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# On the symmetry of Wigner (3-j) and Racah (6-j) coefficients

M. Hage-Hassan

Université Libanaise, Faculté des Sciences Section (1)  
Hadath-Beyrouth

## Abstract

The Clebsh-Gordan coefficient formulas are determined using the integration of the product of three Wigner D-matrix. So we determine the generating function of these coefficients and the explicit expression of Van der Waerden formula for Wigner 3-j coefficient is derived simply. We also determine a new generating function of Wigner 6-j, by the coupling of three angular moments, and we derive without any difficulty Racah's formula for 6-j. We also propose a new method for the determination of the symmetries of the symbols 3-j and 6-j only from the expressions of these formulas. We also prove the connection of the coupling coefficients in bosons basis (Schwinger) and the analytic Fock space.

## 1. Introduction

The angular momentum has been studied for long time ago by many scientists and by different methods [1-11]. Many of the formulas of the Clebsh-Gordan and 6-j coefficients have been found by several authors [7-13] and by different methods but the calculation methods are generally difficult [9, 10]. In addition, the study of symmetries has been studied mainly by Racah, Wigner, Regge and others [5, 9-13].

This work is the continuation of my paper on the angular momentum for the determination of formulas for coefficients 3-j and 6-j symbols [14-15]. Using the triple integral of Wigner D-matrix we determine the generating functions (GF) of 3-j and the Van der Waerden formula [9]. We determine by the coupling of three angular moments in the analytic Fock space, the GF of 6-j, and the Racah formula for 6-j [9].

We propose a new simple method for the study of the symmetries of these coefficients only using the Van der Waerden formula for the symbols 3-j and the Racah formula for the study of the 6-j symmetry.

We also summarize the method for the determination of other expressions of the 3-j symbols from the triple integral of Wigner D-matrix.

The connection between the boson space and the Fock space has been studied for a long time [16] for the study of "Anharmonic vibrations in nuclei" [16-18]. We will apply our method for the connection of the expression of the coupling coefficient in bosons (Schwinger) and in Fock-Bargmann space.

The plan of this paper is as follows: We begin, in sections 2 and 3, by reviewing the angular momentum and the coupling of angular momentum. In section 4, we derive the Van der Waerden formula for 3-j symbols. In sections 5, we study the symmetries of Clebsh-Gordan. In section 6 we show the derivation of some other formulas for the three symbols. In section 7 and 8 we derive the Racah formula for the 6-j and we study the

symmetries of these symbols. In the appendices we deal with the connection between the boson spaces and the analytic Fock space.

## 2. The representation of SU(2) and the generating function of Wigner's D-matrix

We starting from the representation of SU (2) in the analytic Fock's space [7, 14-15] For the determination of Wigner's D-matrix and its generating function.

### 2.1- The Analytic Hilbert space

The Analytic Oscillator space (or Fock space)

Is defined by:

$$u_n(z) = \frac{z^n}{\sqrt{n!}} \quad (2.1)$$

With scalar product is:

$$(u_{n'}, u_n) = \int \frac{\bar{z}_{n'}}{\sqrt{n'!}} \frac{z_n}{\sqrt{n!}} d\mu(z) = \delta_{n,n'}, \quad (2.2)$$

And  $d\mu(z)$  is the measure of integration:

$$d\mu(z) = \left(\frac{1}{\pi}\right) \exp[-\bar{z}z] dx dy, \quad z = x + iy \quad (2.3)$$

Furthermore, it is easy to check the useful relationship:

$$\int \exp[\alpha z + \beta \bar{z}] d\mu(z) = e^{\alpha\beta} \quad (2.4)$$

It is important to note that the functions  $\{u_n(z)e^{-\bar{z}z/2}\}$  are the wave functions of subspaces of the cylindrical basis of the Harmonics oscillators [19].

### 2.2 The representation of SU(2) in the analytic Hilbert space

We define the generators of SU(2) in the analytic Hilbert space,  $u_i, i = 1, 2$ , by:

$$J_1 = \left(u_1 \frac{1}{\partial u_2} - u_2 \frac{1}{\partial u_1}\right), \quad J_2 = \frac{1}{i} \left(u_1 \frac{1}{\partial u_2} - u_2 \frac{1}{\partial u_1}\right), \quad J_3 = \frac{1}{2} \left(u_1 \frac{1}{\partial u_1} - u_2 \frac{1}{\partial u_2}\right) \quad (2.5)$$

And

$$J^2 = N(N+1), \quad N = \frac{1}{2} \left(u_1 \frac{1}{\partial u_1} + u_2 \frac{1}{\partial u_2}\right) \quad (2.6)$$

We find that:

$$J^2 \varphi_{jm}(u) = j(j+1) \varphi_{jm}(u), \quad J_3 \varphi_{jm}(u) = m \varphi_{jm}(u) \quad (2.7)$$

With  $\varphi_{jm}(u)$  is the representation of SU(2) in the analytic Hilbert space with:

$$\varphi_{jm}(u) = \langle u || jm \rangle = \frac{u_1^{(j+m)} u_2^{(j-m)}}{\sqrt{(j+m)!(j-m)!}} \quad (2.8)$$

