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$\mathbb{Z}_4\mathbb{Z}_4[u]$ -ADDITIVE CYCLIC AND CONSTACYCLIC CODES

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ABSTRACT. We study mixed alphabet cyclic and constacyclic codes over the two alphabets \mathbb{Z}_4 , the ring of integers modulo 4, and its quadratic extension $\mathbb{Z}_4[u] = \mathbb{Z}_4 + u\mathbb{Z}_4, u^2 = 0$. Their generator polynomials and minimal spanning sets are obtained. Further, under new Gray maps, we find cyclic, quasi-cyclic codes over \mathbb{Z}_4 as the Gray images of both λ -constacyclic and skew λ -constacyclic codes over $\mathbb{Z}_4[u]$. Moreover, it is proved that the Gray images of $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic and skew $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic codes are generalized quasi-cyclic codes over \mathbb{Z}_4 . Finally, several new quaternary linear codes are obtained from these cyclic and constacyclic codes.

1. INTRODUCTION

Cyclic codes form one of the most important classes of codes, either over finite fields [25], or over finite rings [29], for their properties of encoding, decoding, and ease of generation allowed by their strong algebraic structure. They are defined as linear codes invariant under the cyclic shift of coordinates. The condition of linearity has been relaxed recently and replaced by additivity. Also, the definition has been enlarged to accommodate codes over mixed alphabets. Note that every linear code is additive, but not conversely. In 1973, Delsarte [18] introduced the additive codes in terms of association schemes. Later, Bierbrauer [13] presented these codes as a generalized class of cyclic codes defined as subgroups rather than subspaces. In 2010, Borges et al. [14] studied $\mathbb{Z}_2\mathbb{Z}_4$ -linear codes that generalize both binary and quaternary codes. They have obtained their dual codes as well as their generator matrices. In continuation, Fernandez-Cordoba et al. [21] determined the rank and kernel of $\mathbb{Z}_2\mathbb{Z}_4$ -linear codes. It is worth noting that these codes have a successful engineering application in the area of data hiding, particularly, in steganography [26]. Later, these studies were extended over $\mathbb{Z}_2\mathbb{Z}_{2^s}$ -additive codes and obtained some good binary codes under Gray images in [9]. Subsequently, the natural extensions of above codes are $\mathbb{Z}_p\mathbb{Z}_{p^s}$ -additive codes, $\mathbb{Z}_{p^r}\mathbb{Z}_{p^s}$ -additive codes

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and well studied therefore in [10, 30, 32, 33]. On the other hand, to the progress of cyclic codes on mixed alphabets, in 2014, Abualrub et al. [3] defined $\mathbb{Z}_2\mathbb{Z}_4$ -additive cyclic codes as $\mathbb{Z}_4[x]$ -submodule of $\mathbb{Z}_2[x]/\langle x^r - 1 \rangle \times \mathbb{Z}_4[x]/\langle x^s - 1 \rangle$ and derived the unique set of generators, and minimal spanning set for these codes where s is an odd integer. Also, Borges et al. [15] found generator polynomials and duals for $\mathbb{Z}_2\mathbb{Z}_4$ -additive cyclic codes. After introducing the new mixed alphabets $\mathbb{Z}_2\mathbb{Z}_2[u]$ -additive codes, where $u^2 = 0$ in [6], Aydogdu et al. [7] were also investigated constacyclic codes over mixed alphabets by defining them as $\mathbb{Z}_2[u][x]$ -submodules of $\mathbb{Z}_2[x]/\langle x^\alpha - 1 \rangle \times \mathbb{Z}_2[u][x]/\langle x^\beta - (1+u) \rangle$. They obtained some optimal binary linear codes as the Gray images of $\mathbb{Z}_2\mathbb{Z}_2[u]$ -cyclic codes. Meanwhile, [31, 23] studied the algebraic properties of $\mathbb{Z}_2\mathbb{Z}_2[u]$ -additive cyclic and constacyclic codes with the unit $1+u$, respectively. Therefore, in continuation of these studies, the expected generalization should be $\mathbb{Z}_{2^r}\mathbb{Z}_{2^s}[u]$ -additive cyclic and constacyclic codes, where $u^2 = 0$. If $r = s = 2$, this turns out to be our present study. Obviously, the present work on mixed alphabets cyclic and constacyclic codes is a bridge towards the study of $\mathbb{Z}_{2^r}\mathbb{Z}_{2^s}[u]$ -additive codes which is a stronger and still open problem. Based on the above survey, one would also be agreed that mixed alphabets cyclic, constacyclic codes over different and new alphabets are interesting and promising classes for further study due to their rich algebraic properties and capable to produce several best known codes.

For the sake of strong motivation discussed above, here we introduce the mixed alphabets $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive cyclic and constacyclic codes which lead to generalizing the codes over \mathbb{Z}_4 as well as $\mathbb{Z}_4 + u\mathbb{Z}_4$, $u^2 = 0$. To the best of our knowledge, mixed alphabets codes over $\mathbb{Z}_4\mathbb{Z}_4[u]$ are not considered earlier and also constacyclic codes over mixed alphabets setting are fresh after [7, 23]. We would like to mention that the primary objective of the article is first, to characterize completely these codes in terms of their generator polynomials and minimal spanning sets, etc. Then utilizing these structure and new Gray maps, we are seeking to obtain some new \mathbb{Z}_4 -codes. To do so, for the odd positive integers α, β , we find the complete set of generator polynomials and minimal spanning set for the cyclic codes of length (α, β) . Then we define some new Gray maps and find well-known classes like cyclic, quasi-cyclic, and generalized quasi-cyclic codes over \mathbb{Z}_4 . As a computational result, we construct \mathbb{Z}_4 -codes and some of them improve on the best known [4]. Further, we extend the study to skew constacyclic codes in the sense of [17]. While skew cyclic codes have been studied extensively since that reference (see publications 1,4,5,6,8,9,10 in [35]), it is only the fourth time that they occur in a mixed alphabet setting [11, 12, 28]. The present article shows some algebraic richness of skew codes over the mentioned mixed alphabets. For that, we define mixed skew codes under a non-trivial automorphism θ on $\mathbb{Z}_4 + u\mathbb{Z}_4$. Also, we characterize skew $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic code as a left $\mathbb{Z}_4[u][x; \theta]$ -submodule of $\mathbb{Z}_4[x]/\langle x^\alpha - 1 \rangle \times \mathbb{Z}_4[u][x; \theta]/\langle x^\beta - \lambda \rangle$, where λ is a unit in $\mathbb{Z}_4 + u\mathbb{Z}_4$. Among others, we connect these skew codes under Gray maps to generalized quasi-cyclic codes over \mathbb{Z}_4 .

The manuscript is organized as follows. In Section 2, we discuss some basic definitions and results. Section 3 gives the structure of $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive cyclic codes while Section 4 consider $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic codes. In Section 5, we define some Gray maps and obtain the Gray images of $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic codes. Section 6 and 7 contain skew $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic codes and their Gray images, respectively. In Section 8, we obtain several new linear codes over \mathbb{Z}_4

from these class of codes. The last section is the conclusion of this paper, and also contains long term open problems.

2. PRELIMINARY

Throughout the article, $\mathbb{Z}_4[u]$ denotes the finite commutative ring $\mathbb{Z}_4 + u\mathbb{Z}_4$, where $u^2 = 0$ of size 16 and characteristic 4. Recall from [34], among 6 ideals of $\mathbb{Z}_4[u]$, the unique maximal ideal is $\langle 2, u \rangle$. It is a local Frobenius non-chain ring and quotient ring $\mathbb{Z}_4[u]/\langle 2, u \rangle \cong \mathbb{Z}_2$. Also, $\{a + ub \mid a = 1, \text{ or } 3 \text{ and } b \in \mathbb{Z}_4\}$ is the set of units while the ideal $\langle 2, u \rangle$ is the set of non-units in $\mathbb{Z}_4[u]$. A non-empty subset \mathcal{C} of $\mathbb{Z}_4[u]^n$ is called a linear code of length n if it is a $\mathbb{Z}_4[u]$ -submodule of $\mathbb{Z}_4[u]^n$ and each member of \mathcal{C} is known as codeword.

Definition 2.1. Let \mathcal{C} be a linear code of length $n = st$ over \mathbb{Z}_4 . We define the quasi-cyclic shift operator $\pi_s : \mathbb{Z}_4^n \rightarrow \mathbb{Z}_4^n$ by

$$(1) \quad \pi_s(e_0 \mid e_1 \mid \cdots \mid e_{s-1}) = (\sigma(e_0) \mid \sigma(e_1) \mid \cdots \mid \sigma(e_{s-1})),$$

where $e_i \in \mathbb{Z}_4^t$ for all $i = 0, 1, \dots, (s-1)$ and σ is the cyclic shift operator. Then \mathcal{C} is said to be a quasi-cyclic code of index s if \mathcal{C} is invariant under the map π_s , i.e. $\pi_s(\mathcal{C}) = \mathcal{C}$.

The set $\mathbb{Z}_4\mathbb{Z}_4[u] = \{(a, b) \mid a \in \mathbb{Z}_4, b \in \mathbb{Z}_4[u]\}$ is a commutative group under componentwise addition. For positive integers α, β , we define $\mathbb{Z}_4^\alpha \times \mathbb{Z}_4[u]^\beta = \{(a, b) \mid a = (a_0, a_1, \dots, a_{\alpha-1}) \in \mathbb{Z}_4^\alpha, b = (b_0, b_1, \dots, b_{\beta-1}) \in \mathbb{Z}_4[u]^\beta\}$. Then $\mathbb{Z}_4^\alpha \times \mathbb{Z}_4[u]^\beta$ is a commutative group under the componentwise addition. Now, we define a map $\rho : \mathbb{Z}_4[u] \rightarrow \mathbb{Z}_4$ by $\rho(a + ub) = a$ and a multiplication

$$* : \mathbb{Z}_4[u] \times \mathbb{Z}_4\mathbb{Z}_4[u] \rightarrow \mathbb{Z}_4\mathbb{Z}_4[u]$$

by

$$c * (a, b) = (\rho(c)a, cb), \text{ for } a \in \mathbb{Z}_4, b, c \in \mathbb{Z}_4[u].$$

The extension of the multiplication $*$ to the elements of $\mathbb{Z}_4^\alpha \times \mathbb{Z}_4[u]^\beta$ by the elements of $\mathbb{Z}_4[u]$ defined by

$$c * (a, b) = (\rho(c)a_0, \rho(c)a_1, \dots, \rho(c)a_{\alpha-1}, cb_0, cb_1, \dots, cb_{\beta-1})$$

where $a = (a_0, a_1, \dots, a_{\alpha-1}) \in \mathbb{Z}_4^\alpha, b = (b_0, b_1, \dots, b_{\beta-1}) \in \mathbb{Z}_4[u]^\beta$.

