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Robust material requirement planning with cumulative demand under uncertainty

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In this paper, we deal with the problem of tactical capacitated production planning with the demand under uncertainty modelled by closed intervals. We propose a single-item with backordering model under small uncertainty in the cumulative demand for the Master Production Scheduling (MPS) problem with different rules, namely the Lot For Lot rule and the Periodic Order Quantity rule. Then we study a general multilevel, multi-item, multi-resource model with backordering and the external demand on components for the Material Requirement Planning (MRP) problem under uncertainty in the cumulative demand. In order to choose robust production plans for the above problems that hedge against uncertainty, we adopt the well-known minmax criterion. We propose polynomial methods for evaluating the impact of uncertainty on a given production plan in terms of its cost and for computing optimal robust production plans for both problems (MPS/MRP) under the assumed interval uncertainty representation. We show in this way that the robust problems (MPS/MRP) under this uncertainty representation are not much computationally harder than their deterministic counterparts.

Keywords: supply chain management; MRP; demand uncertainty; robust optimization; linear programming

1. Introduction

Production and capacity planning under uncertainty is a very important issue in production planning and control since uncertainty arises due to the versatility of the market and the bullwhip effect that increases uncertainty through the supply chain (see Lee, Padmanabhan, and Whang 1997). This uncertainty induces supply chain risks such as backordering and obsolete inventory. Therefore taking into account uncertainty in planning process becomes a necessity in order to manage these risks.

As far as manufacturing planning and control are concerned, most companies use *Manufacturing Resource Planning* (MRPII) which is composed of three levels of production and capacity planning processes (Arnold, Chapman, and Clive 2011): the *strategic level* (Production Plan/Resource Plan), the *tactical level* (Master Production Schedule/Rough-cut Capacity Plan and Material Requirement Planning/Capacity Requirements Plan) and the *operational level* (Production Activity Control/Capacity Control). In this paper, we consider the tactical level.

In the literature on production planning (Yeung, Wong, and Ma 1998; Mula et al. 2006; Dolgui and Prodhon 2007; Peidro et al. 2009a; Aloulou, Dolgui, and Kovalyov 2013) three different sources of uncertainty such as: *demand*, *process* and *supply* are distinguished. Moreover, to cope with the unwanted effects of this uncertainty mostly the MRP nervousness, a variety of approaches can be used such as: managing safety stocks, lead times or capacity (see Koh, Saad, and Jones 2002; Hnaïen, Dolgui, and Wu 2016; Li and Disney 2017; Hammami, Frein, and Bahli 2017; Lalami, Frein, and Gayon 2017) or/and freezing the MRP in rolling horizon approaches (see Sahin, Narayanan, and Robinson 2013; Englberger, Herrmann, and Manitz 2016; Díaz-Madroñero et al. 2017). In safety stock or/and lead time approaches, the production planning process applies to each product a lot sizing rule to determine production orders. Many lot sizing rules are used such as: *Fixed Order Quantity* (FOQ), *Lot-for-Lot* (L4L), *Minimal Order Quantity* (MOQ) and *Periodic Order Quantity* (POQ), etc. (2) The so-called *Capacitated Lot Sizing Problem* (CLSP) consists in integrating the production and capacity planning processes (see Trigeiro, Thomas, and McClain 1989). The CLSP problems, which can be considered as *multi-period*, *multi-item* and *multi-level* CLSP (MLCLSP), are mostly formulated as linear or mixed integer programming models (MIP) (see, e.g. Pochet and Wolsey 2006).

In this paper, we focus on uncertainty in the demand (see also Peidro et al. 2009a for a comprehensive review) and extend previous research on MPS reported in Guillaume, Kobylański, and Zieliński (2012) by considering uncertainty on cumulative demands instead of per-period demands, and study the MRP case that was not studied in Guillaume, Kobylański, and Zieliński (2012).

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1.1 Related work

Typically, uncertainty in parameters (data) is modelled by specifying a set, denoted by \mathcal{U} , of all possible realisations of the parameters, called *scenarios*. There are two popular methods of defining the set of scenarios, namely the *discrete* and *interval uncertainty representations* (see, e.g. Kouvelis and Yu 1997). In the discrete uncertainty representation, a scenario set \mathcal{U} is defined by explicitly listing all possible realisations of demands (demand scenarios). Hence, \mathcal{U} is a finite set. In the interval uncertainty representation, a closed interval is assigned to each demand, which means that its possible value will take some value within the interval, but it is not possible to predict which one. Thus \mathcal{U} is the Cartesian product of these intervals. In order to choose a solution, one can adopt some well-known criteria used in *robust optimisation*. Among them there are the *minmax* and *minmax regret* criteria, which assume that the decision-maker, who is risk-averse, seeks a solution minimising the cost or opportunity loss under a worst scenario which may occur (see, e.g. Kouvelis and Yu 1997). The minmax criterion has been applied to CLSP problem under the discrete uncertainty representation of demands in Kouvelis and Yu (1997). Unfortunately, the problem turns out to be NP-hard even for two demand scenarios. It can be solved in pseudo-polynomial time, if the number of scenarios is a constant. When the number of scenarios is part of the input, the problem becomes strongly NP-hard and no solution algorithm is known. The situation is much better for the interval structure of demand uncertainty representation. In Guillaume, Kobylański, and Zieliński (2012), it is shown that the minmax version of lot sizing problems with backorders can be efficiently solved. Namely, the problem of computing a robust production plan with no capacity limits can be done in $O(T)$ time, where T is number of periods. For its capacitated version, an iterative algorithm based on Benders' decomposition (Geoffrion 1972) is provided, where in each iteration a worst-case scenario is computed (the *adversarial problem* is solved – see, e.g. Bienstock and Özbay 2008; Nasrabadi and Orlin 2013) by a dynamic programming-based algorithm. Interestingly, the complexity status of the latter problem is an open question. An effective variant of Benders' decomposition was also proposed in Bienstock and Özbay (2008) for computing optimal basestock policies in the minmax setting under uncertain demand. The minmax regret version of two-stage uncapacitated lot sizing problems is studied in Zhang (2011), i.e. the problem in which at the first stage a scheduling production is determined and at the second stage production and inventory decisions are made, showing that it is solvable in $O(T^6 \log T)$ time. The above-mentioned robust criteria are often regarded as conservative criteria. However the minmax regret is generally believed as less conservative (see, e.g. Kouvelis and Yu 1997; Roy 2010). Some attempts to reduce the conservativeness of minmax (regret) approach were proposed in Jabbarzadeh, Fahimnia, and Sheu (2017), Kouvelis, Kurawarwala, and Gutierrez (1992), Roy (2010). In Kouvelis, Kurawarwala, and Gutierrez (1992), the authors proposed an alternative measure called *p-robustness* and applied it to a single and multiple period layout of a manufacturing systems under demand uncertainty modelled by the discrete uncertainty representation. The approach consists in seeking a solution that simultaneously minimises the total cost under the *nominal scenario*, i.e. a scenario which is the most possible happens, and ensures that its relative deviation from optimality under each scenario does not exceed a prescribed number p that represents the desired percentage deviation in the worst case. Since the constraint on the relative deviation may restrict the set of feasible solutions and may have an effect on the solution quality (a solution can be discarded if it violates the constraint under only one scenario) an improvement called *elastic p-robustness* was introduced in Jabbarzadeh, Fahimnia, and Sheu (2017) with some applications to an integrated production-distribution planning model under supply and demand uncertainties. The elastic *p-robustness* increases flexibility of the relative deviation constraint by adding some variables reflecting the constraint violation whose values are penalised in the objective function. For yet another measure of robustness under the discrete uncertainty representation, the so-called *(b, w)-robustness*, that does not focus only on the worst case and is able to better capture the robustness, we refer the reader to Roy (2010).

A more elaborate approach for dealing with demand uncertainty consists in collecting both intervals and plausible demand values, and modelling uncertain demands by *fuzzy intervals* in the setting of *possibility theory* (see Dubois and Prade 1988). The fuzzy intervals are regarded as *possibility distributions*, describing the sets of more or less plausible values of demands. Consequently, we obtain a joint possibility distribution, induced by these fuzzy intervals, on the set of scenarios \mathcal{U} . Clearly, this approach is a generalisation of the interval uncertainty representation. In this context, the planning processes of MRPII were adapted in Fargier and Thierry (2000), Mula, Poler, and Garcia-Sabater (2007), Tavakkoli-Moghaddam et al. (2007), Guillaume, Kobylański, and Zieliński (2012) to take into account imprecision on demand quantities (MPS and MRP). Moreover, some linear programming formulations of production planning problems such as: *economic order quantity* (Kao and Hsu 2002), *multi-period planning* (Fargier and Thierry 2000; Grabot et al. 2005; Mula, Poler, and Garcia-Sabater 2007; Tavakkoli-Moghaddam et al. 2007; Lan, Liu, and Sun 2009; Mula, Peidro, and Poler 2010; Guillaume, Thierry, and Grabot 2011), and the *problem of supply chain planning* (production distribution, centralised supply chain) (Petrovic, Roy, and Petrovic 1999; Wang and Shu 2005; Peidro et al. 2009b, 2010) were extended to the case of uncertainty in the demand modelled by fuzzy intervals. There are two popular approaches for coping with fuzzy parameters. In the first approach, a *defuzzification* is first performed and then deterministic optimisation methods are used (see Peidro et al. 2009b, 2010). The second one is *possibilistic programming* in which a solution optimises a criterion based on *possibility* measures (see

Tavakkoli-Moghaddam et al. 2007; Mula, Peidro, and Poler 2010; Guillaume, Kobylański, and Zielinski 2012). Interestingly, it is shown in Guillaume, Kobylański, and Zielinski (2012) that there exists a link between the interval uncertainty with the minmax criterion and the fuzzy uncertainty representation with the possibility-based criteria. It turns out that choosing a plan in fuzzy-valued lot sizing problems with backorders is not harder than their interval-valued counterparts, since they are reduced to examining a small number of interval problems with the minmax criterion.

If some additional information about uncertain parameters is provided, then less conservative criteria can be used to choose a solution that enable to model various preferences with respect to a risk. For instance, information about importance of each scenario. In this case, such importance may be expressed by assigning some weights to scenarios in \mathcal{U} and one can apply the *Ordered Weighted Averaging* aggregation operator (OWA for short) proposed by Yager (1988) that utilises this information while computing a solution. It is worth pointing out that the OWA criterion generalises the classical criteria such as the maximum, minimum, average and Hurwicz. If probability distributions of parameters or a probability distribution in \mathcal{U} are available, then the *value at risk* and the *conditional value at risk* criteria are worthy of consideration (see Pflug 2000; Rockafellar and Uryasev 2000). In contrast to the minmax and minmax regret criteria, which consider all scenarios, the above risk criteria take into account a subset of scenarios aggregated with respect to a certain confidence level. In consequence they are less conservative (see Alem and Morabito 2013 for an application of the risk criteria in production planning under stochastic demands and setup times). A criterion that exploits information about both importance and probability of occurrence of each scenario is the *Weighted Ordered Weighted Averaging* aggregation operator proposed in Torra (1997) (see also Ogryczak and Śliwiński 2009). It is a generalization of the expected value and OWA.

Another way of hedging against uncertainty of parameters that does not fall into the aforementioned frameworks of uncertainty handling was given in Bertsimas and Sim (2004). It models uncertain parameters as symmetric random variables, with unknown distributions, centred at their nominal values. The main advantage of this framework is its computational tractability and probabilistic guaranties for constraint violation. Moreover, it flexibly controls the level of conservativeness of a resulting solution by assigning to each constraint a parameter, the so-called *budget of uncertainty* that is an upper bound on the total (scaled) variability of parameters in the constraint from their nominal values. The budgets of uncertainty allow the decision-maker to take his/her attitude towards a risk into account. It is worth pointing out that the approach is a generalisation of a conservative framework earlier proposed in Soyster (1973) which assumes that all parameters may realise at their worst-case values – (the case holds for large values of the budgets of uncertainty). The first direct application of the above-mentioned method to solve lot sizing problems with backorders under uncertain demand with nondecreasing uncertainty budgets in periods was given in Bertsimas and Thiele (2006). Along the same line as the previous application, the method was adapted to periodic inventory control and production planning with uncertain product returns and demand in Wei, Li, and Cai (2011), to lot sizing combined with cutting stock problems under uncertain cost and demand in Alem and Morabito (2012) and to planning in agriculture and processing industry in Bohle, Maturana, and Vera (2010), Alvarez and Vera (2014), Roccoa and Morabito (2016). It was also used in Bienstock and Özbay (2008) to obtain a good approximation of optimal basestock policies under uncertain demand in the minmax setting. Furthermore, the above ideas are introduced in a continuous time fluid model for dynamic pricing and inventory control with demand uncertainty by Adida and Perakis (2006).

1.2 Our results

In this paper, we focus on the tactical level by examining the MPS and MRP planning processes. Namely, we deal with the CLSP (for the MPS process) and MLCLSP (for the MRP process), both with backordering, with uncertain cumulative demand under the interval uncertainty representation of scenario set \mathcal{U} with the minmax criterion for choosing a robust production plan (see, e.g. Zangwill 1969; Pochet and Wolsey 2006; Mula, Poler, and Garcia-Sabater 2007; Albrecht 2010 for models for their deterministic counterparts). Moreover, we assume that deterministic counterparts of the problems under consideration are polynomially solvable and we would like to preserve their computational tractability. Therefore we do not study production processes, for instance, with set-up costs/times, which become NP-hard problems even for many special cases (see, e.g. Florian, Lenstra, and Rinnooy Kan 1980; Chen and Thizy 1990) and with other production parameters under uncertainty, since it may lead to computationally harder problems.

To the best of our knowledge, such a model of uncertainty in cumulative demand has not been investigated in the literature on production planning so far. We distinguish two interval representations of set \mathcal{U} of cumulative demand scenarios, namely for the MPS problems and for MRP ones. They are more realistic than the model of uncertainty in demand in periods (see Section 3 for a detailed explanation). Moreover, we show that they lead to the polynomial solvability of the planning problems under consideration in the minmax setting in contrast to the same problems with uncertainty in demand in periods for which there are few polynomial methods. Accordingly, in Section 4.1, we show that computing a worst-case scenario (the adversarial problem) can be done in $O(T)$ time for the MPS under L4L and POQ rules – note that its computational complexity in the

case of uncertainty in demand in periods is still an open problem (see [Guillaume, Kobylański, and Zieliński \(2012\)](#) and also other hard adversarial problems [Bienstock and Özbay \(2008\)](#)). We provide a linear programme for the MPS under L4L and a strongly polynomial combinatorial algorithm for the MPS under POQ for determining an optimal robust production plan. Interestingly, under uncertainty on demand in periods there is only one iterative algorithm given in [Guillaume, Kobylański, and Zieliński \(2012\)](#) based on Benders' decomposition. A situation is much more involved under the cumulative demand uncertainty for the MRP problems. In Section 4.2 we prove, by characterising optimal scenarios, that computing a worst-case scenario (the adversarial problem) can be done in $O(P \cdot T^3)$ time, where P is the number of products. This allows us to propose a linear programming model for determining an optimal robust production plan. In Section 5, we provide and experimental evaluation of the proposed methods, which show, among others, that the decrease in profit of optimal robust production plans is consistently linear when uncertainty increases.

