1, & x < 0 \\ 2, & x = 0 \\ x+3, & x > 0 \end{cases} \text{ and } g(x) := -x.$$

It is not hard to observe that both functions are transfer lower continuous. However,  $h + g$  fails to be transfer lower continuous. Indeed, the sum function is given by

$$(h+g)(x) = \begin{cases} 1, & x < 0 \\ 2, & x = 0 \\ 3, & x > 0 \end{cases}$$

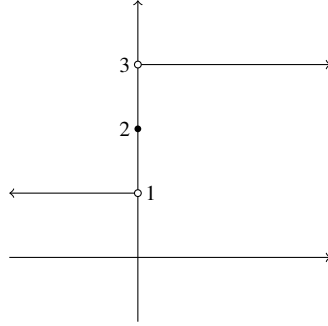
whose graph is represented in Figure 1.

From Figure 1 the function  $h + g$  is not transfer lower continuous at 0.

Given a nonempty subset  $C$  of a topological space  $X$  and a function  $h : C \rightarrow \mathbb{R}$ , we consider the *minimization problem*:

$$\text{Find } x \in C \text{ such that } h(x) \leq h(y), \text{ for all } y \in C.$$

<sup>1</sup> Introduced by R. Baire, see [31] and the references therein.



**Fig. 1** graph of  $h + g$

We denote by  $\operatorname{argmin}_C h$  the solution set of the minimization problem associated to  $h$  and  $C$ . It is important to notice that

$$\operatorname{argmin}_C h = \bigcap_{x \in C} S_h(x) = \bigcap_{\lambda > \alpha} S_h(\lambda), \quad (1)$$

where  $\alpha = \inf_{x \in C} h(x)$ . Additionally,

$$\bigcap_{x \in C} \overline{S_h(x)} = \bigcap_{\lambda > \alpha} \overline{S_h(\lambda)}. \quad (2)$$

The following result is an extension of the celebrated Weierstrass theorem.

**Theorem 2.1** ([32, Theorem 2]) *Let  $C$  be a compact and nonempty subset of a topological space  $X$ , and  $h : C \rightarrow \mathbb{R}$  be a function. Then, the set  $\operatorname{argmin}_C h$  is nonempty and compact if and only if  $h$  is tlc.*

Given a nonempty subset  $C$  of a topological space  $X$ , it is a basic fact from real analysis that every function  $h : C \rightarrow \mathbb{R}$  (not necessarily lsc) admits a *lower semicontinuous regularization*  $\bar{h} : C \rightarrow \mathbb{R} \cup \{-\infty\}$  defined by  $\operatorname{Epi} \bar{h} := \overline{\operatorname{Epi} h}$ , the closure in  $C \times \mathbb{R}$ , or equivalently by  $\bar{h}(x) = \liminf_{y \rightarrow x} h(y) = \sup_U \inf_{y \in U} h(y)$ , where  $U$  runs over all neighbourhoods of  $x$ .

It is well known that for any  $x \in C$  and any  $\lambda \in \mathbb{R}$

- (i)  $\bar{h}(x) = \inf\{\lambda \in \mathbb{R} : x \in \overline{S_\lambda(h)}\}$ ;
- (ii)  $\bar{h}(x) \leq h(x)$ ;
- (iii)  $\overline{S_h(x)} \subset \overline{S_{\bar{h}}(x)}$ ;
- (iv)  $\overline{S_{\bar{h}}(\lambda)} = \bigcap_{\mu > \lambda} \overline{S_h(\mu)}$ ;

We will say that a lower semicontinuous regularization is well-defined if it is real valued, that means  $\bar{h}(x) > -\infty$ , for all  $x \in C$ , or in other words, if  $h$  admits a lsc minorant.

We present now some basic results on transfer lower continuity and on lower semicontinuity regularization.

**Proposition 2.1** *Let  $C$  be a nonempty subset of a topological space  $X$  and  $h : C \rightarrow \mathbb{R}$  be a function. If  $h$  is tlc, then its lower semicontinuous regularization is well-defined.*

**Proof.** It is enough to consider the case when  $h$  is not bounded below. Then, for each  $x \in C$  there exists  $y \in C$  such that  $h(x) > h(y)$ . Since  $h$  is tlc there exist  $V_x$  a neighbourhood of  $x$  and  $y' \in C$  such that  $h(x') > h(y')$ , for all  $x' \in V_x$ , which in turn implies  $\bar{h}(x) \geq h(y')$ . Therefore,  $\bar{h}$  is well-defined.  $\square$

**Proposition 2.2** *Let  $C$  be a nonempty subset of a topological space  $X$  and  $h : C \rightarrow \mathbb{R}$  be a function. Then, the following holds*

$$\inf_{x \in C} h(x) = \inf_{x \in C} \bar{h}(x).$$

**Proof.** It is clear that  $\inf_{x \in C} h(x) \geq \inf_{x \in C} \bar{h}(x)$ . If we suppose that

$$\inf_{x \in C} h(x) > \inf_{x \in C} \bar{h}(x),$$

then, there exists  $x_0 \in C$  such that  $\bar{h}(x_0) < \inf_{x \in C} h(x)$ . Thus, there exists  $\lambda \in \mathbb{R}$  such that  $\lambda < \inf_{x \in C} h(x)$  and  $x_0 \in S_{\bar{h}}(\lambda)$ . Now, for each  $V_{x_0}$  neighbourhood of  $x_0$  there exists  $x' \in V_{x_0} \cap S_h(\lambda)$ , so  $h(x') \leq \lambda$ , which is a contradiction.  $\square$

**Proposition 2.3** *Let  $C$  be a nonempty subset of a topological space  $X$  and  $h : C \rightarrow \mathbb{R}$  be a function. Then  $\operatorname{argmin}_C h = \operatorname{argmin}_C \bar{h}$  if and only if  $h$  is tlc.*

**Proof.** Since  $\overline{S_h(x)} \subset S_{\bar{h}}(x)$ , for all  $x \in C$ , by (1) we have

$$\bigcap_{x \in C} \overline{S_h(x)} \subset \bigcap_{x \in C} S_{\bar{h}}(x) = \operatorname{argmin}_C \bar{h} = \operatorname{argmin}_C h = \bigcap_{x \in C} S_h(x).$$

Hence,  $h$  is transfer lower continuous.

