



Poincaré series and linking of Legendrian knots

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POINCARÉ SERIES AND LINKING OF LEGENDRIAN KNOTS

NGUYEN VIET DANG AND GABRIEL RIVIÈRE

ABSTRACT. On a negatively curved surface, we show that the Poincaré series counting geodesic arcs orthogonal to some pair of closed geodesic curves has a meromorphic continuation to the whole complex plane. When both curves are homologically trivial, we prove that the Poincaré series has an explicit rational value at 0 interpreting it in terms of linking number of Legendrian knots. In particular, for any pair of points on the surface, the lengths of all geodesic arcs connecting the two points determine its genus, and, for any pair of homologically trivial closed geodesics, the lengths of all geodesic arcs orthogonal to both geodesics determine the linking number of the two geodesics.

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1. INTRODUCTION

Let (X, g) be a smooth (\mathcal{C}^∞) , compact, oriented, connected, Riemannian surface which has no boundary and which has *negative curvature*. Given a nontrivial homotopy class $\mathbf{c} \in \pi_1(X)$, one can find a unique geodesic c (parametrized by arc length) in the conjugacy class of \mathbf{c} [55, §3.8]. We say that such a curve c is a geodesic representative of \mathbf{c} and we note that it is naturally oriented by \mathbf{c} . Similarly, any point $c \in X$ will be understood in the following as a geodesic representative of the trivial homotopy class in $\pi_1(X)$.

A classical problem in Riemannian geometry consists in studying the lengths of the geodesic arcs joining two geodesic representatives c_1 and c_2 of two given primitive¹ homotopy classes \mathbf{c}_1 and \mathbf{c}_2 in X . More precisely, for $T > 0$, we denote by $\mathcal{N}_T(c_1, c_2) \in [0, +\infty]$ the number of geodesics γ of length $0 < \ell(\gamma) \leq T$ (parametrized by arc length) that join c_1 to c_2 and that are *directly orthogonal* to c_1 and c_2 . In that framework and when c_1, c_2 are points, Margulis proved, using purely dynamical methods [60, 61], the existence of $A_{c_1, c_2} > 0$ such that

$$(1) \quad \mathcal{N}_T(c_1, c_2) \sim A_{c_1, c_2} e^{Th_{\text{top}}}, \quad \text{as } T \rightarrow +\infty,$$

where $h_{\text{top}} > 0$ is the topological entropy of the geodesic flow. See also [21, 49, 50] for earlier results of Delsarte and Huber in constant negative curvature using the spectral decomposition of the Laplace-Beltrami operator. This asymptotic formula was further generalized by Pollicott in the framework of Axiom A dynamical systems [73]. Parkkonen and Paulin showed that (1) remains true when \mathbf{c}_1 and \mathbf{c}_2 are any elements in $\pi_1(X)$ and when X is not necessarily compact [67]. For smooth compact Riemannian manifolds without any

¹It means that either \mathbf{c}_i is trivial in $\pi_1(X)$, or the equation $\mathbf{c}^p = \mathbf{c}_i$ has no solution for every $p > 1$.

assumption on their curvature and when c_1 and c_2 are points, Mañé proved [58] that

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \log \int_{X \times X} \mathcal{N}_T(c_1, c_2) d\text{vol}_g(c_1) d\text{vol}_g(c_2) = h_{\text{top}},$$

where vol_g is the Riemannian volume induced by g – see also [68, 11, 71, 70].

In this work, we shall focus on the case of negatively curved surfaces as in the works of Margulis. Recall that his strategy consisted in relating the study of these asymptotics with the mixing properties of the measure maximizing the Kolmogorov-Sinai entropy, the so-called Bowen-Margulis measure. We refer to the review of Parkkonen and Paulin for more details on these questions and for some recent developments [66]. In particular, estimates on the size of the remainder in (1) can be derived [67, Th. 27] from quantitative estimates on the rate of mixing of the Bowen-Margulis measure [75, 64, 22, 37, 40]. Such quantitative estimates can for instance be obtained from the spectral analysis of transfer operators on appropriate Banach spaces of currents [22, 37, 40] which is also referred as the study of Pollicott-Ruelle resonances. In the *hyperbolic* case, estimates on the size of the remainder were previously obtained using the spectral decomposition of the Laplacian [72, 44, 83, 38].

Instead of searching for improvements on the size of the remainder in (1), the aim of this work is to study more specifically zeta renormalization of $\mathcal{N}_T(c_1, c_2)$:

$$\forall s \in \mathbb{C}, \quad \mathcal{N}_T(c_1, c_2, s) := \sum_{\gamma \in \mathcal{P}_{c_1, c_2} : 0 < \ell(\gamma) \leq T} e^{-s\ell(\gamma)},$$

where \mathcal{P}_{c_1, c_2} denotes the set of geodesic arcs γ joining c_1 and c_2 and directly orthogonal² to c_1 and c_2 . Note that $\mathcal{N}_T(c_1, c_2, 0) = \mathcal{N}_T(c_1, c_2)$.

1.1. Meromorphic continuation. Thanks to (1), we can define, in the region $\text{Re}(s) > h_{\text{top}}$, the *generalized* Poincaré series³

$$(2) \quad \mathcal{N}_\infty(c_1, c_2, s) := \lim_{T \rightarrow +\infty} \mathcal{N}_T(c_1, c_2, s) = \sum_{\gamma \in \mathcal{P}_{c_1, c_2} : \ell(\gamma) > 0} e^{-s\ell(\gamma)}.$$

This defines a holomorphic function in the region $\text{Re}(s) > h_{\text{top}}$ and we will first prove the following result:

Theorem 1.1. *Let (X, g) be a smooth (\mathcal{C}^∞), compact, oriented, connected, Riemannian surface which has no boundary and which has negative curvature. Then, for every \mathbf{c}_1 and \mathbf{c}_2 in $\pi_1(X)$ and for any of their geodesic representatives c_1 and c_2 , the holomorphic function*

$$s \in \{w : \text{Re}(w) > h_{\text{top}}\} \mapsto \mathcal{N}_\infty(c_1, c_2, s) \in \mathbb{C}$$

has a meromorphic continuation to \mathbb{C} .

²In other words, $\gamma'(0) \perp T_{\gamma(0)}c_1$, $\gamma'(\ell) \perp T_{\gamma(\ell)}c_2$ and $\mathbb{R}\gamma'(0) \oplus T_{\gamma(0)}c_1$, $\mathbb{R}\gamma'(\ell) \oplus T_{\gamma(\ell)}c_2$ have direct orientations.

³This is just the Laplace transform of the measure $\sum_{\gamma \in \mathcal{P}_{c_1, c_2} : \ell(\gamma) > 0} \delta_0(t - \ell(\gamma))$.

Our proof will use the spectral properties of transfer operators for uniformly hyperbolic flows developed by many authors over the last fifteen years [12, 13, 85, 32, 86, 37, 24, 26, 31, 25, 42, 51]. More precisely, we will interpret this Poincaré series in terms of a certain spectral resolvent applied to the conormal cycle of c_1 and c_2 . Then, we will derive this theorem from the meromorphic continuation of this spectral resolvent. Our proof allows to encompass the case of much more general Anosov flows and Poincaré series (see Theorem 4.15) but we limit ourselves to this simplified version for the introduction. As a byproduct of this argument, we will verify that the poles of this meromorphic continuation are included in the set of Pollicott-Ruelle resonances for currents of degree 1. This spectral approach is in some sense close in spirit to what is done when proving the meromorphic continuation of dynamical zeta functions [2]. For instance, Giulietti-Liverani-Pollicott [37] and Dyatlov-Zworski [26] showed by spectral methods the meromorphic continuation of the Ruelle zeta function

$$(3) \quad \zeta_{\text{Ruelle}}(s) = \prod_{\gamma \in \mathcal{P}} (1 - e^{-s\ell(\gamma)})^{-1},$$

where \mathcal{P} denotes the set of primitive *closed* geodesics. See also [76, 78, 34] for earlier results of Ruelle, Rugh and Fried in the analytic case, [2] for a detailed account of Baladi in the case of Axiom A diffeomorphisms or [31, 39] for the semiclassical zeta function of Faure-Tsujii. For all these other zeta functions, the meromorphic continuation was also established by spectral methods and their zeroes and poles were related to the Pollicott-Ruelle resonances on anisotropic spaces of currents as it is the case here.

While there are many results on Ruelle zeta functions, there are not so many works on the meromorphic continuation of Poincaré series. The only results we are aware of concern *hyperbolic* manifolds where one can connect Poincaré series to a certain spectral resolvent of the Laplacian – see for instance [49, Satz A], [50, Satz 2] or [43, Lemme 3.1]. Yet, such a correspondence is not available for general negatively curved manifolds and one has to work directly with the dynamical problem as we shall do here. The only dynamical proof of a meromorphic continuation of Poincaré series we are aware of is due to Paternain [70, p. 138] in the case where (X, g) is hyperbolic. Under these assumptions, he proved by purely geometrical arguments that

$$(4) \quad \lim_{T \rightarrow +\infty} \int_{X \times X} \mathcal{N}_T(c_1, c_2, s) d\text{vol}_g(c_1) d\text{vol}_g(c_2) = \frac{4\pi^2 \chi(X)}{1 - s^2},$$

where c_1, c_2 are points and $\chi(X)$ is the Euler characteristic of X . He obtained this formula by interpreting this integrated Poincaré series via a convenient coarea formula – see also [58, 71] for earlier related results. In some sense, our proof of the meromorphic continuation will use similar ideas but with the addition of the spectral analysis of Anosov flows to compensate the absence of simplifications due to constant curvature and to the integration over $X \times X$. Note that as a corollary of this result, of Theorem 1.1 and of Proposition 6.1 below, we recover the Euler characteristic of an *hyperbolic surface* as a special value of

Poincaré series:

$$(5) \quad \boxed{\int_{X \times X} \mathcal{N}_\infty(c_1, c_2, 0) d\text{vol}_g(c_1) d\text{vol}_g(c_2) = 4\pi^2 \chi(X).}$$

We will now show how to generalize this formula via our spectral approach.

1.2. Value at 0. In a series of recent works, it was observed by Dyatlov-Zworski [27] and the authors [18, 19, 20] that, among Pollicott-Ruelle resonances, the one at 0 plays a special role as its resonant states encode the De Rham cohomology of the manifold where the dynamics takes place. See also [45, 56, 14] for related results of Hadfield, Küster-Weich and Cekić-Paternain. In the case of geodesic flows on Riemannian surfaces, the resonant states were in some sense computed explicitly by Dyatlov and Zworski [27, §3]. As a consequence, they proved

$$\boxed{s^{\chi(X)} \zeta_{\text{Ruelle}}(s)|_{s=0} \neq 0,}$$

and thus they generalized earlier results due to Fried in the case of constant curvature [33].

The behaviour at 0 of Poincaré series will be of slightly different nature as we will consider situations where there will be no pole or zero at $s = 0$ even if there is a Pollicott-Ruelle resonance at 0. Despite this and working out on the ideas introduced in [18, 19, 20, 27], we will verify that the value at 0 still has a topological meaning. To that aim, we set

$$\varepsilon(\mathbf{c}) = 1 \quad \text{if } \mathbf{c} \text{ is trivial in } \pi_1(X), \quad \text{and} \quad \varepsilon(\mathbf{c}) = -1 \quad \text{otherwise,}$$

and the main result of this article reads:

Theorem 1.2. *Let (X, g) be a smooth (\mathcal{C}^∞) , compact, oriented, connected, Riemannian surface which has no boundary and which has negative curvature.*

Then, for every primitive \mathbf{c}_1 and \mathbf{c}_2 in $\pi_1(X)$ which are trivial in homology and for any of their geodesic representatives c_1 and c_2 , one has

$$\boxed{\chi(X) \mathcal{N}_\infty(c_1, c_2, 0) \in \mathbb{Z}.}$$

Moreover, if c_1 and c_2 are embedded and if $X(c_i)$ is the compact surface⁴ whose oriented boundary is given by c_i , then one has

$$(6) \quad \boxed{\mathcal{N}_\infty(c_1, c_2, 0) = \varepsilon(\mathbf{c}_1) \left(\frac{\chi(X(c_1))\chi(X(c_2))}{\chi(X)} - \chi(X(c_1) \cap X(c_2)) + \frac{1}{2}\chi(c_1 \cap c_2) \right),}$$

in the following cases

- \mathbf{c}_1 and \mathbf{c}_2 are distinct nontrivial homotopy classes,
- at least one \mathbf{c}_i is trivial and $c_1 \cap c_2 = \emptyset$.

⁴When c_i is a point, we take the convention that $X(c_i) = c_i$. When c_i is not a point, $X \setminus c_i$ has two connected components (as c_i is homologically trivial and embedded) and $X(c_i)$ is the closure of the component whose oriented boundary is c_i .

Note that as c_1 and c_2 are homologically trivial, they intersect an even number of times and the contribution $\frac{1}{2}\chi(c_1 \cap c_2)$ yields an integer⁵. In paragraph 6.4, we will show that this value at 0 can *always* be expressed via the general formula:

$$\mathcal{N}_\infty(c_1, c_2, 0) = \varepsilon(\mathbf{c}_1) \left(\frac{\chi(X(\tilde{c}_1))\chi(X(\tilde{c}_2))}{\chi(X)} - \chi(X(\tilde{c}_1) \cap X(\tilde{c}_2)) + \frac{1}{2}\chi(\tilde{c}_1 \cap \tilde{c}_2) \right),$$

where \tilde{c}_i is an (arbitrarily small) homotopic deformation of c_i which is obtained by pushing c_i transversally via the geodesic flow. Let us insist that our result in paragraph 6.4 allows the representatives c_i (and their deformation \tilde{c}_i) to be nonembedded and to intersect each other. In that case, we also need to define precisely what is $X(\tilde{c}_i)$ and what we mean by its Euler characteristic. See § 6.4.3 for details. For instance, when $c_1 = c_2$ is a point, we get $\frac{1}{\chi(X)} - 1$ for the value at $s = 0$.

When c_1 and c_2 are *distinct points*, we get

$$\mathcal{N}_\infty(c_1, c_2, 0) = \frac{1}{\chi(X)}.$$

Thus, as a corollary of this result, the Euler characteristic of X can be recovered by the set of lengths of the geodesic arcs joining two points of X . Still in the case of points, we also observe that we recover Paternain's formula (5) without integrating over X and that it can be extended to variable curvature as follows:

$$\int_{X \times X} \mathcal{N}_\infty(c_1, c_2, 0) d\text{vol}_g(c_1) d\text{vol}_g(c_2) = \frac{\text{vol}_g(X)^2}{\chi(X)}.$$

More generally, this Theorem shows that the value at 0 of Poincaré series is rational under homological assumptions on the homotopy classes we consider. As we shall see, the integer $\chi(X)\mathcal{N}_\infty(c_1, c_2, 0)$ has a clear geometric interpretation if we lift the problem to the unit cotangent bundle. It will *correspond to the linking of two Legendrian knots* given by the unit conormal bundles of the geodesic representatives c_1 and c_2 . See §6 for details.