It is important to note that the space of the bosons was introduced by Schwinger for the study of the angular moment with:

$$|jm\rangle = \frac{a_1^{+(j+m)} a_2^{+(j-m)}}{\sqrt{(j+m)!(j-m)!}} |00\rangle \quad (2.9)$$

### 2.3- The Wigner's D-matrix

The Wigner's D-matrix [1-10] of the rotation  $R(\Omega) = e^{i\psi J_z} e^{i\theta J_y} e^{i\varphi J_z}$  is given in Dirac notations by:

$$D_{(m',m)}^j(\Omega) = (jm|R(\Omega)|jm) = e^{im'\psi} d_{(m',m)}^j(\theta) e^{im\varphi} \quad (2.10)$$

With  $(\psi, \theta, \varphi)$  are the angles of Euler.

The four dimensions analytic Hilbert space is:

$$D_{(m',m)}^j(z) = \rho^{2j} D_{(m',m)}^j(\Omega), \quad \Omega = (\psi, \theta, \varphi) \quad (2.11)$$

$$\text{With: } \rho^{2j} R(\Omega) \varphi_{jm}(u) = \frac{(z_1 u_1 + z_2 u_2)^{j+m} (-\bar{z}_2 u_1 + \bar{z}_1 u_2)^{j-m}}{\sqrt{(j+m)!(j-m)!}} \quad (2.12)$$

$$\text{And } z_1 = x_1 + iy_1 = \rho \cos \frac{\theta}{2} e^{i(\psi+\varphi)/2}, \quad z_2 = x_2 + iy_2 = \rho \sin \frac{\theta}{2} e^{i(\psi-\varphi)/2} \quad (2.13)$$

From formula (2.12) we deduce the expression:

$$d_{(m',m)}^j(\theta) = \sqrt{(j+m')!(j-m')!(j+m)!(j-m)!} (\cos(\theta/2))^{-m'+m+2j} (-\sin(\theta/2))^{m'-m} \\ \times \sum_s \frac{(-1)^s}{s!(j-m'-s)!(j+m-s)!(m'-m+s)!} (\tan^2(\theta/2))^s \quad (2.14)$$

We can also write this expression in terms of hypergeometric functions by:

:

$$d_{(m',m)}^j(\theta) = \left[ \frac{(j-m)!(j+m')!}{(j+m)!(j-m')!} \right]^{\frac{1}{2}} \frac{(\cos\theta/2)^{2j+m-m'} (\cos\theta/2)^{m'-m}}{(m'-m)!} \times \\ {}_2F_1\left(- (j-m'), -(j+m); m'-m+1; -\tan^2 \theta/2\right). \quad (2.15)$$

We emphasize that we have identified using Wigner d-matrix symmetries twenty-four formulas [22] which are solutions of the Gauss equation.

### 2.4- The generating function of Wigner's D-matrix

Multiplying (2.12) by  $\varphi_{jm'}(v)$  and summing with respect to  $j, m, m'$  we find the generating function of Wigner's D-matrix

$$G(v, z, u) = \exp \left[ (v_1 \quad v_2) \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right] = \sum_{j,m,m'} \varphi_{jm'}(v) D_{(m',m)}^j(z) \varphi_{jm}(u) \quad (2.16)$$

## 3. The coupling of angular momentum and the Clebsch-Gordan coefficient

We will determine the coupling of two angular moments and the coefficients of the Clebsch-Gordan and 3-j symbols. And we write the triple integral formula of the Wigner D-matrix, in terms of the symbols 3-j, and which is very useful for the calculation of these coefficients.

### 3.1 The coupling of two angular momentums and the coefficient de clebsch-Gordan

The eigenfunctions of total angular momentum  $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2$  is  $\varphi_{(j_1 j_2) j_3 m_3}(u^1, u^2)$ , with:

$$\varphi_{(j_1 j_2) j_3 m_3}(u^1, u^2) = \sum_{m_1 m_2} \langle j_1 m_1 j_2 m_2 | (j_1 j_2) j_3 m_3 \rangle \varphi_{j_1 m_1}(u^1) \varphi_{j_2 m_2}(u^2). \quad (3.1)$$

With  $m_3 = m_1 + m_2$ ,  $j_3 \geq \lfloor m_3 \rfloor$ , and  $j_3 \geq \lfloor j_1 - j_2 \rfloor$

$$\langle j_1 m_1 j_2 m_2 | (j_1 j_2) j_3 m_3 \rangle \text{ or } \langle j_1 m_1 j_2 m_2 | (j_1 - j_2) j_3 m_3 \rangle \quad (3.2)$$

The coefficient (3.2) is the clebsh-Gordan and  $\varphi_{j_1 m_1}(u^1) \varphi_{j_2 m_2}(u^2)$  is the two uncoupled representation of SU(2).

### 3.2 The Wigner 3-j symbols

The study of symmetries of these coefficients is facilitated by the use of the Wigner 3-j symbols [9].