Lemma 2.2. *The set $\mathbb{Z}_4^\alpha \times \mathbb{Z}_4[u]^\beta$ is a $\mathbb{Z}_4[u]$ -module under the multiplication $*$ defined above.*

Proof. Since $\mathbb{Z}_4[u]$ is a commutative ring with unity 1, so there is no distinction between left and right $\mathbb{Z}_4[u]$ -modules. Clearly, $\mathbb{Z}_4^\alpha \times \mathbb{Z}_4[u]^\beta$ is an additive commutative group. Now, to complete the proof we need to check

- (1) $r * [(a, b) + (x, y)] = r * (a, b) + r * (x, y)$,
- (2) $(r + s) * (a, b) = r * (a, b) + s * (a, b)$,
- (3) $(rs) * (a, b) = r * [s * (a, b)]$, and
- (4) $1 * (a, b) = (a, b)$, for all $r, s \in \mathbb{Z}_4[u]$ and $(a, b), (x, y) \in \mathbb{Z}_4^\alpha \times \mathbb{Z}_4[u]^\beta$.

Here, explicitly we prove (1) and the other three points follow similarly. In fact, let $(a, b) = (a_0, a_1, \dots, a_{\alpha-1}, b_0, b_1, \dots, b_{\beta-1}), (x, y) = (x_0, x_1, \dots, x_{\alpha-1}, y_0, y_1, \dots, y_{\beta-1})$

$\in \mathbb{Z}_4^\alpha \times \mathbb{Z}_4[u]^\beta$, and $r = r_1 + ur_2 \in \mathbb{Z}_4[u]$. Then $\rho(r) = r_1$, and

$$\begin{aligned} r * [(a, b) + (x, y)] &= r * (a_0 + x_0, a_1 + x_1, \dots, a_{\alpha-1} + x_{\alpha-1}, \\ &\quad b_0 + y_0, b_1 + y_1, \dots, b_{\beta-1} + y_{\beta-1}) \\ &= (r_1 a_0 + r_1 x_0, r_1 a_1 + r_1 x_1, \dots, r_1 a_{\alpha-1} + r_1 x_{\alpha-1}, \\ &\quad r b_0 + r y_0, r b_1 + r y_1, \dots, r b_{\beta-1} + r y_{\beta-1}) \\ &= (r_1 a_0, r_1 a_1, \dots, r_1 a_{\alpha-1}, r b_0, r b_1, \dots, r b_{\beta-1}) + \\ &\quad (r_1 x_0, r_1 x_1, \dots, r_1 x_{\alpha-1}, r y_0, r y_1, \dots, r y_{\beta-1}) \\ &= r * (a, b) + r * (x, y). \end{aligned}$$

Therefore, $\mathbb{Z}_4^\alpha \times \mathbb{Z}_4[u]^\beta$ is a $\mathbb{Z}_4[u]$ -module with respect to scalar multiplication $*$. \square

Definition 2.3. Any non-empty subset \mathcal{C} of $\mathbb{Z}_4^\alpha \times \mathbb{Z}_4[u]^\beta$ is said to be a $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive code of length (α, β) if \mathcal{C} is a $\mathbb{Z}_4[u]$ -submodule of $\mathbb{Z}_4^\alpha \times \mathbb{Z}_4[u]^\beta$.

Definition 2.4. Let \mathcal{C} be a $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive code of length (α, β) . Then it is said to be a $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive cyclic code if for any $z = (c_0, c_1, \dots, c_{\alpha-1}, r_0, r_1, \dots, r_{\beta-1}) \in \mathcal{C}$, we have $\sigma_{\alpha, \beta}(z) = (c_{\alpha-1}, c_0, \dots, c_{\alpha-2}, r_{\beta-1}, r_0, \dots, r_{\beta-2}) \in \mathcal{C}$.

An extension of the ring homomorphism ρ is

$$\rho : \mathbb{Z}_4[u][x] \longrightarrow \mathbb{Z}_4[x]$$

defined by

$$\rho\left(\sum_{i=0}^n r_i x^i\right) = \sum_{i=0}^n \rho(r_i) x^i.$$

Let $R_{\alpha, \beta} = \mathbb{Z}_4[x]/\langle x^\alpha - 1 \rangle \times \mathbb{Z}_4[u][x]/\langle x^\beta - 1 \rangle$. Then $R_{\alpha, \beta}$ is a $\mathbb{Z}_4[u][x]$ -module under the multiplication defined by

$$s(x) * (c(x), r(x)) = (\rho(s(x))c(x), s(x)r(x)),$$

where $s(x), r(x) \in \mathbb{Z}_4[u][x]$ and $c(x) \in \mathbb{Z}_4[x]$. Let \mathcal{C} be a $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive code of length (α, β) . Then for any codeword $z = (c, r) = (c_0, c_1, \dots, c_{\alpha-1}, r_0, r_1, \dots, r_{\beta-1}) \in \mathcal{C}$, we identify a polynomial $z(x) = (c(x), r(x)) \in R_{\alpha, \beta}$ under the correspondence $z = (c, r) \mapsto (c(x), r(x)) = z(x)$ where $c(x) = c_0 + c_1x + \dots + c_{\alpha-1}x^{\alpha-1} \in \mathbb{Z}_4[x]/\langle x^\alpha - 1 \rangle$, $r(x) = r_0 + r_1x + \dots + r_{\beta-1}x^{\beta-1} \in \mathbb{Z}_4[u][x]/\langle x^\beta - 1 \rangle$.

Lemma 2.5. Let \mathcal{C} be a $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive code of length (α, β) . Then \mathcal{C} is a $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive cyclic code if and only if \mathcal{C} is a $\mathbb{Z}_4[u][x]$ -submodule of $R_{\alpha, \beta}$.

Proof. Let \mathcal{C} be a $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive cyclic code of length (α, β) . Let $s(x) \in \mathbb{Z}_4[u][x]$ and $z(x) = (c(x), r(x)) \in \mathcal{C}$. Then $x * (c(x), r(x)) = (xc(x), xr(x))$ where $xc(x)$ and $xr(x)$ are cyclic shifts of $c(x)$ in $\mathbb{Z}_4[x]/\langle x^\alpha - 1 \rangle$ and $r(x)$ in $\mathbb{Z}_4[u][x]/\langle x^\beta - 1 \rangle$, respectively. Also, $x * (c(x), r(x))$ represents the image of $z(x)$ under the operator $\sigma_{\alpha, \beta}$, therefore, $x * (c(x), r(x)) \in \mathcal{C}$. Similarly, for any positive integer $i \geq 2$, we can show that $x^i * (c(x), r(x)) \in \mathcal{C}$. As \mathcal{C} is a $\mathbb{Z}_4[u]$ -submodule of $\mathbb{Z}_4^\alpha \times \mathbb{Z}_4[u]^\beta$, $s(x) * (c(x), r(x)) \in \mathcal{C}$, which proves that \mathcal{C} is a $\mathbb{Z}_4[u][x]$ -submodule of $R_{\alpha, \beta}$. Conversely, let \mathcal{C} be a $\mathbb{Z}_4[u][x]$ -submodule of $R_{\alpha, \beta}$. For any codeword $z(x) = (c(x), r(x)) \in \mathcal{C}$, $x * (c(x), r(x))$ represents the image of $z(x)$ under the operator $\sigma_{\alpha, \beta}$ and $x * (c(x), r(x)) \in \mathcal{C}$. Thus, \mathcal{C} is a $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive cyclic code of length (α, β) . \square

3. $\mathbb{Z}_4\mathbb{Z}_4[u]$ -ADDITIVE CYCLIC CODES

The present section aims to determine the algebraic structure of additive cyclic codes by means of their generator polynomials and minimal spanning sets. To do so we use the pullback method which applied to find \mathbb{Z}_4 cyclic codes in [1]. Let \mathcal{S} be a cyclic code of odd length β over $\mathbb{Z}_4[u]$. Then the ring homomorphism ρ acts on the polynomial ring $\mathbb{Z}_4[u][x]/\langle x^\beta - 1 \rangle$ by $\rho(\sum_{i=0}^{\beta-1} c_i x^i) = \sum_{i=0}^{\beta-1} \rho(c_i) x^i \in \mathbb{Z}_4[x]/\langle x^\beta - 1 \rangle$. Now, we consider the restriction of ρ on the ideal \mathcal{S} . Clearly, $\rho(\mathcal{S})$ is an ideal of $\mathbb{Z}_4[x]/\langle x^\beta - 1 \rangle$, therefore, by Theorem 1 of [1], $\rho(\mathcal{S}) = \langle g_1 + 2a_1 \rangle$ and $\ker(\rho|_{\mathcal{S}}) = \langle u g_2 + 2u a_2 \rangle$ where a_i, g_i are polynomials such that $a_i \mid g_i \mid (x^\beta - 1) \pmod{4}$ for $i = 1, 2$. Hence, $\mathcal{S} = \langle g_1 + 2a_1 + up, u(g_2 + 2a_2) \rangle$ for some polynomial $p \in \mathbb{Z}_4[x]$.

Now, we define the projection map

$$T : \mathbb{Z}_4[x]/\langle x^\alpha - 1 \rangle \times \mathbb{Z}_4[u][x]/\langle x^\beta - 1 \rangle \rightarrow \mathbb{Z}_4[u][x]/\langle x^\beta - 1 \rangle$$

by

$$T(a(x), b(x)) = b(x), \text{ where } a(x) \in \mathbb{Z}_4[x]/\langle x^\alpha - 1 \rangle, b(x) \in \mathbb{Z}_4[u][x]/\langle x^\beta - 1 \rangle.$$

Clearly T is a $\mathbb{Z}_4\mathbb{Z}_4[u]$ -module homomorphism. Let \mathcal{C} be a $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive cyclic code of length (α, β) where α, β are both odd integers. Then $\ker(T|_{\mathcal{C}}) = \{(a, 0) \mid a \in \mathbb{Z}_4[x]/\langle x^\alpha - 1 \rangle\}$. Let $D = \{a \in \mathbb{Z}_4[x]/\langle x^\alpha - 1 \rangle \mid (a, 0) \in \ker(T|_{\mathcal{C}})\}$. It is easy to check that D is an ideal of $\mathbb{Z}_4[x]/\langle x^\alpha - 1 \rangle$ and ultimately, $D = \langle g_1 + 2a_1 \rangle$ where $a_1 \mid g_1 \mid (x^\alpha - 1) \pmod{4}$. Hence, $\ker(T|_{\mathcal{C}}) = \langle (g_1 + 2a_1, 0) \rangle$ where $a_1 \mid g_1 \mid (x^\alpha - 1) \pmod{4}$. Moreover, $T(\mathcal{C})$ is an ideal of $\mathbb{Z}_4[u][x]/\langle x^\beta - 1 \rangle$, so $T(\mathcal{C}) = \langle g_2 + 2a_2 + up, ug_3 + 2ua_3 \rangle$ with $a_i \mid g_i \mid (x^\beta - 1) \pmod{4}$, for $i = 2, 3$. Thus, the $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive cyclic code \mathcal{C} is given by

$$\mathcal{C} = \langle (g_1 + 2a_1, 0), (f_1, g_2 + 2a_2 + up), (f_2, ug_3 + 2ua_3) \rangle,$$

where $a_i \mid g_i \mid (x^\beta - 1)$ for $i = 2, 3$ and $a_1 \mid g_1 \mid (x^\alpha - 1)$, $f_1, f_2 \in \mathbb{Z}_4[x]$. Therefore, based on the above discussion we have the following result.