The fact that we are on *negatively curved surfaces* is important here. For more general surfaces with Anosov geodesic flows, the result could be generalized when c_1 and c_2 are points but the value at 0 may differ by an integer from the negatively curved case – see Theorem 4.3 and Remark 4.13. Yet, in the case of closed geodesics c_i , the result cannot be extended in general as their conormal bundle may not satisfy the appropriate transversality assumptions required to ensure the meromorphic continuation – see (16) for the precise condition on the curves in terms of unstable Riccati solutions.

The main reason for restricting ourselves to dimension 2 is due to our spectral interpretation of Poincaré series. In particular, their value at 0 is related to the properties of the eigenspace associated to the Pollicott-Ruelle resonance at 0. As already alluded to, a rather precise description of that eigenspace was recently given by Dyatlov and Zworski in dimension 2 [27] and we will crucially use this result (together with some ideas from [19]) in order to identify the value at 0.

⁵When c_1 corresponds to c_2 with its reverse orientation, one has $X(c_2) = \overline{X(c_1)^c}$ and $X(c_1) \cap X(c_2) = c_1 \cap c_2 = c_1$ has 0 Euler characteristic.

1.3. Perspectives and related results. In higher dimensions, very few things are known on the eigenspace at 0 [56, 17] but any progress in that direction should in principle give some insights on the behaviour at 0 of Poincaré series in higher dimensions following the lines of our proof. Recall from [70, Prop. 3.2] that, for a hyperbolic manifold (X, g) of dimension $n_0 \geq 2$ and for trivial homotopy classes, Paternain's formula (4) becomes

$$\lim_{T \rightarrow +\infty} \int_{X \times X} \mathcal{N}_T(c_1, c_2, s) d\text{vol}_g(c_1) d\text{vol}_g(c_2) = \frac{4\pi^{\frac{n_0}{2}} \text{vol}_g(X)}{2^{n_0} \Gamma(\frac{n_0}{2})} \sum_{k=0}^{n_0-1} \frac{(-1)^k \binom{n_0-1}{k}}{s + 2k + 1 - n_0}.$$

In particular, there is a pole at 0 in odd dimensions. Coming back to dimension 2, the results of Hadfield [45] for geodesic flows on surfaces with boundary should allow to find a formula similar to the one from Theorem 1.2 in that case. Yet, this would be much beyond the scope of the present article and we shall not discuss this here.

1.3.1. Relation with analytic number theory. Studying the meromorphic continuation of Poincaré series and their special value at 0 is reminiscent from classical questions in number theory. The most famous example is given by the Riemann zeta function which equals $-\frac{1}{2}$ at 0. More generally, for totally real fields, the Siegel-Klingen Theorem [84, 54] shows that the corresponding zeta function takes a rational value at 0 (in fact at each nonpositive integer). Bergeron, Charollois, Garcia and Venkatesh show that this special value at 0 can be interpreted as a linking number between periodic orbits of the suspension of *hyperbolic* toral automorphisms of the 2-torus [5]. As the geodesic flow on negatively curved surfaces, these are examples of Anosov flows in dimension 3. Coming back to geodesic flows on surfaces, we also mention the works of Ghys. He showed that, on the unit tangent bundle⁶ $\text{PSL}(2, \mathbb{R})/\text{PSL}(2, \mathbb{Z})$ of the modular surface, the linking number of a closed geodesic with the trefoil knot can be identified with the value of the Rademacher function on the given geodesic [35, § 3.3]. The Rademacher function is an integer valued function defined on the set of closed orbits of the geodesic flow. More recently, Duke, Imamoğlu and Tóth showed how to express the linking number of two given (homologically trivial) geodesics of the modular surface as the special value of a certain Dirichlet series [23, Th. 4].

Hence, once reinterpreted in terms of linking between Legendrian knots (see §6), Theorem 1.2 can be viewed as another occurrence of these interactions between knots and dynamics but in a context where no arithmetical tool is available. In the main part of our work, our knots will be Legendrian, thus “orthogonal” to the closed orbits of the geodesic flow. Yet, in Theorem 6.31, we will verify that for two homologically trivial closed geodesics curves c_1, c_2 in X , the number $\mathcal{N}_\infty(c_1, c_2, 0)$ actually computes the linking number of the two geodesics γ_1, γ_2 lifting c_1, c_2 in S^*X , yielding a direct connection with the linking numbers appearing in the above works.

1.3.2. Relation with orthospectrum identities in hyperbolic geometry. Let now X be a hyperbolic surface with nonempty totally geodesic smooth boundary. In that framework, an

⁶It is homeomorphic to the complement of the trefoil knot in the 3-sphere.

orthogeodesic γ is a geodesic arc which is properly immersed in X and which is perpendicular to ∂X at its endpoints. The lengths of these orthogeodesics verify certain identities connecting them to the length of the boundary of X [4] (Basmajian's identity) or to the area of the hyperbolic surface [7] (Bridgeman's identity). For instance, Bridgeman's orthospectrum identity reads [8, Th. B']:

$$(7) \quad \boxed{\sum_{\gamma} \mathcal{R} \left(\operatorname{sech}^2 \left(\frac{\ell(\gamma)}{2} \right) \right) = -\pi^2 \chi(X)}$$

where the sum runs over the orthogeodesics and \mathcal{R} is the Rogers' dilogarithm function. Analogues of these identities were also derived independently by McShane in the case of hyperbolic surfaces with cusps [62, 63]. We refer the reader to [8] for a review of the literature and for further developments of these results. Identity (7) has the same flavour as (6) in the sense that it relates the lengths of geodesics arcs with a purely topological quantity. Yet, let us point three notable differences. First, we work in the closed manifold case, hence the geodesic arcs in \mathcal{P}_{c_1, c_2} can intersect the curves c_1 and c_2 several times. Moreover, while the left hand side of (7) converges in a standard sense, our equation (6) is defined by analytic continuation exactly as the value at 0 of zeta functions in analytic number theory. Finally, our results hold in *variable negative curvature*.

1.3.3. Relation with microlocal index formulas. Our derivation of the topological content of $\mathcal{N}_{\infty}(c_1, c_2, 0)$ relies crucially on the *Poincaré-Hopf index formula* as it was derived by Morse in [65], and we use this formula from a point of view which is inspired by microlocal geometry. In fact, the microlocal index theorems of Brylinski–Dubson–Kashiwara [10] and Kashiwara [52], later revisited by Kashiwara–Schapira [53, p. 384] and Grinberg–McPherson [41], can be understood as generalizations of the Poincaré–Hopf index formula. As it may be helpful to understand our approach in Section 6, we briefly explain their content following the presentation of [41] to which we refer for more details.

First, given a real algebraic manifold X and a stratification \mathcal{S} of X , one says that a function $f : X \mapsto \mathbb{Z}$ is constructible if it is constant on each stratum. The notion of Euler characteristic generalizes to constructible functions [88, 79, 80, 81]:

$$f \mapsto \chi(f) := \int_X f d\chi,$$

and it is referred as Euler (characteristic) integrals – see [3, 15] for an introduction to this notion and various applications. We just say here that, for the characteristic function $\mathbf{1}_{\Omega}$ of a domain Ω , $\chi(\mathbf{1}_{\Omega}) = \int_{\Omega} d\chi$ coincides with the usual Euler characteristic $\chi(\Omega)$ and that the extension to constructible functions follows from \mathbb{Z} –linearity [3, Def 2.6 p. 831].

Now, given any stratum S of \mathcal{S} , one can define its conormal bundle which is a Lagrangian submanifold $\Lambda_S \subset T^*X$. Then, the Lagrangian cycle $\operatorname{Ch}(f)$ of f is defined by assigning to each Lagrangian submanifold Λ_S its multiplicity which roughly speaking is the value of f on S – see [41, p. 277] for details. The constructible function f is then viewed as a *quantization* of the Lagrangian $\Lambda = \operatorname{Ch}(f)$. Then, for every pair f_1, f_2 of constructible

functions on X which satisfy some appropriate transversality conditions, the microlocal index formula reads [41, p. 269]:

$$(8) \quad \underbrace{\chi(f_1 f_2)}_{\text{Euler integral}} = \underbrace{[\text{Ch}(f_1)] \cap [\text{Ch}(f_2)]}_{\text{Lagrangian intersection}}$$

where $[\text{Ch}(f_1)] \cap [\text{Ch}(f_2)]$ is the intersection of the two corresponding Lagrangian cycles. Hence, the microlocal index formula gives an interpretation of Lagrangian intersections as the Euler characteristic of some product of constructible functions.

As we shall see when proving (6), we derive a formula in the spirit of the above microlocal index formula. Instead of computing the intersection of Lagrangian cycles, we rather consider the linking of Legendrian cycles but we also express it in terms of constructible functions. More precisely, for every pair of Legendrian cycles Σ_1, Σ_2 which are small deformations by Hamiltonian isotopies of the unit conormal bundle of our homologically trivial geodesic representatives c_1 and c_2 , we associate a pair (f_1, f_2) of constructible functions quantizing the two knots Σ_1, Σ_2 . Then we prove the microlocal index formula:

$$(9) \quad \underbrace{\frac{\chi(f_1)\chi(f_2)}{\chi(X)} - \chi(f_1 f_2) + \frac{1}{2}\chi(\mathbf{1}_{c_1 \cap c_2})}_{\text{Euler integral}} = \underbrace{\pm \text{Lk}(\Sigma_1, \Sigma_2)}_{\text{Legendrian linking}} = \lim_{s \rightarrow 0} \underbrace{\sum_{\gamma \in \mathcal{P}_{c_1, c_2} : \ell(\gamma) > 0} e^{-\ell(\gamma)s}}_{\text{Poincaré series at zero}}.$$

In the framework of symplectic topology, the Poincaré series is understood as a sum over the Reeb chords of the geodesic flow joining the two Legendrian curves Σ_1 and Σ_2 . Hence, this index formula, which seems to be new⁷, gives an interpretation of some linking of two Legendrian curves in terms of Euler integrals but also as a zeta regularized sum over the Reeb chords from Σ_1 to Σ_2 . While the first equality is obtained by purely topological means, the second one is a consequence of our spectral approach to the problem. In fact, we conjecture that the first equality in this index-type formula should generalize to more general Legendrian knots and also to higher dimensional Legendrian boundaries for the appropriate notion of linking between higher dimensional objects. The generalization of the second equality is more subtle and it is related to the structure of Pollicott-Ruelle resonant states at 0 as we already discussed.

Organization of the article. In Section 2, we review a few standard facts on Riemannian geometry that are used throughout the article. Then, in Section 3, we discuss the notion of Anosov flows and prove some statements related to our problem. Section 4 is the main analytical part where we prove Theorem 1.1 and in fact a much more general statement which is valid for generalized weighted Poincaré series associated with Anosov (not necessarily geodesic) flows. This proof relies on the microlocal methods introduced by Faure-Sjöstrand in [32] and subsequently developed by Dyatlov-Zworski in [26] to study the meromorphic continuation of Ruelle zeta functions. The results in that section could be as well obtained via the geometric approach of Pollicott-Ruelle spectra previously developed

⁷However see [87, Th.4] and [74, Eq. (10)] for related results of Turaev regarding the first equality on $S^*\mathbb{S}^2$.

by Liverani et al. [12, 13, 37]. Yet, the microlocal point of view and more specifically the notion of wavefront sets is quite convenient for the study of Poincaré series and of their value at 0. After that, in Section 5, we briefly review the recent results of Dyatlov and Zworski on Pollicott-Ruelle resonant states at 0 for contact Anosov flows in dimension 3. Then, we apply them to compute the residue of Poincaré series at 0 and we show that, for contact Anosov flows in dimension 3, this residue can be expressed in terms of representatives of the De Rham cohomology. In Section 6, we specify these results for geodesic flows on surfaces and we show that, for homologically trivial homotopy classes, there is no residue at 0. Then, we use the chain homotopy equation derived in our previous work [19] to express the value at 0 as the linking between two Legendrian knots. We conclude by appealing to classical results from differential topology such as the Poincaré-Hopf formula for manifolds with boundary derived by Morse [65]. Finally, in Appendix A, we review some facts on wavefront sets of distributions that are used in this article.

Conventions. All along the article, (M, \tilde{g}) is a smooth, compact, oriented, connected and Riemannian manifold which has no boundary and which has dimension $n \geq 3$. At some point, we will also consider a smooth, compact, oriented, connected and Riemannian manifold (X, g) which has no boundary and which has dimension $n_0 \geq 2$. In that case, we take the unit cotangent bundle S^*X to be equal to M and we take $\tilde{g}^{(1)}$ to be the induced metric on M .

For $0 \leq k \leq n$, we denote by $\Omega^k(M)$ the space of smooth differential (complex-valued) forms of degree k on M . Equivalently, it is the space of smooth complex-valued sections $s : M \rightarrow \Lambda^k(T^*M)$. The topological dual to $\Omega^{n-k}(M)$ (with the topology induced by \mathcal{C}^∞ -topology) is denoted by $\mathcal{D}^k(M)$ and is called the space of currents of degree k – see [82, Ch. 5] for an introduction to the theory of currents.

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2. BACKGROUND ON RIEMANNIAN GEOMETRY

In this preliminary section, we collect some classical results on Riemannian and symplectic geometry following the presentation of [57, App.] – see [6, 69, 89] for a more detailed account. Along the way, we recall classical notations that are used all along this article. Throughout this section, (X, g) is a smooth Riemannian manifold of dimension $n_0 \geq 2$.

Remark 2.1. Recall that the Riemannian metric g on X induces two natural isomorphisms

$$\flat : T_q X \rightarrow T_q^* X, \quad v \mapsto g_q(v, \cdot),$$

and its inverse $\sharp : T_q^*X \rightarrow T_qX$. This natural isomorphism induces a positive definite form on T_q^*X for which these isomorphisms are in fact isometries. We denote by g^* the corresponding metric.

2.1. Horizontal and vertical subbundles. Let $x = (q, p)$ be an element in $T^*X \setminus \underline{0}$, where $\underline{0}$ is the zero section in T^*X . Denote by $\pi : T^*X \setminus \underline{0} \rightarrow X$ the canonical projection $(q, p) \mapsto q$. We introduce the so-called *vertical* subspace:

$$\mathcal{V}_x := \text{Ker}(d_x\pi) \subset T_x T^*X.$$

The fiber T_q^*X is a submanifold of T^*X that contains the point (q, p) . The tangent space to this submanifold at the point (q, p) is the vertical subspace \mathcal{V}_x and it can be identified with T_q^*X .