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_1 - j_2 - m_3}}{\sqrt{2j_3 + 1}} \langle j_1 m_1 j_2 m_2 | (j_1 - j_2) j_3 (-m_3) \rangle \quad (3.3)$$

### 3.3 The triple integral formula for Wigner D-matrix

We deduce from the orthogonality of matrices D ( $\Omega$ ) the relations which are very important for the calculation of the symbols of 3-j.

$$\begin{aligned} \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^\pi \int_0^{2\pi} D_{(m'_1, m_1)}^{j_1}(\Omega) D_{(m'_2, m_2)}^{j_2}(\Omega) D_{(m'_3, m_3)}^{j_3}(\Omega) d\Omega, d\Omega = d\psi \sin \theta d\theta d\varphi \\ = \begin{pmatrix} j_1 & j_2 & j_3 \\ m'_1 & m'_2 & m'_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \end{aligned} \quad (3.4)$$

## 4- The generating function of Wigner 3-j symbol and the Van der Waerden's formula.

There are several formulas for the Clebsh-Gordan [8-12] but we will determine the Van der Waerden formula by developing the generating function of these coefficients.

### 4.1 The generating function of Wigner 3-j symbol

Using (2.16) and (3.4) we find:

$$\begin{aligned} \int \prod_{i=1}^3 [G(u^i, z, v^i)] d\mu(z_1) d\mu(z_2) = \\ \exp \left[ [v^2, v^3] [u^2, u^3] + [v^3, v^1] [u^3, u^1] + [v^1, v^2] [u^1, u^2] \right] \end{aligned} \quad (4.1)$$

or

$$\int \exp[\det(\bar{t}, v)] \times \exp[\det(t, u)] d\mu(t) = \int G_{3j}(\bar{t}, v) \times G_{3j}(t, u) d\mu(t) \quad (4.2)$$

$$\text{With} \quad \exp[\det(t, u)] = \begin{vmatrix} t_1 & t_2 & t_3 \\ u_1^1 & u_1^2 & u_1^3 \\ u_2^1 & u_2^2 & u_2^3 \end{vmatrix} \quad (4.3)$$

C- The function  $G_{3j}(t, u)$  is the generating function of the 3-j symbols [6, 7].

$$\begin{aligned} G_{3j}(t, u) = \exp[\det(t, u)] = \exp[t_1 [u^2, u^3] + t_2 [u^3, u^1] + t_3 [u^1, u^2]] \\ = \sum_{j_1 m_1} \varphi_{j_1 m_1}(u^1) \varphi_{j_2 m_2}(u^2) \varphi_{j_3 m_3}(u^3) \Phi(t_1 t_2 t_3) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \end{aligned} \quad (4.4)$$

And 
$$\Phi(t_1 t_2 t_3) = [(J+1)!]^{1/2} \frac{t_1^{J-2j_1} t_2^{J-2j_2} t_3^{J-2j_3}}{[(J-2j_1)!(J-2j_2)!(J-2j_3)!]^{1/2}} \quad (4.5)$$

With  $J=j_1+j_2+j_3$ ,  $[u^i, u^j] = u_1^i u_2^j - u_2^i u_1^j$  and  $(u^i \cdot u^j) = u_1^i u_2^j + u_1^j u_2^i$  (4.6)

The invariance of the determinant by permutation of the columns and the rows of (4.3) allowed Regge [4] to deduce the symmetries of the symbols 3-j, as we write below.

#### 4.2 The first derivation of the expression of Clebsh-Gordan coefficients

The development of the first member of the expression (4.4) and its identification with the second member gives us:

$$\begin{aligned} & [u^2, u^3]^{J-2j_1} [u^3, u^1]^{J-2j_2} [u^1, u^2]^{J-2j_3} = \sqrt{(J+1)! \Delta(j_1, j_2, j_3)} \times \\ & \sum_{j_m} \varphi_{j_1 m_1}(u^1) \varphi_{j_2 m_2}(u^2) \varphi_{j_3 m_3}(u^3) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \\ & = \sum_{j_m} \frac{(u_1^2 u_2^3)^t (-u_1^3 u_2^2)^{J-2j-t}}{t!(J-2j_1-t)!} \frac{(u_1^3 u_2^1)^p (u_2^3 u_1^1)^{J-2j_2-p}}{p!(J-2j_2-p)!} \frac{(u_1^1 u_2^2)^q (u_2^1 u_1^2)^{J-2j_3-q}}{q!(J-2j_3-q)!} \end{aligned} \quad (4.7)$$

Identification with the first term gives:

$$p = j_1 - j_2 + t + m_3, \quad q = j_1 - j_3 + t - m_2. \quad (4.8)$$

Therefore: 
$$\sqrt{(J+1)! \Delta(j_1, j_2, j_3)} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \quad (4.9)$$

$$\left[ \sum_t (-1)^{J-p-q} \frac{1}{t!(j_1 - j_2 - m_3 + t)!(j_1 - j_3 - m_2 + t)!(j_3 + m_3 - t)!(j_2 + m_2 - t)!(j_3 + j_2 - j_1 - t)!} \right]$$

$$\Delta(abc) = \left[ \frac{(a+b-c)!(a-b+c)!(-a+b+c)!}{(a+b+c+1)!} \right]^{\frac{1}{2}} \quad (4.10)$$