Theorem 3.1. *Let \mathcal{C} be a $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive cyclic code of length (α, β) where α, β are both odd positive integers. Then \mathcal{C} is a $\mathbb{Z}_4[u][x]$ -submodule of $R_{\alpha, \beta}$ given by*

$$\mathcal{C} = \langle (g_1 + 2a_1, 0), (f_1, g_2 + 2a_2 + up), (f_2, ug_3 + 2ua_3) \rangle,$$

where $a_i \mid g_i \mid (x^\beta - 1)$ for $i = 2, 3$ and $a_1 \mid g_1 \mid (x^\alpha - 1)$, $f_1, f_2 \in \mathbb{Z}_4[x]$.

Remark 1. For further calculations, wherever we use Theorem 3.1, it is assumed that $g_i, a_i (i = 1, 2, 3)$ are monic polynomials.

Lemma 3.2. *For any odd positive integers α and β , let \mathcal{C} be a $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive cyclic code of length (α, β) given by*

$$\mathcal{C} = \langle (g_1 + 2a_1, 0), (f_1, g_2 + 2a_2 + up), (f_2, ug_3 + 2ua_3) \rangle,$$

where $a_i \mid g_i \mid (x^\beta - 1)$ for $i = 2, 3$ and $a_1 \mid g_1 \mid (x^\alpha - 1)$, $f_1, f_2 \in \mathbb{Z}_4[x]$. Let $h = \frac{x^\beta - 1}{a_2}, m_1 = \gcd\{hp, x^\beta - 1\}, m_2 = \frac{x^\beta - 1}{m_1}$. Then $(g_1 + 2a_1) \mid m_2 h f_1$ and $(g_1 + 2a_1) \mid f_2 \frac{x^\beta - 1}{a_3}$.

Proof. Here, $T(\frac{x^\beta - 1}{a_3}(f_2, ug_3 + 2ua_3)) = T(\frac{x^\beta - 1}{a_3} f_2, 0) = 0$. Therefore, $(\frac{x^\beta - 1}{a_3} f_2, 0) \in \ker(T)$ and this implies $(g_1 + 2a_1) \mid \frac{x^\beta - 1}{a_3} f_2$. Further, $T(m_2 h(f_1, g_2 + 2a_2 + up)) = T(m_2 h f_1, m_2 h up)$. Since $m_1 \mid hp$, so $hp = m_1 m_3$ for some m_3 and $hpm_2 =$

$m_1 m_2 m_3 = 0$. Hence, $T(m_2 h(f_1, g_2 + 2a_2 + up)) = T(m_2 h f_1, 0) = 0$. Thus, $(m_2 h f_1, 0) \in \ker(T)$ and this implies $(g_1 + 2a_1) \mid m_2 h f_1$. \square

Theorem 3.3. For any odd positive integers α and β , let \mathcal{C} be a $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive cyclic code of length (α, β) given by

$$\mathcal{C} = \langle (g_1 + 2a_1, 0), (f_1, g_2 + 2a_2 + up), (f_2, ug_3 + 2ua_3) \rangle,$$

where $a_i \mid g_i \mid (x^\beta - 1)$ for $i = 2, 3$ and $a_1 \mid g_1 \mid (x^\alpha - 1)$, $f_1, f_2 \in \mathbb{Z}_4[x]$. Let $h = \frac{x^\beta - 1}{a_2}$, $m_1 = \gcd\{hp, x^\beta - 1\}$, $m_2 = \frac{x^\beta - 1}{m_1}$ and

$$\begin{aligned} S_1 &= \bigcup_{i=0}^{\alpha - \deg(a_1) - 1} \{x^i * (g_1 + 2a_1, 0)\}; \\ S_2 &= \bigcup_{i=0}^{\beta - \deg(a_2) - 1} \{x^i * (f_1, g_2 + 2a_2 + up)\}; \\ S_3 &= \bigcup_{i=0}^{\beta - \deg(m_1) - 1} \{x^i * (h f_1, uhp)\}; \\ S_4 &= \bigcup_{i=0}^{\beta - \deg(a_3) - 1} \{x^i * (f_2, ug_3 + 2ua_3)\}. \end{aligned}$$

Then $S = S_1 \cup S_2 \cup S_3 \cup S_4$ is a minimal generating set for the code \mathcal{C} and $|\mathcal{C}| = 4^{\alpha + 4\beta - \deg(a_1) - 2\deg(a_2) - \deg(a_3) - \deg(m_1)}$.

Proof. Let $c \in \mathcal{C}$ be a codeword. Then

$$\begin{aligned} c &= c_1 * (g_1 + 2a_1, 0) + c_2 * (f_1, g_2 + 2a_2 + up) + c_3 * (f_2, ug_3 + 2ua_3) \\ (2) \quad &= (\rho(c_1)(g_1 + 2a_1), 0) + c_2 * (f_1, g_2 + 2a_2 + up) + c_3 * (f_2, ug_3 + 2ua_3) \end{aligned}$$

where $c_i \in \mathbb{Z}_4[u][x]$ for $i = 1, 2, 3$. If $\deg(\rho(c_1)) \leq (\alpha - \deg(a_1) - 1)$, then $(\rho(c_1)(g_1 + 2a_1), 0) \in \text{span}(S_1)$. Otherwise, by division algorithm,

$$\rho(c_1) = \frac{x^\alpha - 1}{a_1} q + r,$$

where $\deg(r) \leq (\alpha - \deg(a_1) - 1)$. Therefore,

$$\begin{aligned} (\rho(c_1)(g_1 + 2a_1), 0) &= \left(\left(\frac{x^\alpha - 1}{a_1} q + r \right) g_1 + 2a_1, 0 \right) \\ &= r(g_1 + 2a_1, 0). \end{aligned}$$

Hence, $(\rho(c_1)(g_1 + 2a_1), 0) \in \text{span}(S_1)$. To prove $c_2 * (f_1, g_2 + 2a_2 + up) \in \text{span}(S_1 \cup S_2 \cup S_3)$, we divide c_2 by h and can write $c_2 = q_2 h + r_2$ where $\deg(r_2) \leq (\beta - \deg(a_2) - 1)$. Therefore, $c_2 * (f_1, g_2 + 2a_2 + up) = (q_2 h + r_2) * (f_1, g_2 + 2a_2 + up) = q_2(h f_1, uhp) + r_2(f_1, g_2 + 2a_2 + up)$. Clearly, $r_2(f_1, g_2 + 2a_2 + up) \in \text{span}(S_2)$. It remains to show $q_2(h f_1, uhp) \in \text{span}(S_1 \cup S_2 \cup S_3)$. Again, by division algorithm, we have $q_2 = q_3 m_2 + r_3$, where $\deg(r_3) \leq (\beta - \deg(m_1) - 1)$. Also, $m_1 \mid hp$, so $hp = m_1 m_3$ for some m_3 and this implies $h p m_2 = m_1 m_2 m_3 = 0$. Hence,

$$\begin{aligned} q_2(h f_1, uhp) &= (q_3 m_2 + r_3)(h f_1, uhp) \\ &= q_3(m_2 h f_1, uhp m_2) + r_3(h f_1, uhp) \\ &= q_3(m_2 h f_1, 0) + r_3(h f_1, uhp). \end{aligned}$$

By Lemma 3.2, $(g_1 + 2a_1) \mid m_2 h f_1$, so $q_3(m_2 h f_2, 0) \in \text{span}(S_1)$. Also, $r_3(h f_1, u h p) \in \text{span}(S_3)$, therefore, $c_2 * (f_1, g_2 + 2a_2 + u p) \in \text{span}(S_1 \cup S_2 \cup S_3)$. Now, to show $c_3 * (f_2, u g_3 + 2u a_3) \in \text{span}(S_1 \cup S_4)$, again applying division algorithm, we have

$$c_3 = \frac{x^\beta - 1}{a_3} q_4 + r_4,$$

where $\deg(r_4) \leq (\beta - \deg(a_3) - 1)$. Therefore,

$$\begin{aligned} c_3 * (f_2, u g_3 + 2u a_3) &= \left(\frac{x^\beta - 1}{a_3} q_4 + r_4 \right) * (f_2, u g_3 + 2u a_3) \\ &= q_4 \left(\frac{x^\beta - 1}{a_3} f_2, 0 \right) + r_4 * (f_2, u g_3 + 2u a_3). \end{aligned}$$

By Lemma 3.2, we have $(g_1 + 2a_1) \mid \frac{x^\beta - 1}{a_3} f_2$, so $q_4(\frac{x^\beta - 1}{a_3} f_2, 0) \in \text{span}(S_1)$. Also, $r_4 * (f_2, u g_3 + 2u a_3) \in \text{span}(S_4)$. Therefore, $c_3 * (f_2, u g_3 + 2u a_3) \in \text{span}(S_1 \cup S_4)$. Hence, from equation 2, we conclude that $c \in \text{span}(S)$, i.e., S generates the code \mathcal{C} . By the construction of S , we can easily check that none of the element of S is a linear combination of the remaining elements of S . Thus, S is a minimal spanning set for the code \mathcal{C} . \square

Corollary 3.4. *Let \mathcal{C} be a $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive cyclic code of length (α, β) given by Theorem 3.3. Then any codeword $c \in \mathcal{C}$ is uniquely expressed as*

$$\begin{aligned} c &= s_1 * (g_1 + 2a_1, 0) + s_2 * (f_1, g_2 + 2a_2 + u p) + s_3 * (h f_1, u h p) \\ &\quad + s_4 * (f_2, u g_3 + 2u a_3), \end{aligned}$$

where $s_1, s_3, s_4 \in \mathbb{Z}_4[x]$, $s_2 \in \mathbb{Z}_4[u][x]$ such that $\deg(s_1) \leq (\alpha - \deg(a_1) - 1)$, $\deg(s_2) \leq (\beta - \deg(a_2) - 1)$, $\deg(s_3) \leq (\beta - \deg(m_1) - 1)$, $\deg(s_4) \leq (\beta - \deg(a_4) - 1)$.