Remark 2.2. Setting $S^*X := \{(q, p) : \|p\|_{g^*(q)} = 1\}$, we can also define the projection map $\Pi := \pi|_{S^*X}$:

$$\Pi : x = (q, p) \in S^*X \mapsto q \in X.$$

This allows to define

$$\mathcal{V}_x^{(1)} := \text{Ker}(d_x\Pi) \subset \mathcal{V}_x,$$

which is the tangent space at the point $x = (q, p)$ to the submanifold S_q^*X .

We will now define the connection map. For that purpose, we fix Z in $T_x T^*X$ and $x(t) = (q(t), p(t))$ a smooth curve in T^*X such that $x(0) = x$ and $x'(0) = Z$. The connection map $\mathcal{K}_x : T_x T^*X \mapsto T_q^*X$ is the following map:

$$\mathcal{K}_x(Z) := \nabla_{q'(0)} p(0),$$

where $\nabla : \Omega^0(X, T^*X) \rightarrow \Omega^1(X, T^*X)$ is the connection induced on T^*X (via the musical isomorphisms) by the Levi-Civita connection $\nabla : \Omega^0(X, TX) \rightarrow \Omega^1(X, TX)$ associated with g . Equivalently, it is the covariant derivative of the section $x(t)$ along the curve $q(t)$. One can verify that this quantity depends only on the initial velocity Z of the curve (and not on the curve itself) and that the map is linear. The *horizontal* space is given by the kernel of this linear map, i.e.

$$\mathcal{H}_x := \text{Ker}(\mathcal{K}_x) \subset T_x T^*X.$$

Remark 2.3. These bundles induce the following decompositions of $T_x T^*X$ and of $T_x S^*X$:

$$T_x T^*X := \mathcal{H}_x \oplus \mathcal{V}_x$$

and

$$T_x S^*X := \mathcal{H}_x \oplus \mathcal{V}_x^{(1)}.$$

Finally, there exists a natural vector bundle isomorphism between the pullback bundle $\pi^*(TX \oplus T^*X) \rightarrow T^*X$ and the canonical bundle $TT^*X \rightarrow T^*X$. The restriction of this isomorphism on the fibers above $x \in T^*X \setminus \underline{0}$ is given by

$$A(x) : T_x T^*X \rightarrow T_{\pi(x)}X \oplus T_{\pi(x)}^*X, \quad Z \mapsto (y, \eta) := (d_x\pi(Z), \mathcal{K}_x(Z)).$$

These coordinates (y, η) allow to express easily the different structures on T^*X . For instance, the Hamiltonian vector field V associated to $\frac{\|q\|_{g^*(p)}^2}{2}$ (*i.e.* the generator of the geodesic flow) satisfies $A(x)V(x) = (d_x\pi(V), 0)$.

2.2. Symplectic, Riemannian and almost complex structures. Recall that the canonical contact form α on T^*X is given by the following expression:

$$\forall x = (q, p) \in T^*X, \forall Z \in T_x T^*X, \alpha_{q,p}(Z) = p(d_x\pi(Z)).$$

The canonical symplectic form on T^*X can then be defined as $\Omega = d\alpha$. Using our natural isomorphism $A(x)$ which identifies TT^*X with $\pi^*(TX \oplus T^*X)$, this symplectic form can be written as

$$\forall Z_1 \cong (y_1, \eta_1) \in T_x T^*X, \forall Z_2 \cong (y_2, \eta_2) \in T_x T^*X, \Omega_x(Z_1, Z_2) = \eta_1(y_2) - \eta_2(y_1).$$

One can also define the following map from $T_x X \oplus T_x^* X$ to itself:

$$\tilde{J}_x(y, \eta) = (\eta^\sharp, -y^\flat).$$

For every $x \in T^*X \setminus \underline{0}$, this map induces an almost complex structure on $T_x T^*X$ through the isomorphism $A(x)$. We denote this almost complex structure by J_x . Finally, the Sasaki metric g^S on $T^*X \setminus \underline{0}$ is defined as

$$g_x^S(Z_1, Z_2) := g_q^*(\mathcal{K}_x(Z_1), \mathcal{K}_x(Z_2)) + g_q(d_x\pi(Z_1), d_x\pi(Z_2)).$$

This is a positive definite bilinear form on $T_x T^*X$.

Remark 2.4. This also induces a Riemannian metric $\tilde{g}^{(1)}$ on S^*X .

This Riemannian metric on $T^*X \setminus \underline{0}$ is compatible with the symplectic structure on T^*X through the almost complex structure. Precisely, one has, for every $(Z_1, Z_2) \in T_x T^*X \times T_x T^*X$,

$$g_x^S(Z_1, Z_2) = \Omega_x(Z_1, J_x Z_2).$$

In fact, using the natural isomorphism $A(x)$, one can write explicitly

$$\Omega_x(Z_1, J_x Z_2) = \eta_1(\eta_2^\sharp) + y_2^\flat(y_1) = g_q^*(\eta_1, \eta_2) + g_q(y_1, y_2).$$

Remark 2.5. Given $x \in T^*X \setminus \underline{0}$, we can define H_x to be the orthogonal complement to the geodesic vector field V inside \mathcal{H}_x . In particular,

$$T_x T^*X = \mathbb{R}V(x) \oplus H_x \oplus \mathcal{V}_x, \quad \text{Ker}(\alpha_x) = H_x \oplus \mathcal{V}_x,$$

and

$$T_x S^*X = \mathbb{R}V(x) \oplus H_x \oplus \mathcal{V}_x^{(1)}, \quad \text{Ker}(\alpha_x|_{T_x S^*X}) = H_x \oplus \mathcal{V}_x^{(1)}.$$

2.3. The case of surfaces. In this paragraph, we suppose that X is an oriented surface and we define a “natural basis” on $T_x T^*X$. Thanks to the fact that the manifold X is oriented with a Riemannian structure, one can define a notion of rotation by $\pi/2$ in every cotangent space T_q^*X (which is of dimension 2). Thus, given any $p \in T_q^*X \setminus \{0\}$ there exists a unique p^\perp such that $\{p, p^\perp\}$ is a direct orthogonal basis with $\|p\|_x = \|p^\perp\|_x$. Using this observation, we can define an orthogonal basis of \mathcal{V}_x for any $x = (q, p) \in T^*X \setminus \underline{0}$:

$$e_2(x) := (A(x))^{-1}(0, p), \text{ and } e_3(x) = (A(x))^{-1}(0, p^\perp).$$

Then, we can define an orthogonal basis of \mathcal{H}_x as follows

$$e_0(x) = J_x e_2(x), \text{ and } e_1(x) = J_x e_3(x).$$

The vector $e_0(x)$ is the geodesic vector field $V(x)$ induced by the Riemannian metric g . The family $\{e_0(x), e_1(x), e_2(x), e_3(x)\}$ forms a direct orthogonal basis of $T_x T^*X$ (it is normalized when $x \in S^*X$). Similarly, $\{e_0(x), e_1(x), e_3(x)\}$ is a direct orthonormal basis of $T_x S^*X$. In the following, we will denote by $\mathbb{R}e_0^*(x) \subset T_x^* S^*X$ the annihilator of $\mathbb{R}e_1(x) \oplus \mathbb{R}e_3(x)$, by $\mathbb{R}e_1^*(x)$ the annihilator of $\mathbb{R}e_0(x) \oplus \mathbb{R}e_3(x)$ and by $\mathbb{R}e_3^*(x)$ the annihilator of $\mathbb{R}e_0(x) \oplus \mathbb{R}e_1(x)$. In each case, we fix e_i^* such that $e_i^*(e_i) = 1$.

Remark 2.6. With these conventions, the kernel of the symplectic form α is generated by $\{e_1, e_2, e_3\}$ and $\alpha_x(e_0(x)) = \|p\|_{g^*(q)}^2$.

3. PRELIMINARIES ON ANOSOV FLOWS

In this section, we briefly review the notion of Anosov vector fields with some emphasis on the example of geodesic flows on negatively curved manifolds. We also prove a few technical statements on the dynamical properties of Anosov flows that will be used later on. All along sections 3 and 4, M will denote a smooth, compact, oriented manifold which has no boundary and which is of dimension $n \geq 3$. We also fix a smooth Riemannian metric \tilde{g} on M .

3.1. Anosov vector fields. Let $V : M \rightarrow TM$ be a smooth (C^∞) Anosov vector field on M generating a smooth flow that we denote by $\varphi^t : M \rightarrow M$. Recall that the Anosov assumption [61, §1] requires the existence of a continuous splitting

$$(10) \quad \forall x \in M, \quad T_x M = \mathbb{R}V(x) \oplus E_u(x) \oplus E_s(x),$$

where $E_u(x) \neq \{0\}$ (resp. $E_s(x) \neq \{0\}$) is the unstable (resp. stable) direction. We shall denote their dimensions by n_u and $n_s = n - n_u - 1$. Moreover the unstable and stable directions are preserved by the tangent map $d_x \varphi^t$ and there exist some constants $C > 0$ and $\lambda_0 > 0$ such that, for every $t \geq 0$,

$$\forall v \in E_u(x), \quad \|d_x \varphi^{-t} v\|_{\tilde{g}(\varphi^{-t}(x))} \leq C e^{-\lambda_0 t} \|v\|_{\tilde{g}(x)},$$

and

$$\forall v \in E_s(x), \quad \|d_x \varphi^t v\|_{\tilde{g}(\varphi^t(x))} \leq C e^{-\lambda_0 t} \|v\|_{\tilde{g}(x)}.$$

All along this article, we will suppose that the vector field V has the Anosov property. For every $x \in M$, one can define the weakly unstable (resp. stable) manifold $W^{u0}(x)$ (resp. $W^{s0}(x)$) which are smooth immersed submanifolds inside M such that, for every $x \in M$,

$$(11) \quad T_x W^{u0}(x) = E_u(x) \oplus \mathbb{R}V(x), \quad \text{and} \quad T_x W^{s0}(x) = E_s(x) \oplus \mathbb{R}V(x).$$

For later purpose, we also define the dual spaces $E_u^*(x)$, $E_s^*(x)$ and $E_0^*(x)$ as the annihilators of $E_u(x) \oplus \mathbb{R}V(x)$, $E_s(x) \oplus \mathbb{R}V(x)$ and $E_u(x) \oplus E_s(x)$.

Remark 3.1. We say that the vector bundle E_u^* is orientable if one can find some continuous, real valued and nonvanishing section $\omega_u : M \rightarrow \Lambda^{n_u}(E_u^*)$. According to [37, App. B] (see also § 3.2 below for the 2-dimensional case), this is for instance the case when V is the geodesic flow over an oriented Riemannian manifold (X, g) with negative sectional curvatures.

3.2. Stable and unstable bundles. When (X, g) has *negative sectional curvatures*, then the geodesic flow⁸ φ^t acting on S^*X enjoys the Anosov property. In that framework, we can describe the stable and unstable bundles in the vertical/horizontal decomposition of TS^*X via the stable and unstable Ricatti solutions. Let us briefly recall this description following [77]. Given $x \in S^*X$, one can define stable Ricatti matrices [77, §3.1.2]:

$$x \in S^*X \mapsto L_x^s \in \mathcal{L}(H_x, \mathcal{V}_x^{(1)}),$$

and unstable ones

$$x \in S^*X \mapsto L_x^u \in \mathcal{L}(H_x, \mathcal{V}_x^{(1)}).$$

These matrices depend continuously on the variable $x \in S^*X$, and they are obtained by considering certain limit solutions of Jacobi equations. For any $x \in S^*X$, one has

$$(12) \quad E_s(x) := \{(W, L_x^s W) : W \in H_x\} \subset \text{Ker}(\alpha_x|_{T_x S^*X}),$$

and

$$(13) \quad E_u(x) := \{(W, L_x^u W) : W \in H_x\} \subset \text{Ker}(\alpha_x|_{T_x S^*X}).$$

Remark 3.2. In the case of surfaces, this has a particular simple expression using the vectors $e_1(x)$ and $e_3(x)$:

$$E_s(x) = \mathbb{R}(e_1(x) + L_x^s e_3(x)) \quad \text{and} \quad E_u(x) = \mathbb{R}(e_1(x) + L_x^u e_3(x)).$$

Note also that $L_x^u > 0$ and $L_x^s < 0$. To see this, observe first that one cannot have $L_x^u \leq 0$ everywhere on S^*X , otherwise the unstable vectors would not be contracted in backward times, see for instance Remark 3.3 below. Suppose now that there exists a point $x_0 \in S^*X$ such that $L_{x_0}^u = 0$. Then, recalling that, for every $x \in S^*X$, $t \mapsto L_{\varphi^t(x)}^u$ is of class \mathcal{C}^∞ and that it solves the Ricatti equation

$$R'(t) + R(t)^2 + K \circ \Pi \circ \varphi^t(x) = 0,$$

where $K < 0$ is the curvature, one finds that $\frac{d}{dt} L_{\varphi^t(x_0)}^u|_{t=0} > 0$. In particular, there exists some point $x_1 \in X$ such that $L_{x_1}^u < 0$. By continuity, this remains true in a small open

⁸This is the flow induced by the vector field V defined in §2.

neighborhood \mathcal{U} of x_1 . Now, using the fact that the geodesic flow is topologically transitive on negatively curved surfaces [28, Th. 3.11], one would find a point x_2 such that $L_{x_2}^u > 0$ whose orbit in forward times enters \mathcal{U} . This would imply the existence of $s_0 > 0$ such that $L_{\varphi^{s_0}(x_2)}^u = 0$ and $\frac{d}{dt}L_{\varphi^t(x_2)}^u|_{t=s_0} \leq 0$. This would contradict the fact that $L_{\varphi^t(x_2)}^u$ solves the Ricatti equation. Hence, $L_x^u > 0$ uniformly on S^*X .