On putting  $s = j_2 + m_2 - t$  and  $m_3$  par  $-m_3$  we derive the well known:

Van der Waerden's formula of the Clebsh-Gordan:

$$\begin{aligned} & \langle j_1 m_1 j_2 m_2 | (j_1 - j_2) j_3 m_3 \rangle = \delta(m_1 + m_2, m_3) \left[ \frac{(2j_3 + 1) \Delta(j_1, j_2, j_3)}{(j_1 + j_2 + j_3 + 1)!} \right]^{\frac{1}{2}} \\ & \times [(j_1 + m_1)!(j_1 - m_1)!(j_2 + m_2)!(j_2 - m_2)!(j_3 + m_3)!(j_3 - m_3)!] \\ & \times \left[ \sum_s (-1)^s \frac{1}{s!(j_1 + j_2 - j - s)!(j_1 - m_1 - s)!(j_2 + m_2 - s)!(j_3 - j_2 + m_1 + s)!(j_3 - j_1 - m_2 + s)!} \right] \end{aligned} \quad (4.11)$$

It is quite clear that our method of deriving the Van der Waerden's formula is very simple by comparison with the complexity of the other methods [9].

## 5- The symmetries of the 3-j symbols

We can determine the symmetries of the Clebsh-Gordan coefficients or the symbols 3-j from all the formulas of these coefficients [4-9] but we will limit ourselves to Van der Waerden's formula (4.11). So we look transformations that leave invariant the product of the summation formula (4.11).

For this we put:

$$AX = B \quad (5.1)$$

$$\begin{pmatrix} 1 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} (j_1 + j_2 - j_3) \\ (j_1 - m_1) \\ (j_2 + m_2) \\ (j_3 - j_2 + m_1) \\ (j_3 - j_1 - m_2) \\ 0 \end{pmatrix} \quad (5.2)$$

$$\text{With} \quad \begin{pmatrix} a & b & c \\ u_1 & u_2 & u_3 \end{pmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \quad (5.3)$$

### 5.1 The symmetries of the three symbols

a- It is simple to verify from (4.9) and (4.11) that:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} \quad (5.4)$$

b- Symmetries by permutation

The symmetries by permutation of the components of  $X = (j_1, j_2, j_3, m_1, m_2, m_3)$  give:

$$\begin{aligned} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} &= (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix} = \\ &(-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_3 & j_2 \\ m_1 & m_3 & m_2 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_3 & j_1 & j_1 \\ m_3 & m_1 & m_1 \end{pmatrix} \end{aligned} \quad (5.5)$$

### 5.2 Regge symmetries

We also find  $(a, b, c, u_1, u_2, u_3)$  from the permutations of the components of the column B which leave the coefficients 3-j invariant.

For exemple:

$$\mathbf{a} \cdot {}^t\mathbf{B} = ((j_1 - m_1) \quad (j_1 + j_2 - j_3) \quad (j_2 + m_2) \quad (j_3 - j_2 + m_1) \quad (j_3 - j_1 - m_2) \quad (0)) \quad (5.6)$$

We derive the formula.

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_1 & \frac{j_2 + j_3 - m_1}{2} & \frac{j_2 + j_3 + m_1}{2} \\ j_3 - j_2 & \frac{j_2 - j_3 - m_1}{2} - m_3 & \frac{j_2 - j_3 + m_1}{2} + m_3 \end{pmatrix} \quad (5.7)$$

$$\mathbf{b} \cdot {}^t\mathbf{B} = ((j_1 - m_1) \quad (j_2 + m_2) \quad (j_1 + j_2 - j_3) \quad (j_3 - j_2 + m_1) \quad (j_3 - j_1 - m_2) \quad (0)) \quad (5.8)$$

We find also:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} \frac{j_2+j_3+m_1}{2} & \frac{j_1+j_3+m_2}{2} & \frac{j_1+j_2+m_3}{2} \\ j_1 - \frac{j_2+j_3-m_1}{2} & j_2 - \frac{j_1+j_3-m_2}{2} & j_3 - \frac{j_1+j_2-m_3}{2} \end{pmatrix} \quad (5.9)$$

Note that Regge has studied the symmetries of the 3-j symbols starting from the invariance of the determinant [4].

## 6- Triple integral of the Wigner D-Matrix and The expressions of the Clebsh-Gordan coefficients

In this part, we present the calculation of the Clebsh-Gordan coefficients by a direct calculation using the triple integral formula of the Wigner D-Matrix and its symmetries.