Conversely, by (2) we have

$$\bigcap_{x \in C} S_{\bar{h}}(x) = \bigcap_{\lambda > \alpha} S_{\bar{h}}(\lambda) = \bigcap_{\lambda > \alpha} \left( \bigcap_{\mu > \lambda} \overline{S_h(\mu)} \right) = \bigcap_{\lambda > \alpha} \overline{S_h(\lambda)} = \bigcap_{x \in C} \overline{S_h(x)},$$

where  $\alpha = \inf_{x \in C} h(x)$ . The result follows from (1) and the transfer lower continuity of  $h$ .  $\square$

The following example shows that transfer lower continuity is essential in the previous result.

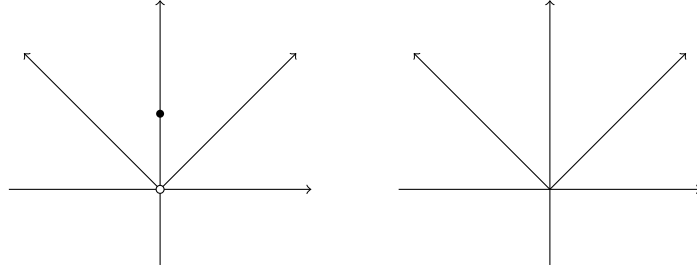
*Example 2.2* Let  $h : \mathbb{R} \rightarrow \mathbb{R}$  be the function defined as

$$h(x) := \begin{cases} |x|, & x \neq 0 \\ 1, & x = 1 \end{cases}$$

Clearly the lower semicontinuous regularization of  $h$  is  $\bar{h} : \mathbb{R} \rightarrow \mathbb{R}$  defined as  $\bar{h}(x) = |x|$ . Figure 2 shows the graphs of  $h$  and its regularization  $\bar{h}$ , respectively.

Moreover,

$$\operatorname{argmin}_{\mathbb{R}} h = \emptyset \quad \text{and} \quad \operatorname{argmin}_{\mathbb{R}} \bar{h} = \{0\}.$$



**Fig. 2** graphs of  $h$  and  $\bar{h}$

**Proposition 2.4** *Let  $C$  be a nonempty subset of a topological space  $X$  and  $h : C \rightarrow \mathbb{R}$  be a tlc function. If there exists  $x \in C$  such that*

$$\bar{h}(x) \leq h(y), \text{ for all } y \in C;$$

*then  $x \in \operatorname{argmin}_C h$ .*

**Proof.** Since  $h$  is tlc, its lower semicontinuous regularization  $\bar{h}$  is well-defined, due to Proposition 2.1. Now, it is clear that  $\operatorname{Epi}(h) \subset C \times [\bar{h}(x), +\infty[$ . Thus, we deduce that  $x \in \operatorname{argmin}_C \bar{h}$ . The result follows from Proposition 2.3.  $\square$

Example 2.2 shows that the transfer lower semi-continuity can not be dropped in the previous result.

Now, we recall some definitions for bifunctions. Given a topological space  $X$  and  $C \subset X$ , a bifunction  $f : C \times C \rightarrow \mathbb{R}$  is said:

- to have the *triangle inequality property* on  $C$  if, for all  $x, y, z \in C$  the following holds

$$f(x, y) \leq f(x, z) + f(z, y);$$

- to be *cyclically monotone* on  $C$  if, for all  $n \in \mathbb{N}$  and all  $x_0, x_1, \dots, x_n \in C$  the following holds ,

$$\sum_{i=0}^n f(x_i, x_{i+1}) \leq 0,$$

with  $x_{n+1} = x_0$ ;

- to be *monotone* on  $C$  if, for all  $x, y \in C$  the following holds

$$f(x, y) + f(y, x) \leq 0;$$

- to be *pseudo-monotone* on  $C$  if, for all  $x, y \in C$  the following implication holds

$$f(x, y) \geq 0 \implies f(y, x) \leq 0;$$

The concept of cyclic monotonicity for bifunctions appeared first in [6]. Many authors studied its properties, see for instance [2, 13, 24]. Recently in [13, 23] the authors used cyclic monotonicity in order to solve equilibrium and quasi-equilibrium problems.

It is clear that cyclic monotonicity implies monotonicity which in turn implies pseudo-monotonicity. Important instances of these kinds of bifunctions are given below.

*Example 2.3* Let  $C$  be a nonempty subset of a topological space  $X$  and let  $h : C \rightarrow \mathbb{R}$  be a function with a well-defined lower semicontinuous regularization. Consider bifunctions  $g, f : C \times C \rightarrow \mathbb{R}$  defined as

$$g(x, y) := h(y) - h(x) \quad \text{and} \quad f(x, y) := \bar{h}(y) - h(x). \quad (3)$$

It is clear that  $f$  is cyclically monotone and  $g$  satisfies the triangular inequality property. Moreover, the following inequality holds:  $g \geq f^2$ .