Attached to every point $x \in S^*X$ and to these Ricatti matrices are the stable and unstable Jacobi matrices

$$J_x^s(t), J_x^u(t) \in \mathcal{L}(H_x, H_{\varphi^t(x)}),$$

which are the solutions to the (matrix-valued) differential equations

$$\mathbb{J}'_x(t) = L_{\varphi^t(x)}^s \mathbb{J}_x(t) \quad \text{and} \quad \mathbb{J}'_x(t) = L_{\varphi^t(x)}^u \mathbb{J}_x(t).$$

They allow to describe the action of $d_x\varphi^t$ on $E_s(x)$:

$$\forall x \in S^*X, \forall W \in H_x, \quad d_x\varphi^t(W, L_x^s W) = (J_x^s(t)W, L_{\varphi^t(x)}^s J_x^s(t)W),$$

and on $E_u(x)$:

$$\forall x \in S^*X, \forall W \in H_x, \quad d_x\varphi^t(W, L_x^u W) = (J_x^u(t)W, L_{\varphi^t(x)}^u J_x^u(t)W),$$

Remark 3.3. In the case of surfaces, the expressions are one more time simpler. More precisely, for every $x \in S^*X$, one has

$$d_x\varphi^t(e_1(x) + L_x^s e_3(x)) = e^{\int_0^t L_{\varphi^\tau(x)}^s d\tau} (e_1(\varphi^t(x)) + L_{\varphi^t(x)}^s e_3(\varphi^t(x))),$$

and

$$d_x\varphi^t(e_1(x) + L_x^u e_3(x)) = e^{\int_0^t L_{\varphi^\tau(x)}^u d\tau} (e_1(\varphi^t(x)) + L_{\varphi^t(x)}^u e_3(\varphi^t(x))).$$

In particular, $d_x\varphi^t$ can be expressed in the basis $(e_1(x), e_3(x)) \rightarrow (e_1(\varphi^t(x)), e_3(\varphi^t(x)))$ defined in §2.3:

$$\frac{1}{L_x^u - L_x^s} \begin{pmatrix} L_x^u e^{\int_0^t L_{\varphi^\tau(x)}^s d\tau} - L_x^s e^{\int_0^t L_{\varphi^\tau(x)}^u d\tau} & e^{\int_0^t L_{\varphi^\tau(x)}^u d\tau} - e^{\int_0^t L_{\varphi^\tau(x)}^s d\tau} \\ L_x^u L_{\varphi^t(x)}^s e^{\int_0^t L_{\varphi^\tau(x)}^s d\tau} - L_x^s L_{\varphi^t(x)}^u e^{\int_0^t L_{\varphi^\tau(x)}^u d\tau} & L_{\varphi^t(x)}^u e^{\int_0^t L_{\varphi^\tau(x)}^u d\tau} - L_{\varphi^t(x)}^s e^{\int_0^t L_{\varphi^\tau(x)}^s d\tau} \end{pmatrix}.$$

Using Ricatti equation and the fact that the curvature is < 0 , we observe that the anti-diagonal terms become positive for $t > 0$ while they become negative for $t < 0$. Note also that, in the case of constant negative curvature $K = -1$, this matrix is equal to

$$\begin{pmatrix} \text{ch}(t) & \text{sh}(t) \\ \text{sh}(t) & \text{ch}(t) \end{pmatrix}.$$

We note that the description in this paragraph can be extended to more general Riemannian manifolds with Anosov geodesic flows except for certain positivity properties (like $L_x^u > 0$ everywhere) that made use of the curvature hypothesis.

3.3. Transversal submanifolds. We now fix two smooth embedded submanifolds Σ_1 and Σ_2 that we suppose to be oriented and boundaryless. We make the following transversality assumptions which already appeared in the seminal work of Margulis [61, p. 49]:

$$(14) \quad \forall x \in \Sigma_1, \quad T_x M = T_x \Sigma_1 \oplus T_x W^{u0}(x),$$

and

$$(15) \quad \forall x \in \Sigma_2, \quad T_x M = T_x \Sigma_2 \oplus T_x W^{s0}(x).$$

Note that these assumptions imply that $\dim \Sigma_1 = n_s$ and $\dim \Sigma_2 = n_u$.

The simplest example of a submanifold Σ verifying either (14) or (15) is $S_q^* X$ where V is the geodesic vector field over a negatively curved Riemannian manifold (X, g) . This follows from the fact that $T_x(S_q^* X)$ is the vertical space at x which is transversal to the weakly unstable/stable manifold at x thanks to (12) and (13).

Remark 3.4. We note that each submanifold $S_q^* X$ is orientable. In dimension 2, we choose to orient it using the one form $e_3^*(x)$ (or equivalently with the vector $e_3(x)$) defined in Section 2. It means that we orient it in the trigonometric sense relative to the orientation on X .

3.3.1. The case of geodesic flows on negatively curved surfaces. Besides the fiber $S_q^* X$ (and small perturbations of it), we can consider $c : t \in \mathbb{R}/\ell\mathbb{Z} \mapsto q(t) \in X$, $\ell > 0$ to be a *smooth* curve such that $q'(t) \neq 0$ for every $t \in \mathbb{R}/\ell\mathbb{Z}$. Up to reparametrization, we can choose $q'(t)$ to be of norm 1 for every $t \in \mathbb{R}/\ell\mathbb{Z}$. Note that q may have self intersections. Then, we define the (unit) conormal bundle to c :

$$\Sigma := N_1^*(c) := \{(q(t), p) \in S^* X : t \in \mathbb{R}/\ell\mathbb{Z}, p(q'(t)) = 0\}.$$

This defines a smooth submanifold inside $S^* X$ and we can verify that, for every $x \in \Sigma$, $T_x \Sigma$ is contained inside the kernel of the canonical Liouville one form α . Indeed, let $x = (q(t_0), \tilde{p}(t_0))$ be an element of Σ and let $t \in (t_0 - \epsilon, t_0 + \epsilon) \mapsto (q(t), \tilde{p}(t)) \in \Sigma$ be a smooth curve passing through x . We denote by $p(t_0)$ the covector associated to $q'(t_0)$ via the Riemannian metric g – see paragraph 2. Then, one has

$$Z(t_0) = (q'(t_0), \tilde{p}'(t_0)) \in \text{Ker}(\alpha_{q(t_0), \tilde{p}(t_0)}) \iff g_{q(t_0)}^*(p(t_0), \tilde{p}(t_0)) = 0.$$

As the tangent space to the submanifold Σ is contained inside the kernel of the contact form, we say that Σ is a *Legendrian submanifold*.

Remark 3.5. Note that the same discussion could be carried out if c is replaced by a smooth submanifold inside X and if we consider the (unit) conormal bundle to this submanifold.

Suppose now that $\dim X = 2$. In that case, Σ consists of two connected components since there are two conormal co-vectors above each point of the closed curve. Given $t \in \mathbb{R}/\ell\mathbb{Z}$, we denote these two covectors by $p^\perp(t)$ and $-p^\perp(t)$ and we set

$$Z(t) = (q'(t), (p^\perp)'(t)) \in T_{q(t), p^\perp(t)} S^* X \cap \text{Ker}(\alpha_{q(t), p^\perp(t)}).$$

In order to check the transversality assumption (14), we can compute the horizontal and the vertical components (introduced in § 2) of $Z(t)$ – see also [69, Ch. 1]. The horizontal

component is given by $d_{q(t), p^\perp(t)}\pi(Z(t)) = q'(t) = p(t)^\sharp$. For the vertical component, it is given by the covariant derivative $\nabla_{q'(t)}p(t)^\perp$ induced by g . We can now remark that

$$0 = \frac{d}{dt}g_{q(t)}^*(p(t)^\perp, p(t)^\perp) = 2g_{q(t)}^*(p(t)^\perp, \nabla_{q'(t)}p(t)^\perp).$$

Hence, the horizontal component of $Z(t)$ can be rewritten as

$$\nabla_{q'(t)}p(t)^\perp = g_{q(t)}^*(\nabla_{q'(t)}p(t)^\perp, p(t))p(t).$$

Now, we can use the explicit expression of the unstable bundle in the horizontal/vertical bundles to rewrite the transversality assumption (14) for $\Sigma = N_1^*(c)$. More precisely, it is equivalent to the following property of the closed curve γ :

$$(16) \quad \forall t \in \mathbb{R}/\ell\mathbb{Z}, \quad g_{q(t)}^*(\nabla_{q'(t)}p(t)^\perp, p(t)) \neq L_{q(t), p(t)^\perp}^u,$$

where $L_x^u > 0$ is the unstable Ricatti solution [77, Ch. 3] – see paragraph 3.2.

Remark 3.6. One can now consider the case where $c : t \in \mathbb{R}/\ell\mathbb{Z} \mapsto q(t) \in X$ is a closed geodesic. To that aim, we can observe that

$$0 = \frac{d}{dt}g_{q(t)}^*(p(t), p(t)^\perp) = g_{q(t)}^*(\nabla_{q'(t)}p(t), p(t)^\perp) + g_{q(t)}^*(p(t), \nabla_{q'(t)}p(t)^\perp) = g_{q(t)}^*(p(t), \nabla_{q'(t)}p(t)^\perp),$$

where we used the geodesic equation $\nabla_{q'(t)}p(t) = 0$ to write the last equality. Hence, property (16) is immediately verified since $L_x^u > 0$ on S^*X on *negatively curved surfaces*. Yet, it may fail for more general surfaces with Anosov geodesic flows as L_x^u may vanish at some points.

Remark 3.7. In the following, a running example will be to consider the case where $c(t)$ is a geodesic representative of a homotopy class $\mathbf{c} \in \pi_1(X)$ – see §3.4 below. In that case, we will take $\Sigma(c)$ to be the connected component of $N_1^*(c)$ consisting of the covectors directly orthogonal to $p(t)$. We choose to orient $\Sigma(c)$ via the orientation of the geodesic, i.e. with the one-form $-e_1^*(x)$ (or equivalently with the vector $-e_1(x)$).

3.3.2. Orientations in a toy model. Let us illustrate our choices of orientation in a toy model on $\mathbb{R}^2 \times \mathbb{S}^1$, oriented by $dq_1 \wedge dq_2 \wedge d\phi$, that will be useful for our computations in Section 6.

- **Example 1.** We consider the horizontal line $c_1 = \{(q_1, 0), q_1 \in \mathbb{R}\}$ oriented by dq_1 in \mathbb{R}^2 . Recalling now that $\mathbf{1}'_{[0, +\infty)}(q) = \delta_0(q)$, therefore $\partial[\mathbb{R} \times \mathbb{R}_+] = \partial\mathbf{1}_{\mathbb{R}_+}(q_2) = -d\mathbf{1}_{\mathbb{R}_+}(q_2) = -\delta_0(q_2)dq_2 = [c_1]$. In other words, c_1 is the oriented boundary of $\mathbb{R} \times \mathbb{R}_+$. Then, $\Sigma(c_1) := \{(q_1, 0, \pi/2) : q_1 \in \mathbb{R}\}$ and we oriented it using dq_1 . This yields the following representation of its current of integration

$$[\Sigma(c_1)] = \delta_0(q_2)\delta_0\left(\phi - \frac{\pi}{2}\right) dq_2 \wedge d\phi.$$

Introduce now the surface $S := \{(q_1, q_2, \pi/2) : q_2 \geq 0\}$ whose (topological) boundary is $\Sigma(c_1)$. This surface is naturally oriented by $dq_1 \wedge dq_2$ and thus it can be represented as

$$[S] = \mathbf{1}_{q_2 \geq 0}(q_1, q_2)\delta_0\left(\phi - \frac{\pi}{2}\right) d\phi.$$

One finds that $[\Sigma(c_1)]$ is a coboundary:

$$d[S] = \delta_0(q_2)\delta_0\left(\phi - \frac{\pi}{2}\right) dq_2 \wedge d\phi = [\Sigma(c_1)].$$

• **Example 2.** Consider now the point $c_0 = (0, 0)$. One has

$$S_{c_0}^* \mathbb{R}^2 = \{c_0\} \times \mathbb{S}^1 := \{(0, 0, \phi) : 0 \leq \phi \leq 2\pi\}.$$

Our choice of orientation on this curve is to take $d\phi$. Hence, the current of integration on the fiber $S_{c_0}^* \mathbb{R}^2$ reads $[S_{c_0}^* \mathbb{R}^2] = \delta_0(q_1, q_2) dq_1 \wedge dq_2$. As in the above example, introduce the following submanifold S in $\mathbb{R}^2 \times \mathbb{S}^1$:

$$S := \left\{ (q_1, q_2, \phi) : (q_1, q_2) \in \mathbb{R}^2 \setminus \{c_0\}, \cos \phi = \frac{q_1}{\sqrt{q_1^2 + q_2^2}}, \sin \phi = \frac{q_2}{\sqrt{q_1^2 + q_2^2}} \right\},$$

whose topological boundary is $S_{c_0}^* \mathbb{R}^2$. Endowing S with the orientation $dq_1 \wedge dq_2$ yields the following representation of $[S]$ in $\mathbb{R}^2 \setminus \{0\} \times (-\pi/2, \pi/2)$:

$$\begin{aligned} [S] &= \mathbf{1}_{\mathbb{R}_+}(q_1) \delta_0(q_2 - q_1 \tan(\phi)) d(q_1 \tan \phi - q_2) \\ &= \mathbf{1}_{\mathbb{R}_+}(q_1) \delta_0(q_2 - q_1 \tan(\phi)) \left(\frac{q_1 d\phi}{1 + \phi^2} + \tan(\phi) dq_1 - dq_2 \right). \end{aligned}$$

This current can be extended into a well-defined current on $\mathbb{R}^2 \times (-\pi/2, \pi/2)$. Hence, by a partition of unity in the ϕ variable, $[S]$ defines a current on $\mathbb{R}^2 \times \mathbb{S}^1$. Finally, in $\mathbb{R}^2 \times (-\pi/2, \pi/2)$, one has

$$d[S] = -\delta_0(q_1) \delta_0(q_2 - q_1 \tan \phi) dq_1 \wedge dq_2 = -[S_{c_0}^* \mathbb{R}^2].$$

Performing the same argument in every half plane, one finds that $[S_{c_0}^* \mathbb{R}^2]$ is a coboundary: $d[S] = -[S_{c_0}^* \mathbb{R}^2]$. Using polar coordinates (r, θ, ϕ) , this manifold can also be viewed as the boundary of the manifold $\mathbb{R}_{>0} \times \mathbb{T}^2$ oriented by $r dr \wedge d\theta \wedge d\phi$. In these coordinates, the current of integration on the fiber $S_{c_0}^* \mathbb{R}^2$, oriented by $d\phi$, reads

$$[S_{c_0}^* \mathbb{R}^2] = \delta_0(r) \delta_0(\phi - \theta) dr \wedge (d\theta - d\phi).$$

If we now form the surface

$$S := \{(r, \theta, \theta) : r > 0, 0 \leq \theta \leq 2\pi\},$$

endowed with the orientation $r dr \wedge d\theta$, then

$$[S] = \mathbf{1}_{\mathbb{R}_{>0}}(r) \delta_0(\phi - \theta) (d\phi - d\theta).$$

In particular,

$$d[S] = \delta_0(r) \delta_0(\phi - \theta) dr \wedge (d\phi - d\theta) = -[S_{c_0}^* \mathbb{R}^2].$$

3.3.3. *Pieces of unstable/stable manifolds.* Beyond the case of geodesic flows, we can consider Anosov flows on a 3-dimensional manifold. If the unstable and stable vector bundles are orientable, then one can find some nonvanishing \mathcal{C}^1 vector fields⁹ V_s and V_u such that

$$\forall x \in M, \quad E_u(x) = \mathbb{R}V_u(x) \quad \text{and} \quad E_s(x) = \mathbb{R}V_s(x).$$

These vector fields generate two complete flows $(\varphi_u^t)_{t \in \mathbb{R}}$ and $(\varphi_s^t)_{t \in \mathbb{R}}$ which are referred as W^u and W^s flows (or horocycle flows in the case of geodesic flows). It was proved by Marcus [59] that these flows are uniquely ergodic. In particular, if we fix any point x_0 in M and some small $\epsilon > 0$, one can find some $\tau_0 > 0$ such that $\varphi_u^{\tau_0}(x_0)$ is within a distance ϵ of x_0 . The smooth curve

$$\tau \in [0, \tau_0] \mapsto \varphi_u^\tau(x_0)$$

satisfies the transversality assumption (15). The only problem is that it is not boundaryless as we required it for Σ_2 . In order to fix this problem, we just need to (smoothly) modify the curve in a small neighborhood of x_0 by keeping the transversality assumption (15). This gives rise to a closed curve Σ_2 satisfying the above requirements. The same procedure with φ_s^t allows to construct a closed curve Σ_1 satisfying (14).