2. Deterministic MPS/MRP problems

In this section, we recall the deterministic counterparts of the MPS and MRP problems (see, e.g. [Albrecht 2010](#); [Pochet and Wolsey 2006](#)). We provide slightly modified linear programmes and some compact formulations for them that allow us to propose and analyse, in Section 4, methods and linear programming models for these problems under uncertainty.

2.1 Single-item model (MPS) – Lot for Lot rule (L4L)

We now discuss a single-item problem under *capacity and cumulative capacity constraints*. Given T periods, for period $t \in [T]$ ($[T]$ denotes the set $\{1, \dots, T\}$), let $d_t \geq 0$ be the demand in period t , let $x_t \geq 0$ be the production amount in period t . Set $\mathbf{D}_t = \sum_{i=1}^t d_i$ and $\mathbf{X}_t = \sum_{i=1}^t x_i$, \mathbf{D}_t and \mathbf{X}_t stand for the cumulative demand up to period t and the cumulative production up to period t , respectively. Obviously, $\mathbf{X}_{t-1} \leq \mathbf{X}_t$ and $\mathbf{D}_{t-1} \leq \mathbf{D}_t$, $t = 2, \dots, T$. The production x_t in each period $t \in [T]$ can be under two kinds of limits: *capacity limits* l_t and u_t such that $l_t \leq x_t \leq u_t$ and *cumulative capacity limits* \mathbf{L}_t and \mathbf{U}_t on the cumulative production up to period t such that $\mathbf{L}_t \leq \sum_{i=1}^t x_i \leq \mathbf{U}_t$. Thus the set of feasible production amounts $\mathbb{X} \subseteq \mathbb{R}_+^T$ can be defined as follows:

$$\mathbb{X} = \{x = (x_1, \dots, x_T) : l_t \leq x_t \leq u_t, \mathbf{L}_t \leq \sum_{i=1}^t x_i \leq \mathbf{U}_t, t \in [T]\}. \quad (1)$$

Moreover, at this (tactical) level of planning for MPS, we assume that production, inventory and backordering cost coefficients and the selling price coefficient do not depend on the period t . These cost and selling price coefficients are denoted, respectively, by c^I , c^B and b^P . We do not consider in the production process setup times and costs. Thus the problem, here studied, is a version of the *capacitated single-item lot sizing model with backordering* (see, e.g. [Pochet and Wolsey 2006](#)) and it can be formulated as a linear programming model as follows – the decision variables and parameters in model (2)–(7) are defined in Table 1:

$$\min \sum_{t=1}^T (c^I I_t + c^B B_t - b^P s_t) \quad (2)$$

$$\text{s.t. } B_t - I_t = \mathbf{D}_t - \sum_{i=1}^t x_i \quad t \in [T] \quad (3)$$

$$\sum_{i=1}^t s_i = \mathbf{D}_t - B_t \quad t \in [T] \quad (4)$$

$$l_t \leq x_t \leq u_t \quad t \in [T] \quad (5)$$

$$\mathbf{L}_t \leq \sum_{i=1}^t x_i \leq \mathbf{U}_t \quad t \in [T] \quad (6)$$

$$B_t, I_t, s_t, x_t \geq 0 \quad t \in [T] \quad (7)$$

Since we have only one product and production cost coefficients are constant over the planning horizon, they can be ignored in the formulation (2)–(7). Constraints (3) are the flow constraints of production with backordering and (4) specify the potential sales of the product in the planning horizon. It is easily seen that (4) can be also rewritten as: $\sum_{i=1}^t s_i = \sum_{i=1}^t x_i - I_t$. Constraints (5) are the capacity limits in each period and (6) are the limits on the cumulative production up to period t . We

Table 1. The parameters and decision variables in model (2)–(7).

| | Definition |
|---------------------------|---|
| <i>Set of indices</i> | |
| $[T]$ | Set of periods in the planning horizon $\{1, \dots, T\}$ |
| <i>Parameters</i> | |
| T | Number of periods |
| c^I | Inventory cost of one unit of the product |
| c^B | Backorder cost for one unit of the product |
| b^P | Selling price of one unit of the product |
| d_t | External demand in period $t \in [T]$, cumulative demand $\mathbf{D}_t = \sum_{i=1}^t d_i$ |
| l_t | Lower bound on the production in period $t \in [T]$ |
| u_t | Upper bound on the production in period $t \in [T]$ |
| \mathbf{L}_t | Lower bound on the cumulative production in period $t \in [T]$ |
| \mathbf{U}_t | Upper bound on the cumulative production in period $t \in [T]$ |
| <i>Decision variables</i> | |
| x_t | Production amount of the product in period $t \in [T]$ |
| I_t | Inventory level of the product at the end of period $t \in [T]$ |
| B_t | Backordering level of the product at the end of period $t \in [T]$ |
| s_t | Sales of the product at the end of period $t \in [T]$ |

have assumed that an initial inventory and an initial backorder are equal to zero. This assumption is not particularly restrictive. An easy computation shows that (2)–(7) may be written in an equivalent compact form. Indeed, define \mathcal{C} and \mathcal{P} to be the nonnegative real functions that represent either the cost of storing inventory from period t to period $t + 1$ or the cost of backordering the product amount from period $t + 1$ to period t , and the benefit from selling the product in the planning horizon, respectively. The functions have the forms: $\mathcal{C}(\mathbf{X}_t, \mathbf{D}_t) = \max\{c^I(\mathbf{X}_t - \mathbf{D}_t), c^B(\mathbf{D}_t - \mathbf{X}_t)\}$ (see (2) and (3)) and $\mathcal{P}(\mathbf{X}_T, \mathbf{D}_T) = b^P \min\{\mathbf{X}_T, \mathbf{D}_T\}$ (see (2), (3) and (4)). We can now write (2)–(7) in the following equivalent form:

$$\text{MPS-L4L: } \min_{x \in \mathbb{X}} F(x) = \min_{x \in \mathbb{X}} \sum_{t=1}^T \mathcal{C}(\mathbf{X}_t, \mathbf{D}_t) - \mathcal{P}(\mathbf{X}_T, \mathbf{D}_T) \quad (8)$$

Therefore, our production planning consists in finding a feasible production plan $x \in \mathbb{X}$, subject to the conditions of satisfying each demand, which minimises the total cost of storage and backordering minus the benefit from selling the product. The compact form (8) allows us to formulate (2)–(7) as the following linear programme which will be exploited in Section 4:

$$\min \sum_{t=1}^T \pi_t \quad (9)$$

$$\text{s.t. } \pi_t \geq c^I \left(\sum_{i=1}^t x_i - \mathbf{D}_t \right) \quad t = 1, \dots, T-1 \quad (10)$$

$$\pi_t \geq c^B \left(\mathbf{D}_t - \sum_{i=1}^t x_i \right) \quad t = 1, \dots, T-1 \quad (11)$$

$$\pi_T \geq c^I \left(\sum_{i=1}^T x_i - \mathbf{D}_T \right) - b^P \mathbf{D}_T \quad (12)$$

$$\pi_T \geq c^B \left(\mathbf{D}_T - \sum_{i=1}^T x_i \right) - b^P \sum_{i=1}^T x_i \quad (13)$$

$$\pi_t \text{ unrestricted} \quad t \in [T] \quad (14)$$

$$\mathbf{x} \in \mathbb{X} \quad (15)$$

where the optimal values of π_t , $t = 1, \dots, T-1$, are such that $\pi_t = \mathcal{C}(\mathbf{X}_t, \mathbf{D}_t)$, $\pi_T = \mathcal{C}(\mathbf{X}_T, \mathbf{D}_T) - \mathcal{P}(\mathbf{X}_T, \mathbf{D}_T)$ and the value of $\sum_{t=1}^T \pi_t$ equals $F(x)$ for an optimal robust production plan \mathbf{x} .

2.2 Single-item model (MPS) – periodic order quantity rule (POQ)

We now turn to a single-item problem under *periodic capacity constraints* in which $u_t = \infty$ if t is the production period; and $u_t = 0$ otherwise. Obviously, it is a special case of the single product problem with the Lot-For-Lot rule (see problem (8)). Furthermore, we are given a *periodicity* ℓ being a positive integer such that: $1 < \ell < T$ and $x_t \geq 0$ if $t \bmod \ell = 1$; $x_t = 0$ otherwise (if $t \bmod \ell \neq 1$) for $t \in [T]$. We assume without loss of generality that $(T - 1) \bmod \ell = 0$. Now, the set of feasible production amounts $\mathbb{X} \subseteq \mathbb{R}_+^T$ can be defined as follows:

$$\mathbb{X} = \{x = (x_1, \dots, x_T) : x_t \geq 0 \text{ for } t \bmod \ell = 1, x_t = 0 \text{ for } t \bmod \ell \neq 1, t \in [T]\}. \quad (16)$$

The optimisation problem consists in finding a feasible production plan $x \in \mathbb{X}$, subject to the conditions of satisfying each demand, which minimises the total cost of storage and backordering, that is:

$$\text{MPS- POQ: } \min_{x \in \mathbb{X}} F(x) = \min_{x \in \mathbb{X}} \sum_{t=1}^T \mathcal{C}(\mathbf{X}_t, \mathbf{D}_t). \quad (17)$$

Clearly, problem (17) is a version of the *uncapacitated single-item lot sizing model with backordering* (see, e.g. Zangwill 1969) with the periodic order quantity policy. Thus, it can be formulated as follows:

$$\min \sum_{t=1}^T (c^I I_t + c^B B_t) \quad (18)$$

$$\text{s.t. } B_t - I_t = \mathbf{D}_t - \sum_{i=1}^t x_i \quad t \in [T] \quad (19)$$

$$x_t = 0 \quad t \bmod \ell \neq 1, t \in [T] \quad (20)$$

$$B_t, I_t, x_t \geq 0 \quad t \in [T] \quad (21)$$

Taking into account the periodicity in the problem MPS- POQ and its network nature, one can adapt the results of Ahuja, Magnanti, and Orlin (1993, Chapter 19.9) for problem (18)–(21) and obtain the following proposition:

PROPOSITION 1 *The problem (18)–(21) can be solved in $O(T)$.*

Proof. See Appendix 1. □

2.3 Multi-Item, multi-level model (MRP)

We focus on the most involved problem. We are given T periods, P products and R resources. For period $t \in [T]$ and product $p \in [P]$ let $d_{t,p}$ be the external demand for product p in period t , $d_{t,p} \geq 0$, let $x_{t,p}$ be the production amount of product p in period t , $x_{t,p} \geq 0$. Set $\mathbf{D}_{t,p} = \sum_{i=1}^t d_{i,p}$ and $\mathbf{X}_{t,p} = \sum_{i=1}^t x_{i,p}$, $\mathbf{D}_{t,p}$ and $\mathbf{X}_{t,p}$ stand for the cumulative demand for product p up to period t and the cumulative production of product p up to period t , respectively. Obviously, $\mathbf{X}_{t-1,p} \leq \mathbf{X}_{t,p}$ and $\mathbf{D}_{t-1,p} \leq \mathbf{D}_{t,p}$, $t = 2, \dots, T$. In this problem, making products requires resources. Let $a_{p,r}$ be the required amount of resource r , $r \in [R]$, to produce one unit of product p , $p \in [P]$. Hence the production of product p in period t , $x_{t,p}$, will satisfy two kinds of *resource constraints*: the capacity limits of each resource r in each period t , denoted by $l_{t,r}$ and $u_{t,r}$, respectively, such that $l_{t,r} \leq \sum_{p=1}^P a_{p,r} x_{t,p} \leq u_{t,r}$ and the cumulative capacity limits of each resource r in each period t , denoted by $\mathbf{L}_{t,r}$ and $\mathbf{U}_{t,r}$, respectively, such that $\mathbf{L}_{t,r} \leq \sum_{p=1}^P a_{p,r} \mathbf{X}_{t,p} \leq \mathbf{U}_{t,r}$. Besides the external demand $d_{t,p}$, a given amount of product $p \in [P]$ is required to make one unit of product $p' \in [P]$, denoted by $b_{p,p'}$ ($b_{p,p'} = 0$ if $p = p'$), and so $\sum_{j=1}^P b_{p,j} x_{t+Ld_j,j}$ is the *internal demand* for product p in period t , where Ld_j is the *lead-time* of product j . Furthermore, if one accepts backordering, this means that an external demand for product p in period t can be fulfilled in period t' such that $t' > t$. Of course the backordering concerns only external demands. Therefore, the set of feasible production amounts $\mathbb{X} \subseteq \mathbb{R}_+^{T \times P}$ has the following form:

$$\mathbb{X} = \{x = (x_{t,p}) : l_{t,r} \leq \sum_{j=1}^P a_{j,r} x_{t,j} \leq u_{t,r}, \mathbf{L}_{t,r} \leq \sum_{j=1}^P a_{j,r} \sum_{i=1}^t x_{i,j} \leq \mathbf{U}_{t,r}, \sum_{i=1}^t \left(x_{i,p} - \sum_{j=1}^P b_{p,j} x_{i+Ld_j,j} \right) \geq 0, t \in [T], r \in [R], p \in [P]\}, \quad (22)$$

Table 2. The parameters and decision variables in model (23)–(28).