Given a bifunction  $f : C \times C \rightarrow \mathbb{R}$ , we consider the bifunction  $\hat{f} : C \times C \rightarrow \mathbb{R}$  given by

$$\hat{f}(x, y) := -f(y, x).$$

Due to [6, Remark 2.2], we note that if  $f$  verifies the triangle inequality property, then  $\hat{f}$  is cyclically monotone. By [24, Proposition 5.1] cyclic monotonicity of  $\hat{f}$  is equivalent to the existence of a function  $h : C \rightarrow \mathbb{R}$  such that

$$\hat{f}(x, y) \leq h(y) - h(x), \quad \forall x, y \in C.$$

Moreover, it is not difficult to check that

$$f(x, y) \geq h(y) - h(x) \geq \hat{f}(x, y).$$

Additionally, if  $f$  is monotone, then  $f(x, y) = h(y) - h(x)$ , for all  $x, y \in C$ . Hence,  $f$  is cyclically monotone.

The following result says that there is a strong relationship between monotonicity and pseudo-monotonicity.

**Proposition 2.5** *Let  $C$  be a nonempty subset of a topological space  $X$  and  $f : C \times C \rightarrow \mathbb{R}$  be a bifunction. Then,  $f$  is pseudo-monotone if and only if there are bifunctions  $f_1, f_2 : C \times C \rightarrow \mathbb{R}$  satisfying*

$$f(x, y) = f_1(x, y)f_2(x, y), \quad (4)$$

where  $f_1$  is strictly positive and  $f_2$  is monotone.

<sup>2</sup> By  $g \geq f$  we mean that  $g(x, y) \geq f(x, y)$  for all  $x, y \in C$ .



**Proof.** Assume that  $f$  is pseudo-monotone. We denote by  $D$  the subset of  $C$  where  $f$  vanishes. Define bifunctions  $f_1, f_2 : C \times C \rightarrow \mathbb{R}$  by

$$f_1(x, y) := \begin{cases} 1, & (x, y) \in D \\ |f(x, y)|, & \text{otherwise} \end{cases} \quad \text{and} \quad f_2(x, y) := \text{sign}(f(x, y)).$$

It is clear that  $f_1$  is strictly positive. We affirm that  $f_2$  is monotone. Indeed, for each  $x, y \in C$  we have  $f_2(x, y) \in \{-1, 0, 1\}$ . So, if  $f_2(x, y) = -1$ , then it is obvious that  $f_2(x, y) + f_2(y, x) \leq 0$ . If  $f_2(x, y) = 0$  then  $f(x, y) = 0$ , which in turn implies  $f(y, x) \leq 0$  due the pseudo-monotonicity of  $f$ . Thus,  $f_2(y, x) \in \{-1, 0\}$  and this allows us to conclude that  $f_2(x, y) + f_2(y, x) \leq 0$ . Finally, if  $f_2(x, y) = 1$ , that means  $f(x, y) > 0$ , then again by pseudo-monotonicity of  $f$  we have  $f(y, x) < 0$ , in other words  $f_2(y, x) = -1$ . Hence,  $f_2(x, y) + f_2(y, x) = 0$ .

The converse is not difficult to check.  $\square$

*Remark 2.1* In a similar way to [8, Theorem 2.1], let  $C$  be a subset of a topological space  $X$ , and let  $f : C \times C \rightarrow \mathbb{R}$  be a bifunction. If  $f$  and  $\hat{f}$  both are pseudo-monotone on  $C$ , then for all  $x, y \in C$  the following equivalence holds

$$f(x, y) = 0 \iff \hat{f}(x, y) = 0.$$

Hence,  $(x, y)$  is an element of the set  $D$ , where  $f$  vanishes, if and only if  $(y, x) \in D$ . In other words,  $D$  is symmetric.

However, contrary to [8, Theorem 2.1], the converse does not hold, even if we assume the continuity of  $f$ . Indeed, consider the bifunction  $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  defined as

$$f(x, y) := (y - x)^2.$$

Clearly,  $f$  is continuous. Moreover,  $f(x, y) = 0$  if and only if  $y = x$ , which in turn is equivalent to  $f(y, x) = 0$ . But, the bifunction  $f$  is not pseudo-monotone, due to the fact that  $f(1, 0) = f(0, 1) = 1$ .

As a direct consequence of the result above, we have the following corollary, which was inspired from [5, Theorem 1.4] for maps in the setting of finite dimensional spaces.

**Corollary 2.1** *Let  $C$  be a subset of a topological space  $X$  and  $f : C \times C \rightarrow \mathbb{R}$  be a bifunction. Then,  $f, \hat{f}$  are pseudo-monotone if and only if there are bifunctions  $f_1, f_2 : C \times C \rightarrow \mathbb{R}$  such that  $f_1$  is strictly positive and  $f_2, \hat{f}_2$  are monotone, satisfying (4).*

### 3 The Ekeland variational principle

We begin this section recalling the celebrated Ekeland variational principle and then its extension to equilibrium problems.