3.4. **First properties.** Using the Anosov property and our transversality assumptions, one can show the following:

Lemma 3.8. *Suppose that (14) and (15) hold. Then, there exists some $T_0 > 0$ such that, for every $t \geq T_0$, $\varphi^{-t}(\Sigma_1)$ and Σ_2 intersect transversally with respect to the flow in the sense that, for every $x \in \varphi^{-t}(\Sigma_1) \cap \Sigma_2$,*

$$T_x M = T_x \varphi^{-t}(\Sigma_1) \oplus T_x \Sigma_2 \oplus \mathbb{R}V(x).$$

In particular, for every fixed $t \geq T_0$, the set of points lying in $\varphi^{-t}(\Sigma_1) \cap \Sigma_2$ is finite. Moreover, the times $t \geq T_0$ for which this intersection is not empty is discrete and has no accumulation points.

Proof. We fix $(v_1(x, t), \dots, v_{n_s}(x, t))$ generating $T_{\varphi^t(x)}\Sigma_1$. Thanks to (14), one has then

$$T_{\varphi^t(x)}M = \text{Span}(v_1(x, t), \dots, v_{n_s}(x, t)) \oplus E_u(\varphi^t(x)) \oplus \mathbb{R}V(\varphi^t(x)).$$

Using the Anosov property and the fact that Σ_1 is a closed embedded submanifold satisfying (14), one can find some T_0 such that, for $t \geq T_0$ and for every $x \in \varphi^{-t}(\Sigma_1)$, $d_{\varphi^t(x)}\varphi^{-t}(v_j(x, t))$ lies in a fixed (small) conical neighborhood of $E_s(x)$. Moreover, one has

$$T_x M = d_{\varphi^t(x)}\varphi^{-t}\text{Span}(v_1(x, t), \dots, v_{n_s}(x, t)) \oplus E_u(x) \oplus \mathbb{R}V(x).$$

Hence, we have shown that the family $(V(x), (d_{\varphi^t(x)}\varphi^{-t}(v_j(x, t)))_{1 \leq j \leq n_s})$ generates a vector space of dimension $n_s + 1$ lying in a conical neighborhood of $E_s(x) \oplus \mathbb{R}V(x)$ that does not intersect $T_x \Sigma_2$ thanks to (15). This concludes the first part of the lemma

Now, we fix $t_1 < t_2$ both greater than T_0 and we consider the two submanifolds Σ_2 and $(\varphi^{-t}(\Sigma_1))_{t_1 < t < t_2}$ of M . They intersect transversally thanks to the first part of the lemma. Hence, as they are respectively of dimension n_u and $n - n_u$, one finds that the intersection

⁹The fact that the unstable and stable bundles are of class \mathcal{C}^1 is due to the fact that $\dim E_{u/s} = 1$.

between these two submanifolds consists in a finite number of points which concludes the proof. \square

In the case of the geodesic flow on a negatively curved surface (X, g) , one can in fact be slightly more precise. To that aim, we fix \mathbf{c} a primitive element in the fundamental group $\pi_1(X)$ of X . We know that when \mathbf{c} is nontrivial, there exists a unique (closed) geodesic c parametrized by arc length and lying in the class \mathbf{c} . We denote it by

$$c(t) : t \in [0, \ell_{\mathbf{c}}] \mapsto q(t) \in X,$$

where $\ell_{\mathbf{c}} > 0$ is the length of the closed geodesic. As already explained, to this geodesic curve $c(t)$, we associate a Legendrian curve

$$\Sigma(c) := \{(q(t), p(t)^\perp) : t \in [0, \ell_{\mathbf{c}}]\},$$

where $p(t)^\perp$ is the unit co-vector which is directly orthogonal to the geodesic covector $p(t) = q'(t)^\flat$. When \mathbf{c} is trivial, we can fix $c = q$ to be any point inside X and we set

$$\Sigma(c) := S_q^* X.$$

In both cases, we say that c is a *geodesic representative* of the homotopy class \mathbf{c} and one has:

Lemma 3.9. *Suppose that $M = S^*X$ where (X, g) is a Riemannian surface and V is the geodesic vector field. Let \mathbf{c}_1 and \mathbf{c}_2 be two elements in $\pi_1(X)$ and let c_1 and c_2 be two of their geodesic representatives. Then, the conclusion of Lemma 3.8 holds for $\Sigma(c_1)$ and $\Sigma(c_2)$ with*

- $T_0 = 0$ if $\Sigma(c_1) \cap \Sigma(c_2) = \emptyset$,
- any $T_0 > 0$ otherwise.

Proof. The proof is the same as for Lemma 3.8 combined with the explicit expressions of the unstable space given in Remark 3.2 and of the tangent map given in Remark 3.3. In fact, one can also pick $T_0 = 0$ when \mathbf{c}_1 is trivial and \mathbf{c}_2 is not (even if c_1 lies on c_2). \square

3.5. Wavefront sets of the currents $\varphi^{T*}[\Sigma_i]$. Let us now define more precisely the currents associated with the oriented submanifolds Σ_1 and Σ_2 and their pullback under the Anosov flow φ^t . All along the article, we adopt the following conventions for the currents of integration on the submanifolds Σ_i :

$$S_0 := [\Sigma_1] \quad \text{and} \quad U_0 := [\Sigma_2],$$

In particular, S_0 defines an element in $\mathcal{D}'^{n-n_s}(M)$ whose wavefront set¹⁰ is contained inside the conormal bundle to Σ_1 :

$$N^*(\Sigma_1) := \{(x, \xi) \in T^*M \setminus \underline{0} : x \in \Sigma_1 \text{ and } \forall v \in T_x \Sigma_1, \xi(v) = 0\}.$$

¹⁰See appendix A for a brief reminder on wavefront sets.

Similarly, one finds that U_0 defines an element in $\mathcal{D}^{n-n_u}(M)$ whose wavefront set is contained inside the conormal set $N^*(\Sigma_2)$ to Σ_2 . Hence, if we denote by $\mathcal{D}_\Gamma^k(M)$ the set of currents of degree k having their wavefront set inside a fixed conic set Γ , one has

$$(17) \quad S_0 \in \mathcal{D}_{N^*(\Sigma_1)}^{n_u+1}(M) \quad \text{and} \quad U_0 \in \mathcal{D}_{N^*(\Sigma_2)}^{n-n_u}(M).$$

If we apply the pullback by the flow, we find the following in terms of wavefront properties:

Lemma 3.10. *Let Γ_u (resp. Γ_s) be a closed conic set of $T^*M \setminus \underline{0}$ which contains $E_u^* \oplus E_0^* \setminus \underline{0}$ (resp. $E_s^* \oplus E_0^* \setminus \underline{0}$) in its interior. Then, there exists $T_0 > 0$ such that, for every $T \geq T_0$,*

$$S_T := \varphi^{T*}(S_0) \in \mathcal{D}_{\Gamma_s}^{n_u+1}(M) \quad \text{and} \quad U_{-T} := \varphi^{-T*}(U_0) \in \mathcal{D}_{\Gamma_u}^{n-n_u}(M).$$

Proof. We already said that S_0 belongs to $\mathcal{D}_{N^*(\Sigma_1)}^{n_u+1}(M)$. According to the properties of the pullback on the space $\mathcal{D}_\Gamma^k(M)$ (see Appendix A.3), we know that for every $t \geq 0$, $\varphi^{t*}(S_0)$ will belong to $\mathcal{D}_{\Gamma(t)}^{n_u+1}(M)$, where

$$\Gamma(t) := \{(\varphi^{-t}(x), ((d_x \varphi^{-t})^T)^{-1} \xi) : (x, \xi) \in N^* \Sigma_1\}.$$

From the transversality assumption (14), the hyperbolicity of the flow and the compactness of Σ_1 , we know that for t large enough, the set $\Gamma(t)$ will be uniformly close to $E_s^* \oplus E_0^*$ (thus inside Γ_s). This follows for instance from the existence of a continuous norm on T^*M and of families of cones both adapted to the Anosov dynamics – see e.g. [17, § 3.1]. This concludes the proof for S_0 and the same argument works for U_0 . \square

3.6. A priori bounds on the growth of intersection points. According to Lemma 3.8, one knows that there exists some $T_0 \geq 0$ such that

$$\mathcal{P}_{\Sigma_1, \Sigma_2} := \{t \geq T_0 : \varphi^{-t}(\Sigma_1) \cap \Sigma_2 \neq \emptyset\}$$

defines a discrete subset of $[T_0, \infty]$ with no accumulation points. Moreover, for every $t \geq T_0$, we can define

$$(18) \quad m_{\Sigma_1, \Sigma_2}(t) := |\{x \in \varphi^{-t}(\Sigma_1) \cap \Sigma_2\}| < +\infty,$$

which is thus equal to 0 outside a discrete subset of $[T_0, +\infty)$. We begin with the following a priori upper bound on these quantities:

Lemma 3.11. *Let Σ_1 and Σ_2 satisfying respectively assumptions (14) and (15). Then, for every $h > h_{\text{top}}$, one can find some constant $C_h > 0$ such that, for every¹¹ $T \geq T_0$,*

$$\sum_{t \in [T, T+1)} m_{\Sigma_1, \Sigma_2}(t) \leq C_h e^{hT}.$$

The proof of this Lemma could be extracted from Margulis' arguments in [61, §7]. Yet, for the sake of completeness, we give a short proof of it.

¹¹Recall that the constant T_0 comes from Lemma 3.8.

Proof. We fix some $h > h_{\text{top}}$. For every $x \in M$ and for every $\epsilon, T > 0$, we define the Bowen ball centered at x :

$$B(x, \epsilon, T) := \{y \in M : \forall 0 \leq t \leq T, d_{\tilde{g}}(\varphi^t(x), \varphi^t(y)) < \epsilon\},$$

where $d_{\tilde{g}}$ is the distance induced by the Riemannian metric. From the definition of the topological entropy, one can find some $\epsilon_0 > 0$ such that, for every $0 < \epsilon < \epsilon_0$, one can find some constant $C_\epsilon > 0$ so that

$$\forall T > 0, \quad \inf \left\{ |F| : F \subset M \text{ and } \bigcup_{x \in F} B(x, \epsilon, T) = M \right\} \leq C_\epsilon e^{hT}.$$

Fix now some $T > 0$ and some $0 < \epsilon < \epsilon_0$. We let $F \subset M$ so that the infimum is attained in the previous inequality. We decompose Σ_2 as follows

$$\Sigma_2 = \bigcup_{x \in F} \Sigma_2(x, \epsilon, T),$$

where

$$\Sigma_2(x, \epsilon, T) := \Sigma_2 \cap B(x, \epsilon, T).$$

We fix some conic neighborhood \mathbf{C}_u of $E_u \setminus \underline{0}$ so that it does not intersect $T\Sigma_1$. From the Anosov assumption and from the transversality assumption (15), we know that there exists some $T_1 > 0$ such that for every $t \geq T_1$, $d\varphi^t(T\Sigma_2) \subset \mathbf{C}_u$. Observe that for ε small enough, for every *small* piece of submanifold $\tilde{\Sigma}$ (of dimension $\leq n_u$) so that $T(\tilde{\Sigma})$ is contained in \mathbf{C}_u and $\tilde{\Sigma}$ is contained in a ball of radius ε , then $\tilde{\Sigma}$ intersects Σ_1 at most at one point thanks to (14). Still thanks to our transversality assumptions, one can verify that there exists some integer p_0 (depending only on the cone and on Σ_1) so that, for every $\tilde{\Sigma}$ such that $T(\tilde{\Sigma})$ is contained in \mathbf{C}_u

$$(19) \quad \tilde{\Sigma} \cap \Sigma_1 \neq \emptyset \implies \forall 0 < t \leq p_0^{-1}, \varphi^t(\tilde{\Sigma}) \cap \Sigma_1 = \emptyset.$$

In particular, if $\Sigma_2(x, \epsilon, T) \cap \varphi^{-t}(\Sigma_1) \neq \emptyset$ for some $t \geq \max\{T_0, T_1\}$, then

$$\left| \left(\Sigma_2(x, \epsilon, T) \cap \varphi^{-t}(\Sigma_1) \right)_{T \leq t < T+1} \right| \leq p_0.$$

As the cardinal of F is $\leq C_\epsilon e^{hT}$, we finally find that, for every $T \geq \max\{T_0, T_1\}$,

$$\sum_{t \in [T, T+1)} m_{\Sigma_1, \Sigma_2}(t) \leq C_\epsilon p_0 e^{hT},$$

which concludes the proof of the Lemma. \square

4. MEROMORPHIC CONTINUATION OF ZETA FUNCTIONS

All along this section, we fix Σ_1 and Σ_2 as in (14) and (15), and some large $T_0 > 0$ so that the properties of Section 3 are satisfied. Using the conventions of Section 3, we can

define the following zeta function

$$\zeta_{\Sigma_1, \Sigma_2}(z) := \sum_{t \geq 0: \varphi^{-T_0}(\Sigma_1) \cap \varphi^{t+T_0}(\Sigma_2) \neq \emptyset} e^{-zt} \left(\sum_{x \in \varphi^{-T_0}(\Sigma_1) \cap \varphi^{t+T_0}(\Sigma_2)} \epsilon_t(x) \right),$$

where $\epsilon_t(x) = 1$ if

$$d_{\varphi^{T_0}(x)} \varphi^{-T_0} (T_{\varphi^{T_0}(x)} \Sigma_1) \oplus \mathbb{R}V(x) \oplus d_{\varphi^{-T_0-t}(x)} \varphi^{T_0+t} (T_{\varphi^{-T_0-t}(x)} \Sigma_2)$$

has the same orientation as $T_x M$, and to -1 otherwise. Recall from (18) that $\# \varphi^{-T_0}(\Sigma_1) \cap \varphi^{t+T_0}(\Sigma_2)$ is finite for every $t \geq 0$ if T_0 is large enough. Moreover, thanks to Lemma 3.11, the function $\zeta_{\Sigma_1, \Sigma_2}$ is well defined and holomorphic for $\operatorname{Re}(z) > h_{\text{top}}$. We emphasize that this function depends implicitly on some parameter T_0 and that changing the value of T_0 may amount to modify the factors $\epsilon_t(x)$ by a constant factor in $\{\pm 1\}$.