| | Definition |
|---------------------------|---|
| <i>Set of indices</i> | |
| $[P]$ | Set of products $\{1, \dots, P\}$ |
| $[R]$ | Set of resources $\{1, \dots, R\}$ |
| $[T]$ | Set of periods in the planning horizon $\{1, \dots, T\}$ |
| <i>Parameters</i> | |
| P | Number of products |
| R | Number of resources |
| T | Number of periods |
| $a_{p,r}$ | Required quantity of resource $r \in [R]$ to produce one unit of product $p \in [P]$ |
| $b_{i,j}$ | Amount of product $i \in [P]$ required to produce one unit of product $j \in [P]$ |
| b_p^P | Selling price of one unit of product $p \in [P]$ |
| c_p^I | Production cost of one unit of product $p \in [P]$ |
| c_p^S | Inventory cost of one unit of product $p \in [P]$ |
| c_p^B | Backorder cost for one unit of product $p \in [P]$ |
| $d_{t,p}$ | External demand of product $p \in [P]$ in period $t \in [T]$, cumulative demand $\mathbf{D}_{t,p} = \sum_{i=1}^t d_{i,p}$ |
| $l_{t,r}$ | Lower capacity limit of resource $r \in [R]$ in period $t \in [T]$ |
| $u_{t,r}$ | Upper capacity limit of resource $r \in [R]$ in period $t \in [T]$ |
| $\mathbf{L}_{t,r}$ | Lower cumulative capacity limit of resource $r \in [R]$ in period $t \in [T]$ |
| $\mathbf{U}_{t,r}$ | Upper cumulative capacity limit of resource $r \in [R]$ in period $t \in [T]$ |
| Ld_p | Lead-time of product $p \in [P]$ |
| <i>Decision variables</i> | |
| $x_{t,p}$ | Production amount of product $p \in [P]$ in period $t \in [T]$ |
| $I_{t,p}$ | Inventory level of product $p \in [P]$ at the end of period $t \in [T]$ |
| $B_{t,p}$ | Backordering level of product $p \in [P]$ at the end of period $t \in [T]$ |
| $s_{t,p}$ | Sales of product $p \in [P]$ at the end of period $t \in [T]$ |

where $\sum_{i=1}^t (x_{i,p} - \sum_{j=1}^P b_{p,j} x_{i+Ld_j,j})$ is the cumulative production of product p up to period t minus the internal demand for product p . At the (tactical) level of planning, one assumes that the inventory, production and backordering cost coefficients and selling prices only depend on product p and not on period t . We are now ready to give the linear programming model of the problem under consideration, the decision variables and parameters are defined in Table 2 (we do not take into account setup times and costs in the production process):

$$\min \sum_{p=1}^P \sum_{t=1}^T (c_p^I I_{t,p} + c_p^B B_{t,p} + c_p^P x_{t,p} - b_p^P s_{t,p}) \quad (23)$$

$$\text{s.t. } B_{t,p} - I_{t,p} = \mathbf{D}_{t,p} - \sum_{i=1}^t \left(x_{i,p} - \sum_{j=1}^P b_{p,j} x_{i+Ld_j,j} \right) \quad t \in [T], p \in [P] \quad (24)$$

$$\sum_{i=1}^t s_{i,p} = \mathbf{D}_{t,p} - B_{t,p} \quad t \in [T], p \in [P] \quad (25)$$

$$l_{t,r} \leq \sum_{j=1}^P a_{j,r} x_{t,j} \leq u_{t,r} \quad t \in [T], r \in [R] \quad (26)$$

$$\mathbf{L}_{t,r} \leq \sum_{j=1}^P a_{j,r} \sum_{i=1}^t x_{i,j} \leq \mathbf{U}_{t,r} \quad t \in [T], r \in [R] \quad (27)$$

$$B_{t,p}, I_{t,p}, s_{t,p}, x_{t,p} \geq 0 \quad t \in [T], p \in [P] \quad (28)$$

Note that the model (23)–(28) admits that each product can have external and internal demands simultaneously, thus it is a generalisation of ones proposed in Albrecht (2010), Mula, Poler, and Garcia-Sabater (2007), Pochet and Wolsey

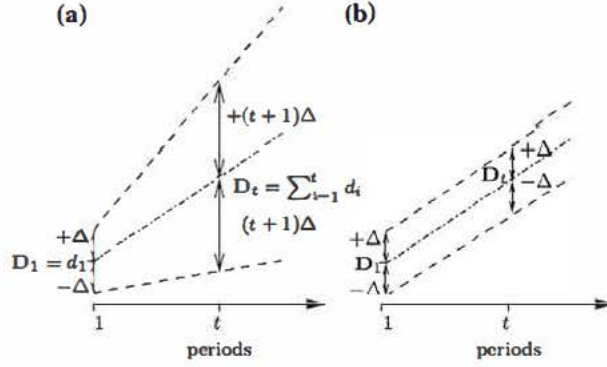


Figure 1. (a) Case with uncertainty in demand d_t . (b) Case with uncertainty in cumulative demand D_t .

(2006). Constraints (24) are flow conservation constraints of production with backordering. In order to guarantee that the backordering is only possible for the external demands, constraints (25) have been added. Note that constraints (25) can be also rewritten as: $\sum_{i=1}^t s_{i,p} = \sum_{i=1}^t (x_{i,p} - \sum_{j=1}^P b_{p,j} x_{i+Ld_{j,j}})$ $I_{t,p}$. Constraints (26) and (27) are the capacity ones. Similarly, as previously, we assume that an initial inventory and an initial backorder are equal to zero. We now rewrite (23)–(28) in the following equivalent compact form:

$$\text{MRP: } \min_{x \in \mathbb{X}} F(x) = \min_{x \in \mathbb{X}} \sum_{p=1}^P \sum_{t=1}^T C_p(X_{t,p}^{int}, D_{t,p}) + c_p^P X_{T,p} \quad \mathcal{P}_p(X_{T,p}^{int}, D_{T,p}), \quad (29)$$

where $X_{t,p}^{int} = \sum_{i=1}^t (x_{i,p} - \sum_{j=1}^P b_{p,j} x_{i+Ld_{j,j}})$, $X_{t,p}^{int} \geq 0$, C_p and \mathcal{P}_p are the nonnegative real functions that represent either the cost of storing inventory from period t to period $t + 1$ or the cost of backordering product $p \in [P]$ from period $t + 1$ to period t and the benefit from selling product $p \in [P]$ in the planning horizon, respectively. They have the forms: $C_p(X_{t,p}^{int}, D_{t,p}) = \max\{c_p^I(X_{t,p}^{int} - D_{t,p}), c_p^B(D_{t,p} - X_{t,p}^{int})\}$ and $\mathcal{P}_p(X_{T,p}^{int}, D_{T,p}) = b_p^P \min\{X_{T,p}^{int}, D_{T,p}\}$. Thus the production planning problem with the deterministic demands consists in finding a feasible production plan $x \in \mathbb{X}$, that minimises the total cost of storage, backordering and production minus the benefit from selling products.

3. Demand uncertainty in MPS/MRP problems

In Section 2, we have assumed all the input parameters in MPS- L4L, MPS- POQ and MRP problems are precisely known. However, in real life this is rarely the case. Here, we admit that the external demands are subject to uncertainty. For clarity of presentation, we address now the single-item model.

Let us examine one of the simplest representations of uncertainty, i.e. modelling uncertain demands \bar{d}_t , $t \in [T]$, as closed intervals $[\underline{d}_t, \bar{d}_t]$, $\underline{d}_t \geq 0$, where \underline{d}_t and \bar{d}_t are the minimum and the maximum possible values of demand \bar{d}_t in period t , respectively. It is worth pointing out that in the above model of uncertainty, the imprecision of cumulative demand $D_t = \sum_{i=1}^t d_i$ increases in subsequent periods, $D_t \in [\sum_{i=1}^t \underline{d}_i, \sum_{i=1}^t \bar{d}_i]$. In fact, practitioners often express knowledge about uncertainty in a demand as $\pm \Delta$, where Δ is a possible deviation from its nominal value. The demand can be interpreted in two different ways: a demand in a period or a cumulative demand. The former interpretation is often assumed. Accordingly, the assumption that this uncertainty Δ is interpreted as uncertainty in the demand in each period leads to the cumulative demand shown in Figure 1(a), where the demand uncertainty Δ cumulates in the subsequent periods due to the interval addition, which is not realistic comparing with uncertainty in the cumulative demand presented in Figure 1(b). Therefore in this paper, the uncertainty of demand \bar{d}_t lies in the cumulative demand, modelled by interval $[\underline{D}_t, \bar{D}_t]$, instead of uncertainty in the demand in periods, modelled by $[\underline{d}_t, \bar{d}_t]$. Obviously, $\underline{D}_t \leq \underline{D}_t$ and $\bar{D}_t \leq \bar{D}_t$, $t = 2, \dots, T$.

A vector $S = (D_1, \dots, D_T)$, $D_t \in [\underline{D}_t, \bar{D}_t]$, $D_{t-1} \leq D_t$, that is an assignment of cumulative demands D_t to periods t , $t \in [T]$, is called a *scenario*. It is easy to check that scenario $S = (D_1, \dots, D_T)$ induces a vector of demands in periods t , i.e. $d_1 = D_1$, $d_t = D_t - D_{t-1}$, $t = 2, \dots, T$. Let $\mathcal{U} \subseteq \mathbb{R}_+^T$ be the set of all scenarios. The cumulative demand and the demand in period t under scenario S are denoted by $D_t(S)$, $D_t(S) \in [\underline{D}_t, \bar{D}_t]$, and $d_t(S)$, respectively, $d_1(S) = D_1(S)$, $d_t(S) = D_t(S) - D_{t-1}(S)$, $t = 2, \dots, T$. Clearly, for every $S \in \mathcal{U}$ inequalities $D_{t-1}(S) \leq D_t(S)$, $t = 2, \dots, T$, hold.

For the practitioners, cumulative demand intervals have the following form $D_t \in [D_t^a - \underline{\Delta}_t, D_t^a + \bar{\Delta}_t]$, where $\underline{\Delta}_t$ and $\bar{\Delta}_t$ stand for possible deviations from the nominal cumulative demand D_t^a in period t , $t \in [T]$. Therefore with our proposed

model, the practitioners are able to express different types of uncertainty on the demand effortlessly. Indeed, the uncertainty can be constant all over a planning horizon, or can increase proportionally to the demand or can be due to shifts of a part of the demand between two consecutive periods. The above three cases can be expressed as uncertainty on the cumulative demand. In the first one, this uncertainty is expressed as an uncertainty of $\pm K$ on the cumulative demand. More formally, $\underline{\Delta}_t = \bar{\Delta}_t = K$, where K is a constant. In the second case, $\underline{\Delta}_t = \bar{\Delta}_t = \alpha \mathbf{D}_t^n$, where $\alpha \in [0, 1]$ is a prescribed uncertainty parameter being the maximum percentage deviation of cumulative demands in periods from their nominal values. In the third case $\underline{\Delta}_t = \alpha d_t$, and $\bar{\Delta}_t = \alpha d_{t+1}$, where $\alpha \in [0, 1]$, which is the maximum percentage deviation of the demand that can be moved to its adjacent periods. For instance, $\alpha = 1$ means that all the demand in a period can be moved to its adjacent periods and $\alpha = 0.5$ means that half of the demand in a period can be moved to its adjacent periods. Moreover, in contrast with the period-specific uncertainty that cannot model uncertainty reductions in parts of a planning horizon, defining uncertainty on the cumulative demand allows the decision-maker to capture a variation of uncertainty throughout a planning horizon.

3.1 Model of uncertainty for the MPS problems

For MPS- L4L, MPS- POQ problems studied in Sections 2.1 and 2.2, we adopt a model in which uncertainty in the cumulative demand is small with respect to the minimum demand. In other words, a demand in a period can be equal to 0 only if the previous cumulative demand is at its maximum value. In fact, at this level of planning, the lengths of time periods are big enough (for example, one month) to have the minimum periodic demand greater than zero. Thus this assumption is realistic. More formally, cumulative demand intervals are such that $\bar{\mathbf{D}}_{t-1} \leq \underline{\mathbf{D}}_t$ for every $t = 2, \dots, T$. Hence the set of cumulative demand scenarios \mathcal{U} is the Cartesian product of the cumulative demands intervals, i.e.

$$\mathcal{U} = [\underline{\mathbf{D}}_1, \bar{\mathbf{D}}_1] \times \dots \times [\underline{\mathbf{D}}_T, \bar{\mathbf{D}}_T], \text{ where } \bar{\mathbf{D}}_{t-1} \leq \underline{\mathbf{D}}_t. \quad (30)$$

3.2 Model of uncertainty for the MRP problem

For the MRP problem, formulated in Section 2.3, one assumes that the length of periods is smaller than the length of periods of the MPS problems, for instance, the length of a period for MPS is one month while the length of a period for MRP is one week. Moreover, the demand has more versatility in the short range. For such a length of periods of MRP, it is possible that the demand is delayed by a period, what is not possible for a longer length of period. For both reasons, it is hard to assume that for MRP problem, uncertainty in the cumulative demand is small with respect to the minimum demand. Furthermore, we extend now the previously described model of uncertainty for the single-item model to the one for the multi-item model. Namely, for period $t, t \in [T]$, and product $p, p \in [P]$, uncertainty in the external demand $\tilde{d}_{t,p}$ is described by uncertainty in the cumulative demands modelled by intervals $[\underline{\mathbf{D}}_{t,p}, \bar{\mathbf{D}}_{t,p}]$ assuming that the external demands are unrelated. We drop the constraints on the cumulative demands in periods $\bar{\mathbf{D}}_{t-1,p} \leq \underline{\mathbf{D}}_{t,p}$. Accordingly, the cumulative demand scenario S is a matrix of the form $S = (\mathbf{D}(S)_{t,p})_{t \in [T], p \in [P]}$, $\mathbf{D}(S)_{t,p} \in [\underline{\mathbf{D}}_{t,p}, \bar{\mathbf{D}}_{t,p}]$, $\mathbf{D}(S)_{t-1,p} \leq \mathbf{D}(S)_{t,p}$. Obviously, every scenario S induces the matrix of external demands of each product $p \in [P]$ in periods $t \in [T]$, i.e. $d_{1,p} = \mathbf{D}(S)_{1,p}$, $d_{t,p} = \mathbf{D}(S)_{t,p} - \mathbf{D}(S)_{t-1,p}$, $t = 2, \dots, T$, and so the set of cumulative demand scenarios $\mathcal{U} \subseteq \mathbb{R}_+^{T \times P}$ has the following form:

$$\mathcal{U} = \{S = (\mathbf{D}(S)_{t,p}) : \mathbf{D}(S)_{t,p} \in [\underline{\mathbf{D}}_{t,p}, \bar{\mathbf{D}}_{t,p}], t \in [T], \mathbf{D}(S)_{t-1,p} \leq \mathbf{D}(S)_{t,p}, t = 2, \dots, T, p \in [P]\}. \quad (31)$$

4. Robust MPS/MRP problems

Assuming that demand uncertainty is represented as in Section 3, we wish to find a robust production plan $x \in \mathbb{X}$ in MPS- L4L, MPS- POQ, MRP problems with the best worst-case cost guarantee. In order to this, one of common robust criteria called the *minmax* can be adopted (see, e.g. Kouvelis and Yu 1997). In other words, we seek a production plan that minimises the maximum production plan cost $F(x, S)$ over all the cumulative demand scenarios from a fixed set \mathcal{U} (see (30), (31)). This leads to the following *general robust problem* \mathcal{P} :

$$\text{ROB } \mathcal{P}: \min_{x \in \mathbb{X}} A(x) = \min_{x \in \mathbb{X}} \max_{S \in \mathcal{U}} F(x, S),$$

where $\mathcal{P} \in \{\text{MPS- L4L, MPS- POQ, MRP}\}$, $F(x, S)$ is the cost of production plan x (see (8), (17), (29)) under scenario $S \in \mathcal{U}$, $A(x)$ is the *maximum cost of production plan* $x \in \mathbb{X}$, $A(x) = \max_{S \in \mathcal{U}} F(x, S)$, and the set of feasible production plans \mathbb{X} is the one of the forms: (1), (16) and (22). An optimal solution x^r to the problem ROB \mathcal{P} is called *optimal robust production plan* for \mathcal{P} .