By posing in expression (3.4)  $m'_1 = j_1$ ,  $m'_2 = j_2$ ,  $m'_1 + m'_2 + m'_3 = 0$  we find:

$$\begin{aligned} \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^\pi \int_0^{2\pi} d_{(j_1, m_1)}^{j_1}(\theta) d_{(j_2, m_2)}^{j_2}(\theta) d_{(-j_1-j_2, m_3)}^{j_3}(\theta) d\Omega \\ = \begin{pmatrix} j_1 & j_2 & j_3 \\ j_1 & j_2 & -j_1-j_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \end{aligned} \quad (6.1)$$

From expression (2.15) we find that:

$$D_{(j, m_1)}^j(\theta) = \sqrt{\frac{(2j)!}{(j+m)!(j-m)!}} (\cos(\theta/2))^{j+m} (\sin(\theta/2))^{j-m} \quad (6.2)$$

$$\text{And } D_{(-j, m_1)}^j(\theta) = (-1)^{j+m} \sqrt{\frac{(2j)!}{(j+m)!(j-m)!}} (\cos(\theta/2))^{j-m} (\sin(\theta/2))^{j+m} \quad (6.3)$$

By setting  $m_1 = j_1$ ,  $m_2 = -j_2$  and  $m_3 = -m_1 - m_2$  and using (4.8) and (6.2), (6.3) we find that:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ j_1 & -j_2 & -j_1+j_2 \end{pmatrix} = \sqrt{\frac{(2j_1)!(2j_2)!}{(j_1+j_2-j_3)!(j_1+j_2+j_3+1)!}} \quad (6.4)$$

As we have already pointed out that the d-matrix can be expressed using several formulas [22]. Then we get several expressions of the symbols 3-j.

In addition the calculation of (6.1) is carried out using a Beta functions [21-22]:

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = 2 \int_0^{\frac{\pi}{2}} \sin^{2x-1} \varphi \sin^{2y-1} \varphi d\varphi \quad (6.5)$$

Note that the symmetries of the 3-j symbols can also be deduced using these formulas.

## 7- The Racah's formula for 6-j symbols

We summarize the coupling of three angular moments [6, 14, 15], and the derivation of the Racah coefficients and its compact form in terms of 6-j.



### 7.1 The coupling of three angular momentums and Racah coefficients

The coupling of three angular moments is carried out by two different ways and we arrive at two different states. And the Racah coefficients are the coefficient of passage from one representation to another.

The coupling of three angular moments can be done in two ways:

$$\vec{J} = (\vec{J}_1 + \vec{J}_2) + \vec{J}_3 = \vec{J}_1 + (\vec{J}_2 + \vec{J}_3) \text{ With } \vec{J}_{12} = \vec{J}_1 + \vec{J}_2, \text{ And } \vec{J}_{23} = \vec{J}_2 + \vec{J}_3 \quad (7.1)$$

1- The base of the first representation is:  $|(j_1 j_2) j_{12} j_3; j\rangle$

2- The basis of the second representation is:  $|j_1 (j_2 j_3) j_{23}; j\rangle$

The Racah coefficient is defined by:

$$\langle (j_1 j_2) j_{12} j_3; j | j_1 (j_2 j_3) j_{23}; j \rangle = \sqrt{(2j_{12} + 1)(2j_{23} + 1)} W(j_1 j_2 j_3 j; j_{12} j_{23}) \quad (7.2)$$

### 7.2 The generating function of the non-coupled representation

The generating function of the first representation is deduced from the non-coupled representation

$$\Psi = \exp \left[ (t^1 u^1) + (t^2 u^2) + (t^3 u^3) \right] \quad (7.3)$$

By application on  $\Psi$

$$\exp \left[ \alpha_3 \left[ \frac{\partial}{\partial t^1}, \frac{\partial}{\partial t^2} \right] + \alpha_2 (x.u^1) + \alpha_1 (x.t^2) \right] \quad (7.4)$$

And

$$\exp \left[ \beta_3 \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial t^3} \right] + \beta_2 (y.x) + \beta_1 (y.t^3) \right] \quad (7.5)$$

So we get the generating function of the base  $|(j_a j_b) j_{ab} j_c; j\rangle$  :

$$\begin{aligned} \Psi_1(u) = \sum_j (2j+1)^{1/2} \Phi_{j_1 j_2 j_{12}}(\alpha) \Phi_{j_{12} j_3 j}(\beta) \phi_{jm}(z) \Phi(j_1 j_2) j_{12} j_3; j(u) = \\ \exp \left[ \alpha_3 [u^1, u^2] + \beta_3 \alpha_2 [u^1, u^3] + \beta_3 \alpha_1 [u^2, u^3] + \right. \\ \left. + \beta_2 \alpha_2 (y.u^1) + \beta_2 \alpha_1 (y.u^2) + \beta_1 (y.u^3) \right] = \exp \{Q_1(u)\} \end{aligned} \quad (7.6)$$

Likewise the generating function of the representation  $|j_a (j_b j_c) j_{bc}; j\rangle$  is:

$$\begin{aligned} \Psi_2 = \sum_j (2j+1)^{1/2} \Phi_{j_2 j_3 j_{23}}(\alpha') \Phi_{j_1 j_{23} j}(\beta') \phi_{jm}(\bar{z}) |j_1, (j_2 j_3) j_{23}; j\rangle = \\ \exp \left[ \alpha'_3 [u^2, u^3] + \beta'_3 \alpha'_2 [u^2, u^1] + \beta'_3 \alpha'_1 [u^3, u^1] \right. \\ \left. + \beta'_2 \alpha'_2 (y.u^2) + \beta'_2 \alpha'_1 (y.u^3) + \beta'_1 (y.u^1) \right] = \exp \{Q_2(u)\} \end{aligned} \quad (7.7)$$