**Theorem 3.1 (The Ekeland variational principle [20])** *Let  $C$  be a nonempty closed subset of the complete metric space  $(X, d)$ , and  $h : C \rightarrow \mathbb{R}$  be a lsc function bounded from below. For every  $\varepsilon > 0$ , and for any  $x_0 \in C$ , there exists  $\hat{x} \in C$  such that*

$$\begin{aligned} h(\hat{x}) + \varepsilon d(x_0, \hat{x}) &\leq h(x_0), \text{ and} \\ h(x) + \varepsilon d(x, \hat{x}) &> h(\hat{x}), \text{ for all } x \in C \setminus \{\hat{x}\}. \end{aligned}$$

**Theorem 3.2** ([6, Theorem 2.1] and [1, Theorem 2.2]) *Let  $C$  be a nonempty closed subset of a complete metric space  $(X, d)$  and  $f : C \times C \rightarrow \mathbb{R}$  be a bifunction. Assume that the following conditions hold*

- (i)  $f$  is bounded from below and lsc with respect to its second argument;
- (ii)  $f(x, x) = 0$ , for all  $x \in C$ ;
- (iii)  $f$  satisfies the triangle inequality property.

Then, for all  $\varepsilon > 0$  and all  $x_0 \in C$ , there exists  $\hat{x} \in C$  such that

$$\begin{aligned} f(x_0, \hat{x}) + \varepsilon d(x_0, \hat{x}) &\leq 0, \\ f(\hat{x}, x) + \varepsilon d(x, \hat{x}) &> 0 \text{ for every } x \in C \setminus \{\hat{x}\}. \end{aligned}$$

Let us restate Theorem 3.1 in terms of lower semicontinuous regularizations.

**Theorem 3.3** *Let  $C$  be a nonempty closed subset of the complete metric space  $(X, d)$ , and  $h : C \rightarrow \mathbb{R}$  be a function bounded from below. For every  $\varepsilon > 0$ , and for any  $x_0 \in C$ , there exists  $\hat{x} \in C$  such that*

$$\begin{aligned} \bar{h}(\hat{x}) + \varepsilon d(x_0, \hat{x}) &\leq h(x_0), \text{ and} \\ h(x) + \varepsilon d(x, \hat{x}) &> \bar{h}(\hat{x}), \text{ for all } x \in C \setminus \{\hat{x}\}. \end{aligned}$$

For the sake of completeness, we give a self-contained proof of Theorem 3.3 which mimics the proof of Ekeland's Theorem.

**Proof.** Since  $h$  is bounded from below, its lsc regularization  $\bar{h}$  is well-defined. Without loss of generality we consider  $\varepsilon = 1$ . Denote by  $H(x)$  the set

$$H(x) := \{y \in C : \bar{h}(y) + d(y, x) \leq \bar{h}(x)\}.$$

Since the distance is continuous and  $\bar{h}$  is lsc, the set  $H(x)$  is closed, for every  $x \in C$ . Moreover,  $x \in H(x)$ . For each  $y \in H(x)$  and any  $z \in H(y)$ , it is easy to verify that  $z \in H(x)$ . Hence  $y \in H(x)$  implies  $H(y) \subset H(x)$ . Define  $r(x) := \inf_{z \in H(x)} \bar{h}(z)$ . For each  $z \in H(x)$  we have  $\bar{h}(z) + d(z, x) \leq \bar{h}(x)$ , which in turn implies  $d(x, z) \leq \bar{h}(x) - r(x)$ . So, for any  $z_1, z_2 \in H(x)$

$$d(z_1, z_2) \leq d(z_1, x) + d(x, z_2) \leq 2(\bar{h}(x) - r(x)).$$

Thus the diameter  $\text{diam}(H(x))$  of  $H(x)$  satisfies:

$$\text{diam}(H(x)) \leq 2(\bar{h}(x) - r(x)).$$

For  $x_0 \in C$ , there exists  $x_1 \in H(x_0)$  such that

$$\bar{h}(x_1) \leq r(x_0) + \frac{1}{2}.$$

Now, for this  $x_1$  there exists  $x_2 \in H(x_1)$  such that

$$\bar{h}(x_2) \leq r(x_1) + \frac{1}{2^2}.$$

Inductively, we define a sequence  $\{x_n\}$  of points in  $C$  such that  $x_{n+1} \in H(x_n)$  and

$$\bar{h}(x_{n+1}) \leq r(x_n) + \frac{1}{2^{n+1}}. \quad (5)$$

On the other hand, we note that

$$r(x_{n+1}) = \inf_{z \in H(x_{n+1})} \bar{h}(z) \geq \inf_{z \in H(x_n)} \bar{h}(z) = r(x_n). \quad (6)$$

Combining (6) and (5) we obtain

$$r(x_n) \leq r(x_{n+1}) \leq \bar{h}(x_{n+1}) \leq r(x_n) + \frac{1}{2^{n+1}}.$$

Therefore,  $\text{diam}(H(x_{n+1})) \leq 2(\bar{h}(x_{n+1}) - r(x_{n+1})) \leq \frac{1}{2^n}$ , for all  $n \in \mathbb{N}$ . As a consequence we deduce that

$$\bigcap_{n \in \mathbb{N}} H(x_n) = \{\hat{x}\}.$$

Since  $\hat{x} \in H(x_0)$ , we have  $\bar{h}(\hat{x}) + d(\hat{x}, x_0) \leq \bar{h}(x_0) \leq h(x_0)$ . Moreover  $\hat{x} \in H(x_n)$  for all  $n \in \mathbb{N}$  and since  $H(\hat{x}) \subset H(x_n)$  we deduce that  $H(\hat{x}) = \{\hat{x}\}$ . As a result,

$$x \notin H(\hat{x}) \quad \text{if and only if} \quad x \neq \hat{x}.$$

Therefore, for any  $x \in C \setminus \{\hat{x}\}$ ,

$$\bar{h}(\hat{x}) < \bar{h}(x) + d(x, \hat{x}) \leq h(x) + d(x, \hat{x}).$$

This completes the proof.  $\square$

The next theorem shows that the previous results are equivalent.