We are aware of the conservativeness of the minmax. Fortunately, using this criterion, with the assumed cumulative demand uncertainty representation (see (30), (31)), enables us to construct polynomial methods for the problems under consideration. Moreover, applying other less conservative criteria may not guarantee the preservation of polynomial solvability and requires additional information about uncertain demands (see Section 1 for comments).

A natural optimisation problem associated with ROB \mathcal{P} , and simpler than ROB \mathcal{P} , is the one of *evaluating a given production plan* $x^* \in \mathbb{X}$. A scenario $S^o \in \mathcal{U}$ that minimises the total cost $F(x^*, S)$ of the production plan x^* is called *optimistic scenario*. A scenario $S^w \in \mathcal{U}$ that maximises the total cost $F(x^*, S)$ of the production plan x^* is called *the worst-case scenario*. The problem of computing a worst-case scenario is often called *adversarial* problem (see, e.g. Bienstock and Özbay 2008; Nasrabadi and Orlin 2013), in which an *adversary* maliciously wants to increase the total cost $F(x^*, S)$ of the production plan x^* . Therefore in order to evaluate a given production plan x^* , we look for the interval, $F_{x^*} = [\underline{f}_{x^*}, \overline{f}_{x^*}]$, containing possible values of costs of x^* , such that

$$\underline{f}_{x^*} = F(x^*, S^o) = \min_{S \in \mathcal{U}} F(x^*, S), \quad (32)$$

$$\overline{f}_{x^*} = F(x^*, S^w) = \max_{S \in \mathcal{U}} F(x^*, S). \quad (33)$$

4.1 Robust MPS problems

In this section, we focus on the problem of evaluating a fixed production plan and the problem of computing an optimal robust production plan to ROB \mathcal{P} , where $\mathcal{P} \in \{\text{MPS-L4L}, \text{MPS-POQ}\}$, under the uncertainty representation \mathcal{U} defined in (30). We start by proving the following proposition that we substantially use in this section.

PROPOSITION 2 *The objective function $F(x^*, S)$ for any fixed production plan $x^* \in \mathbb{X}$ in ROB \mathcal{P} , $\mathcal{P} \in \{\text{MPS-L4L}, \text{MPS-POQ}\}$, is convex in \mathcal{U} .*

Proof. See Appendix 1. □

The convexity of $F(x^*, S)$ in \mathcal{U} is a crucial fact. For problem (32), it implies that a locally optimal scenario is also globally optimal. For problem (33), it implies the maximum of F is attained at a *vertex* of convex polytope \mathcal{U} . We recall that a scenario $S \in \mathcal{U}$ is a vertex of \mathcal{U} if it is not the strict convex combination of scenarios of \mathcal{U} .

4.1.1 Evaluating a given production plan

We now examine the problem of evaluating a given production plan $x^* \in \mathbb{X}$. Namely, we wish to find the interval $F_{x^*} = [\underline{f}_{x^*}, \overline{f}_{x^*}]$ (see problems (32) and (33)). Interval F_{x^*} determined provides valuable information for the decision-maker, since it contains possible values of costs of x^* . It may be useful, for instance, in a risk analysis in the setting of possibility theory (Dubois and Prade 1988), where it helps in evaluating degrees of possibility and necessity that a cost of a given plan x^* does not exceed a given threshold. Thus we need to compute its optimistic and worst-case scenarios. We consider the case ROB MPS-L4L. All the results presented in this section apply also to ROB MPS-POQ – it is sufficient to set b^p to zero.

We first deal with the problem of determining an optimistic scenario $S^o \in \mathcal{U}$ for a production plan x^* , i.e. the problem (32). Fortunately, for every $x^* \in \mathbb{X}$ one can give the explicit form of an optimistic scenario S^o :

$$\mathbf{D}(S^o)_t = \begin{cases} \underline{\mathbf{D}}_t & \text{if } \mathbf{X}_t^* < \underline{\mathbf{D}}_t, \\ \mathbf{X}_t^* & \text{if } \underline{\mathbf{D}}_t \leq \mathbf{X}_t^* \leq \overline{\mathbf{D}}_t, \\ \overline{\mathbf{D}}_t & \text{if } \mathbf{X}_t^* > \overline{\mathbf{D}}_t, \end{cases} \quad t \in [T]. \quad (34)$$

It is easy to check that S^o in the form of (34) is feasible and locally optimal for the problem (32). Hence and from Proposition 2, scenario S^o is also globally optimal, $\underline{f}_{x^*} = F(x^*, S^o)$. Thus computing an optimistic scenario can be done in $O(T)$ time.

We now study the problem of computing a worst-case scenario $S^w \in \mathcal{U}$ for a given production plan $x^* \in \mathbb{X}$ (the adversarial problem), i.e. the problem (33). It turns out that, for problem (33), the set \mathcal{U} can be reduced to the set of *extreme scenarios* denoted by \mathcal{U}_{ext} , $\mathcal{U}_{\text{ext}} \subseteq \mathcal{U}$,

$$\mathcal{U}_{\text{ext}} = \{\underline{\mathbf{D}}_1, \overline{\mathbf{D}}_1\} \times \cdots \times \{\underline{\mathbf{D}}_T, \overline{\mathbf{D}}_T\}, \text{ where } \overline{\mathbf{D}}_{t-1} \leq \underline{\mathbf{D}}_t. \quad (35)$$

The following proposition characterises a worst-case scenario S^w .

PROPOSITION 3 A worst-case scenario S^w is an extreme one, i.e. $S^w \in \mathcal{U}_{\text{ext}}$.

Proof. See Appendix 1. □

Using Proposition 3 and the fact that the bounds are such that $\bar{\mathbf{D}}_{t-1} \leq \underline{\mathbf{D}}_t$, we can rewrite problem (33), equivalently, as

$$\begin{aligned} \bar{f}_{x^*} = F(x^* S^w) &= \max_{S \in \mathcal{U}} F(x^*, S) = \max_{S \in \mathcal{U}_{\text{ext}}} F(x^*, S) \\ &= \sum_{t=1}^{T-1} \max_{\mathbf{D}(S)_t \in \{\underline{\mathbf{D}}_t, \bar{\mathbf{D}}_t\}} \mathcal{C}(\mathbf{X}_t^*, \mathbf{D}(S)_t) + \max_{\mathbf{D}(S)_T \in \{\underline{\mathbf{D}}_T, \bar{\mathbf{D}}_T\}} \mathcal{C}(\mathbf{X}_T^*, \mathbf{D}(S)_T) - \mathcal{P}(\mathbf{X}_T^*, \mathbf{D}(S)_T). \end{aligned} \quad (36)$$

Thus a special form of optimisation problem (36) and the forms of functions \mathcal{C} and \mathcal{P} allow us to give an explicit formula for an optimal solution – a worst-cost scenario S^w for every $x^* \in \mathbb{X}$. That is for $t \in [T]$:

$$\mathbf{D}(S^w)_t = \begin{cases} \underline{\mathbf{D}}_t & \text{if } c^I(\mathbf{X}_t^* - \underline{\mathbf{D}}_t) > c^B(\bar{\mathbf{D}}_t - \mathbf{X}_t^*) \text{ and } t = 1, \dots, T-1, \\ \underline{\mathbf{D}}_t & \text{if } c^I(\mathbf{X}_t^* - \underline{\mathbf{D}}_t) - b^P \underline{\mathbf{D}}_t > c^B(\bar{\mathbf{D}}_t - \mathbf{X}_t^*) - b^P \mathbf{X}_t^* \text{ and } t = T, \\ \bar{\mathbf{D}}_t & \text{otherwise.} \end{cases} \quad (37)$$

Therefore, a worst-cost scenario S^w for a given production plan x^* can be computed in $O(T)$ time.

4.1.2 Computing an optimal robust production plan

In this section, we address the problem of computing an optimal robust production plan $x^r \in \mathbb{X}$ to ROB \mathcal{P} , $\mathcal{P} \in \{\text{MPS-L4L}, \text{MPS-POQ}\}$. We first consider the problem ROB MPS-L4L. It is easy to see that the problem contains as subproblem, the one of computing a worst-case scenario for a fixed $x \in \mathbb{X}$ – problem (36). Hence, we get the decomposition of ROB MPS-L4L:

$$\begin{aligned} \min_{x \in \mathbb{X}} \max_{S \in \mathcal{U}} F(x, S) &= \min_{x \in \mathbb{X}} \max_{S \in \mathcal{U}_{\text{ext}}} F(x, S) = \min_{x \in \mathbb{X}} \sum_{t=1}^{T-1} \max_{\mathbf{D}(S)_t \in \{\underline{\mathbf{D}}_t, \bar{\mathbf{D}}_t\}} \mathcal{C}(\mathbf{X}_t, \mathbf{D}(S)_t) \\ &\quad + \max_{\mathbf{D}(S)_T \in \{\underline{\mathbf{D}}_T, \bar{\mathbf{D}}_T\}} \mathcal{C}(\mathbf{X}_T, \mathbf{D}(S)_T) - \mathcal{P}(\mathbf{X}_T, \mathbf{D}(S)_T). \end{aligned} \quad (38)$$

Using model (9)–(14) and the fact that $\mathbf{D}(S)_t \in \{\underline{\mathbf{D}}_t, \bar{\mathbf{D}}_t\}$, $t \in [T]$, we may formulate problem (36) as the following linear programme:

$$\begin{aligned} \min \quad & \sum_{t=1}^T \pi_t & (39) \\ \text{s.t.} \quad & \pi_t \geq c^I(\mathbf{X}_t^* - \underline{\mathbf{D}}_t) & t = 1, \dots, T-1 \\ & \pi_t \geq c^B(\bar{\mathbf{D}}_t - \mathbf{X}_t^*) & t = 1, \dots, T-1 \\ & \pi_T \geq c^I(\mathbf{X}_T^* - \underline{\mathbf{D}}_T) - b^P \underline{\mathbf{D}}_T \\ & \pi_T \geq c^B(\bar{\mathbf{D}}_T - \mathbf{X}_T^*) - b^P \mathbf{X}_T^* \\ & \pi_t \text{ unrestricted} & t \in [T] \end{aligned}$$

For an optimal solution to (39), the value of $\sum_{t=1}^T \pi_t$ equals $\max_{S \in \mathcal{U}} F(x^*, S)$ (problem (36)), the values of π_t , $t \in [T]$, are such that $\pi_t = \max_{\mathbf{D}(S)_t \in \{\underline{\mathbf{D}}_t, \bar{\mathbf{D}}_t\}} \mathcal{C}(\mathbf{X}_t^*, \mathbf{D}(S)_t)$ and $\pi_T = \max_{\mathbf{D}(S)_T \in \{\underline{\mathbf{D}}_T, \bar{\mathbf{D}}_T\}} \mathcal{C}(\mathbf{X}_T^*, \mathbf{D}(S)_T) - \mathcal{P}(\mathbf{X}_T^*, \mathbf{D}(S)_T)$. Adding now the constraints describing the feasible production amounts (1) to (39) and developing \mathbf{X}_t , $t \in [T]$ yield a linear programming model for computing an optimal robust production plan x^r to ROB MPS-L4L:

$$\min \sum_{t=1}^T \pi_t \quad (40)$$

$$\text{s.t. } \pi_t \geq c^I \left(\sum_{i=1}^t x_i - \underline{\mathbf{D}}_t \right) \quad t = 1, \dots, T-1 \quad (41)$$

$$\pi_t \geq c^B \left(\bar{\mathbf{D}}_t - \sum_{i=1}^t x_i \right) \quad t = 1, \dots, T-1 \quad (42)$$

$$\pi_T \geq c^I \left(\sum_{i=1}^T x_i - \underline{\mathbf{D}}_T \right) - b^P \underline{\mathbf{D}}_T \quad (43)$$

$$\pi_T \geq c^B \left(\overline{\mathbf{D}}_T - \sum_{i=1}^T x_i \right) - b^P \sum_{i=1}^T x_i \quad (44)$$

$$\pi_t \text{ unrestricted} \quad t \in [T] \quad (45)$$

$$\mathbf{x} \in \mathbb{X} \quad (46)$$

The vector of decision variables $\mathbf{x} = (x_t)_{t \in [T]}$ describes an optimal robust production plan x^r to ROB MPS- L4L and $\sum_{t=1}^T \pi_t$ provides $\max_{S \in \mathcal{U}} F(\mathbf{x}, S)$. The interpretation of variables $\pi_t, t \in [T]$, is the same as for model (39). A worst-case scenario S^w for x^r can be determined by formula (37) and corresponding inventory and backordering levels and sales at the end of each period can be easily computed by (3) and (4). The problem ROB MPS- L4L with no capacity and cumulative capacity limits is a trivial one. Namely, from results presented in [Guillaume, Kobylański, and Zieliński \(2012, Section 4.2\)](#) and the decomposed form (38) of the problem, we immediately deduce the following $O(T)$ method for computing an optimal robust production plan x^r , for $t \in [T]$:

$$\mathbf{X}_t^r = \begin{cases} \frac{c^B \overline{\mathbf{D}}_t + c^I \underline{\mathbf{D}}_t}{c^B + c^I} & \text{if } t = 1, \dots, T-1, \\ \frac{c^B \overline{\mathbf{D}}_t + (c^I + b^P) \underline{\mathbf{D}}_t}{c^B + c^I + b^P} & \text{if } t = T. \end{cases} \quad (47)$$

Of course, $\mathbf{X}_{t-1}^r \leq \mathbf{X}_t^r$ for $t = 2, \dots, T$. Thus one can set $x_1^r = \mathbf{X}_1^r$ and $x_t^r = \mathbf{X}_t^r - \mathbf{X}_{t-1}^r$ for $t = 2, \dots, T$.