### 7.3 The generating function of the coupled representation

The generating function of the 6-j symbols is:

$$G_{6j} = \langle \Psi_2 | \Psi_1 \rangle = \int d\mu(y) \left[ \bar{\Psi}_2 \Psi_1 \prod_{i=1, j=1, 2}^3 du_j^i \right] \quad (7.8)$$

we make the change of the variables  $u_j^i$  by  $\bar{u}_2^i$   $j=1..3$  and we denote the column (u) by  $(u_1^1 \ u_1^2 \ u_1^3 \ u_2^1 \ u_2^2 \ u_2^3)$  then we perform the integration with respect to y,

We find:

$$G_{6j} = \left(\frac{1}{\pi}\right)^n \int \prod_{i=1, j=1,2}^3 du_j^i \exp(-(\bar{u})^t M(u))$$

The calculation of this expression can be done using the Gauss formula:

$$\left(\frac{1}{\pi}\right)^n \int \prod_{i=1}^n dx_i dy_i \exp(-\bar{z}^t M z) = (\det(M))^{-1} \quad (7.9)$$

We find:

$$G_{6j} = \left(\frac{1}{\pi}\right)^n \int \prod_{i=1, j=1,2}^3 du_j^i \exp(-(\bar{u})^t M(u)) = (\det(M))^{-1}$$

With:

$$M = \begin{pmatrix} 1-xp & -xy & -xt & 0 & -\mu_{12} & -\mu_{13} \\ -up & 1-uy & -ut & \mu_{12} & 0 & -\mu_{23} \\ -vp & -vy & 1-vt & \mu_{13} & \mu_{23} & 0 \\ 0 & \lambda_{12} & \lambda_{13} & 1-xp & -up & -vp \\ -\lambda_{12} & 0 & \lambda_{23} & -xy & 1-uy & -vy \\ -\lambda_{13} & -\lambda_{23} & 0 & -xt & -ut & 1-vt \end{pmatrix} \quad (7.10)$$

And

$$\begin{pmatrix} \lambda_{12} = \alpha_3 & \lambda_{13} = \beta_3 \alpha_2 & \lambda_{23} = \beta_3 \alpha_1 \\ \mu_{12} = \beta'_3 \alpha'_2 & \mu_{13} = \beta'_3 \alpha'_1 & \mu_{23} = \alpha'_3 \end{pmatrix}, \begin{pmatrix} p = \beta_2 \alpha_2 & y = \beta_2 \alpha_1 & t = \beta_1 \\ x = \beta'_1 & u = \beta'_2 \alpha'_2 & v = \beta'_2 \alpha'_1 \end{pmatrix} \quad (7.11)$$

#### 7.4 The generating function of the 6-j symbols

The determinant of the coupling matrix of angular momentum (M) is:

$$\det(M) = [1 - xp - uy - vt - \mu_{12}\lambda_{12} - \mu_{13}\lambda_{13} - \mu_{23}\lambda_{23} + \\ + (p\lambda_{23} + t\lambda_{12} - y\lambda_{13})(x\mu_{23} + v\mu_{12} - u\mu_{13})]^2 \quad (7.12)$$

Using (7, 11) we find the generator function of the symbols 6-j:

$$\sum_{j_1 j_2 j_3 j_{12} j_{23} j} \Phi_{j_1 j_2 j_{12}}(\alpha) \Phi_{j_{12} j_3 j}(\beta) \Phi_{j_2 j_3 j_{23}}(\alpha') \Phi_{j_1 j_{23} j}(\beta') W(j_1 j_2 j_3; j_{12} j_{23}) \quad (7.13) \\ = (1 - \beta'_1 \beta_2 \alpha_2 - \beta'_2 \alpha'_2 \beta_2 \alpha_1 - \beta'_2 \alpha'_1 \beta_1 - \alpha_3 \beta'_3 \alpha'_2 - \beta_3 \alpha_2 \beta'_3 \alpha'_1 - \beta_3 \alpha_1 \alpha'_3 + \beta'_1 \alpha'_3 \alpha_3 \beta_1)^{-2}$$

It is important to note that we can also make the coupling in terms of bosons, Schwinger [6] (appendiceI)

#### 7.5 The formula of 6-j

We can develop  $G_{6j}$  in the form:

$$G_{6j} = \frac{(z+1)! (\beta'_1 \beta_2 \alpha_2)^\lambda (\beta'_2 \alpha'_2 \beta_2 \alpha_1)^m (\beta'_2 \alpha'_1 \beta_1)^n (\alpha_3 \beta'_3 \alpha'_2)^p}{(\lambda)!(m)!(n)!(p)!} \times$$

$$\times \frac{(\beta_3 \alpha_2 \beta'_3 \alpha'_1)^q (\beta_3 \alpha_1 \alpha'_3)^r (\beta'_1 \alpha'_3 \alpha_3 \beta_1)^s}{(q)!(r)!(s)!} \quad (7.14)$$