**Theorem 3.4** *Theorems 3.1 through 3.3 are equivalent.*

**Proof.** Theorem 3.1  $\iff$  Theorem 3.2.

It is clear that Theorem 3.2 implies Theorem 3.1. Reciprocally, for each  $\varepsilon > 0$  and  $x_0 \in C$ , Ekeland's variational principle applied to the function  $f(x_0, \cdot)$  gives the existence of  $\hat{x} \in C$  such that

$$f(x_0, \hat{x}) \leq f(x_0, x_0) - \varepsilon d(x_0, \hat{x}) \quad \text{and} \quad (7)$$

$$f(x_0, x) > f(x_0, \hat{x}) - \varepsilon d(\hat{x}, x), \quad \forall x \in C \setminus \{\hat{x}\}. \quad (8)$$

Since  $f$  vanishes on the diagonal of  $C \times C$ , inequality (7) reduces to

$$f(x_0, \hat{x}) + \varepsilon d(x_0, \hat{x}) \leq 0.$$

On the other hand, according to the triangle inequality property we have  $f(x_0, x) \leq f(x_0, \hat{x}) + f(\hat{x}, x)$ . Thus, inequality (8) reduces to

$$f(\hat{x}, x) + \varepsilon d(\hat{x}, x) > 0, \quad \forall x \in C \setminus \{\hat{x}\}.$$

For the equivalence of Theorem 3.1 and Theorem 3.3, we show both implications.

The first implication follows from  $\bar{h} \leq h$  and by applying Theorem 3.1 to  $\bar{h}$ . The converse follows from the fact that  $\bar{h} = h$ .  $\square$

As a direct consequence of Theorem 3.2 we have the following corollary.

**Corollary 3.1** *Let  $C$  be a nonempty closed subset of a complete metric space  $(X, d)$  and  $f : C \times C \rightarrow \mathbb{R}$  be a bifunction. Assume that there exists a bifunction  $g : C \times C \rightarrow \mathbb{R}$  such that:*

- (i)  $f \geq g$ ;
- (ii)  $g$  is bounded from below and lsc with respect to its second argument;
- (iii)  $g$  vanishes on the diagonal of  $C \times C$ ;
- (iv)  $g$  satisfies the triangle inequality property.

Then, for all  $\varepsilon > 0$ , and all  $x_0 \in C$ , there exists  $\hat{x} \in C$  such that

$$\begin{aligned} g(x_0, \hat{x}) + \varepsilon d(x_0, \hat{x}) &\leq 0, \text{ and} \\ f(\hat{x}, x) + \varepsilon d(x, \hat{x}) &> 0, \text{ for every } x \in C \setminus \{\hat{x}\}. \end{aligned}$$

The conclusion of Corollary 3.1 is similar to the one in [13, Theorem 2.4], where instead of supposing that  $g$  satisfies the triangle inequality property, the authors considered  $g$  defined as in (3).

#### 4 New existence results of equilibria and quasi-equilibria

We begin this section with recalling the definitions of equilibrium and Minty equilibrium problems, respectively.

##### 4.1 Equilibrium problems

Let  $C$  be a nonempty subset of a topological space  $X$  and  $f : C \times C \rightarrow \mathbb{R}$ , be a given bifunction. We denote by  $EP(f, C)$  the *solution set* of the *equilibrium problem*, introduced by Blum and Oettli in [9],

$$\text{Find } x \in C \text{ such that } f(x, y) \geq 0, \text{ for all } y \in C. \quad (9)$$

In a similar way,  $MEP(f, C)$  denotes the *solution set* of the so-called *Minty equilibrium problem*

$$\text{Find } x \in C \text{ such that } f(y, x) \leq 0, \text{ for all } y \in C. \quad (10)$$

Clearly, they satisfy

$$EP(f, C) = MEP(\hat{f}, C) \text{ and } EP(\hat{f}, C) = MEP(f, C).$$

Provided that  $f(x, y) \geq h(y) - h(x)$ , which implies that  $\hat{f}$  is cyclically monotone, we may observe that

$$MEP(f, C) \subset \underset{C}{\operatorname{argmin}} h \subset EP(f, C). \quad (11)$$

Moreover, if  $f$  is pseudo-monotone, then the above inclusions are actually equalities.

*Remark 4.1* If the bifunction  $f$  vanishes on the diagonal of  $C \times C$ , then

$$x \in \text{EP}(f, C) \Leftrightarrow x \in \underset{C}{\operatorname{argmin}} f(x, \cdot) \text{ and } x \in \text{MEP}(f, C) \Leftrightarrow x \in \underset{C}{\operatorname{argmin}} \hat{f}(x, \cdot).$$

Moreover,

$$\text{EP}(f, C) \subset \bigcup_{x \in C} \underset{C}{\operatorname{argmin}} f(x, \cdot) \text{ and } \text{MEP}(f, C) \subset \bigcup_{y \in C} \underset{C}{\operatorname{argmin}} \hat{f}(y, \cdot).$$

**Theorem 4.1** Let  $C$  be a compact and nonempty subset of a topological space  $X$ , and  $f : C \times C \rightarrow \mathbb{R}$  be a bifunction. If there exists a tlc function  $h : C \rightarrow \mathbb{R}$  with

$$f(x, y) \geq h(y) - h(x), \text{ for all } x, y \in C;$$

then, the set  $\text{EP}(f, C)$  is nonempty.