Remark 4.1. The reason for introducing $\epsilon_t(x)$ in the definition is that the unstable bundle may not be orientable. If it was orientable (e.g. for Anosov geodesic flows), then one could verify that $\epsilon_t(x)$ becomes constant for t large enough.

Remark 4.2. Thanks to Lemma 3.9, we know that we can take any $T_0 > 0$ (and even $T_0 = 0$ in most cases) when M is the unit cotangent bundle of a Riemannian surface (X, g) , φ^t is the geodesic flow and $\Sigma_i = \Sigma(c_i)$ with c_i being geodesic representatives of elements in $\pi_1(X)$. In that case, it also follows from the explicit expression of the tangent map given in paragraph 3.2 that $\epsilon_t(x)$ is independent of t and x . In fact, it only depends on the homotopy classes \mathbf{c}_1 and \mathbf{c}_2 .

The main result of this section is

Theorem 4.3. *Let (Σ_1, Σ_2) be two smooth, compact, embedded and oriented submanifolds satisfying respectively the transversality assumptions (14) and (15).*

Then, $\zeta_{\Sigma_1, \Sigma_2}(z)$ has a meromorphic continuation to \mathbb{C} .

We shall in fact prove a slightly stronger result for zeta functions involving an exponential weight for each flow line between Σ_1 and Σ_2 . This Theorem is the main analytical result of this article and its proof relies on microlocal methods that were initiated in [30, 32, 26]. Yet, it is plausible that a similar result could be derived using the geometric methods from [12, 37] or the coherent states approach of [86, 31].

Remark 4.4. As we shall see in our proof, the poles of this meromorphic function are included in the set of Pollicott-Ruelle resonances for currents of degree $n_s + 1$ [12, 13, 32, 37, 26]. Recall from [37, Prop. 4.9] that the real parts of the resonances are in that case $\leq h_{\text{top}}$. Moreover, if the flow is topologically transitive, then it is a simple eigenvalue. If the flow is topologically mixing, then h_{top} is the only resonance on the axis $h_{\text{top}} + i\mathbb{R}$. Using the inverse Laplace transform, this would allow to recover Margulis' asymptotic formula (1) in that framework.

4.1. Anisotropic Sobolev spaces and wavefront properties of the resolvent. Let $W : M \rightarrow \mathbb{C}$ be a smooth map and denote by $\mathcal{L}_V = d\iota_V + \iota_V d$ the Lie derivative along the vector field V . For every $0 \leq k \leq n$, the map

$$R_k(z) := (\mathcal{L}_V + W + z)^{-1} = \int_0^{+\infty} e^{-tz} e^{-t(\mathcal{L}_V + W)} |dt| : \Omega^k(M) \rightarrow \mathcal{D}^k(M)$$

is well defined and holomorphic in some region $\operatorname{Re}(z) \geq C_0$ for some $C_0 > 0$ depending on W , M and V . Here $|dt|$ is understood as the Lebesgue measure on \mathbb{R} in order to distinguish with currents of integration. It follows from the works of Butterley-Liverani [12, 13], Faure-Sjöstrand [32], Giulietti-Liverani-Pollicott [37] and Dyatlov-Zworski [26] that this resolvent admits a meromorphic extension to the whole complex plane. The poles are the so-called Pollicott-Ruelle resonances and the residues are given by spectral projectors. Such a property was obtained by defining appropriate Banach spaces with anisotropic regularity properties and we briefly describe in this paragraph the anisotropic Sobolev spaces introduced by Faure-Sjöstrand [32] via microlocal methods – see also [30] for an earlier construction of Faure-Roy-Sjöstrand in the case of diffeomorphisms and [26] for the extension to the case of currents by Dyatlov-Zworski as we need here.

4.1.1. Escape functions. In the following, we denote by \mathring{E}_j^* the closed conic set $E_j^* \setminus \underline{0}$ inside $T^*M \setminus \underline{0}$, where $j = u, s, 0$. The key step in this microlocal approach is the construction of an escape function with nice enough properties with respect to the flow [32, Lemma 1.2]. Let us recall some of the properties of these escape functions we will need in our proof. It follows from this reference (see also [17, Lemma 3.2] for a formulation close to the one given here) that there exist a function $f \in C^\infty(T^*M, \mathbb{R}_+)$ which is 1-homogeneous for $\|\xi\|_x \geq 1$ and a (small) closed conic neighborhood Γ_{uu} (resp. Γ_{ss}) of \mathring{E}_u^* (resp. \mathring{E}_s^*) such that the following properties hold:

- (1) $f(x, \xi) = \|\xi\|_x$ for $\|\xi\|_x \geq 1$ and $(x, \xi) \notin \Gamma_{uu} \cup \Gamma_{ss}$,
- (2) for every $N_1 > 16N_0 > 0$ and for every (small) closed conic neighborhood Γ'_{uu} (resp. Γ'_{ss}) contained in the interior of Γ_{uu} (resp. Γ_{ss}), there exist $\tilde{\Gamma}_{uu}$ (resp. $\tilde{\Gamma}_{ss}$) contained in the interior of Γ'_{uu} (resp. Γ'_{ss}) and a smooth function

$$m_{N_0, N_1} : T^*M \rightarrow [-2N_0, 2N_1]$$

with the following requirements

- m_{N_0, N_1} is 0-homogeneous for $\|\xi\|_x \geq 1$,
- $m_{N_0, N_1}(x, \xi/\|\xi\|_x) \geq N_1$ on $\tilde{\Gamma}_{ss}$, $m_{N_0, N_1}(x, \xi/\|\xi\|_x) \leq -N_0$ on $\tilde{\Gamma}_{uu}$,
- $m_{N_0, N_1}(x, \xi/\|\xi\|_x) \geq \frac{N_1}{8}$ outside Γ'_{uu} .

These functions have in fact more properties that we shall not need explicitly here¹². In particular, the corresponding escape function

$$G_{N_0, N_1}(x, \xi) := m_{N_0, N_1}(x, \xi) \ln(1 + f(x, \xi))$$

¹²Note however that these decaying properties are crucial to prove the meromorphic continuation of the resolvent in [32].

decays along the lifted flow on T^*M :

$$\forall(x, \xi) \in T^*M, \quad \Phi^t(x, \xi) := (\varphi^t(x), ((d_x \varphi^t)^T)^{-1} \xi).$$

4.1.2. *Anisotropic Sobolev spaces.* Let $0 \leq k \leq n$. Recall that we have a scalar product on $\Omega^k(M)$ by setting, for every $(\psi_1, \psi_2) \in \Omega^k(\mathcal{M})$,

$$\langle \psi_1, \psi_2 \rangle_{L^2} := \int_M \langle \psi_1, \overline{\psi_2} \rangle_{g^*} \mathrm{dvol}_g,$$

where g^* is the metric induced by g on k -forms. We set $L^2(M, \Lambda^k(T^*M))$ (or $L^2(M)$ if there is no ambiguity) to be the completion of the (complex-valued) k -forms $\Omega^k(M)$ for this scalar product. Recall that the set of De Rham currents of degree k (the topological dual to $\Omega^{n-k}(M)$) is denoted by $\mathcal{D}^k(M)$. It was shown in [12, 13, 32, 37, 26] that $\mathcal{L}_V + W$ has a discrete spectrum when acting on convenient Banach spaces of currents of degree k .

Let us recall the definition of these spaces in the microlocal framework of Faure and Sjöstrand. Given an appropriate escape function G_{N_0, N_1} as above and following¹³ [30, 32], one can find some invertible pseudo-differential operator \widehat{A}_{N_0, N_1} with variable order [30, App.] and with principal symbol equal to $e^{G_{N_0, N_1}} \mathrm{Id}_{\Lambda^k(T^*M)}$. We then define the corresponding *anisotropic Sobolev space*:

$$\mathcal{H}_k^{m_{N_0, N_1}}(M) := \widehat{A}_{N_0, N_1}^{-1} L^2(M; \Lambda^k(T^*M)).$$

These spaces are related to the usual Sobolev spaces

$$H_k^N(M) := (1 + \Delta_{\tilde{g}}^{(k)})^{-N/2} L^2(M, \Lambda^k(T^*M))$$

as follows ($\Delta_{\tilde{g}}$ is the Laplace Beltrami operator induced by \tilde{g}):

$$(20) \quad H_k^{2N_1}(M) \subset \mathcal{H}_k^{m_{N_0, N_1}}(M) \subset H_k^{-2N_0}(M),$$

with continuous injections. Elements in $\mathcal{H}_k^{m_{N_0, N_1}}(M)$ have positive Sobolev regularity outside a conic neighborhood of E_u^* and negative Sobolev regularity inside a slightly smaller conic neighborhood. In [32, Th. 1.4] (see [26, §3.2] for the case of currents), it is shown that $(\mathcal{L}_V + W + z) : \mathcal{D}(\mathcal{L}_V) \rightarrow \mathcal{H}_k^{m_{N_0, N_1}}(M)$ is a family of Fredholm operators of index 0 depending analytically on z in the region $\{\mathrm{Re}(z) > C_0 - c_0 N_0\}$ for some constants $C_0, c_0 > 0$ independent of N_0 and N_1 . Then, the poles of the meromorphic extension are the eigenvalues of $-\mathcal{L}_V - W$ on $\mathcal{H}_k^{m_{N_0, N_1}}(M)$, the so-called *Pollicott-Ruelle resonances*. The residues at each pole are the corresponding spectral projectors, and the range of each spectral projector generates the *Pollicott-Ruelle* resonant states.

Remark 4.5. We note that it is implicitly shown in these references [32, Proof of Lemma 3.3] (adapted to the case of currents and to the case of a nontrivial W) that, for $\mathrm{Re}(z) > C_0$, one has

$$\|(\mathcal{L}_V + W + z)^{-1}\|_{\mathcal{H}_k^{m_{N_0, N_1}} \rightarrow \mathcal{H}_k^{m_{N_0, N_1}}} \leq \frac{1}{\mathrm{Re}(z) - C_0}.$$

¹³The proof in this reference is given for currents of degree 0 and $W = 0$. Yet, the proof can be adapted in higher degrees (see for instance [26]) and can handle a lower order term as W as long as it is smooth.

Then, combining this observation with [32, Th. 1.4] (still adapted to the case of currents) and with the Hille-Yosida Theorem [29, Cor. 3.6, p. 76], one finds that, for every $t \geq 0$,

$$(21) \quad \left\| e^{-\int_{-t}^0 W \circ \varphi^s ds} \varphi^{-t*} \right\|_{\mathcal{H}_k^{m_{N_0, N_1}} \rightarrow \mathcal{H}_k^{m_{N_0, N_1}}} \leq e^{tC_0}.$$

4.1.3. *Dual spaces.* The dual space to $\mathcal{H}_k^{m_{N_0, N_1}}(M)$ is given by

$$(\mathcal{H}_k^{m_{N_0, N_1}}(M))' = \widehat{A}_{N_0, N_1} L^2(M; \Lambda^k(T^*M)).$$

Via the Hodge star map, it can be identified with a Hilbert space of currents of degree $n-k$, and, with a slight abuse of notations, we denote it by $\mathcal{H}_{n-k}^{-m_{N_0, N_1}}(M)$. Elements in the dual have positive Sobolev regularity in a small conic neighborhood of E_u^* and negative Sobolev regularity outside a slightly bigger neighborhood. The duality pairing is then given, for every $(\psi_1, \psi_2) \in \mathcal{H}_k^{m_{N_0, N_1}}(M) \times \mathcal{H}_{n-k}^{-m_{N_0, N_1}}(M)$,

$$\langle \psi_1, \psi_2 \rangle_{\mathcal{H}_k^{m_{N_0, N_1}}(M) \times \mathcal{H}_{n-k}^{-m_{N_0, N_1}}(M)} = \int_M \psi_1 \wedge \overline{\psi_2}.$$

The operator dual to $-\mathcal{L}_V + W$ is given by $-\mathcal{L}_{-V} + \overline{W}$. Finally, using Appendix A, if we choose N_0 large enough and Γ'_{uu} small enough (depending on T_0) in the construction of the escape function, then

$$(22) \quad \forall T \geq T_0, \quad S_T \in \mathcal{H}_{n_u+1}^{-m_{N_0, N_1}}(M),$$

where S_T is the current constructed in Lemma 3.10.

Remark 4.6. So far, we have discussed the construction of anisotropic spaces by taking V as a reference but we could also have worked out the same procedure with $-V$ instead of V (this would exchange the role of the stable and unstable bundles). For later purpose, we denote by \tilde{m}_{N_0, N_1} the order function adapted to $-V$. This gives rise to the anisotropic spaces $\mathcal{H}_k^{\tilde{m}_{N_0, N_1}}(M)$ and its dual $\mathcal{H}_{n-k}^{-\tilde{m}_{N_0, N_1}}(M)$. Elements in $\mathcal{H}_k^{\tilde{m}_{N_0, N_1}}(M)$ (resp. $\mathcal{H}_{n-k}^{-\tilde{m}_{N_0, N_1}}(M)$) have now positive Sobolev regularity outside (resp. inside) a conic neighborhood of E_s^* and negative Sobolev regularity inside (resp. outside) a slightly smaller (resp. larger) conic neighborhood. In particular, provided N_0 is chosen large enough and Γ'_{ss} small enough, one has

$$(23) \quad \forall T \geq T_0, \quad U_{-T} \in \mathcal{H}_{n_s+1}^{-\tilde{m}_{N_0, N_1}}(M),$$

where U_{-T} is the current constructed in Lemma 3.10.

From (23) and from the boundedness of the resolvent for large real parts of z (see Remark 4.5) on the anisotropic Sobolev space, one finds that

$$(\mathcal{L}_V + W + z)^{-1} \iota_V(U_{-T}) \in \mathcal{H}_{n_s}^{-\tilde{m}_{N_0, N_1}}(M),$$

for $\text{Re}(z) \gg 1$. In the following, we will need to pair this new current with S_T using product rules for currents with specified wavefront sets – see Appendix A.2. Yet, regarding (22), it seems that we cannot a priori pair this current with S_T (even for large real parts of z). This is due to the fact that they both seem to have negative Sobolev regularity near E_0^* .