Proof. By Theorem 3.3, we know that S is a minimal spanning set of \mathcal{C} , so any codeword $c \in \mathcal{C}$ is uniquely expressed as a linear combination of elements of S . \square

4. $\mathbb{Z}_4\mathbb{Z}_4[u]$ -ADDITIVE CONSTACYCLIC CODES

In this section, by using the structure of additive cyclic codes obtained in the previous section we discuss the algebraic properties of additive constacyclic codes.

Definition 4.1. Let λ be a unit in $\mathbb{Z}_4[u]$. A linear code \mathcal{S} of length β over $\mathbb{Z}_4[u]$ is said to be a λ -constacyclic code if $\tau_\lambda(c) = (\lambda c_{\beta-1}, c_0, \dots, c_{\beta-2}) \in \mathcal{C}$ whenever $c = (c_0, c_1, \dots, c_{\beta-1}) \in \mathcal{S}$. The operator τ_λ is known as λ -constacyclic shift.

Let $\lambda = 3 + 2u$. Then λ is a unit in $\mathbb{Z}_4[u]$. Define a map

$$\gamma : \mathbb{Z}_4[u][x] / \langle x^\beta - 1 \rangle \rightarrow \mathbb{Z}_4[u][x] / \langle x^\beta - \lambda \rangle$$

by

$$\gamma(a(x)) = a(\lambda x), \text{ where } a(x) \in \mathbb{Z}_4[u][x] / \langle x^\beta - 1 \rangle.$$

Then it is easy to check that γ is a ring isomorphism if β is an odd integer. Also, by above ring isomorphism γ , we can characterize the λ -constacyclic codes over $\mathbb{Z}_4[u]$ with odd length β as follows.

Theorem 4.2. *Let \mathcal{S} be a λ -constacyclic code of odd length β over $\mathbb{Z}_4[u]$. Then generators of \mathcal{S} are given by*

$$\mathcal{S} = \langle g_1(y) + 2a_1(y) + up(y), ug_2(y) + 2ua_2(y) \rangle,$$

where $y = \lambda x$ and $a_i \mid g_i \mid (x^\beta - 1) \pmod{4}$.

Definition 4.3. The $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive code \mathcal{C} of length (α, β) is said to be a $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic code if

$$\tau_{\lambda, \beta}(z) = (c_{\alpha-1}, c_0, \dots, c_{\alpha-2}, \lambda r_{\beta-1}, r_0, \dots, r_{\beta-2}) \in \mathcal{C}$$

whenever $z = (c_0, c_1, \dots, c_{\alpha-1}, r_0, r_1, \dots, r_{\beta-1}) \in \mathcal{C}$.

Lemma 4.4. *A $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive code \mathcal{C} of length (α, β) is a $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic code if and only if it is a $\mathbb{Z}_4[u][x]$ -submodule of $R_{\alpha, \beta, \lambda} = \mathbb{Z}_4[x]/\langle x^\alpha - 1 \rangle \times \mathbb{Z}_4[u][x]/\langle x^\beta - \lambda \rangle$.*

Proof. Let \mathcal{C} be a $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic code. Similar to Lemma 2.5, for $(a(x), b(x)) \in \mathcal{C}$, we have $x^i * (a(x), b(x)) \in \mathcal{C}$ for $i \geq 1$. Therefore, $r(x) * (a(x), b(x)) \in \mathcal{C}$ for any $r(x) \in \mathbb{Z}_4[u][x]$. Hence, \mathcal{C} is a $\mathbb{Z}_4[u][x]$ -submodule of $R_{\alpha, \beta, \lambda}$. Conversely, if \mathcal{C} is a $\mathbb{Z}_4[u][x]$ -submodule of $R_{\alpha, \beta, \lambda}$, then for any codeword $z = (a, b) \in \mathcal{C}$, we have $x * (a(x), b(x)) = \tau_{\lambda, \beta}(z) \in \mathcal{C}$, where $(a(x), b(x))$ is the polynomial representation of the codeword $z = (a, b) \in \mathcal{C}$. Therefore, \mathcal{C} is a $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic code. \square

Here, we define a projection map

$$P : \mathbb{Z}_4[x]/\langle x^\alpha - 1 \rangle \times \mathbb{Z}_4[u][x]/\langle x^\beta - \lambda \rangle \rightarrow \mathbb{Z}_4[u][x]/\langle x^\beta - \lambda \rangle$$

by

$$P(a(x), b(x)) = b(x), \text{ where } a(x) \in \mathbb{Z}_4[x]/\langle x^\alpha - 1 \rangle, b(x) \in \mathbb{Z}_4[u][x]/\langle x^\beta - \lambda \rangle.$$

Then P is a $\mathbb{Z}_4[u][x]$ -module homomorphism. Let \mathcal{C} be a $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic code of length (α, β) . Then, by similar arguments as given in case of cyclic codes, $\ker(P|_{\mathcal{C}}) = \langle (g_1 + 2a_1, 0) \rangle$ with $a_1 \mid g_1 \mid (x^\alpha - 1)$ and also $P(\mathcal{C})$ is an ideal of $\mathbb{Z}_4[u][x]/\langle x^\beta - \lambda \rangle$. Therefore, by Theorem 4.2, we have

$$P(\mathcal{C}) = \langle g_1(y) + 2a_1(y) + up(y), ug_2(y) + 2ua_2(y) \rangle,$$

with $y = \lambda x$ and $a_i \mid g_i \mid (x^\beta - 1) \pmod{4}$. Thus, by above discussion, we characterize the $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic code of length (α, β) as follows.

Theorem 4.5. *Let \mathcal{C} be a $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic code of length (α, β) where α, β both are odd integers. Then \mathcal{C} is a $\mathbb{Z}_4[u][x]$ -submodule of $R_{\alpha, \beta, \lambda}$ given by*

$$\begin{aligned} \mathcal{C} = & \langle (g_1(x) + 2a_1(x), 0), (f_1(x), g_2(\lambda x) + 2a_2(\lambda x) + up(\lambda x)), \\ & (f_2(x), ug_3(\lambda x) + 2ua_3(\lambda x)) \rangle, \end{aligned}$$

where $a_1(x) \mid g_1(x) \mid (x^\alpha - 1)$, $a_i(x) \mid g_i(x) \mid (x^\beta - 1) \pmod{4}$ for $i = 2, 3$ and $f_1(x), f_2(x), p(x) \in \mathbb{Z}_4[x]$.

5. GRAY MAPS AND \mathbb{Z}_4 -IMAGES OF CONSTACYCLIC CODES

In this section, we define some Gray maps and discuss their \mathbb{Z}_4 -images. Towards this, we first define a map

$$\phi_1 : \mathbb{Z}_4[u] \longrightarrow \mathbb{Z}_4^2$$

by

$$(3) \quad \phi_1(a + ub) = (2a, 2b), \quad a, b \in \mathbb{Z}_4.$$

The map ϕ_1 is linear and its extension

$$\phi_1 : \mathbb{Z}_4[u]^\beta \longrightarrow \mathbb{Z}_4^{2\beta}$$

is defined by

$$\phi_1(r_0, r_1, \dots, r_{\beta-1}) = (2a_0, 2a_1, \dots, 2a_{\beta-1}, 2b_0, 2b_1, \dots, 2b_{\beta-1}),$$

where $r_i = a_i + ub_i \in \mathbb{Z}_4[u]$ for $i = 0, 1, \dots, \beta - 1$. We define the Gray weight $w_G(x) = w_L(\phi_1(x))$, for $x \in \mathbb{Z}_4[u]^\beta$, where w_L is the Lee weight over \mathbb{Z}_4 and hence, the Gray distance is $d_G(x, y) = w_G(x - y)$ for $x, y \in \mathbb{Z}_4[u]^\beta$. In this regards, the map ϕ_1 is a linear isometric map from $\mathbb{Z}_4[u]^\beta$ (Gray distance) to $\mathbb{Z}_4^{2\beta}$ (Lee distance).

Lemma 5.1. *Let ϕ_1 be the Gray map defined in equation (3), τ_λ , the λ -constacyclic shift and π_2 , the quasi-cyclic shift defined in equation (1). Then $\phi_1\tau_\lambda = \pi_2\phi_1$.*

Proof. Let $r_i = a_i + ub_i \in \mathbb{Z}_4[u]$ for $i = 0, 1, \dots, \beta - 1$ and $r = (r_0, r_1, \dots, r_{\beta-1}) \in \mathbb{Z}_4[u]^\beta$. Now,

$$\begin{aligned} \phi_1\tau_\lambda(r) &= \phi_1(\lambda r_{\beta-1}, r_0, \dots, r_{\beta-2}) \\ &= (2a_{\beta-1}, 2a_0, \dots, 2a_{\beta-2}, 2b_0, 2b_1, \dots, 2b_{\beta-2}). \end{aligned}$$

On the other side, we have

$$\begin{aligned} \pi_2\phi_1(r) &= \pi_2(2a_0, 2a_1, \dots, 2a_{\beta-1}, 2b_0, 2b_1, \dots, 2b_{\beta-1}) \\ &= (2a_{\beta-1}, 2a_0, \dots, 2a_{\beta-2}, 2b_0, 2b_1, \dots, 2b_{\beta-2}). \end{aligned}$$

Hence, $\phi_1\tau_\lambda = \pi_2\phi_1$. □

Theorem 5.2. *Let \mathcal{S} be a λ -constacyclic code of length β over $\mathbb{Z}_4[u]$. Then $\phi_1(\mathcal{S})$ is a quasi-cyclic code of length 2β and index 2 over \mathbb{Z}_4 .*

Proof. Since \mathcal{S} is a λ -constacyclic code, so $\tau_\lambda(\mathcal{S}) = \mathcal{S}$. By Lemma 5.1, we have $\phi_1\tau_\lambda(\mathcal{S}) = \phi_1(\mathcal{S}) = \pi_2(\phi_1(\mathcal{S}))$. Therefore, $\phi_1(\mathcal{S})$ is a quasi-cyclic code of length 2β and index 2 over \mathbb{Z}_4 . □