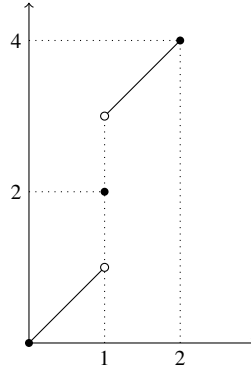
**Proof.** From Theorem 2.1, the set  $\underset{C}{\operatorname{argmin}} h$  is nonempty. The result follows from (11).  $\square$

The previous result was given in [13, Theorem 3.4], but instead of considering the transfer lower continuity of  $h$ , the authors assumed lower semicontinuity.

*Example 4.1* Let  $h : [0, 2] \rightarrow \mathbb{R}$  be defined as

$$h(x) := \begin{cases} x, & 0 \leq x < 1 \\ 2, & x = 1 \\ x + 2, & 1 < x \leq 2 \end{cases}$$

Its graph is shown in Figure 3.



**Fig. 3** graph of  $h$

Clearly,  $h$  is not lsc. However, it is tlc. Indeed, it follows from  $S(x) = [0, x]$ , for all  $x \in [0, 2]$ . Since  $[0, 2]$  is a compact set, for any bifunction  $f : [0, 2] \times [0, 2] \rightarrow \mathbb{R}$ , which satisfies

$$f(x, y) \geq h(y) - h(x),$$

the set  $\text{EP}(f, [0, 2])$  is nonempty, due to Theorem 4.1. It is important to notice that we cannot apply [13, Theorem 3.4].

As a direct consequence we have the following corollary, which is a generalization of [23, Theorem 3.1].

**Corollary 4.1** *Let  $C$  be a compact and nonempty subset of a topological space  $X$ , and  $f : C \times C \rightarrow \mathbb{R}$  be a bifunction. If there exists a tlc function  $h : C \rightarrow \mathbb{R}$  with*

$$f(x, y) \leq h(y) - h(x), \text{ for all } x, y \in C;$$

*then, the set  $\text{MEP}(f, C)$  is nonempty.*

#### 4.2 Quasi-equilibrium problems

Given a nonempty subset  $C$  of a complete metric space  $(X, d)$ , a bifunction  $f : C \times C \rightarrow \mathbb{R}$  and a set-valued mapping  $K : C \rightrightarrows C$ , we denote by  $\text{QEP}(f, K)$  the solution set of the so-called *quasi-equilibrium problem*:

$$\text{Find } x \in C \text{ such that } x \in K(x) \wedge f(x, y) \geq 0, \text{ for all } y \in K(x). \quad (12)$$

**Lemma 4.1** *Let  $C$  be a nonempty subset of a complete metric space  $(X, d)$ ,  $K : C \rightrightarrows C$  be a set-valued mapping and  $h : C \rightarrow \mathbb{R}$  be a function bounded from below. We assume that for every  $\varepsilon > 0$ , and for any  $x_0 \in C$  the following implication holds:*

$$\bar{h}(x) + \varepsilon d(x, x_0) \leq h(x_0) \implies \exists y \in K(x), h(y) + \varepsilon d(x, y) \leq \bar{h}(x).$$

*Then, there exists  $\hat{x} \in \text{Fix}(K)$ <sup>3</sup> satisfying*

$$\begin{aligned} &\bar{h}(\hat{x}) + \varepsilon d(x_0, \hat{x}) \leq h(x_0), \text{ and} \\ &h(x) + \varepsilon d(x, \hat{x}) > \bar{h}(\hat{x}), \text{ for all } x \in C \setminus \{\hat{x}\}. \end{aligned}$$

**Proof.** By Theorem 3.3, for each  $\varepsilon > 0$  and  $x_0$ , there exists  $\hat{x} \in C$  such that

$$\begin{aligned} &\bar{h}(\hat{x}) + \varepsilon d(x_0, \hat{x}) \leq h(x_0), \text{ and} \\ &h(x) + \varepsilon d(x, \hat{x}) > \bar{h}(\hat{x}), \text{ for all } x \in C \setminus \{\hat{x}\}. \end{aligned}$$

It is enough to show that  $\hat{x}$  is a fixed point of  $K$ . According to the first inequality, there exists  $y \in K(\hat{x})$  such that

$$h(y) + \varepsilon d(\hat{x}, y) \leq \bar{h}(\hat{x}).$$

Supposing  $y \neq \hat{x}$  leads to a contradiction with the second inequality, and therefore, we derive that  $y = \hat{x} \in K(\hat{x})$ .  $\square$

Associated to  $f$  and  $K$ , we notice that if there exists a function  $h : C \rightarrow \mathbb{R}$  such that  $f(x, y) \geq h(y) - h(x)$  (in other words,  $\hat{f}$  is cyclically monotone), then

$$\underset{C}{\operatorname{argmin}} h \cap \text{Fix}(K) \subset \text{QEP}(f, K). \quad (13)$$

The following result is an extension of [13, Theorem 3.11] to the transfer lower continuous case.

<sup>3</sup>  $\text{Fix}(K)$  denotes the set of fixed points of  $K$ .

**Theorem 4.2** *Let  $C$  be a nonempty closed subset of a complete metric space  $(X, d)$ , let  $K : C \rightrightarrows C$  be a set-valued mapping, and let  $f : C \times C \rightarrow \mathbb{R}$  be a bifunction. Assume that the following conditions hold.*

- (i)  $\text{Fix}(K)$  is compact and nonempty;
- (ii) there exists a tlc function bounded below  $h : C \rightarrow \mathbb{R}$  such that

$$f(x, y) \geq h(y) - h(x), \text{ for all } x, y \in C.$$

Suppose that for each  $\varepsilon > 0$  and each  $x_0 \in X$  the following implication holds:

$$\bar{h}(x) + \varepsilon d(x, x_0) \leq h(x_0) \implies \exists y \in K(x), h(y) + \varepsilon d(x, y) \leq \bar{h}(x).$$

Then, the set  $\text{QEP}(f, K)$  is nonempty.