This problem can be overcome by taking advantage of the specific form of our currents and by using finer properties of the resolvent that we will now describe.

4.1.4. *Wavefront properties.* Besides its meromorphic continuation, the main property of the resolvent that we shall use is a characterization of the wavefront set of its distributional kernel [26] – see also [32, Th. 1.7] for earlier results in that direction. More precisely, we shall use the following result which was proved by Dyatlov and Zworski [26, Prop. 3.3]:

Proposition 4.7. *Let $0 \leq k \leq n$. Near each pole z_0 of the resolvent $R_k(z)$, one has*

$$R_k(z) = R_k^H(z) - \sum_{j=1}^{d(z_0)} \frac{(\mathcal{L}_V + W - z_0)^{j-1} \pi_{z_0}^{(k)}}{(z - z_0)^j},$$

where

- $\pi_{z_0}^{(k)} : \mathcal{D}'_{\hat{E}_u^*}(M) \mapsto \mathcal{D}'_{\hat{E}_s^*}(M)$ verifies

$$(\pi_{z_0}^{(k)})^2 = \pi_{z_0}^{(k)} \text{ and } \text{WF}'(\pi_{z_0}^{(k)}) \subset E_u^* \times E_s^*,$$

with WF' being understood as the wavefront set of the distributional kernel¹⁴

$$\pi_{z_0}^{(k)}(x, y, dx, dy) \in \mathcal{D}'(M \times M, \Lambda^k(T^*M) \times \Lambda^{n-k}(T^*M)),$$

- $R_k^H(z)$ is holomorphic near z_0 and one has

$$\text{WF}'(R_k^H(z)) \subset \Delta(T^*M) \cup \Omega_+ \cup E_u^* \times E_s^*,$$

where $\Delta(T^*M)$ is the diagonal of $T^*M \times T^*M$ and

$$\Omega_+ := \{(\Phi^t(x, \xi), x, \xi) \in T^*M \times T^*M : t \geq 0 \text{ and } \xi(V(x)) = 0\},$$

where $\Phi^t(x, \xi) = (\varphi^t(x), ((d_x \varphi^t)^T)^{-1} \xi)$.

4.2. **Application to our problem.** We now apply this result to the currents S_T and U_{-T} that were constructed in Lemma 3.10. To that aim, we define the following current on $M \times M$:

$$S_T \otimes U_{-T},$$

which is of degree $n+1$. As a direct application of Proposition 4.7, we obtain the following lemma:

Lemma 4.8. *There exists $T_0 > 0$ such that, for every $T \geq T_0$ satisfying*

$$(24) \quad \text{supp}(S_T) \cap \text{supp}(U_{-T}) = \emptyset,$$

the map

$$z \mapsto R_{n_s}(z) \wedge (S_T \otimes \iota_V(U_{-T}))$$

defines a meromorphic function on \mathbb{C} with values in $\mathcal{D}'^{2n}(M \times M)$, where $R_{n_s}(z)$ is understood as the distributional kernel of the resolvent.

¹⁴See Remark A.2 or [26, App. C.1].

The condition (24) may sound a little bit restrictive but, thanks to Lemma 3.8, this is satisfied outside a discrete subset of $[T_0, +\infty)$. For any such T , it allows us to define the following meromorphic function on \mathbb{C} :

$$(25) \quad Z_T(z) := \langle R_{n_s}(z) \wedge (S_T \otimes \iota_V(U_{-T})), 1 \rangle.$$

Below, we will show that, for $\operatorname{Re}(z)$ large enough, this quantity can in fact be rewritten as a zeta function for the “lengths” of the orbits joining $\varphi^{-T}(\Sigma_1)$ to $\varphi^T(\Sigma_2)$. Before discussing this more precisely, let us give the proof of Lemma 4.8.

Proof. We fix some small enough conic neighborhood Γ_u (resp. Γ_s) of $E_u^* \oplus E_0^*$ (resp. $E_s^* \oplus E_0^*$). Thanks to Lemma 3.10, we know that, for $T \geq T_0$, $S_T \otimes \iota_V(U_{-T})$ belongs to $\mathcal{D}_{\Gamma_s \times \Gamma_u \setminus \underline{Q}}^n(M \times M)$. Proposition 4.7 completely determines the wavefront of the distributional kernel of the resolvent $R_{n_s}(z)$. Hence, thanks to Appendix A.2, in order to prove this Lemma, we just need to show that $\Gamma_s \times \Gamma_u \setminus \underline{Q}$ does not intersect the wavefront set of this distributional kernel in order to make sense of the product of these two currents. To see this, we first note that $\Gamma_s \times \Gamma_u \setminus \underline{Q}$ does not intersect $E_u^* \times E_s^* \setminus \underline{Q}$. Thanks to assumption (24), we also know that $S_T \otimes \iota_V(U_{-T})$ vanishes in a small neighborhood of the diagonal $\Delta \subset M \times M$. Hence, it remains to handle the part of phase space given by Ω_+ . To that aim, suppose that there exists a point (y, η, x, ξ) lying in $\Gamma_s \times \Gamma_u \setminus \underline{Q}$ and in Ω_+ . This means that $\eta \in \Gamma_s \cap \operatorname{Ker}(\iota_V)$ and $\xi \in \Gamma_u \cap \operatorname{Ker}(\iota_V)$. Moreover, there exists some $t \geq 0$ such that

$$\eta = ((d_x \varphi^t)^T)^{-1} \xi,$$

which is not possible as we can choose Γ_u and Γ_s such that, for every $t \geq 0$, $\Phi^t(\Gamma_u \cap \operatorname{Ker}(\iota_V))$ is disjoint from $\Gamma_s \cap \operatorname{Ker}(\iota_V)$ – see for instance [26, Lemma C.1] or [17, Lemma 3.1]. \square

One more time, we can be slightly more precise when M is the unit cotangent bundle of a Riemannian surface (X, g) :

Lemma 4.9. *Suppose that $M = S^*X$ where (X, g) is a Riemannian surface and V is the geodesic vector field. Let \mathbf{c}_1 and \mathbf{c}_2 be two elements in $\pi_1(X)$ and let c_1, c_2 be two geodesic representatives. Then, for every $T_0 \geq 0$ such that $\varphi^{-T_0}(\Sigma(c_1)) \cap \varphi^{T_0}(\Sigma(c_2)) = \emptyset$, the map*

$$z \mapsto R_1(z) \wedge (\varphi^{T_0*}([\Sigma(c_1)]) \otimes \iota_V \varphi^{-T_0*}([\Sigma(c_2)]))$$

defines a meromorphic function on \mathbb{C} with values in $\mathcal{D}^6(M \times M)$, where $R_1(z)$ is understood as the distributional kernel of the resolvent.

Proof. The proof is one more time the same as in the general case except that we take advantage of the structure to determine some optimal value for T_0 . Recall that we computed the tangent space to $\Sigma(c_i)$ in paragraph 3.3.1 when we verified that the transversality assumptions are satisfied. In particular, if \mathbf{c}_i is trivial in $\pi_1(X)$, then the tangent space to $\Sigma(c_i)$ is the vertical bundle induced by the Riemannian metric – see Section 2 for the definition. Then one can verify that the wavefront set of the current of integration $[\Sigma(c_i)]$ is in the annihilator set to this vertical bundle, i.e. $\mathbb{R}e_0^* \oplus \mathbb{R}e_1^*$ – see Appendix A for a brief reminder on wavefront sets. Similarly, when \mathbf{c}_i is nontrivial, the tangent space $T\Sigma(c_i)$ is given by the intersection of the horizontal bundle with the kernel of the contact form α .

Then, we obtain that the wavefront set of the current of integration is contained inside the annihilator of this direction, i.e. $\mathbb{R}e_0^* \oplus \mathbb{R}e_3^*$. For $T_0 > 0$, we find, as an application of these two observations and of the explicit expressions of Remark 3.3, that the wavefront set of $(\varphi^{T_0*}([\Sigma(c_1)]) \otimes \iota_V \varphi^{-T_0*}([\Sigma(c_2)]))$ does not intersect $\Delta(T^*M) \cup E_u^* \times E_s^*$ provided that $\varphi^{-T_0}(\Sigma(c_1)) \cap \varphi^{T_0}(\Sigma(c_2)) = \emptyset$. It remains to check that it also does not intersect Ω_+ and this follows from the explicit computation of the tangent map given in Remark 3.3. \square

4.3. Truncating the integral in time. We saw how to make sense of the pairing between the resolvent $R_{n_s}(z)$ and our current of integration using the wavefront properties of $R_{n_s}(z)$. We will now connect this quantity to the zeta function we are interested in. To that aim, we first truncate the integral in time defining the resolvent and we show that

Proposition 4.10. *Let $W \in \mathcal{C}^\infty(M, \mathbb{C})$. There exists $T_0 > 0$ such that, for every $T \geq T_0$, one can find some $t_0 > 0$ such that, for every $\chi \in \mathcal{C}_c^\infty((-t_0, +\infty))$,*

$$I_T(\chi) := (-1)^{n_s} \int_M S_T(x, dx) \wedge \int_{\mathbb{R}} \chi(t) e^{-\int_{-t}^0 W \circ \varphi^s(x) |ds|} \varphi^{-t*} \iota_V(U_{-T})(x, dx) |dt|$$

is well defined and it is equal to

$$\sum_{t \geq -t_0 : \varphi^{-T}(\Sigma_1) \cap \varphi^{T+t}(\Sigma_2) \neq \emptyset} \left(\sum_{x \in \varphi^{-T}(\Sigma_1) \cap \varphi^{T+t}(\Sigma_2)} \epsilon_t(x) \chi(t) e^{-\int_{-t}^0 W \circ \varphi^s(x) |ds|} \right).$$

where $\epsilon_t(x)$ is equal to 1 if

$$d_{\varphi^T(x)} \varphi^{-T} (T_{\varphi^T(x)} \Sigma_1) \oplus \mathbb{R}V(x) \oplus d_{\varphi^{-T-t}(x)} \varphi^{T+t} (T_{\varphi^{-T-t}(x)} \Sigma_2)$$

has the same orientation as $T_x M$, and to -1 otherwise.

We note that $\epsilon_t(x)$ depends implicitly on $T \geq T_0$ and that it may differ by a uniform factor ± 1 from the one appearing in the definition of $\zeta_{\Sigma_1, \Sigma_2}$ (depending on the value of n_s and n).

The proof of the above proposition relies on the following fundamental geometric lemma which expresses a certain counting measure associated to our geometric problem as an integral formula.

Lemma 4.11. *Let N_1, N_2 be smooth, compact, embedded and oriented submanifolds without boundary and let Y be a nonsingular vector field which generates a flow φ_Y^t and which is transverse to N_2 , i.e.*

$$\forall x \in N_2, Y(x) \notin T_x N_2.$$

Assume that

- $\dim(N_1) + \dim(N_2) + 1 = \dim(M)$;
- $N_1 \cap N_2 = \emptyset$;
- for all $T \geq 0$ such that $N_1 \cap \varphi_Y^T(N_2) = \emptyset$, the submanifolds N_1 and $\{\varphi_Y^t(x) : t \in [0, T], x \in N_2\}$ intersect transversally.

Let $\mu \in \mathcal{D}'(\mathbb{R}_{>0})$ be defined as

$$(26) \quad \mu(t) = \sum_{0 \leq \tau \leq T: N_1 \cap \varphi_Y^\tau(N_2) \neq \emptyset} \left(\sum_{x \in N_1 \cap \varphi_Y^\tau(N_2)} \epsilon_\tau(x) \right) \delta(t - \tau),$$

where $\epsilon_\tau(x)$ is equal to 1 if

$$T_x N_1 \oplus \mathbb{R}Y(x) \oplus d_{\varphi_Y^{-\tau}(x)} \varphi_Y^\tau T_{\varphi_Y^{-\tau}(x)} N_2$$

has the same orientation as $T_x M$, and to -1 otherwise. Then, we have

$$(27) \quad \boxed{\mu(t) = (-1)^{\dim(N_1)} \int_M [N_1] \wedge (\iota_Y \varphi_Y^{-t*} [N_2])},$$

where both sides should be understood as **distributions** of t .

Note that our assumptions on N_1 and N_2 ensure that the intersection of the two submanifolds $\{\varphi_Y^t(x) : t \in [0, T], x \in N_2\}$ and N_1 consists of a finite number of points. Equation (27) is similar to the Atiyah–Bott–Guillemin trace formula for counting periodic orbits used in [26, Eq. (2.4)] where one writes a distribution involving the lengths of the periodic orbits as the (distributional) flat trace of φ^{-t*} , denoted by $\text{Tr}^b(\varphi^{-t*})$. As for the flat trace of φ^{-t*} , the right-hand side of (27) makes sense as a *distribution* in the variable t . A more conceptual definition of this quantity reads as follows. Define $F : (t, x) \in \mathbb{R}_{>0} \times M \mapsto (t, \varphi_Y^t(x)) \in \mathbb{R}_{>0} \times M$ and $\mathbf{p}_1 : \mathbb{R}_{>0} \times M \mapsto \mathbb{R}_{>0}$, $\mathbf{p}_2 : \mathbb{R}_{>0} \times M \mapsto M$ the two canonical projections. Then we can write:

$$(28) \quad \int_M [N_1] \wedge (\iota_Y \varphi^{-t*} [N_2]) = \mathbf{p}_{1*} (\mathbf{p}_2^* [N_1] \wedge F_* [\mathbb{R} \times N_2]) \in \mathcal{D}'(\mathbb{R}_{>0}).$$

Proof. By a partition of unity argument, we only need to work locally near some point $x \in N_1$ such that $\varphi_Y^{-\tau}(x) \in N_2$ for some τ . Up to replacing N_2 by $\varphi_Y^\tau(N_2)$, we may also assume that $\tau = 0$ and that we work on a small interval of time centered around 0. Using our transversality assumptions on N_1 , N_2 and Y , we may assume without loss of generality that there are local coordinates $(x_1, \dots, x_k, x_{k+1}, \dots, x_n)$ near x such that

- N_1 (resp. N_2) is given by the equations $x_{k+1} = \dots = x_n = 0$ (resp. $x_1 = \dots = x_{k+1} = 0$);
- x is given by $x_1 = \dots = x_n = 0$;
- the vector field Y reads $\frac{\partial}{\partial x_{k+1}}$.