We define another map

$$\phi_2 : \mathbb{Z}_4[u] \longrightarrow \mathbb{Z}_4$$

by

$$(4) \quad \phi_2(a + ub) = (2a + b, 3b), \quad a, b \in \mathbb{Z}_4.$$

Lemma 5.3. *Let ϕ_2 be the Gray map defined in equation (4), τ_λ be the λ -constacyclic and σ be the cyclic shift operator. Then $\phi_2\tau_\lambda = \sigma\phi_2$.*

Proof. Let $r_i = a_i + ub_i \in \mathbb{Z}_4[u]$ for $i = 0, 1, \dots, \beta - 1$ and $r = (r_0, r_1, \dots, r_{\beta-1}) \in \mathbb{Z}_4[u]^\beta$. Now,

$$\begin{aligned}\phi_2\tau_\lambda(r) &= \phi_2(\lambda r_{\beta-1}, r_0, \dots, r_{\beta-2}) \\ &= (3b_{\beta-1}, 2a_0 + b_0, \dots, 2a_{\beta-2} + b_{\beta-2}, 2a_{\beta-1} + b_{\beta-1}, 3b_0, \dots, 3b_{\beta-2}).\end{aligned}$$

On the other side,

$$\begin{aligned}\sigma\phi_2(r) &= \sigma(2a_0 + b_0, 2a_1 + b_1, \dots, 2a_{\beta-1} + b_{\beta-1}, 3b_0, 3b_1, \dots, 3b_{\beta-1}) \\ &= (3b_{\beta-1}, 2a_0 + b_0, \dots, 2a_{\beta-2} + b_{\beta-2}, 2a_{\beta-1} + b_{\beta-1}, 3b_0, \dots, 3b_{\beta-2}).\end{aligned}$$

Hence, $\phi_2\tau_\lambda = \sigma\phi_2$. \square

Theorem 5.4. *Let \mathcal{S} be a λ -constacyclic code of length β over $\mathbb{Z}_4[u]$. Then $\phi_2(\mathcal{S})$ is a cyclic code of length 2β over \mathbb{Z}_4 .*

Proof. Since \mathcal{S} is a λ -constacyclic code, so $\tau_\lambda(\mathcal{S}) = \mathcal{S}$. Also, by Lemma 5.3, we have $\phi_2\tau_\lambda(\mathcal{S}) = \phi_2(\mathcal{S}) = \sigma(\phi_2(\mathcal{S}))$. This shows that $\phi_2(\mathcal{S})$ is a cyclic code of length 2β over \mathbb{Z}_4 . \square

Definition 5.5. (Definition 1, [20]) Let m_1, m_2, \dots, m_l be positive integers and set $R_i = \frac{\mathbb{Z}_4[x]}{\langle x^{m_i} - 1 \rangle}$, for $i = 1, 2, \dots, l$. Then any $\mathbb{Z}_4[x]$ -submodule of $\mathbb{Z}_4[x]$ -module $R' := R_1 \times R_2 \times \dots \times R_l$ is said to be generalized quasi-cyclic (GQC) code of block length (m_1, m_2, \dots, m_l) . Note that a GQC code of block length (m_1, m_2, \dots, m_l) with $m_1 = m_2 = \dots = m_l$ is a QC code of length lm . Further, if $l = 1$ then it is a cyclic code of length m .

Now, we define a map Φ_1 with the help of the map ϕ_1 as follows:

$$\Phi_1 : \mathbb{Z}_4 \times \mathbb{Z}_4[u] \longrightarrow \mathbb{Z}_4^3$$

by

$$(5) \quad \Phi_1(c, a + ub) = (c, \phi_1(a + ub)) = (c, 2a, 2b), \quad a, b, c \in \mathbb{Z}_4.$$

The map Φ_1 can also be extended as

$$\Phi_1 : \mathbb{Z}_4^\alpha \times \mathbb{Z}_4[u]^\beta \longrightarrow \mathbb{Z}_4^{\alpha+2\beta}$$

defined by

$$\begin{aligned}\Phi_1(c_0, c_1, \dots, c_{\alpha-1}, r_0, r_1, \dots, r_{\beta-1}) \\ = (c_0, c_1, \dots, c_{\alpha-1}, 2a_0, 2a_1, \dots, 2a_{\beta-1}, 2b_0, 2b_1, \dots, 2b_{\beta-1})\end{aligned}$$

where $r_i = a_i + ub_i \in \mathbb{Z}_4[u]$ and $a_i, b_i, c_i \in \mathbb{Z}_4$ for $i = 0, 1, \dots, \beta - 1$.

Theorem 5.6. *Let \mathcal{C} be a $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic code of length (α, β) . Then $\Phi_1(\mathcal{C})$ is a generalized QC code of block length (α, β, β) over \mathbb{Z}_4 .*

Proof. Let $(c_0, c_1, \dots, c_{\alpha-1}, 2a_0, 2a_1, \dots, 2a_{\beta-1}, 2b_0, 2b_1, \dots, 2b_{\beta-1}) \in \Phi_1(\mathcal{C})$. Then there exists $(c_0, c_1, \dots, c_{\alpha-1}, r_0, r_1, \dots, r_{\beta-1}) \in \mathcal{C}$ such that

$$\begin{aligned}\Phi_1(c_0, c_1, \dots, c_{\alpha-1}, r_0, r_1, \dots, r_{\beta-1}) \\ = (c_0, c_1, \dots, c_{\alpha-1}, 2a_0, 2a_1, \dots, 2a_{\beta-1}, 2b_0, 2b_1, \dots, 2b_{\beta-1})\end{aligned}$$

where $r_i = a_i + ub_i \in \mathbb{Z}_4[u]$ and $a_i, b_i, c_i \in \mathbb{Z}_4$ for $i = 0, 1, \dots, \beta - 1$. Since \mathcal{C} is a $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic code, $(c_{\alpha-1}, c_0, \dots, c_{\alpha-2}, \lambda r_{\beta-1}, r_0, \dots, r_{\beta-2}) \in \mathcal{C}$. Therefore,

$$\begin{aligned} & \Phi_1(c_{\alpha-1}, c_0, \dots, c_{\alpha-2}, \lambda r_{\beta-1}, r_0, \dots, r_{\beta-2}) \\ &= (c_{\alpha-1}, c_0, \dots, c_{\alpha-2}, 2a_{\beta-1}, 2a_0, \dots, 2a_{\beta-2}, 2b_{\beta-1}, 2b_0, \dots, 2b_{\beta-2}) \in \Phi(\mathcal{C}). \end{aligned}$$

This shows that $\Phi_1(\mathcal{C})$ is a generalized QC code of block length (α, β, β) over \mathbb{Z}_4 . \square

Corollary 5.7. *If \mathcal{C} is a $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic code of length (α, α) , then $\Phi(\mathcal{C})$ is a quasi-cyclic code of length 3α and index 3 over \mathbb{Z}_4 .*

Proof. It follows from the Theorem 5.6 with $\alpha = \beta$. \square

Now, we define another map Φ_2 as follows:

$$\Phi_2 : \mathbb{Z}_4^\alpha \times \mathbb{Z}_4[u]^\beta \longrightarrow \mathbb{Z}_4^{\alpha+2\beta}$$

by

$$\begin{aligned} & \Phi_2(c_0, c_1, \dots, c_{\alpha-1}, r_0, r_1, \dots, r_{\beta-1}) \\ &= (c_0, c_1, \dots, c_{\alpha-1}, 2a_0 + b_0, 2a_1 + b_1, \dots, 2a_{\beta-1} + b_{\beta-1}, 3b_0, 3b_1, \dots, 3b_{\beta-1}) \end{aligned}$$

where $r_i = a_i + ub_i \in \mathbb{Z}_4[u]$ and $a_i, b_i, c_i \in \mathbb{Z}_4$ for $i = 0, 1, \dots, \beta - 1$.

Theorem 5.8. *Let \mathcal{C} be a $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic code of length (α, β) . Then $\Phi_2(\mathcal{C})$ is a generalized QC code of block length $(\alpha, 2\beta)$ over \mathbb{Z}_4 .*

Proof. Let $(c_0, c_1, \dots, c_{\alpha-1}, 2a_0 + b_0, 2a_1 + b_1, \dots, 2a_{\beta-1} + b_{\beta-1}, 3b_0, 3b_1, \dots, 3b_{\beta-1}) \in \Phi_2(\mathcal{C})$. Then there exist $(c_0, c_1, \dots, c_{\alpha-1}, r_0, r_1, \dots, r_{\beta-1}) \in \mathcal{C}$ such that

$$\begin{aligned} & \Phi_2(c_0, c_1, \dots, c_{\alpha-1}, r_0, r_1, \dots, r_{\beta-1}) \\ &= (c_0, c_1, \dots, c_{\alpha-1}, 2a_0 + b_0, 2a_1 + b_1, \dots, 2a_{\beta-1} + b_{\beta-1}, 3b_0, 3b_1, \dots, 3b_{\beta-1}) \end{aligned}$$

where $r_i = a_i + ub_i \in \mathbb{Z}_4[u]$ and $a_i, b_i, c_i \in \mathbb{Z}_4$ for $i = 0, 1, \dots, \beta - 1$. Since \mathcal{C} is a $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic code, so $(c_{\alpha-1}, c_0, \dots, c_{\alpha-2}, \lambda r_{\beta-1}, r_0, \dots, r_{\beta-2}) \in \mathcal{C}$. Therefore,

$$\begin{aligned} & \Phi_2(c_{\alpha-1}, c_0, \dots, c_{\alpha-2}, \lambda r_{\beta-1}, r_0, \dots, r_{\beta-2}) \\ &= (c_{\alpha-1}, c_0, \dots, c_{\alpha-2}, 3b_{\beta-1}, 2a_0 + b_0, \dots, 2a_{\beta-1} + b_{\beta-1}, 3b_0, \dots, 3b_{\beta-2}) \in \Phi_2(\mathcal{C}). \end{aligned}$$

This shows that $\Phi_2(\mathcal{C})$ is a generalized QC code of block length $(\alpha, 2\beta)$ over \mathbb{Z}_4 . \square

Corollary 5.9. *If \mathcal{C} is a $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic code of length $(2\delta, \delta)$, then $\Phi_2(\mathcal{C})$ is a quasi-cyclic code of length 4δ and index 2 over \mathbb{Z}_4 .*

Proof. It follows from the proof of the Theorem 5.8 with $\alpha = 2\delta = 2\beta$. \square