It is worth noting that linear programme (40)–(46) for ROB MPS- L4L can be also obtained if we adapt a more general approach proposed in [Soyster \(1973\)](#) to model (9)–(15) for MPS- L4L under cumulative demand uncertainty. Indeed, this approach copes with uncertainty in cumulative demands by considering their worst-case values. Furthermore, it is equivalent to the robust approach to linear optimisation problems in the sense of [Bertsimas and Sim \(2004\)](#) if cumulative demand \mathbf{D}_t in each period t is admitted to be uncertain. In order to maximise the values of the right-hand side of constraints (10)–(13) over cumulative demand scenario set \mathcal{U} , we set cumulative demand \mathbf{D}_t in (10), (12) and (11), (13) at its lower possible value $\underline{\mathbf{D}}_t$ and its upper possible value $\overline{\mathbf{D}}_t$, respectively, and get constraints (41)–(44). The above equivalence between Soyster's method applied to MPS- L4L under cumulative demand uncertainty and ROB MPS- L4L holds only under scenario set \mathcal{U} defined by (30), when $\overline{\mathbf{D}}_{t-1} \leq \underline{\mathbf{D}}_t, t = 2, \dots, T$. If $\overline{\mathbf{D}}_{t-1} \leq \underline{\mathbf{D}}_t$ is dropped then Soyster's approach considers infeasible cumulative demand scenarios, i.e. scenarios S which do not satisfy $\mathbf{D}(S)_{t-1} \leq \mathbf{D}(S)_t, t = 2, \dots, T$. This follows from the fact that the approach assumes that the value of each cumulative demand may vary independently of the values of the remaining cumulative demands. Therefore, Soyster's method and in consequence the method proposed in [Bertsimas and Sim \(2004\)](#) can no longer be applied to problem MRP under cumulative demand scenario set \mathcal{U} defined by (31) considered in Section 4.2.

We proceed with the study of the problem ROB MPS- POQ. An approach similar to (40)–(46) can be applied to obtain a linear programming model for computing an optimal robust production plan x^r to ROB MPS- POQ. However, a more efficient combinatorial algorithm for determining x^r can be proposed. From Proposition 3, the fact that the bounds fulfil inequalities $\overline{\mathbf{D}}_{t-1} \leq \underline{\mathbf{D}}_t$ and the periodicity in the problem ROB MPS- POQ, it follows that one can decompose the problem into $N + 1$ separate subproblems, where $N = (T - 1)/\ell$, that is:

$$\begin{aligned} \min_{\mathbf{x} \in \mathbb{X}} \max_{S \in \mathcal{U}_{\text{ext}}} \sum_{t=1}^T \mathcal{C}(\mathbf{X}_t, \mathbf{D}_t(S)) &= \sum_{k=1}^N \min_{\mathbf{X}_{(k-1)\cdot\ell+1} \in \mathcal{X}^k} \max_{S \in \mathcal{U}_{\text{ext}}^k} \sum_{t=(k-1)\cdot\ell+1}^{k\cdot\ell} \mathcal{C}(\mathbf{X}_{(k-1)\cdot\ell+1}, \mathbf{D}_t(S)) \\ &+ \min_{\mathbf{X}_T \in \mathcal{X}^T} \max_{S \in \mathcal{U}_{\text{ext}}^T} \mathcal{C}(\mathbf{X}_T, \mathbf{D}_T(S)), \end{aligned} \quad (48)$$

where $\mathcal{X}^k = [\underline{\mathbf{D}}_{(k-1)\cdot\ell+1}, \overline{\mathbf{D}}_{(k-1)\cdot\ell+1}] \cup \dots \cup [\underline{\mathbf{D}}_{k\cdot\ell}, \overline{\mathbf{D}}_{k\cdot\ell}]$, $\mathcal{U}_{\text{ext}}^k = \{\underline{\mathbf{D}}_{(k-1)\cdot\ell+1}, \overline{\mathbf{D}}_{(k-1)\cdot\ell+1}\} \times \dots \times \{\underline{\mathbf{D}}_{k\cdot\ell}, \overline{\mathbf{D}}_{k\cdot\ell}\}$, $k \in [N]$, $\mathcal{X}^T = [\underline{\mathbf{D}}_T, \overline{\mathbf{D}}_T]$ and $\mathcal{U}_{\text{ext}}^T = \{\underline{\mathbf{D}}_T, \overline{\mathbf{D}}_T\}$; $\mathcal{U}_{\text{ext}} = \mathcal{U}^1 \times \dots \times \mathcal{U}^N \times \mathcal{U}^T$. Obviously, the above possible cumulative production levels are nondecreasing sequence of their values, $\mathbf{X}_1 \leq \mathbf{X}_{\ell+1} \leq \dots \leq \mathbf{X}_T$. Therefore, we need only to solve the $N + 1$ separate subproblems. The last subproblem is trivial, i.e. an optimal cumulative production level \mathbf{X}_T^r can be computed by formula:

$$\mathbf{X}_T^r = \frac{c^B \overline{\mathbf{D}}_T + c^I \underline{\mathbf{D}}_T}{c^B + c^I}. \quad (49)$$

It remains to solve the subproblems for $k \in [N]$, i.e.

$$\min_{\mathbf{X}_{(k-1)\cdot\ell+1} \in \mathcal{X}^k} \max_{S \in \mathcal{U}_{\text{ext}}^k} \sum_{t=(k-1)\cdot\ell+1}^{k\cdot\ell} C(\mathbf{X}_{(k-1)\cdot\ell+1}, \mathbf{D}_t(S)). \quad (50)$$

Consider the k th subproblem. Since $\mathbf{X}_t = \mathbf{X}_{(k-1)\cdot\ell+1}$ in period t , $t \in \{(k-1)\cdot\ell+1, \dots, k\cdot\ell\}$, worst-case scenarios $S^w \in \mathcal{U}_{\text{ext}}^k$ for $\mathbf{X}_{(k-1)\cdot\ell+1}$ belong to the set $\{\underline{\mathbf{D}}_{(k-1)\cdot\ell+1}\} \times \dots \times \{\underline{\mathbf{D}}_{h-1}\} \times \{\underline{\mathbf{D}}_h, \overline{\mathbf{D}}_h\} \times \{\overline{\mathbf{D}}_{h+1}\} \times \dots \times \{\overline{\mathbf{D}}_{k\cdot\ell}\} \subseteq \mathcal{U}_{\text{ext}}^k$, assuming that $\mathbf{X}_{(k-1)\cdot\ell+1} \in [\underline{\mathbf{D}}_h, \overline{\mathbf{D}}_h]$.

For each $h = (k-1)\cdot\ell+1, \dots, k\cdot P$, we compute a possible cumulative production level

$$\mathbf{X}_{(k-1)\cdot\ell+1}^h = \frac{c^B \overline{\mathbf{D}}_h + c^I \underline{\mathbf{D}}_h}{c^B + c^I} \in [\underline{\mathbf{D}}_h, \overline{\mathbf{D}}_h].$$

Note that for $\mathbf{X}_{(k-1)\cdot\ell+1}^h$ equality $C(\mathbf{X}_{(k-1)\cdot\ell+1}^h, \underline{\mathbf{D}}_h) = C(\mathbf{X}_{(k-1)\cdot\ell+1}^h, \overline{\mathbf{D}}_h)$ holds and its worst-case scenario has the form $(\underline{\mathbf{D}}_{(k-1)\cdot\ell+1}, \dots, \underline{\mathbf{D}}_{h-1}, \overline{\mathbf{D}}_h, \dots, \overline{\mathbf{D}}_{k\cdot\ell})$. Thus the worst-case cost for $\mathbf{X}_{(k-1)\cdot\ell+1}^h$ over the set of scenarios $\mathcal{U}_{\text{ext}}^k$, denoted by C^h , is as follows:

$$\begin{aligned} C^h(\mathbf{X}_{(k-1)\cdot\ell+1}^h) &= \max_{S \in \mathcal{U}_{\text{ext}}^k} \sum_{t=(k-1)\cdot\ell+1}^{k\cdot\ell} C(\mathbf{X}_{(k-1)\cdot\ell+1}^h, \mathbf{D}_t(S)) \\ &= \sum_{t=(k-1)\cdot\ell+1}^{h-1} C(\mathbf{X}_{(k-1)\cdot\ell+1}^h, \underline{\mathbf{D}}_t) + \sum_{t=h}^{k\cdot\ell} C(\mathbf{X}_{(k-1)\cdot\ell+1}^h, \overline{\mathbf{D}}_t). \end{aligned}$$

We then determine an optimal cumulative production level $\mathbf{X}_{(k-1)\cdot\ell+1}^r$ for the k th subproblem (50) by formula:

$$\mathbf{X}_{(k-1)\cdot\ell+1}^r = \arg \min_{h \in \{(k-1)\cdot\ell+1, \dots, k\cdot\ell\}} C^h(\mathbf{X}_{(k-1)\cdot\ell+1}^h). \quad (51)$$

Using (49) and (51), one can easily compute an optimal robust production plan \mathbf{x}^r . Thus the problem (48) (ROB MPS- POQ) can be solved in $O(\ell \cdot T)$ time.

4.2 Robust MRP problem

In this section, we investigate the problems of evaluating a given production plan and of determining an optimal robust production plan to ROB MRP under the uncertainty representation \mathcal{U} defined in (31).

We begin by proving a convexity property of the objective function of ROB MRP, which allows us to characterise optimal scenarios to problems (32) and (33).

PROPOSITION 4 *The objective function $F(\mathbf{x}^*, S)$ of any fixed production plan $\mathbf{x}^* \in \mathbb{X}$ in ROB MRP is convex in \mathcal{U} .*

Proof. See Appendix 1. □

4.2.1 Evaluating a given production plan

We now look for the interval $F_{\mathbf{x}^*} = [f_{\mathbf{x}^*}, \overline{f}_{\mathbf{x}^*}]$ containing possible values of costs of \mathbf{x}^* . Let us consider the problem of determining an optimistic scenario $S^o \in \mathcal{U}$ for a production plan \mathbf{x}^* in ROB MRP, i.e. the problem (32). Such a scenario can be easily computed by solving a linear programme consisted of (23), (24), (25) and additional constraints: $\mathbf{D}_{t-1,p} \leq \mathbf{D}_{t,p}$, $t = 2, \dots, T$, $p \in [P]$ and $\underline{\mathbf{D}}_{t,p} \leq \mathbf{D}_{t,p} \leq \overline{\mathbf{D}}_{t,p}$, $t \in [T]$, $p \in [P]$, in which $x_{t,p}$ are fixed, i.e. $x_{t,p} = x_{t,p}^*$ and $\mathbf{D}_{t,p}$, $t \in [T]$, $p \in [P]$ are decision variables describing scenario S^o .

We now consider the problem of computing a worst-case scenario $S^w \in \mathcal{U}$ for a given production plan $\mathbf{x}^* \in \mathbb{X}$, in ROB MRP, i.e. the problem (33). This problem is more difficult than the one of computing an optimistic scenario and thus a solution algorithm is much more involved. We check at once that the set of cumulative demand scenarios (see (31)) may be written as follows: $\mathcal{U} = \mathcal{U}^1 \times \dots \times \mathcal{U}^P$, where

$$\mathcal{U}^p = \{S = (\mathbf{D}(S)_{t,p})_{t \in [T]} : \mathbf{D}(S)_{t,p} \in [\underline{\mathbf{D}}_{t,p}, \overline{\mathbf{D}}_{t,p}], \mathbf{D}(S)_{t-1,p} \leq \mathbf{D}(S)_{t,p}, \\ t = 2, \dots, T, p \in [P]\}. \quad (52)$$

Hence problem (33) has the equivalent separable form:

$$\begin{aligned}\bar{f}_{x^*} &= \max_{S \in \mathcal{U}} F(x^*, S) = \max_{S \in \mathcal{U}} \sum_{p=1}^P \sum_{t=1}^T C_p(\mathbf{X}_{t,p}^{*int}, \mathbf{D}(S)_{t,p}) - \mathcal{P}_p(\mathbf{X}_{T,p}^{*int}, \mathbf{D}(S)_{T,p}) + c_p^P \mathbf{X}_{T,p}^* \\ &= \sum_{p=1}^P \left(c_p^P \mathbf{X}_{T,p}^* + \max_{S \in \mathcal{U}^p} \sum_{t=1}^T C_p(\mathbf{X}_{t,p}^{*int}, \mathbf{D}(S)_{t,p}) \right) - \mathcal{P}_p(\mathbf{X}_{T,p}^{*int}, \mathbf{D}(S)_{T,p}).\end{aligned}\quad (53)$$

Accordingly, the problem of computing a worst-case scenario $S^w \in \mathcal{U}$ for a given production plan $x^* \in \mathbb{X}$ boils down to solving the P separated problems. Namely,

$$\max_{S \in \mathcal{U}^p} \sum_{t=1}^T C_p(\mathbf{X}_{t,p}^{*int}, \mathbf{D}(S)_{t,p}) - \mathcal{P}_p(\mathbf{X}_{T,p}^{*int}, \mathbf{D}(S)_{T,p}) + c_p^P \mathbf{X}_{T,p}^*, \quad p \in [P]. \quad (54)$$

Let us regard p as fixed. It is evident that the objective function in (54) is convex in \mathcal{U}^p for a given production plan $x^* \in \mathbb{X}$ (see the proof of Proposition 4). Consequently, it attains its maximum value at a vertex of convex polytope \mathcal{U}^p (see, e.g. Martos 1975). Hence it suffices to examine only vertices of \mathcal{U}^p .

We now characterise vertices (vertex scenarios) of scenario set \mathcal{U}^p . We recall first that $v \in \mathcal{U}^p$ is a vertex (vertex scenario) of \mathcal{U}^p if v cannot be the strict convex combination of elements (scenarios) of \mathcal{U}^p . Thus the following lemma is obvious.