With  $z = \lambda + m + n + p + q + r + s$  more

1- The powers of  $\alpha_i$  are :  $P(\alpha_1) = m + r$ ,  $P(\alpha_2) = \lambda + q$  and  $P(\alpha_3) = p + s$

We deduce that:  $\sum_{i=1}^3 P(\alpha_i) = z - n = j_1 + j_2 + j_{12}$  then  $n = z - (j_1 + j_2 + j_{12})$

2- The powers of  $\beta_i$  are:  $P(\beta_1) = n + s$ ,  $P(\beta_2) = l + m$  et  $P(\beta_3) = q + r$   $P(\alpha_3) = p + s$

We deduce that:  $\sum_{i=1}^3 P(\beta_i) = z - p = j_1 + j_{12} + j$  then  $p = z - (j_1 + j_{12} - j)$

3- The powers of  $\alpha'_i$  are.  $P(\alpha'_1) = n + q$ ,  $P(\alpha'_2) = m + p$  and  $P(\alpha'_3) = r + s$

We deduce that:  $\sum_{i=1}^3 P(\alpha'_i) = z - l = j_1 + j_2 + j_{12}$  then  $l = z - (j_2 + j_3 + j_{23})$

4 The powers of  $\beta'_i$  are:  $P(\beta'_1) = \lambda + s$ ,  $P(\beta'_2) = m + n$  and  $P(\beta'_3) = p + q$

We find that:  $\sum_{i=1}^3 P(\beta'_i) = z - r = z - (j_1 + j_{23} + j)$  donc  $r = z - (j_1 + j_{23} - j)$

And  $q = j_1 + j_{23} + j_{12} + j_3 - z$   $m = j_{12} + j + j_2 + j_{23} - z$   $s = j_1 + j_2 + j_3 + j - z$

Finally we find the Racah formula for the coefficients 6-j that we will write in the usual notations [4,9].

$$\left\{ \begin{matrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{matrix} \right\} = \left\{ \begin{matrix} a & b & c \\ d & e & f \end{matrix} \right\} = (-1)^{a+b+c+d} W(abcde, f) \quad (7.15)$$

### 7.6 Racah's formula

Racah's formula is:

$$\left\{ \begin{matrix} a & b & c \\ d & e & f \end{matrix} \right\} = \Delta(abc) \Delta(aef) \Delta dbc) \Delta(dec) \sum_z (-1)^z \left\{ \frac{(z+1)!}{(z-a-b-c)! (z-a-e-f)!} \times \right. \quad (7.16)$$

$$\left. \frac{1}{(z-b-d-f)! (z-d-e-c)! (a+b+d+e-z)! (b+c+e+f-z)! (a+e+d+f-z)!} \right\}$$

With  $\Delta(abc)$  is given in (4.10).

## 8- The symmetries of 6-j coefficients

We are looking for transformations which leave the summation invariant (7.16)

### 8.1 The symmetries of 6-j coefficients

We put again:  $AX = B$

$$\begin{pmatrix} -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & -1 & 0 & -1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -a-b-c \\ -a-e-f \\ -b-d-f \\ a+b+c+d \\ b+c+e+f \\ a+e+d+f \end{pmatrix} \quad (8.1)$$

The symbol 6-j is invariant by permutation of the composantes of:

a- the column of  $X = (abcdef)$ ,

b- or the composantes of the column B.

**8.2 The symmetries by permutation of the components of  $X = (abcdef)$ .**

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \begin{Bmatrix} a & c & b \\ d & f & e \end{Bmatrix} = \begin{Bmatrix} a & e & f \\ d & b & c \end{Bmatrix} = \begin{Bmatrix} a & f & e \\ d & c & b \end{Bmatrix} \quad (8.2)$$