**Proof.** Fix  $x_0 \in C$ . By Lemma 4.1, for each  $n \in \mathbb{N}$ , there exists  $x_n \in \text{Fix}(K)$  such that

$$h(x) + \frac{1}{n}d(x, x_n) \geq \bar{h}(x_n), \text{ for all } x \in C.$$

Since  $\text{Fix}(K)$  is compact, without loss of generality, we can assume that  $(x_n)_{n \in \mathbb{N}}$  converges to  $\hat{x} \in \text{Fix}(K)$ . We claim that  $\hat{x} \in \text{argmin}_C h$ . Indeed, as the distance function is continuous and  $\bar{h}$  is lsc, we have

$$h(x) \geq \bar{h}(\hat{x}), \text{ for all } x \in C.$$

By Proposition 2.4,  $\hat{x} \in \text{argmin}_C h$ . The result follows from (13).  $\square$

As a direct consequence of Theorem 4.2 we derive.

**Corollary 4.2** *Let  $C$  be a nonempty closed subset of a complete metric space  $(X, d)$ ,  $K : C \rightrightarrows C$  be a set-valued mapping, and let  $h : C \rightarrow \mathbb{R}$  be a function. Assume that the following conditions hold.*

- (i)  $\text{Fix}(K)$  is compact and nonempty;
- (ii)  $h$  is a tlc function bounded from below.

Suppose that for each  $\varepsilon > 0$ , and each  $x_0 \in X$  the following implication holds:

$$\bar{h}(x) + \varepsilon d(x, x_0) \leq h(x_0) \implies \exists y \in K(x), h(y) + \varepsilon d(x, y) \leq \bar{h}(x).$$

Then, there exists  $\hat{x} \in \text{Fix}(K)$  such that

$$h(\hat{x}) \leq h(x), \text{ for all } x \in K(\hat{x}).$$

The previous result is known as the existence of solutions to a *quasi-optimization* problem. Important results about the existence of solution of this kind of problem were presented in [3, Propositions 4.2 and 4.5] and [18, Corollary 3.2] under continuity and quasi-convexity assumptions.

## 5 System of quasi-equilibrium problems

Let  $I$  be an index set. For each  $i \in I$ , we consider a complete metric space  $(X_i, d_i)$ , a nonempty closed subset  $C_i$  of  $X_i$  and a set-valued mapping  $K_i : C_i \rightrightarrows C_i$ . We define the set-valued mapping  $K : C \rightrightarrows C$  by

$$K(x) := \prod_{i \in I} K_i(x_i),$$

where  $C = \prod C_i$  and  $x = (x^i)_{i \in I}$ . By a *system of quasi-equilibrium problems* we understand the problem of finding

$$\hat{x} \in \text{Fix}(K) \text{ such that } f_i(\hat{x}, y^i) \geq 0 \text{ for all } y \in K(\hat{x}), \quad (14)$$

where the  $f_i : C \times C_i \rightarrow \mathbb{R}$  are given. It is important to see that

$$\text{Fix}(K) = \prod_{i \in I} \text{Fix}(K_i).$$

When for each  $i \in I$ ,  $K_i(x^i) = C_i$ , for all  $x^i \in C_i$ , we obtain the known *system of equilibrium problems*

The following result generalizes [13, Theorem 4.2], [1, Proposition 4.2] and [6, Proposition 2].

**Theorem 5.1** *For each  $i \in I$ , let  $C_i$  be a nonempty closed subset of a topological space  $X_i$ , and let each  $f_i : C \times C_i \rightarrow \mathbb{R}$  be a bifunction such that*

$$f_i(x, y^i) \geq h_i(y^i) - h_i(x^i), \quad \forall x, y \in C \quad (15)$$

*holds for some transfer lower continuous function  $h_i : C_i \rightarrow \mathbb{R}$  that is also bounded from below. Then, the system of equilibrium problems admits at least a solution.*

**Proof.** For each  $i \in I$ , we apply Theorem 2.1 and obtain  $\hat{x}^i \in \text{argmin}_{C_i} h_i$ . Thus, from (15),  $\hat{x} = (\hat{x}^i)$  is a solution of the system of equilibrium problems.  $\square$

*Remark 5.1* Condition (15) is equivalent to the following: for any  $x_1, x_2, \dots, x_m \in C$  it holds

$$\sum_{j=1}^m f_i(x_j, x_{j+1}^i) \geq 0 \quad (16)$$

where  $x_{m+1} = x_1$ . It follows from the same steps of the proof of [24, Proposition 5.1].

We denote by  $\text{SEP}(f_i, C_i, I)$  the solution set of (14), when  $K_i(x^i) = C_i$ , for all  $x^i \in C_i$ . If  $I$  is a finite index set, as a particular case, we define the bifunction  $f : C \times C \rightarrow \mathbb{R}$  by

$$f(x, y) := \sum_{i \in I} f_i(x, y^i). \quad (17)$$

The next result says that a system of equilibrium problems is equivalent to a particular equilibrium problem under suitable assumptions.