LEMMA 1 *Let $v \in \mathcal{U}^p \cap \prod_{t \in [T]} \{\underline{\mathbf{D}}_{t,p}, \bar{\mathbf{D}}_{t,p}\}$. Then v is a vertex of \mathcal{U}^p .*

LEMMA 2 *Suppose that $v = (v_t)_{t \in [T]} \in \mathcal{U}^p$ and there exists at least one component v_k of v such that $v_k \in (\underline{\mathbf{D}}_{k,p}, \bar{\mathbf{D}}_{k,p})$. Then v is a vertex of \mathcal{U}^p if and only if for each component v_k of v , $v_k \in (\underline{\mathbf{D}}_{k,p}, \bar{\mathbf{D}}_{k,p})$, in the subsequence $(v_i, \dots, v_{k-1}, v_k, v_{k+1}, \dots, v_l)$ of v , where $i = \min\{t \in [T] : v_t = v_k, t \leq k\}$ and $l = \max\{t \in [T] : v_t = v_k, k \leq t\}$, the first and the last elements are such that $v_i = \bar{\mathbf{D}}_{i,p}$ or $v_l = \underline{\mathbf{D}}_{l,p}$.*

Proof. See Appendix 1. □

The above lemma shows, among others, that there may exist a vertex scenario $S = (\mathbf{D}(S)_{t,p}) \in \mathcal{U}^p$, a candidate for an optimal solution to (54), with at least one cumulative demand $\mathbf{D}(S)_{k,p}$ in some period, say k , such that $\mathbf{D}(S)_{k,p} \in (\underline{\mathbf{D}}_{k,p}, \bar{\mathbf{D}}_{k,p})$. Therefore, for instance, Soyster's framework (Soyster 1973), which is a particular case of the method for robust linear problems proposed in Bertsimas and Sim (2004) and focuses also on a worst-case analysis, cannot be applied here. This framework assumes that every cumulative demand $\mathbf{D}_{t,p}$, $t \in [T]$ has to be taken its worst-case value in the interval $[\underline{\mathbf{D}}_{t,p}, \bar{\mathbf{D}}_{t,p}]$. Thus it may omit the vertex scenario described above or lead to a scenario S that violates conditions $\mathbf{D}(S)_{t-1,p} \leq \mathbf{D}(S)_{t,p}$, $t = 2, \dots, T$.

Using Lemmas 1 and 2, we can construct a layered graph $G^p = (V^p, A^p)$ that represents a subset of scenarios of set \mathcal{U}^p that contains all its vertex scenarios and during examining no vertex scenario can be omitted. The set V^p is partitioned into $T + 2$ disjoint layers $V_0^p, V_1^p, \dots, V_T^p, V_{T+1}^p$ in which $V_0^p = \{\mathfrak{s}\}$ and $V_{T+1}^p = \{\mathfrak{t}\}$ contain two distinguished nodes, \mathfrak{s} and \mathfrak{t} , and each node $u \in V_t^p$, $t \in [T]$, corresponds to exactly one possible value of the t th component, denoted by v_u^p , $v_u^p \in [\underline{\mathbf{D}}_{t,p}, \bar{\mathbf{D}}_{t,p}] \cap \bigcup_{t \in [T]} \{\underline{\mathbf{D}}_{t,p}, \bar{\mathbf{D}}_{t,p}\}$, of all the vertices of \mathcal{U}^p . Obviously, $|V_t^p| = |[\underline{\mathbf{D}}_{t,p}, \bar{\mathbf{D}}_{t,p}] \cap \bigcup_{t \in [T]} \{\underline{\mathbf{D}}_{t,p}, \bar{\mathbf{D}}_{t,p}\}|$, $t \in [T]$. Let $A^p = A_1^p \cup \dots \cup A_T^p \cup A_{T+1}^p$. Arc $(u, w) \in A_1^p$ if $u \in V_0^p$ and $w \in V_1^p$; $(u, w) \in A_{T+1}^p$ if $u \in V_T^p$ and $w \in V_{T+1}^p$; and arc $(u, w) \in A_t^p$, $t = 2, \dots, T$, if $u \in V_{t-1}^p$, $w \in V_t^p$ and $v_u^p \leq v_w^p$. An easy computation shows that A^p has $O(T^3)$ arcs and V^p has $O(T^2)$ nodes. We associate with each arc $(u, w) \in A^p$ length $l_{u,w}^p$ in the following way:

$$l_{u,w}^p = \begin{cases} C_p(\mathbf{X}_{t,p}^{*int}, v_w^p) & \text{if } (u, w) \in A_t^p, t = 1, \dots, T-1, \\ C_p(\mathbf{X}_{T,p}^{*int}, v_w^p) - \mathcal{P}_p(\mathbf{X}_{T,p}^{*int}, v_w^p) + c_p^P \mathbf{X}_{T,p}^* & \text{if } (u, w) \in A_T^p, \\ 0 & \text{if } (u, w) \in A_{T+1}^p \end{cases} \quad (55)$$

for a fixed $x^* \in \mathbb{X}$. The graph G^p models a subset of \mathcal{U}^p . Indeed, if $\mathfrak{s} = u_0 \rightsquigarrow u_1 \rightsquigarrow \dots \rightsquigarrow u_T \rightsquigarrow u_{T+1} = \mathfrak{t}$ is a path from \mathfrak{s} to \mathfrak{t} , then it corresponds to $v = (v_{u_t}^p)_{t \in [T]} \in \mathcal{U}^p$, $u_t \in V_t^p$, $t \in [T]$, and the length of this path, $\sum_{t \in [T]} l_{u_{t-1}, u_t}^p$, is equal to the value of the objective function in (54) for v . Since G^p has been built according to Lemmas 1 and 2, the subset modelled contains all the vertices of \mathcal{U}^p . Thus by the convexity of the objective function in (54) over \mathcal{U}^p , this subset contains all the optimal scenarios to problem (54). Hence solving (54) boils down to finding a longest path σ_p from \mathfrak{s} to \mathfrak{t} in G^p , which can be done in $O(T^3)$ (see, e.g. Ahuja, Magnanti, and Orlin 1993). From the decomposition of (53), we have an $O(P \cdot T^3)$ algorithm for the determining a worst-case scenario $S^w \in \mathcal{U}$ for a given production plan $x^* \in \mathbb{X}$ (the problem (33)). Worst-case scenario S^w corresponds to the concatenation $\sigma_1 \rightsquigarrow \dots \rightsquigarrow \sigma_P$ of each longest path σ_p in G^p , $p \in [P]$.

4.2.2 Computing an optimal robust production plan

We now deal with the problem of computing an optimal robust production plan \mathbf{x}^r to ROB MRP. Using (53), it can be decomposed in the following way:

$$\min_{x \in \mathbb{X}} \max_{S \in \mathcal{U}} F(x, S) = \min_{x \in \mathbb{X}} \sum_{p=1}^P c_p^P \mathbf{X}_{T,p} + \max_{S \in \mathcal{U}^P} \sum_{t=1}^T \mathcal{C}_t(\mathbf{X}_{t,p}^{int}, \mathbf{D}(S)_{t,p}) - \mathcal{P}_p(\mathbf{X}_{T,p}^{int}, \mathbf{D}(S)_{T,p}). \quad (56)$$

The inner problem in (56) corresponds to the problem of computing a worst-case scenario for a fixed $x^* \in \mathbb{X}$. The latter problem can be reduced to find the value of a longest path σ_p in each of the P layered weighted graphs $G^p = (V^p, A^p)$, $p \in [P]$ (see Section 4.2.1). Finding the length of a longest path σ_p from \mathfrak{s} to \mathfrak{t} in $G^p = (V^p, A^p)$ for a fixed p may be modelled by the following linear programming problem:

$$\begin{aligned} \min \quad & \pi_{\mathfrak{t}}^p & (57) \\ \text{s.t.} \quad & \pi_w^p - \pi_u^p \geq l_{u,w}^p & (u, w) \in A_t^p, t \in [T] \\ & \pi_{\mathfrak{t}}^p - \pi_u^p \geq 0 & u \in V_T^p \\ & \pi_{\mathfrak{s}}^p = 0 \\ & \pi_u^p \text{ unrestricted} & u \in V^p \end{aligned}$$

where lengths $l_{u,w}^p$, $(u, w) \in A^p$, are determined according to (55) for a given $\mathbf{x}^* \in \mathbb{X}$. The optimal value of decision variable $\pi_{\mathfrak{t}}^p$ is the length of longest path σ_p from \mathfrak{s} to \mathfrak{t} in G^p , i.e. $\pi_{\mathfrak{t}}^p = \max_{S \in \mathcal{U}^P} \sum_{t=1}^T \mathcal{C}_p(\mathbf{X}_{t,p}^{*int}, \mathbf{D}(S)_{t,p}) - \mathcal{P}_p(\mathbf{X}_{T,p}^{*int}, \mathbf{D}(S)_{T,p}) + c_p^P \mathbf{X}_{T,p}^*$. Substituting (55) into model (57), we obtain an equivalent linear programming model of the longest path problem in $G^p = (V^p, A^p)$ for fixed p and $\mathbf{x}^* \in \mathbb{X}$:

$$\begin{aligned} \min \quad & \pi_{\mathfrak{t}}^p & (58) \\ \text{s.t.} \quad & B_w^p - I_w^p = v_w^p - \mathbf{X}_{t,p}^{*int} & w \in V_t^p, t \in [T] & (59) \\ & \pi_w^p - \pi_u^p \geq c_p^I I_w^p + c_p^B B_w^p & (u, w) \in A_t^p, t = 1, \dots, T-1 & (60) \\ & \pi_w^p - \pi_u^p \geq c_p^I I_w^p + c_p^B B_w^p - b_p^P z_w^p + c_p^P \mathbf{X}_{T,p}^* & (u, w) \in A_T^p & (61) \\ & \pi_{\mathfrak{t}}^p - \pi_u^p \geq 0 & u \in V_T^p & (62) \\ & \pi_{\mathfrak{s}}^p = 0 & & (63) \\ & z_w^p \leq v_w^p & w \in V_T^p & (64) \\ & z_w^p \leq \mathbf{X}_{T,p}^{*int} & w \in V_T^p & (65) \\ & B_w^p, I_w^p \geq 0 & w \in V_t^p, t \in [T] & (66) \\ & z_w^p \geq 0 & w \in V_T^p & (67) \\ & \pi_u^p \text{ unrestricted} & u \in V^p & (68) \end{aligned}$$

where v_w^p , $w \in V_t^p$, $t \in [T]$ is a parameter, i.e. it is a possible value of the t th component, $v_w^p \in [\underline{\mathbf{D}}_{t,p}, \overline{\mathbf{D}}_{t,p}] \cap \bigcup_{t \in [T]} \{\underline{\mathbf{D}}_{t,p}, \overline{\mathbf{D}}_{t,p}\}$, of all the vertices of \mathcal{U}^p . In other words, v_w^p is a possible value of the cumulative demand in period t of product p . Constraint (59) and the right-hand side of (60) express the value of $\mathcal{C}_p(\mathbf{X}_{t,p}^{*int}, v_w^p)$. Constraint (59) together with the right-hand side of (61) and constraints (64), (65) represent the value of $\mathcal{C}_p(\mathbf{X}_{T,p}^{*int}, v_w^p) - \mathcal{P}_p(\mathbf{X}_{T,p}^{*int}, v_w^p) + c_p^P \mathbf{X}_{T,p}^*$, where v_w^p , $w \in V_T^p$, is a possible value of the cumulative demand in period T of product p , $v_w^p \in [\underline{\mathbf{D}}_{T,p}, \overline{\mathbf{D}}_{T,p}] \cap \bigcup_{t \in [T]} \{\underline{\mathbf{D}}_{t,p}, \overline{\mathbf{D}}_{t,p}\}$, (64) and (65) help to model the value of $\mathcal{P}_p(\mathbf{X}_{T,p}^{*int}, v_w^p)$. Solving linear programming model (58)–(68) for each $p \in [P]$ yields P longest paths $\sigma_1, \dots, \sigma_P$ whose concatenation $\sigma_1 \rightsquigarrow \dots \rightsquigarrow \sigma_P$ corresponds to worst case scenario S^w for a fixed production plan \mathbf{x}^* . Namely, if node $w \in \sigma_p$ and $w \in V_t^p$, $t \in [T]$, then $\mathbf{D}(S^w)_{t,p} = v_w^p$ and the values of decisions variables I_w^p and B_w^p are corresponding inventory and backordering levels, respectively, at the end of period t for product p , clearly if $I_w^p > 0$ (resp. $B_w^p > 0$) then $B_w^p = 0$ (resp. $I_w^p = 0$). Such a longest path in G^p can be identified (after solving (58)–(68)), if needed, by performing a breadth-first search of G^p using the arcs satisfying the equalities: $\pi_w^p - \pi_u^p = c_p^I I_w^p + c_p^B B_w^p$ or $\pi_w^p - \pi_u^p = c_p^I I_w^p + c_p^B B_w^p - b_p^P \min\{v_w^p, \mathbf{X}_{T,p}^{*int}\} + c_p^P \mathbf{X}_{T,p}^*$ or $\pi_{\mathfrak{t}}^p - \pi_u^p = 0$.

Making use of (56), combining (58)–(68) with the definition of the set of feasible production amounts \mathbb{X} (see (22)) and developing $\mathbf{X}_{t,p}^{int}, \mathbf{X}_{T,p}, t \in [T], p \in [P]$, we obtain a linear programming model for computing an optimal robust production plan x^r to ROB MRP:

$$\begin{aligned}
\min \quad & \sum_{p=1}^P \pi_t^p & (69) \\
\text{s.t.} \quad & B_w^p - I_w^p = v_w^p - \sum_{i=1}^t (x_{i,p} - \sum_{j=1}^P b_{p,j} x_{i+Ld_j,j}) & w \in V_t^p, t \in [T], p \in [P] \\
& \pi_w^p - \pi_u^p \geq c_p^I I_w^p + c_p^B B_w^p & (u, w) \in A_t^p, t = 1, \dots, T-1, p \in [P] \\
& \pi_w^p - \pi_u^p \geq c_p^I I_w^p + c_p^B B_w^p - b_p^P z_w^p + c_p^P \sum_{i=1}^T x_{i,p} & (u, w) \in A_T^p, p \in [P] \\
& \pi_t^p - \pi_u^p \geq 0 & u \in V_T^p, p \in [P] \\
& \pi_s^p = 0 & p \in [P] \\
& z_w^p \leq v_w^p & w \in V_T^p, p \in [P] \\
& z_w^p \leq \sum_{i=1}^t (x_{i,p} - \sum_{j=1}^P b_{p,j} x_{i+Ld_j,j}) & w \in V_T^p, p \in [P] \\
& B_w^p, I_w^p \geq 0 & w \in V_t^p, t \in [T], p \in [P] \\
& z_w^p \geq 0 & w \in V_T^p, p \in [P] \\
& \pi_u^p \text{ unrestricted} & u \in V^p, p \in [P] \\
& x \in \mathbb{X}
\end{aligned}$$

The decision variables $x_{j,p}, p \in [P], t \in [T]$, describe an optimal robust production plan x^r to ROB MRP and $\sum_{p=1}^P \pi_t^p$ provides $\max_{S \in \mathcal{U}} F(x^r, S)$. Meanings of the constraints and the rest of decision variables are the same as for model (58)–(68).