**8.3 The symmetries by permutation of the components of the column B or**

**Regge symmetries**

$$\begin{aligned} \begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} &= \begin{Bmatrix} a & \frac{b+c+e-f}{2} & \frac{c+b+f-e}{2} \\ d & \frac{e+b+f-c}{2} & \frac{f+c+e-b}{2} \end{Bmatrix} = \begin{Bmatrix} a & \frac{e+b+f-c}{2} & \frac{f+c+e-b}{2} \\ d & \frac{b+c+e-f}{2} & \frac{c+b+f-e}{2} \end{Bmatrix} \quad (8.3) \\ &= \begin{Bmatrix} a & \frac{c+b+f-e}{2} & \frac{b+c+e-f}{2} \\ d & \frac{f+c+e-b}{2} & \frac{e+b+f-c}{2} \end{Bmatrix} = \begin{Bmatrix} a & \frac{f+c+e-b}{2} & \frac{e+b+f-c}{2} \\ d & \frac{c+b+f-e}{2} & \frac{b+c+e-f}{2} \end{Bmatrix} \\ &= \begin{Bmatrix} \frac{a+c+f-d}{2} & \frac{e+b+f-c}{2} & \frac{a+d+e-b}{2} \\ \frac{c+d+f-a}{2} & \frac{b+c+e-f}{2} & \frac{a+d+f-e}{2} \end{Bmatrix} = \begin{Bmatrix} \frac{a+d+f-c}{2} & \frac{a+e+b-d}{2} & \frac{e+f+c-b}{2} \\ \frac{a+c+d-f}{2} & \frac{b+d+e-a}{2} & \frac{b+c+f-e}{2} \end{Bmatrix} \\ &= \begin{Bmatrix} \frac{b+e+f-c}{2} & \frac{a+b+d-e}{2} & \frac{c+d+f-a}{2} \\ \frac{b+c+e-f}{2} & \frac{a+d+e-b}{2} & \frac{a+c+f-d}{2} \end{Bmatrix} = \begin{Bmatrix} \frac{a+b+e-d}{2} & \frac{b+c+f-e}{2} & \frac{a+c+d-f}{2} \\ \frac{b+d+e-a}{2} & \frac{c+f+e-b}{2} & \frac{a+d+f-c}{2} \end{Bmatrix} \\ &= \begin{Bmatrix} \frac{b+e+f-c}{2} & \frac{a+d+e-b}{2} & \frac{a+c+f-d}{2} \\ \frac{b+c+e-f}{2} & \frac{a+b+d-e}{2} & \frac{c+d+f-a}{2} \end{Bmatrix} = \begin{Bmatrix} \frac{c+d+f-a}{2} & \frac{b+c+e-f}{2} & \frac{a+b+d-e}{2} \\ \frac{a+c+f-d}{2} & \frac{b+e+f-c}{2} & \frac{a+d+e-b}{2} \end{Bmatrix} \\ &= \begin{Bmatrix} \frac{c+d+f-a}{2} & \frac{a+d+e-b}{2} & \frac{b+c+e-f}{2} \\ \frac{a+c+f-d}{2} & \frac{a+b+d-e}{2} & \frac{b+e+f-c}{2} \end{Bmatrix} = \begin{Bmatrix} \frac{a+c+f-d}{2} & \frac{a+d+e-b}{2} & \frac{b+e+f-c}{2} \\ \frac{c+d+f-a}{2} & \frac{a+b+d-e}{2} & \frac{b+c+e-f}{2} \end{Bmatrix} \end{aligned}$$

$$= \left\{ \begin{array}{ccc} \frac{a+d+e-b}{2} & \frac{b+c+e-f}{2} & \frac{c+d+f-a}{2} \\ \frac{a+b+d-e}{2} & \frac{b+e+f-c}{2} & \frac{a+c+f-d}{2} \end{array} \right\} \quad (8.4)$$

## 9- Appendix: The connection between the boson space And the analytic Fock space

The connection between the boson space and the analytic Fock space has been studied for a long time [16].

### *a- The coupling coefficients in bosons basis (Schwinger)*

Schwinger in his famous paper [6] took as a starting point the base of bosons for the study of angular momentum:

$$|n\rangle = \frac{(a^+)^n}{\sqrt{n!}} |0\rangle \quad (9.1)$$

The unitary operator [16] is written with the help of analytic Fock space by:

$$I = \sum_{n=0}^{\infty} |n\rangle \langle n| = \int \exp(\bar{z}a^+) |0\rangle \langle 0| \exp(za) d\mu(z) \quad (9.2)$$

b- The generating functions of the couplings in bosons space are:

1- The generating function of  $|(j_a j_b) j_{ab} j_c; j\rangle$  is  $\exp(Q_1(a^+)|0\rangle$  (7,6)

2- The generating function of  $|j_a (j_b j_c) j_{bc}; j\rangle$  is  $\exp(Q_2(a^+)|0\rangle$  (7,7)

### *c- The generating function of Racah coefficients*

The expression of  $G_{6j}$  in the base of the bosons is then written using the unit operator by:

$$G_{6j} = \langle 0 | \exp(Q_1(a)) \exp(Q_1(a^+)) | 0 \rangle \quad (9.3)$$

By introducing the unit operator we find:

$$\begin{aligned} & \langle 0 | \exp(Q_1(a)) I \exp(Q_1(a^+)) | 0 \rangle = \\ & \langle 0 | \exp(Q_1(a)) [\exp(\bar{u}a^+) | 0 \rangle \langle 0 | \exp(ua)] \exp(Q_1(a^+)) | 0 \rangle = \langle \Psi_2 | \Psi_1 \rangle \\ & = \int d\mu(y) \left[ \prod_{i=1, j=1,2}^3 du_j^i \bar{\Psi}_2 \Psi_1 \right] \prod_{i=1, j=1,2}^3 du_j^i \end{aligned} \quad (9.4)$$

We thus find the passage from the bosons representation to the analytical representation and the computations are carried out simply using the Gauss integral of the determinant of the couplings matrix of angular momentum (M).

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