6. SKEW CONSTACYCLIC CODES

In this section, we discuss skew constacyclic codes over $\mathbb{Z}_4[u]$ and $\mathbb{Z}_4\mathbb{Z}_4[u]$, respectively. It is worth mentioning that skew constacyclic codes over $\mathbb{Z}_4[u]$ with a different unit $\lambda = 1 + 2u$ are studied in [27]. Here, we use the unit $\lambda = 3 + 2u$ which allows to define suitable Gray maps to characterize these codes as quasi-cyclic or cyclic codes over \mathbb{Z}_4 . Now, we define an automorphism θ on $\mathbb{Z}_4[u]$ as $\theta(a + ub) = a + 3ub$ for all $a, b \in \mathbb{Z}_4$. Here, the order of the automorphism is 2 and the set of polynomials

$$\mathbb{Z}_4[u][x; \theta] = \{a_0 + a_1x + \dots + a_{\beta-1}x^{\beta-1} \mid a_i \in \mathbb{Z}_4[u], \text{ for all } i\}$$

is a non-commutative ring under usual addition of polynomials and multiplication of polynomials define with respect to the condition $(ax^s)(bx^t) = a\theta^s(b)x^{s+t}$. Under the left multiplication defined by

$$z(x)(a(x) + \langle x^\beta - \lambda \rangle) = z(x)a(x) + \langle x^\beta - \lambda \rangle, \text{ where } z(x), a(x) \in \mathbb{Z}_4[u][x; \theta],$$

the set $R_{\lambda, \theta, \beta} = \mathbb{Z}_4[u][x; \theta] / \langle x^\beta - \lambda \rangle$ is a left $\mathbb{Z}_4[u][x; \theta]$ -module. We identify each codeword $z = (z_0, z_1, \dots, z_{\beta-1}) \in \mathbb{Z}_4[u]^\beta$ by a polynomial $z(x) = z_0 + z_1x + \dots + z_{\beta-1}x^{\beta-1} \in R_{\lambda, \theta, \beta}$.

Definition 6.1. A linear code \mathcal{S} of length β over $\mathbb{Z}_4[u]$ is said to be a skew λ -constacyclic code if for any $c = (c_0, c_1, \dots, c_{\beta-1}) \in \mathcal{S}$, we have $\tau_{\lambda, \theta}(c) = (\theta(\lambda c_{\beta-1}), \theta(c_0), \dots, \theta(c_{\beta-2})) \in \mathcal{S}$. The operator $\tau_{\lambda, \theta}$ is known as the skew λ -constacyclic shift.

Lemma 6.2. Let \mathcal{S} be a linear code of length β over $\mathbb{Z}_4[u]$. Then \mathcal{S} is a skew λ -constacyclic code if and only if \mathcal{S} is a left $\mathbb{Z}_4[u][x; \theta]$ -submodule of $R_{\lambda, \theta, \beta}$.

Proof. Let \mathcal{S} be a skew λ -constacyclic code of length β . Let $a(x) \in \mathcal{S}$, where $a(x) = a_0 + a_1x + \dots + a_{\beta-1}x^{\beta-1}$. Now,

$$\begin{aligned} xa(x) &= \theta(a_0)x + \theta(a_1)x^2 + \dots + \theta(a_{\beta-2})x^{\beta-1} + \theta(a_{\beta-1})x^\beta \\ &= \lambda\theta(a_{\beta-1}) + \theta(a_0)x + \theta(a_1)x^2 + \dots + \theta(a_{\beta-2})x^{\beta-1} \\ &= \tau_{\lambda, \theta}(a) \in \mathcal{S}. \end{aligned}$$

Therefore, for all $i \geq 2$, we have $x^i a(x) \in \mathcal{S}$. Hence, for any $r(x) \in \mathbb{Z}_4[x; \theta]$, we have $r(x)a(x) \in \mathcal{S}$. Therefore, \mathcal{S} is a left $\mathbb{Z}_4[u][x; \theta]$ -submodule of $R_{\lambda, \theta, \beta}$.

On the other hand, if \mathcal{S} is a left $\mathbb{Z}_4[u][x; \theta]$ -submodule of $R_{\lambda, \theta, \beta}$, then for any $a(x) \in \mathcal{S}$ we must have $\tau_{\lambda, \theta}(a) = xa(x) \in \mathcal{S}$. Hence, \mathcal{S} is a skew λ -constacyclic code of length β over $\mathbb{Z}_4[u]$. □

To consider an algebraic structure on $R_{\lambda, \theta, \alpha, \beta} = \mathbb{Z}_4[x] / \langle x^\alpha - 1 \rangle \times \mathbb{Z}_4[u][x; \theta] / \langle x^\beta - \lambda \rangle$, we define the left multiplication by $z(x)(a(x), b(x)) = (\rho(z(x))a(x), z(x)b(x))$, where $z(x) \in \mathbb{Z}_4[u][x; \theta]$, $a(x) \in \mathbb{Z}_4[x] / \langle x^\alpha - 1 \rangle$ and $b(x) \in \mathbb{Z}_4[u][x; \theta] / \langle x^\beta - \lambda \rangle$. Under this multiplication, $R_{\lambda, \theta, \alpha, \beta}$ is a left $\mathbb{Z}_4[u][x; \theta]$ -module. We identify each codeword $z = (c_0, c_1, \dots, c_{\alpha-1}, r_0, r_1, \dots, r_{\beta-1})$ in $\mathbb{Z}_4^\alpha \times \mathbb{Z}_4[u]^\beta$ by a polynomial $z(x) = (c(x), r(x)) \in R_{\lambda, \theta, \alpha, \beta}$ where $c(x) = c_0 + c_1x + \dots + c_{\alpha-1}x^{\alpha-1}$ and $r(x) = r_0 + r_1x + \dots + r_{\beta-1}x^{\beta-1}$.

Definition 6.3. Let \mathcal{C} be a $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive code of length (α, β) . Then \mathcal{C} is said to be skew $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic code if for any

$$z = (c_0, c_1, \dots, c_{\alpha-1}, r_0, r_1, \dots, r_{\beta-1}) \in \mathcal{C},$$

we have

$$\tau_{\lambda, \theta, \beta}(z) = (c_0, c_1, \dots, c_{\alpha-1}, \theta(\lambda r_{\beta-1}), \theta(r_0), \dots, \theta(r_{\beta-2})) \in \mathcal{C}.$$

Lemma 6.4. Let \mathcal{C} be a $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive code of length (α, β) . Then \mathcal{C} is skew $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic code if and only if \mathcal{C} is a left $\mathbb{Z}_4[u][x; \theta]$ -submodule of $R_{\lambda, \theta, \alpha, \beta}$.

Proof. Let \mathcal{C} be a skew $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic code of length (α, β) and $t(x) \in \mathbb{Z}_4[u][x; \theta]$. Let $z(x) = (c(x), r(x))$ be any codeword of \mathcal{C} where $c(x) \in \mathbb{Z}_4[x] / \langle x^\alpha - 1 \rangle$ and $r(x) \in \mathbb{Z}_4[u][x; \theta] / \langle x^\beta - \lambda \rangle$. It is easy to verify that $xc(x)$ is

the cyclic shift of $c(x)$ and $xr(x)$ is the skew constacyclic shift of $r(x)$. Therefore, $xz(x) = x(c(x), r(x))$ is the image of $z(x)$ under the operation $\tau_{\lambda, \theta, \beta}$. Since \mathcal{C} is a skew $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic code, so $xz(x) \in \mathcal{C}$. Similarly, for any integer $i \geq 2$, we can show $x^i z(x) \in \mathcal{C}$. As \mathcal{C} is $\mathbb{Z}_4[u]$ -submodule of $\mathbb{Z}_4^\alpha \times \mathbb{Z}_4[u]^\beta$, we have $t(x)z(x) \in \mathcal{C}$. Hence, \mathcal{C} is a left $\mathbb{Z}_4[u][x; \theta]$ -submodule of $R_{\lambda, \theta, \alpha, \beta}$. Conversely, let \mathcal{C} be a left $\mathbb{Z}_4[u][x; \theta]$ -submodule of $R_{\lambda, \theta, \alpha, \beta}$. Let $z(x) = (c(x), r(x)) \in \mathcal{C}$ where $c(x) \in \mathbb{Z}_4[x]/\langle x^\alpha - 1 \rangle$ and $r(x) \in \mathbb{Z}_4[u][x; \theta]/\langle x^\beta - \lambda \rangle$. Since $xz(x) = x(c(x), r(x))$ is the image of $z(x)$ under the map $\tau_{\lambda, \theta, \beta}$, and $xz(x) \in \mathcal{C}$, therefore \mathcal{C} is a skew $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic code of length (α, β) . \square

Theorem 6.5. *Let \mathcal{C} be a skew $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic code of length $(\alpha, 1)$ where α is any odd positive integer. Then \mathcal{C} is a $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive cyclic code of length $(\alpha, 1)$.*

Proof. Let $z = (c_0, c_1, \dots, c_{\alpha-1}, r_0) \in \mathcal{C}$. Since the polynomial representation of z is $z(x) = (c_0 + c_1x + \dots + c_{\alpha-1}x^{\alpha-1}, r_0)$, therefore,

$$x^{\alpha+1}z(x) = (c_{\alpha-1} + c_0x + \dots + c_{\alpha-2}x^{\alpha-2}, \theta^{\alpha+1}(r_0)x^{\alpha+1}).$$

Also, α is an odd integer, so $\theta^{\alpha+1}(r_0) = r_0$. Moreover, in $\mathbb{Z}_4[u][x]/\langle x - \lambda \rangle$, we have $x = \lambda$, which implies $x^{\alpha+1} = \lambda^{\alpha+1} = 1$. Hence,

$$\begin{aligned} x^{\alpha+1}z(x) &= (c_{\alpha-1} + c_0x + \dots + c_{\alpha-2}x^{\alpha-2}, \theta^{\alpha+1}(r_0)x^{\alpha+1}) \\ &= (c_{\alpha-1} + c_0x + \dots + c_{\alpha-2}x^{\alpha-2}, r_0) \in \mathcal{C}. \end{aligned}$$

This shows that \mathcal{C} is a $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive cyclic code of length $(\alpha, 1)$. \square

Corollary 6.6. *For any odd positive integer α , every left $\mathbb{Z}_4[u][x; \theta]$ -submodule of $\mathbb{Z}_4[x]/\langle x^\alpha - 1 \rangle \times \mathbb{Z}_4[u][x; \theta]/\langle x - \lambda \rangle$ is also a $\mathbb{Z}_4[u][x]$ -submodule of $\mathbb{Z}_4[x]/\langle x^\alpha - 1 \rangle \times \mathbb{Z}_4[u][x]/\langle x - \lambda \rangle$.*

7. GRAY MAPS AND \mathbb{Z}_4 -IMAGES OF SKEW CONSTACYCLIC CODES

In the present section, like codes over commutative rings, we discuss the \mathbb{Z}_4 -images of skew constacyclic codes. These codes turn out to be quasi-cyclic or generalized quasi-cyclic codes under different Gray maps.