5. Experimental results

In this section, we present experiments illustrating our robust approach to the MRP problem with uncertain cumulative demand (ROB MRP, see Section 4.2) under the uncertainty representation \mathcal{U} defined in (31). We are given 12 products and 3 resources, $P = 12$ and $R = 3$. The bill of materials (the product structures) with the product costs and the lead times of the products are presented in Figure 2. The selling prices of the products are as follows: $b_1^P = 10,000$, $b_2^P = 5500$ and $b_p^P = 0$ for $p = 3, \dots, 12$. The backorder cost c_p^B of each product p is equal to 15% of its selling price. The inventory cost c_p^I of each product p is equal to 5% of the sum of the production cost of product p and all the components consumed when product p is produced. For instance, using the product structures shown in Figure 2, the inventory cost of product 5 is computed as follows: $c_5^I = 0.05(c_5^P + b_{12,5}c_{12}^P + b_{11,5}c_{11}^P) = 22.5$. The planning horizon consists of 23 periods, $T = 23$, with the capacity limits: $l_{t,1} = l_{t,2} = l_{t,3} = 0$ for $t = 1, 2$ and $l_{t,1} = 1000$, $l_{t,2} = l_{t,3} = 2000$ for $t = 3, \dots, 23$; $u_{t,1} = 10,000$ for $t = 1, \dots, 23$, $u_{t,2} = 9100$ for $t = 1, \dots, 14$, $u_{t,2} = 11,200$ for $t = 15, \dots, 23$ and $u_{t,3} = 10,300$ for $t = 1, \dots, 8$, $u_{t,3} = 9300$ for $t = 9, \dots, 14$, $u_{t,3} = 6800$ for $t = 15, \dots, 17$, $u_{t,3} = 10,800$ for $t = 18, \dots, 23$. There are no cumulative capacity limits in this example.

Regarding the uncertainty representation (31) of scenario set \mathcal{U} , cumulative demands are uncertain and modelled by intervals (see Section 3). In our experiments, the cumulative demand intervals have the following form $[\mathbf{D}_{t,p}^n - \underline{\Delta}_{t,p}, \mathbf{D}_{t,p}^n + \overline{\Delta}_{t,p}]$, where $\mathbf{D}_{t,p}^n = \sum_{i=1}^t d_{i,p}$ is the nominal cumulative demand and $d_{t,p}$ is given demand in period t . We assume that product 1 and 2 have external demand so we only provide in Table 3 demands $d_{t,p}$ for $p = 1, 2$. The rest of $d_{t,p}$, and in consequence $\mathbf{D}_{t,p}^n$, for $t = 1, \dots, 23$ and $p = 3, \dots, 12$ are equal to zero. The values of $\underline{\Delta}_{t,p}$ and $\overline{\Delta}_{t,p}$ are determined according to two kinds of uncertainty on the cumulative demand. In the first type $\underline{\Delta}_{t,p} = \overline{\Delta}_{t,p} = \alpha \mathbf{D}_{t,p}^n$, $t = 1, \dots, 23$, $p = 1, 2$, where $\alpha \in [0, 1]$ is a prescribed *uncertainty parameter* being the maximum percentage deviation of cumulative demands in periods from their nominal values. In the second type $\underline{\Delta}_{t,p} = \alpha d_{t,p}$, $t = 1, \dots, 23$, and $\overline{\Delta}_{t,p} = \alpha d_{t+1,p}$, $t = 1, \dots, 22$ and $\overline{\Delta}_{23,p} = 0$, $p = 1, 2$, where $\alpha \in [0, 1]$, which is the maximum percentage deviation of demands in periods from their nominal values. This type reflects uncertainty about occurring demands in periods, i.e. some of them may advanced or delayed. For instance, $\alpha = 0.5$ means half demand in a period can be moved to its adjacent periods. Therefore,

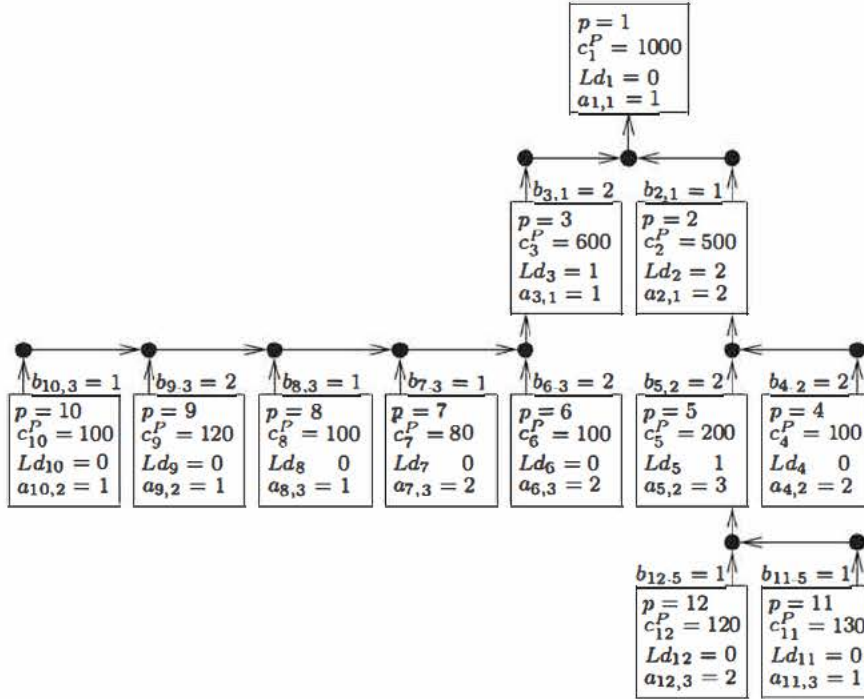


Figure 2. Bill of materials.

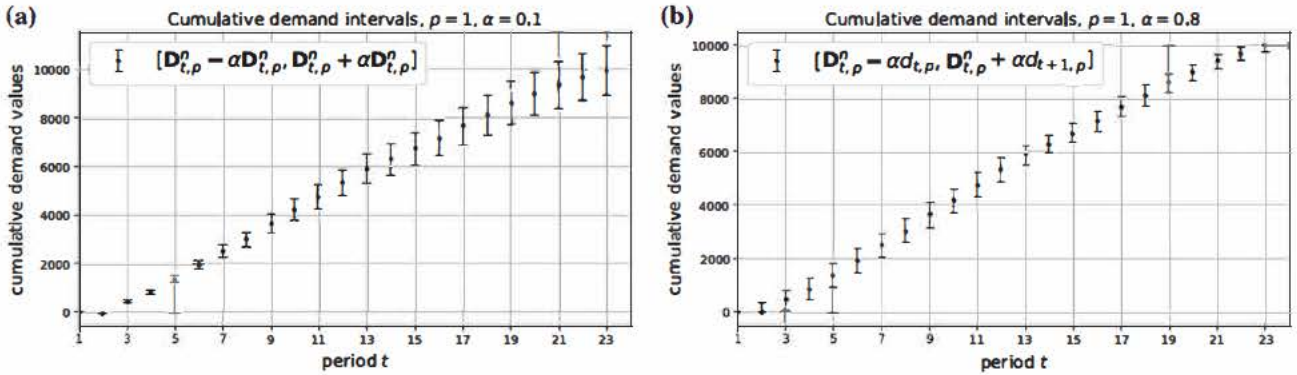


Figure 3. Cumulative demand intervals in periods for product $p=1$, where dots are the nominal cumulative demands $D_{t,1}^n$, $t=1, \dots, 23$. (a) The first type of uncertainty. (b) The second type of uncertainty.

such an amount of uncertainty is not too high. A sample of the cumulative demand intervals of product 1 for some values of α is depicted in Figure 3.

Solving linear programme (23)–(28) for the demands shown in Table 3 gives a *nominal production plan*, denoted by x^n , with the value of the objective function equal to -7.3935×10^7 . Thus the *nominal profit* of x^n equals 7.3935×10^7 . We will denote it by $P(x^n)$. The cumulative demand scenario induced by the demands in periods may be seen as a *nominal scenario*. We examined various values of uncertainty parameter α for two kinds of uncertainty in cumulative demand. Namely, for the first type, parameter $\alpha \in [0.01, 0.3]$ was generated with a step h equal to 0.01 as follows: $\alpha_i = 0.01 + ih$, $i = 0, \dots, 29$ (30 instances of the problem). For second type, $\alpha \in [0.1, 0.99]$ was generated with $h = 0.0307$ as follows: $\alpha_i = 0.1 + ih$, $i = 0, \dots, 29$ (30 instances of the problem).

Table 3. Demands in periods for $p = 1, 2$.

| t | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
|-----------|---|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $d_{t,1}$ | 0 | 0 | 457 | 434 | 511 | 564 | 540 | 527 | 622 | 585 | 550 | 592 | 527 | 400 | 420 | 473 | 473 | 486 | 478 | 387 | 360 | 317 | 284 |
| $d_{t,2}$ | 0 | 0 | 379 | 386 | 400 | 319 | 386 | 300 | 387 | 154 | 144 | 166 | 151 | 191 | 156 | 115 | 150 | 182 | 100 | 140 | 281 | 263 | 257 |

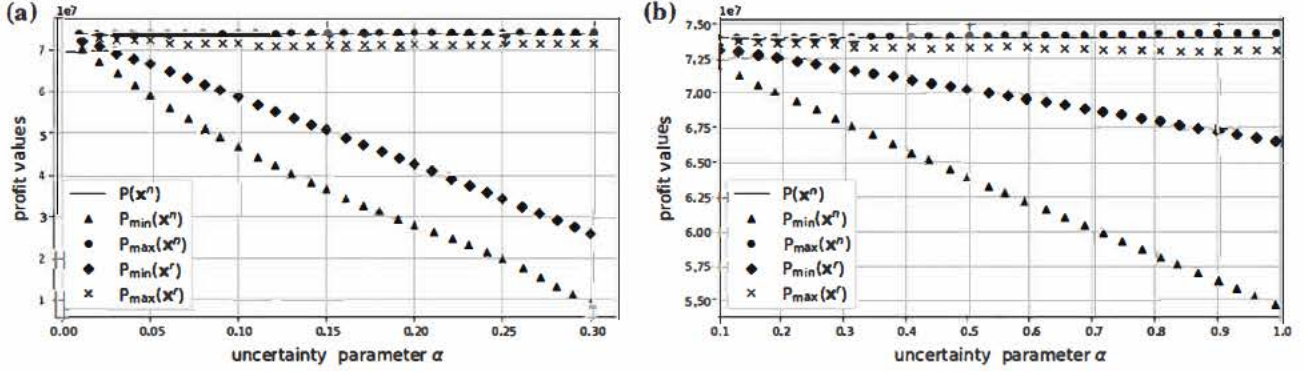


Figure 4. Profits of nominal and optimal robust production plans x^n and x^r , respectively, as functions of uncertainty parameter α . (a) The first type of uncertainty. (b) The second type of uncertainty.

For both kinds of uncertainty and for each value of α_i , we found optimal robust production plans x^r to ROB MRP by solving model (69) whose objective value equals $F(x^r, S^w)$, where $S^w \in \mathcal{U}$ is a worst-case scenario for x^r . Thus the *minimum profit* of x^r over scenario set \mathcal{U} , denoted by $P_{\min}(x^r)$ is equal to $-F(x^r, S^w)$. We computed an optimistic scenario $S^o \in \mathcal{U}$ for x^r , which determines its *maximum profit* over set \mathcal{U} , i.e. $P_{\max}(x^r) = -F(x^r, S^o)$ (see Section 4.2.1). Hence $[P_{\min}(x^r), P_{\max}(x^r)]$ is an interval containing possible values of profits for x^r over \mathcal{U} . Similarly for x^n , we determined its minimum and maximum profits, denoted by $P_{\min}(x^n)$ and $P_{\max}(x^n)$, respectively, by methods presented in Section 4.2.1, i.e. $P_{\min}(x^n) = -F(x^n, S^w)$ and $P_{\max}(x^n) = -F(x^n, S^o)$. The interval containing possible profits has the form $[P_{\min}(x^n), P_{\max}(x^n)]$. The profits of x^n and x^r as functions of uncertainty parameter α are illustrated in Figure 4. Furthermore, by means of the presented profits the intervals of possible profits: $[P_{\min}(x^r), P_{\max}(x^r)]$ and $[P_{\min}(x^n), P_{\max}(x^n)]$ can be easily identified for all the values of parameter α .

From Figure 4, it is obvious that, in both cases, even if the robust plan slightly decreases the maximal possible profit (the best case), the minimal profit (the worst case) is significantly improved. In other words, for this application the minmax criterion is not too conservative, since the maximum profit is relatively close to the profit of the nominal production plan. It remains the problem of choosing a robust plan (for what value of α). The decision-maker must be careful to take into account only possible scenarios. Therefore she/he must analyse the worst and the best scenarios in order to verify if they seem to be realistic. If they are not, then the decision-maker can modify the value of α or directly modify the upper and/or lower bound of the cumulative demands so that the scenarios become realistic. The flexibility of our uncertainty model allows the decision-maker to customise a distribution of uncertainty throughout a planning horizon.