**Proposition 5.1** *Assume that  $I$  is a finite index set and  $f$  is defined as (17). Then  $\text{SEP}(f_i, C_i, I) \subset \text{EP}(f, C)$ . The equality holds provided that  $f_i(x, x^i) = 0$ , for all  $i \in I$ .*

**Proof.** Let  $x \in \text{SEP}(f_i, C_i, I)$  and  $y \in C$ . For each  $i \in I$ , we have

$$f_i(x, y^i) \geq 0.$$

Thus  $f(x, y) \geq 0$ . Hence  $x \in \text{EP}(f, C)$ .

Conversely, let  $x \in \text{EP}(f, C)$ ,  $i \in I$  and  $y^i \in C_i$ . We set

$$y = (y^i, x^{-i}) \in C.$$

So,

$$0 \leq f(x, y) = \sum_{j \in I} f_j(x, y^j) = f_i(x, y^i).$$

Therefore,  $x \in \text{SEP}(f_i, C_i, I)$ . □

Given a finite index set  $I$  and for each  $i \in I$ , we consider a subset  $C_i$  of a topological space and a function  $f_i : C \times C_i \rightarrow \mathbb{R}$ . We say that the family of functions  $\{f_i\}_{i \in I}$  have the *transfer lower continuity property* if there exists a tlc function  $h : C \rightarrow \mathbb{R}$  such that the bifunction  $f$  defined in (17) satisfies

$$f(x, y) \geq h(y) - h(x).$$

*Remark 5.2* Two remarks are needed.

- (i) The bifunction  $\hat{f}$ , where  $f$  is defined in (17) associated to a family of functions with the transfer lower continuity property, is cyclically monotone.
- (ii) If for each  $i \in I$  the function  $f_i$  satisfies condition (16), then the function  $\hat{f}$  is cyclically monotone, where  $f$  is defined as (17). Moreover, if for each  $i \in I$ , the function  $f_i$  is usc in its second argument; then the family of function  $\{f_i\}_{i \in I}$  has the transfer lower continuity property. This is due to [13, Theorem 2.16].

Below we present a result similar to Theorem 5.1.

**Theorem 5.2** *Assume that  $I$  is a finite index set and the family of functions  $\{f_i\}_{i \in I}$  has the transfer lower continuity property. If  $f_i(x, x^i) = 0$ , for all  $x = (x^i, x^{-i}) \in C$  and all  $i \in I$ , then the set  $\text{SEP}(f_i, C_i, I)$  admits at least an element.*

**Proof.** It follows from Theorem 4.1 and Proposition 5.1. □

The following example shows us that the previous result is not a consequence of Theorem 5.1.

*Example 5.1* Consider  $C_1 = C_2 = C$  both compact and nonempty subsets of  $\mathbb{R}$  and the functions  $f_1, f_2 : C^2 \times C \rightarrow \mathbb{R}$  defined as

$$f_1(x^1, x^2, y^1) := y^1 - x^2 \quad \text{and} \quad f_2(x^1, x^2, y^2) := y^2 - x^1.$$

It is clear that neither  $f_1$  nor  $f_2$  satisfy condition (15). However, the bifunction  $f$  defined in (17) is given by

$$f_1(x^1, x^2, y^1) + f_2(x^1, x^2, y^2) = \sum_{i=1}^2 y^i - \sum_{i=1}^2 x^i.$$

Therefore, the existence of solution of the system of equilibrium problems follows from Theorem 5.2, but not from Theorem 5.1.

Finally, using the same steps in the proof of Theorem 13 we can guarantee the existence of solution of the system of quasi-equilibrium problems in complete metric spaces.

**Theorem 5.3** *For each  $i \in I$ , let  $C_i$  be a nonempty closed subset of a complete metric space  $(X_i, d_i)$ ,  $K_i : C_i \rightrightarrows C_i$  be a set-valued mapping, and let  $f_i : C \times C_i \rightarrow \mathbb{R}$  be a function such that (15) holds for some transfer lower continuous function  $h_i : C_i \rightarrow \mathbb{R}$  that is also bounded below. If  $\text{Fix}(K)$  is compact and, for any  $\varepsilon > 0$ , any  $x_0 \in C$ , and any  $i \in I$  such that the following implication holds*

$$\bar{h}_i(x^i) + \varepsilon d_i(x^i, x_0^i) \leq h_i(x_0^i) \implies \exists y^i \in K_i(x^i), h_i(y^i) + \varepsilon d_i(x^i, y^i) \leq \bar{h}_i(x^i);$$

*then there exists a solution of (14).*

**Proof.** For each  $i \in I$ ,  $x_0 \in C$ , and  $n \in \mathbb{N}$ , we apply Lemma 4.1 and we obtain the existence of a fixed point of  $K_i$ , say  $\hat{x}_n^i$ , such that

$$h_i(x^i) + \frac{1}{n} d_i(x^i, \hat{x}_n^i) \geq \bar{h}_i(\hat{x}_n^i), \text{ for all } x^i \in C_i.$$

Since  $\text{Fix}(K_i)$  is compact, without loss of generality, assume that  $(\hat{x}_n^i)_{n \in \mathbb{N}}$  converges to  $\hat{x}^i \in \text{Fix}(K_i)$ . By continuity of  $d_i$  and lower semicontinuity of  $\bar{h}_i$ , we have

$$h_i(x^i) \geq \bar{h}_i(\hat{x}^i), \text{ for all } x^i \in C_i.$$

We deduce  $\hat{x}^i \in \text{argmin}_{C_i} h_i$ , due to Proposition 2.4. The result follows from considering  $\hat{x} = (\hat{x}^i) \in \text{Fix}(K)$  and (15).  $\square$

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