Lemma 7.1. *Let ϕ_1 be the Gray map define in equation (3), $\tau_{\lambda, \theta}$ be the skew λ -constacyclic shift and π_2 be the quasi-cyclic shift defined in equation (1). Then $\phi_1\tau_{\lambda, \theta} = \pi_2\phi_1$.*

Proof. Let $r_i = a_i + ub_i \in \mathbb{Z}_4[u]$ for $i = 0, 1, \dots, \beta - 1$ and $r = (r_0, r_1, \dots, r_{\beta-1})$. Now $\lambda r_{\beta-1} = (3 + 2u)(a_{\beta-1} + ub_{\beta-1}) = 3a_{\beta-1} + u(2a_{\beta-1} + 3b_{\beta-1})$ and $\theta(\lambda r_{\beta-1}) = 3a_{\beta-1} + u(2a_{\beta-1} + b_{\beta-1})$. Therefore,

$$\begin{aligned} \phi_1\tau_{\lambda, \theta}(r) &= \phi_1(\theta(\lambda r_{\beta-1}), \theta(r_0), \dots, \theta(r_{\beta-2})) \\ &= (2a_{\beta-1}, 2a_0, \dots, 2a_{\beta-2}, 2b_{\beta-1}, 2b_0, \dots, 2b_{\beta-2}). \end{aligned}$$

Also, from the proof of the Lemma 3.2, we have

$$\pi_2\phi_2(r) = (2a_{\beta-1}, 2a_0, \dots, 2a_{\beta-2}, 2b_{\beta-1}, 2b_0, \dots, 2b_{\beta-2}).$$

Hence, $\phi_1\tau_{\lambda, \theta} = \pi_2\phi_1$. \square

Theorem 7.2. *If \mathcal{S} is a skew λ -constacyclic code of length β over $\mathbb{Z}_4[u]$, then $\phi_1(\mathcal{S})$ is a quasi-cyclic code of length 2β and index 2 over \mathbb{Z}_4 .*

Proof. Let \mathcal{S} be a skew λ -constacyclic code of length β over $\mathbb{Z}_4[u]$. Then $\tau_{\lambda,\theta}(\mathcal{S}) = \mathcal{S}$. Now, by Lemma 7.1, $\phi_1\tau_{\lambda,\theta}(\mathcal{C}) = \phi_1(\mathcal{S}) = \pi_2\phi_1(\mathcal{S})$. Therefore, $\phi_1(\mathcal{S})$ is a quasi-cyclic code of length 2β and index 2 over \mathbb{Z}_4 . \square

We define a map

$$\phi_3 : \mathbb{Z}_4[u] \longrightarrow \mathbb{Z}_4^2$$

by

$$(6) \quad \phi_3(a + ub) = (a + 2b, 3a + 2b), \quad a, b \in \mathbb{Z}_4.$$

Lemma 7.3. *Let ϕ_3 be a Gray map defined in equation (6), $\tau_{\lambda,\theta}$ be skew λ -constacyclic shift and σ be the cyclic shift operator. Then $\phi_3\tau_{\lambda,\theta} = \sigma\phi_3$.*

Proof. Let $r_i = a_i + ub_i \in \mathbb{Z}_4[u]$ for $i = 0, 1, \dots, \beta - 1$ and $r = (r_0, r_1, \dots, r_{\beta-1})$. Now, $\lambda r_{\beta-1} = (3 + 2u)(a_{\beta-1} + ub_{\beta-1}) = 3a_{\beta-1} + u(2a_{\beta-1} + 3b_{\beta-1})$ and $\theta(\lambda r_{\beta-1}) = 3a_{\beta-1} + u(2a_{\beta-1} + b_{\beta-1})$. Therefore,

$$\begin{aligned} \phi_3\tau_{\lambda,\theta}(r) &= \phi_3(\theta(\lambda r_{\beta-1}), \theta(r_0), \dots, \theta(r_{\beta-2})) \\ &= (3a_{\beta-1} + 2b_{\beta-1}, a_0 + 2b_0, \dots, a_{\beta-2} + 2b_{\beta-2}, a_{\beta-1} + 2b_{\beta-1}, \\ &\quad 3a_0 + 2b_0, 3a_1 + 2b_1, \dots, 3a_{\beta-2} + 2b_{\beta-2}). \end{aligned}$$

On the other side,

$$\begin{aligned} \sigma\phi_3(r) &= \sigma(a_0 + 2b_0, a_1 + 2b_1, \dots, a_{\beta-1} + 2b_{\beta-1}, 3a_0 + 2b_0, 3a_1 + 2b_1, \\ &\quad \dots, 3a_{\beta-1} + 2b_{\beta-1}) \\ &= (3a_{\beta-1} + 2b_{\beta-1}, a_0 + 2b_0, \dots, a_{\beta-2} + 2b_{\beta-2}, a_{\beta-1} + 2b_{\beta-1}, \\ &\quad 3a_0 + 2b_0, 3a_1 + 2b_1, \dots, 3a_{\beta-2} + 2b_{\beta-2}). \end{aligned}$$

Hence, $\phi_3\tau_{\lambda,\theta} = \sigma\phi_3$. \square

Theorem 7.4. *If \mathcal{S} is a skew λ -constacyclic code of length β over $\mathbb{Z}_4[u]$, then $\phi_3(\mathcal{S})$ is a cyclic code of length 2β over \mathbb{Z}_4 .*

Proof. If \mathcal{S} is a skew λ -constacyclic code of length β over $\mathbb{Z}_4[u]$, then $\tau_{\lambda,\theta}(\mathcal{S}) = \mathcal{S}$. Also, by Lemma 7.3, $\phi_3\tau_{\lambda,\theta}(\mathcal{S}) = \phi_3(\mathcal{S}) = \sigma\phi_3(\mathcal{S})$. Therefore, $\phi_3(\mathcal{S})$ is a cyclic code of length 2β over \mathbb{Z}_4 . \square

Theorem 7.5. *Let Φ_1 be the Gray map defined in equation (5) and \mathcal{C} be a skew $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic code of length (α, β) . Then $\Phi_1(\mathcal{C})$ is a generalized QC code of block length (α, β, β) over \mathbb{Z}_4 .*

Proof. It is similar to the proof of Theorem 5.6. \square

Corollary 7.6. *If \mathcal{C} is a skew $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic code of length (α, α) , then $\Phi_1(\mathcal{C})$ is a quasi-cyclic code of length 3α and index 3 over \mathbb{Z}_4 .*

Now, we define another map Φ_3 as

$$\Phi_3 : \mathbb{Z}_4^\alpha \times \mathbb{Z}_4[u]^\beta \longrightarrow \mathbb{Z}_4^{\alpha+2\beta}$$

by

$$(7) \quad \begin{aligned} &\Phi_3(c_0, c_1, \dots, c_{\alpha-1}, r_0, r_1, \dots, r_{\beta-1}) \\ &= (c_0, c_1, \dots, c_{\alpha-1}, a_0 + 2b_0, a_1 + 2b_1, \dots, a_{\beta-1} + 2b_{\beta-1}, 3a_0 + 2b_0, \\ &\quad 3a_1 + 2b_1, \dots, 3a_{\beta-1} + 2b_{\beta-1}) \end{aligned}$$

where $r_i = a_i + ub_i \in \mathbb{Z}_4[u]$ and $a_i, b_i, c_i \in \mathbb{Z}_4$ for $i = 0, 1, \dots, \beta - 1$.

Theorem 7.7. *Let Φ_3 be a Gray map defined in equation (7) and \mathcal{C} be a skew $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic code of length (α, β) . Then $\Phi_3(\mathcal{C})$ is a generalized QC code of block length $(\alpha, 2\beta)$ over \mathbb{Z}_4 .*

Proof. Let $(c_0, c_1, \dots, c_{\alpha-1}, a_0 + 2b_0, a_1 + 2b_1, \dots, a_{\beta-1} + 2b_{\beta-1}, 3a_0 + 2b_0, 3a_1 + 2b_1, \dots, 3a_{\beta-1} + 2b_{\beta-1}) \in \Phi_3(\mathcal{C})$. Then there exist $(c_0, c_1, \dots, c_{\alpha-1}, r_0, r_1, \dots, r_{\beta-1}) \in \mathcal{C}$ such that

$$\begin{aligned} & \Phi_3(c_0, c_1, \dots, c_{\alpha-1}, r_0, r_1, \dots, r_{\beta-1}) \\ &= (c_0, c_1, \dots, c_{\alpha-1}, a_0 + 2b_0, a_1 + 2b_1, \dots, a_{\beta-1} + 2b_{\beta-1}, 3a_0 + 2b_0, 3a_1 + 2b_1, \\ & \quad \dots, 3a_{\beta-1} + 2b_{\beta-1}) \end{aligned}$$

where $r_i = a_i + ub_i \in \mathbb{Z}_4[u]$ and $a_i, b_i, c_i \in \mathbb{Z}_4$ for $i = 0, 1, \dots, n - 1$. Since \mathcal{C} is a skew $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic code, so $(c_{\alpha-1}, c_0, \dots, c_{\alpha-2}, \theta(\lambda r_{\beta-1}), \theta(r_0), \dots, \theta(r_{\beta-2})) \in \mathcal{C}$. Therefore,

$$\begin{aligned} & \Phi_3(c_{\alpha-1}, c_0, \dots, c_{\alpha-2}, \theta(\lambda r_{\beta-1}), \theta(r_0), \dots, \theta(r_{\beta-2})) \\ &= (c_{\alpha-1}, c_0, \dots, c_{\alpha-2}, 3a_{\beta-1} + 2b_{\beta-1}, a_0 + 2b_0, a_1 + 2b_1, \dots, a_{\beta-1} + 2b_{\beta-1}, \\ & \quad 3a_0 + 2b_0, 3a_1 + 2b_1, \dots, 3a_{\beta-2} + 2b_{\beta-2}) \in \Phi_3(\mathcal{C}). \end{aligned}$$

Hence, $\Phi_3(\mathcal{C})$ is a generalized QC code of block length $(\alpha, 2\beta)$ over \mathbb{Z}_4 . □

Corollary 7.8. *If \mathcal{C} is a skew $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic code of length $(2\alpha, \alpha)$, then $\Phi_3(\mathcal{C})$ is a quasi-cyclic code of length 4α and index 2 over \mathbb{Z}_4 .*

8. EXAMPLES

Here, we present some new quaternary linear codes from above obtained cyclic and constacyclic codes. At first, we consider the $(3 + 2u)$ -constacyclic code of odd length β over $\mathbb{Z}_4[u]$, which is found in Theorem 4.2 as follows

$$\mathcal{S} = \langle g_1(y) + 2a_1(y), ug_2(y) + 2ua_2(y) \rangle,$$

where $y = \lambda x = (3 + 2u)x$ and $a_i \mid g_i \mid (x^\beta - 1) \pmod{4}$, for $i = 1, 2$. Here, we represent \mathcal{S} by $\langle h(x), k(x) \rangle$, where $h(x) = g_1(\lambda x) + 2a_1(\lambda x)$, $k(x) = ug_2(\lambda x) + 2ua_2(\lambda x)$. In Table 1, column 2 and column 3 represent the generator polynomials $h(x)$ and $k(x)$, respectively while column 4 and column 5 include the Gray images of constacyclic codes under the maps ϕ_1 and ϕ_2 , respectively. We write coefficients of generator polynomials in decreasing order, for example, we write $[3 + 2u, 3, 0, 1, 1 + 2u, 0, 1 + 2u, 1 + u]$ to represent the polynomial $(3 + 2u)x^7 + 3x^6 + x^4 + (1 + 2u)x^3 + (1 + 2u)x + 1 + u$.