Figure 5 shows relative reductions in the profits computed (in %), i.e. the ratios: $\text{red}^n = 100 \times (P(x^n) - P_{\min}(x^r))/P(x^n)$ and $\text{red}^r = 100 \times (P_{\min}(x^r) - P_{\min}(x^n))/P_{\min}(x^r)$, as uncertainty parameter α increases. It is interesting to note that for both types of uncertainty if we admit maximum protection against uncertainty, i.e. we choose an optimal robust production x^r , the profit of x^r linearly decreases as α increases (see Figure 4). Of course, it is normal to expect a decrease of profit of the robust solution as uncertainty increases, since we must protect ourselves more so the worst-case will be worse, but our experiments show that this decrease is consistently linear. It is obvious that parameter α reflects a percentage uncertainty introduced. Thus it is easy to check that the relative reduction of the nominal profit red^n approximately equals twice and 1/10 of the percentage uncertainty introduced for the first and the second types, respectively (Figure 5). For instance, for the first type for $\alpha = 0.05$ (the amount of uncertainty equals 5% of the nominal cumulative demands) the nominal profit 7.3935×10^7 is reduced by 9% to 6.6609×10^7 . For the second type for $\alpha = 0.22$ (the amount of uncertainty equals 22% of the nominal demands), the nominal profit is reduced by 2% to 7.2583×10^7 . Accordingly, the nominal profit is marginally affected in this case. The same conclusion can be drawn for the whole range of α (Figure 5(b)). If the decision-maker is

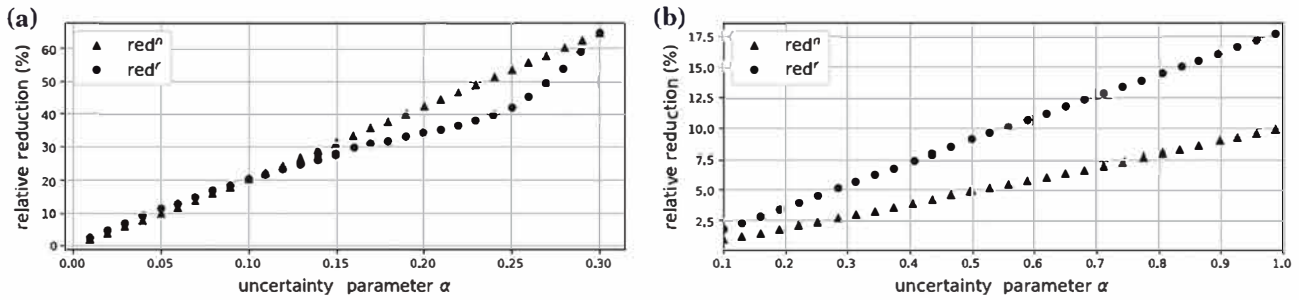


Figure 5. Relative reductions as functions of uncertainty parameter α . (a) The first type of uncertainty. (b) The second type of uncertainty.

optimistic and chooses the nominal production plan x^n , instead of an optimal robust plan x^r , then its profit may decrease when a worst case scenario reveals. The relative reduction of the minimum profit of optimal robust production plans red^r approximately equals twice and 1/5 of the percentage uncertainty introduced for the first and the second types, respectively (see Figure 5). For the latter type, the reduction red^n is twice higher than red^r (see Figure 5(b)). For instance, for the first type for $\alpha = 0.05$ the minimum profit of an optimal robust production plan 6.6609×10^7 is reduced by 11% to 5.8980×10^7 . For the second type for $\alpha = 0.22$, the minimum profit of an optimal robust production plan 7.2367×10^7 is reduced by 4% to 6.9449×10^7 .

All the experiments, including the methods runs (see Section 4.2), were implemented using the modelling package JuMP (Lubin and Dunning 2015) embedded in the programming language Julia. Within our implementations, we called IBM ILOG CPLEX 12.5 optimiser. The experiments were executed on a computer equipped with Intel Core 2 Duo 2.4 GHz. Computation time required to find nominal production plan x^n was 9.06 s. Average computation times for computing optimal robust production plans x^r under the first and second types of cumulative demand uncertainty were 10.6 and 9.88 s, respectively (over 30 problem instances for each type of uncertainty). Thus the ROB MRP problems are still computationally tractable.

6. Conclusions

In the context of supply chains, the problem of tactical production and capacity planning under uncertainty in the demand is a central concern for the production planning systems. In this paper, we have dealt with the MPS (CLSP) under small uncertainty in the cumulative demand and with MRP (MLCLSP) problems under uncertainty in the cumulative demand. The model of uncertainty in the cumulative demand enables to take into account jointly the imprecision on order quantities and dates. For both problems, linear programming models, which include backordering and the cumulative demand, have been presented. Further on, we have proposed efficient methods for evaluating the impact of uncertainty on production plans and linear programmes for computing optimal robust production plans for MPS and MRP problems under the uncertainty in the cumulative demand. Thus we have shown that introducing uncertainty in the cumulative demand to the above optimisation processes with the minmax criterion does not significantly increase their computational complexity, comparing with the deterministic counterparts, – they still remain polynomially solvable. Therefore, they can be applied in industrial context, i.e. in the manufacturing planning tools using linear programming solvers.

There are some opportunities for further research in production planning in the robust optimisation setting. Namely, analysing the above problems under less conservative robust criteria or considering the second source of uncertainty in production planning, pointed out in the literature, that is the uncertainty in the process (uncertain lead time and scrap of components) are good topics for further research.

Disclosure statement

No potential conflict of interest was reported by the authors.

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Appendix 1. Some proofs

Proof of Proposition 1 It is clear that problem (18)–(21) is a network flow problem. Thus using the results presented in Ahuja, Magnanti, and Orlin (1993, Chapter 19.9) one can model problem (18)–(21) as the shortest path problem in a network that has $T + 1$ nodes, where node 0 represents the source and the t th node for each $t \in [T]$ corresponds to the t th period. There are zero cost production arcs $(0, t)$ if $t \bmod \ell = 1$, $t \in [T]$; the inventory carrying arcs $(t - 1, t)$, $t = 2, \dots, T$, with cost c^I ; and the backorder carrying arcs $(t, t - 1)$, $t = 2, \dots, T$, with cost c^B in the network. It turns out (see Ahuja, Magnanti, and Orlin 1993, Chapter 19.9) that an optimal production

plan corresponds to a tree of shortest paths from the source to every node $t \in [T]$. Indeed, for each $k \in [N]$, where $N = (T - 1)/\ell$, we find period nodes:

$$h^k = \max\{t : t \in \{(k-1) \cdot \ell + 1, \dots, k \cdot \ell\}, c^I(t - (k-1) \cdot \ell - 1) \leq c^B(k \cdot \ell + 1 - t)\}.$$

The node h^k determines two paths. The first one $0 \rightsquigarrow (k-1)\ell + 1 \rightsquigarrow (k-1)\ell + 2 \rightsquigarrow \dots \rightsquigarrow h^k$ that goes through production and inventory carrying arcs and the second path $0 \rightsquigarrow k\ell + 1 \rightsquigarrow kP \rightsquigarrow \dots \rightsquigarrow h^k + 1$ that traverses production and backorder carrying arcs. Using a tree of shortest paths in the network, induced by $h^k, k \in [N]$, an optimal production plan to (18)–(21) can be determined by the following formula:

$$x_t = \begin{cases} \mathbf{D}_{h^1} & \text{if } t = 1, \\ \mathbf{D}_{h^k} - \mathbf{D}_{h^{k-1}} & \text{if } t = (k-1) \cdot \ell + 1 \text{ and } 2 \leq k \leq N, \\ \mathbf{D}_{h^T} - \mathbf{D}_{h^N} & \text{if } t = T, \\ 0 & \text{otherwise,} \end{cases} \quad t \in [T].$$

□

Proof of Proposition 2 We will show the convexity of $F(x^*, S)$ in \mathcal{U} for ROB MPS- L4L. Similar arguments apply to the objective function $F(x^*, S)$ in ROB MPS- POQ. Observe that the objective function for production plan x^* under scenario $S = (\mathbf{D}(S)_t)_{t \in [T]} \in \mathcal{U}$ can be rewritten as follows (see (8)): $\sum_{t=1}^T \mathcal{C}(\mathbf{X}_t^*, \mathbf{D}(S)_t) + (-\mathcal{P}(\mathbf{X}_T^*, \mathbf{D}(S)_T))$, where $\mathcal{C}(\mathbf{X}_t^*, \mathbf{D}(S)_t) = \max\{c^I(\mathbf{X}_t^* - \mathbf{D}(S)_t), c^B(\mathbf{D}(S)_t - \mathbf{X}_t^*)\}$ and $-\mathcal{P}(\mathbf{X}_T^*, \mathbf{D}(S)_T) = b^P \max\{-\mathbf{X}_T^*, -\mathbf{D}(S)_T\}$. Clearly, functions: $c^I(\mathbf{X}_t^* - \mathbf{D}(S)_t)$, $c^B(\mathbf{D}(S)_t - \mathbf{X}_t^*)$ and $-\mathbf{D}(S)_T$ are convex in \mathcal{U} . Hence the maximum and the sum of them in \mathcal{U} is also a convex function. □

Proof of Proposition 3 Proposition follows from the same as in Guillaume, Kobylański, and Zieliński (2012) reasoning. Scenarios $S \in \mathcal{U}_{\text{ext}}$ are easily seen to be the vertices of convex polytope \mathcal{U} . Hence and Proposition 2, we get that $F(x^*, S)$ attains the maximum value at a vertex of \mathcal{U} (see, e.g. Martos 1975). □

Proof of Proposition 4 Note that term $c_p^P \mathbf{X}_{T,p}^*$, $p \in [P]$, in $F(x^*, S)$ (see (29)) is a constant. Applying Proposition 2 for each $p \in [P]$ (taking into account slightly modified functions \mathcal{C}_p and \mathcal{P}_p), we deduce that $F(x^*, S)$ is the sum of P convex functions in \mathcal{U} and so $F(x^*, S)$ is also convex in \mathcal{U} . □

Proof of Lemma 2 Suppose contrary that there exists a component $v_k, v_k \in (\underline{\mathbf{D}}_{k,p}, \bar{\mathbf{D}}_{k,p})$, for which the subsequence $(v_i, \dots, v_{k-1}, v_k, v_{k+1}, \dots, v_l)$ has the first and the last elements such that $v_i \in [\underline{\mathbf{D}}_{i,p}, \bar{\mathbf{D}}_{i,p}]$ and $v_l \in (\underline{\mathbf{D}}_{l,p}, \bar{\mathbf{D}}_{l,p}]$. Fix $v_i = \underline{\mathbf{D}}_{i,p}$. Thus $\underline{\mathbf{D}}_{i,p} = v_i = v_k > \underline{\mathbf{D}}_{k,p}$, a contradiction with the fact that $\underline{\mathbf{D}}_{i,p} \leq \underline{\mathbf{D}}_{k,p}$. Similarly, fix $v_l = \bar{\mathbf{D}}_{l,p}$. Then $\bar{\mathbf{D}}_{l,p} = v_l = v_k < \bar{\mathbf{D}}_{k,p}$, a contradiction with $\bar{\mathbf{D}}_{l,p} \geq \bar{\mathbf{D}}_{k,p}$. Therefore $v_i \in (\underline{\mathbf{D}}_{i,p}, \bar{\mathbf{D}}_{i,p})$ and $v_l \in (\underline{\mathbf{D}}_{l,p}, \bar{\mathbf{D}}_{l,p})$. We need consider four cases:

- (i) $1 < i \leq l < T$. Thus $v_{i-1} < v_i$ and $v_l < v_{l+1}$. We construct $v^{(1)} = (v_t^{(1)})_{t \in [T]}$, $v^{(2)} = (v_t^{(2)})_{t \in [T]} \in \mathcal{U}^P$ and $v \neq v^{(1)}$, $v \neq v^{(2)}$ such that v lies between them. Set $v_t^{(1)} = v_t^{(2)} = v_t$ for $t = 1, \dots, i-1, l+1, \dots, T$; $v_i^{(1)} = \min\{\bar{\mathbf{D}}_{i,p}, v_{l+1}\}$ and $v_i^{(2)} = \max\{v_{i-1}, \underline{\mathbf{D}}_{l,p}\}$ for $t = i, \dots, l$. Clearly, $v^{(1)}, v^{(2)} \in \mathcal{U}^P$ and $v \neq v^{(1)}$, $v \neq v^{(2)}$. Moreover, it is easily seen that there exist $\lambda' \in (0, 1)$, $\lambda' = (v_i - v_i^{(2)}) / (v_i^{(1)} - v_i^{(2)})$, such that $v = \lambda' v^{(1)} + (1 - \lambda') v^{(2)}$ which contradicts that v is a vertex of \mathcal{U}^P .
- (ii) $1 < i \leq l = T$ and so $v_{i-1} < v_i$. Set $v_t^{(1)} = v_t^{(2)} = v_t$ for $t = 1, \dots, i-1$; $v_i^{(1)} = \bar{\mathbf{D}}_{i,p}$ and $v_i^{(2)} = \max\{v_{i-1}, \underline{\mathbf{D}}_{l,p}\}$ for $t = i, \dots, T$. Obviously, $v^{(1)}, v^{(2)} \in \mathcal{U}^P$ and $v \neq v^{(1)}$, $v \neq v^{(2)}$. It easy to check that there exist $\lambda' \in (0, 1)$ such that $v = \lambda' v^{(1)} + (1 - \lambda') v^{(2)}$, a contradiction with the fact that v is a vertex of \mathcal{U}^P .
- (iii) $1 = i \leq l < T$ and so $v_l < v_{l+1}$. Set $v_t^{(1)} = v_t^{(2)} = v_t$ for $t, l+1, \dots, T$; $v_i^{(1)} = \min\{\bar{\mathbf{D}}_{i,p}, v_{l+1}\}$ and $v_i^{(2)} = \underline{\mathbf{D}}_{l,p}$ for $t = 1, \dots, l$. Again a trivial verification shows that v lies between $v^{(1)}$ and $v^{(2)}$, a contradiction.
- (iv) $1 = i \leq l = T$. Set $v_t^{(1)} = \bar{\mathbf{D}}_{i,p}$ and $v_t^{(2)} = \underline{\mathbf{D}}_{l,p}$ for $t \in [T]$. Similarly to the above cases, we obtain a contradiction.

Suppose that v is not a vertex of \mathcal{U}^P . Thus there exist $v^{(1)}, v^{(2)} \in \mathcal{U}^P$ and $v \neq v^{(1)}$, $v \neq v^{(2)}$ and $\lambda' \in (0, 1)$ such that $v = \lambda' v^{(1)} + (1 - \lambda') v^{(2)}$. By assumption, for each sequence $(v_i, \dots, v_{k-1}, v_k, v_{k+1}, \dots, v_l)$ associated with a component $v_k \in (\underline{\mathbf{D}}_{k,p}, \bar{\mathbf{D}}_{k,p})$ we have $v_i = \bar{\mathbf{D}}_{i,p}$ or $v_l = \underline{\mathbf{D}}_{l,p}$. Hence $v_t^{(1)} = v_t^{(2)} = v_t$ for $t = i, \dots, l$. The other components of v are at their boundary values. Therefore $v = v^{(1)}$ and $v = v^{(2)}$, a contradiction. □