Again, we recall from Theorem 3.3 that for the odd positive integers α and β the $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive cyclic code is given by

$$\mathcal{C} = \langle (g_1 + 2a_1, 0), (f_1, g_2 + 2a_2 + up), (f_2, ug_3 + 2ua_3) \rangle$$

where $a_i \mid g_i \mid (x^\beta - 1)$ for $i = 2, 3$ and $a_1 \mid g_1 \mid (x^\alpha - 1)$, $f_1, f_2 \in \mathbb{Z}_4[x]$. In Table 2, we give the generator polynomials for $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive cyclic codes and also obtain their \mathbb{Z}_4 -Gray images which give several new linear codes. In both, Table 1 and Table 2, '*' denotes the new code and '#'' denotes the best known code according to the online database [4].

TABLE 1. Gray images of $(3 + 2u)$ -constacyclic codes over $\mathbb{Z}_4[u]$

β	$h(x)$	$k(x)$	$\phi_1(\mathcal{S})$	$\phi_2(\mathcal{S})$
3	$[1, 3 + 2u, 3 + u]$	$[3u, u]$	$[6, 4^0 2^4, 2]^*$	$[6, 4^4 2^0, 2]^\#$
7	$[1, 3 + 2u, 3, 0, 1 + u]$	$[3u, 3u, 2u, u]$	$[14, 4^3 2^7, 2]$	$[14, 4^7 2^3, 4]^\#$
7	$[3 + 2u, 2, 3 + 2u, 1 + u]$	$[3u, 3u, 2u, u]$	$[14, 4^0 2^{11}, 2]$	$[14, 4^7 2^4, 2]$
9	$[3 + 2u, 0, 2, 1 + u]$	$[u, 0, 0, 3u, 0, 0, 3u]$	$[18, 4^0 2^{15}, 2]$	$[18, 4^9 2^6, 2]$
9	$[3 + 2u, 3, 0, 1, 1 + 2u, 0, 1 + 2u, 1 + u]$	$[u, 3u, 3u]$	$[18, 4^0 2^{11}, 2]^*$	$[18, 4^9 2^2, 2]$
15	$[3 + 2u, 3, 2, 1, 2 + u]$	$[u, 2u, 2u, 3u, u, u, 3u]$	$[30, 4^0 2^{26}, 2]$	$[30, 4^{15} 2^{11}, 2]^*$
15	$[1, 0, 2, 2, 1, 3 + 2u, 1, 3 + 2u, 3 + u]$	$[u, 3u, 3u]$	$[30, 4^0 2^{20}, 4]$	$[30, 4^{15} 2^5, 4]^*$
17	$[3 + 2u, 3, 2, 1, 2, 2, 1 + 2u, 2, 1 + 2u, 2, 1 + 2u, 1 + u]$	$[u, 3u, 3u, 0, 3u, 0, 3u, 3u, 3u]$	$[34, 4^0 2^{25}, 2]^*$	$[34, 4^{17} 2^8, 2]^*$
17	$[1, 0, 2, 1 + 2u, 1, 1 + 2u, 2, 0, 3 + u]$	$[u, 3u, 3u, 0, 3u, 0, 3u, 3u, 3u]$	$[34, 4^0 2^{26}, 2]^*$	$[34, 4^{17} 2^9, 2]^*$
21	$[3 + 2u, 0, 0, 3, 0, 0, 0, 0, 2, 1 + u]$	$[u, 3u, 3u, 0, 3u, 2u, 3u]$	$[42, 4^0 2^{33}, 2]^*$	$[42, 4^{21} 2^{18}, 2]^*$
21	$[3 + 2u, 3, 2, 2, 3 + 2u, 1, 2, 1, 3 + 2u, 3 + u]$	$[3u, 0, 2u, u]$	$[42, 4^0 2^{33}, 2]^*$	$[42, 4^{21} 2^{12}, 2]^*$

TABLE 2. \mathbb{Z}_4 -Gray images of $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive cyclic codes of length (α, β)

(α, β)	Generators	$\Phi_1(\mathcal{C})$	$\Phi_2(\mathcal{C})$
(3, 3)	$g_1 = g_2 = x^2 + x + 1, g_3 = x + 1, a_1 = a_2 = a_3 = 1, p = 1, f_1 = f_2 = x + 1$	$[9, 4^3 2^3, 1]^*$	$[9, 4^6 2^0, 1]$
(3, 3)	$g_1 = g_2 = g_3 = x^3 - 1, a_1 = x^2 + x + 1, a_2 = a_3 = x + 3, p = 1, f_1 = f_2 = x^2 + 2$	$[9, 4^3 2^2, 1]^*$	$[9, 4^5 2^1, 1]^*$
(3, 7)	$g_1 = x^3 - 1, g_2 = x^4 + x^3 + 3x^2 + 2x + 1, g_3 = x^7 - 1,$ $a_1 = x^2 + x + 1, a_2 = x + 3, a_3 = x^3 + 3x^2 + 2x + 3, p = 1, f_1 = f_2 = x + 1$	$[17, 4^2 2^8, 2]^*$	$[17, 4^9 2^1, 2]^*$
(7, 3)	$g_1 = x^7 - 1, g_2 = x^3 - 1, g_3 = x + 3,$ $a_1 = x^4 + 2x^3 + 3x^2 + x + 1, a_2 = x^2 + x + 1, a_3 = 1, p = 1, f_1 = f_2 = x + 3$	$[13, 4^3 2^5, 2]^*$	$[13, 4^6 2^2, 2]^*$
(7, 7)	$g_1 = x^4 + x^3 + 3x^2 + 2x + 1, g_2 = x^4 + 2x^3 + 3x^2 + x + 1, g_3 = x + 3,$ $a_1 = x^3 + 2x^2 + x + 3, a_2 = x^3 + 3x^2 + 2x + 3, a_3 = 1, p = 1, f_1 = f_2 = x + 3$	$[21, 4^6 2^8, 2]^*$	$[21, 4^{13} 2^2, 2]^*$
(9, 9)	$g_1 = x^2 + x + 1, g_2 = x^7 + 3x^6 + x^4 + 3x^3 + x + 3, g_3 = x^6 + x^3 + 1,$ $a_1 = 1, a_2 = x^6 + x^3 + 1, a_3 = 1, p = 1, f_1 = f_2 = x + 3$	$[27, 4^9 2^9, 2]^*$	$[27, 4^{16} 2^4, 2]^*$
(3, 9)	$g_1 = x^2 + x + 1, g_2 = x^7 + 3x^6 + x^4 + 3x^3 + x + 3, g_3 = x^3 + 3,$ $a_1 = 1, a_2 = a_3 = x + 3, p = 1, f_1 = f_2 = x + 3$	$[21, 4^3 2^9, 1]^*$	$[21, 4^{12} 2^0, 1]$
(9, 3)	$g_1 = x^3 + 3, g_2 = x^3 - 1, g_3 = x + 3,$ $a_1 = x^2 + x + 1, a_2 = x + 3, a_3 = 1, p = 1, f_1 = f_2 = x + 3$	$[15, 4^8 2^4, 2]^*$	$[15, 4^{11} 2^1, 2]^*$
(7, 9)	$g_1 = x^4 + x^3 + 3x^2 + 2x + 1, g_2 = x^7 + 3x^6 + x^4 + 3x^3 + x + 3, g_3 = x^2 + x + 1,$ $a_1 = x^3 + 2x^2 + x + 3, a_2 = x^6 + x^3 + 1, a_3 = 1, p = 1, f_1 = f_2 = x + 1$	$[25, 4^6 2^{10}, 2]^*$	$[25, 4^{15} 2^1, 2]^*$

9. CONCLUSION AND OPEN PROBLEMS

In the present paper, we have studied codes over the mixed alphabet $(\mathbb{Z}_4, \mathbb{Z}_4[u])$ with respect to cyclicity. Thus, we have determined the algebraic structure of both constacyclic and skew constacyclic codes, and of their Gray images, which turn out to be quasi-cyclic or generalized quasi-cyclic. Examples in modest lengths show that the resulting \mathbb{Z}_4 -codes are sometimes as good as the best known. Although algebraic properties of skew $\mathbb{Z}_4\mathbb{Z}_4[u]$ -additive constacyclic codes are presented their computational burden is still under consideration. Apart from the above, the following problems are worth to work on in the future:

1. The natural open question is to study the codes just replacing \mathbb{Z}_4 by some other ring R , and $\mathbb{Z}_4[u]$ by some quadratic extension of R in this study. It would also be rewarding to consider mixed codes over $(\mathbb{Z}_4, \mathbb{Z}_4[v])$ where v satisfies different condition than $v^2 = 0$. For instance, $v^2 = v + 1$ leads to $\mathbb{Z}_4[v] = GR(4, 2)$, the Galois ring of characteristic 4 and order 16, already studied in [17].
2. As stated in the introduction the mixed alphabets codes over the alphabets $\mathbb{Z}_2\mathbb{Z}_{2^s}[u]$ or in more general, $\mathbb{Z}_{2^r}\mathbb{Z}_{2^s}[u]$, where $u^2 = 0$ are promising and open.

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