Here, one has $\dim(N_1) = k$ and $\dim(N_2) = n - (k + 1)$. In these local coordinates, the currents of integration on N_1, N_2 read:

$$[N_1] = \delta_0(x_{k+1}, \dots, x_n) dx_{k+1} \wedge \dots \wedge dx_n \quad \text{and} \quad [N_2] = \delta_0(x_1, \dots, x_{k+1}) dx_1 \wedge \dots \wedge dx_{k+1}.$$

In this representation, N_1 is oriented by $(-1)^{(n-k)k} dx_1 \wedge \dots \wedge dx_k$ and N_2 by $dx_{k+2} \wedge \dots \wedge dx_n$, where we assume that M is oriented¹⁵ by $\text{Or}_M = dx_1 \wedge \dots \wedge dx_n$. In particular, at $x = 0$,

¹⁵Recall that the integration current on any submanifold depends on some choice of orientation of the submanifold and the ambient manifold.

the tangent space $T_x N_1 \oplus \mathbb{R}Y(x) \oplus T_x N_2$ is oriented by the volume form $(-1)^{(n-k)k} \text{Or}_M$. Let now χ_1 be a smooth function compactly supported near $t = 0$ and $x = 0$. In order to conclude, we need to compute

$$\int_{\mathbb{R} \times M} \chi_1[N_1] \wedge \iota_Y \varphi_Y^{-t*}([N_2]) |dt|.$$

Using the above explicit formulas, this is in fact equal to

$$(-1)^{k(n-k+1)} \int_{\mathbb{R} \times M} \chi_1 \delta_0(x_{k+1}, \dots, x_n) \delta_0(x_1, \dots, x_k, x_{k+1} - t) dx_1 \wedge \dots \wedge dx_n |dt|.$$

This can be rewritten as

$$\begin{aligned} \int_{\mathbb{R} \times M} \chi_1[N_1] \wedge \iota_Y \varphi_Y^{-t*}([N_2]) |dt| &= (-1)^k (-1)^{(n-k)k} \chi_1(0, 0) \\ &= (-1)^{k+(n-k)k} \int_{M \times \mathbb{R}} \chi(t, x) \delta_0(x_1, \dots, x_n, t) dx_1 \wedge \dots \wedge dx_n |dt|, \end{aligned}$$

which implies the expected result (by partition of unity). Working in these local coordinates, one could also verify that $\int_0^T \iota_Y(\varphi_Y^{-t*}[N_2]) |dt|$ is the current of integration on the submanifold $(\varphi_Y^t(N_2))_{0 \leq t \leq T}$. In fact, with the above conventions for local coordinates, one has, locally near $x = 0$ and for some small enough $t_0 > 0$,

$$\int_{-t_0}^{t_0} \iota_Y(\varphi_Y^{-t*}[N_2]) |dt| = (-1)^k \delta_0(x_1, \dots, x_k) dx_1 \wedge \dots \wedge dx_k.$$

In other words, $\int_0^T \iota_Y(\varphi_Y^{-t*}[N_2]) |dt|$ is the current of integration on the submanifold $(\varphi_Y^t(N_2))_{0 < t < T}$. \square

Proof of Proposition 4.10. We fix some T larger than the T_0 appearing in Lemma 4.8. In particular, it implies that the transversality assumptions of Lemma 4.11 are satisfied. Thanks to Lemma 3.8, the set of times t such that $\text{supp } \varphi^{-t*}(U_{-T}) \cap \text{supp}(S_T) \neq \emptyset$ forms a discrete subset of $[T_0, +\infty)$ without any accumulation point. We now fix $T > T_0$, $0 < t_0 < T - T_0$ and some smooth function $\chi \in \mathcal{C}_c^\infty((-t_0, +\infty))$. Then, we proceed as in the proof of Lemma 4.11 with a test function $\chi_1(t, x) = \chi(t) e^{-\int_{-t}^0 W \circ \varphi^s(x) |ds|}$ in order to obtain the desired result. \square

In the case of geodesic flows on Riemannian surfaces, the situation is again slightly simpler:

Proposition 4.12. *Suppose that $M = S^*X$ where (X, g) is a negatively curved Riemannian surface and that V is the geodesic vector field. Let $W \in \mathcal{C}^\infty(M, \mathbb{C})$. Let \mathbf{c}_1 and \mathbf{c}_2 be two elements in $\pi_1(X)$ and let c_1, c_2 be two geodesic representatives. Then, for every $T_0 \geq 0$ satisfying*

$$\varphi^{-T_0}(\Sigma(c_1)) \cap \varphi^{T_0}(\Sigma(c_2)) = \emptyset,$$

one can find some $t_0 > 0$ such that, for every $\chi \in \mathcal{C}_c^\infty((2T_0 - t_0, +\infty))$,

$$I(\chi) := - \int_M \varphi^{T_0*}([\Sigma(c_1)])(x, dx) \wedge \left(\int_{\mathbb{R}} \chi(t - 2T_0) e^{-\int_{-t}^0 W \circ \varphi^s(x) |ds|} \varphi^{-(t+T_0)*} \iota_V[\Sigma(c_2)](x, dx) |dt| \right)$$

is well defined and it is equal to

$$\varepsilon(\mathbf{c}_2) \sum_{t \geq 2T_0 - t_0 : \Sigma(c_1) \cap \varphi^t(\Sigma(c_2)) \neq \emptyset} \left(\sum_{x \in \Sigma(c_1) \cap \varphi^t(\Sigma(c_2))} \chi(t - 2T_0) e^{-\int_{-t+T_0}^{-T_0} W \circ \varphi^s(x) |ds|} \right),$$

where

$$\varepsilon(\mathbf{c}_2) := 1, \text{ if } \mathbf{c}_2 \text{ is trivial,}$$

and

$$\varepsilon(\mathbf{c}_2) := -1, \text{ otherwise.}$$

Proof. One more time, the proof is the same as in the general case and it takes advantage of the explicit structure of the tangent map described in paragraph 3.2. In particular, from this explicit expression, one has that $\epsilon_t(x)$ is independent of (t, x) in that case and that it is equal to $\varepsilon(\mathbf{c}_2)$. In fact, let us for instance treat the case where \mathbf{c}_1 and \mathbf{c}_2 are both trivial. Using the conventions of Section 2, we have put the orientation induced by e_3 on $\Sigma(c_i)$ when \mathbf{c}_i is trivial. Then, the orientation on

$$T_x \Sigma(c_1) \oplus \mathbb{R}V(x) \oplus d_{\varphi^{-t}(x)} \varphi^t(T_{\varphi^{-t}(x)} \Sigma(c_2))$$

induced by the orientation on each subspace is given by

$$e_3(x) \wedge e_0(x) \wedge (d_{\varphi^{-t}(x)} \varphi^t(e_3(\varphi^{-t}(x)))) = \gamma(t, x) e_0(x) \wedge e_1(x) \wedge e_3(x),$$

for some function $\gamma(t, x)$ that can be made explicit thanks to Remark 3.3:

$$\gamma(t, x) = \frac{e^{\int_0^t L_{\varphi^{\tau-t}(x)}^u |d\tau|} - e^{\int_0^t L_{\varphi^{\tau-t}(x)}^s |d\tau|}}{L_{\varphi^{-t}(x)}^u - L_{\varphi^{-t}(x)}^s}.$$

In that same remark¹⁶, we said that $\gamma(t, x) > 0$ for $t > 0$. Hence, we find that $\epsilon_t(x) = 1$ in that case. The same kind of calculations yield the other cases. \square

Remark 4.13. Note that, for surfaces with an Anosov geodesic flow (but not necessarily negatively curved), the function $\gamma(t, x)$ may not be positive for every $t > 0$ if c_1 and c_2 are points. Yet, it will be positive for $t \geq t_0$ for some large enough t_0 . In particular, $\epsilon_t(x)$ will be constant equal to 1 for $t \geq t_0$ even if it may be equal to -1 for some values of $0 \leq t \leq t_0$.

¹⁶Note that this observation used the negative curvature assumption.

4.4. Proof of Theorem 4.3. We are now in position to prove Theorem 4.3 if we are able to replace the compactly supported function in Proposition 4.10 by $e^{-zt}\mathbf{1}_{\mathbb{R}_+}$. We fix $T_0 > 0$ as in the statement of Proposition 4.10, some $T \geq T_0$ and some $t_0 > 0$ such that

$$(29) \quad \forall t \in [-t_0, t_0], \quad \varphi^t(\text{supp}(S_T)) \cap \text{supp}(U_{-T}) = \emptyset.$$

We let $\theta : \mathbb{R} \rightarrow [0, 1]$ be a smooth cutoff function which equals 1 on $(-t_0/4, t_0/4)$, θ vanishes outside $(-3t_0/4, 3t_0/4)$ and verifies

$$(30) \quad \forall t \in \mathbb{R}, \quad \sum_{k \in \mathbb{Z}} \theta(t - kt_0) = 1.$$

Then, for every $N \geq 1$, we set

$$\chi_N(t) := \sum_{k=0}^N \theta(t - kt_0),$$

and we apply Proposition 4.10 to $\chi_N(t)e^{-zt}$. Recalling (29), it yields

$$(31) \quad I_T(\chi_N e^{-z \cdot}) = \sum_{t \geq -t_0 : \varphi^{-T}(\Sigma_1) \cap \varphi^{T+t}(\Sigma_2) \neq \emptyset} \left(\sum_{x \in \varphi^{-T}(\Sigma_1) \cap \varphi^{T+t}(\Sigma_2)} \epsilon_t(x) e^{-zt} \chi_N(t) e^{-\int_{-t}^0 W \circ \varphi^s(x) |ds|} \right).$$

where $I_T(\chi_N e^{-z \cdot})$ is defined as

$$I_T(\chi_N e^{-z \cdot}) = (-1)^{n_s} \int_M S_T(x, dx) \wedge \int_{\mathbb{R}} e^{-zt} \chi_N(t) e^{-\int_{-t}^0 W \circ \varphi^s(x) |ds|} \varphi^{-t*} \iota_V(U_{-T})(x, dx) |dt|.$$

Using the anisotropic Sobolev spaces $\mathcal{H}_{n_s}^m(M)$ of Faure and Sjöstrand [32], we would now like (if it makes sense) to rewrite this quantity as a duality pairing in the anisotropic Sobolev spaces:

$$I_T(\chi_N e^{-z \cdot}) = (-1)^{n_s} \left\langle S_T, \left(\int_0^{+\infty} \chi_N(t) e^{-zt} e^{-\int_{-t}^0 W \circ \varphi^s |ds|} \varphi^{-t*} |dt| \right) \iota_V(U_{-T}) \right\rangle_{\mathcal{H}_{n-n_s}^{-m} \times \mathcal{H}_{n_s}^m},$$

or equivalently

$$(32) \quad I_T(\chi_N e^{-z \cdot}) = (-1)^{n_s} \sum_{k=1}^N e^{-kt_0 z} \left\langle S_T, e^{-\int_{-kt_0}^0 W \circ \varphi^s |ds|} \varphi^{-kt_0*} A_\chi \iota_V(U_{-T}) \right\rangle_{\mathcal{H}_{n-n_s}^{-m} \times \mathcal{H}_{n_s}^m},$$

where

$$A_\chi := \int_{-t_0/2}^{t_0/2} \theta(t) e^{-zt} e^{-\int_{-t}^0 W \circ \varphi^s |ds|} \varphi^{-t*} |dt| : \mathcal{D}_{\Gamma_u}^{n_s}(M) \rightarrow \mathcal{D}^{n_s}(M),$$

where Γ_u is a closed conic set containing $E_u^* \oplus E_0^* \setminus \underline{0}$ in its interior as in Lemma 3.10. Here, we note that we omit the term $k = 0$ as it is equal to 0 thanks to (29). In order to justify this expression, we will study more specifically the properties of the operator A_χ . The main observation is the following:

Lemma 4.14. *Let $N_1 > 16N_0 \gg 1$ and let Γ'_{uu} and $\tilde{\Gamma}_{uu}$ be the conic neighborhoods of \mathring{E}_u^* appearing in the definition of the order function m_{n_0, N_1} . Then, there exists $T_0 > 0$ such that, for every $T \geq T_0$, one can find some constant $C_T > 0$ such that*

$$\|A_\chi \iota_V(U_{-T})\|_{\mathcal{H}_{n_s}^{m_{N_0, N_1}}} \leq C_T.$$

Note that we always pick T large enough to ensure that the conditions of Proposition 4.10 and Lemmas 3.10 and 4.14 are satisfied. In particular, according to (23), the current $\iota_V(U_{-T})$ also belongs to some anisotropic Sobolev space $\mathcal{H}_{n_s}^{-m_{N_0, N_1}}(M)$. Hence, this Lemma shows that averaging over a small interval of time yields a gain of regularity in the flow direction.

Proof. Let $T \geq T_0 > 0$ where T_0 is the constant appearing in the preceding Lemmas. Recall that the integration current on the submanifold $(\varphi^{T+t}(\Sigma_2))_{-\frac{t_0}{2} < t < \frac{t_0}{2}}$ reads $\int_{-t_0/2}^{t_0/2} \varphi^{-t*} \iota_V(U_{-T}) |dt|$ (up to a sign) as we explained at the end of the proof of Lemma 4.11. One can thus remark that $A_\chi \iota_V(U_{-T})$ is just a truncated (and weighted) version of this current of integration. In particular, it is a current of order 0 (in the sense that its action on continuous form is bounded) whose wavefront is carried by the conormal to this submanifold. In particular, if we fix N_0, N_1 large enough and if we take T_0 large enough to ensure that the wavefront set of $(\varphi^{T+t}(\Sigma_2))_{-\frac{t_0}{2} < t < \frac{t_0}{2}}$ lies inside¹⁷ the cones Γ'_{uu} and $\tilde{\Gamma}_{uu}$ used in our definition of the anisotropic space $\mathcal{H}_{n_s}^{m_{N_0, N_1}}(M)$, then the current $A_\chi \iota_V(U_{-T})$ will belong to this anisotropic space – see Remark A.6 for the definition of wavefront sets via pseudodifferential operators. \square

As a consequence of Lemma 4.14 and of (21), we deduce that there exists some $C_0 > 0$ such that, for $\text{Re}(z) > C_0$, the sum

$$\sum_{k=1}^N e^{-kt_0 z} e^{-\int_{-kt_0}^0 W \circ \varphi^s |ds|} \varphi^{-kt_0*} A_\chi \iota_V(U_{-T})$$

converges (uniformly in z) in the anisotropic Sobolev space $\mathcal{H}_{n_s}^{m_{N_0, N_1}}(M)$. On the other hand, we also know that $\iota_V(U_T)$ belongs to some other anisotropic Sobolev space $\mathcal{H}_{n_s}^{-m_{N'_0, N'_1}}(M)$. Hence, this sum also converges in that space and, working in that second space, it can be identified with $R_{n_s}(z) \iota_V(U_T)$ (as the resolvent is well defined in that space for $\text{Re}(z)$ large enough). Hence, we have shown that

$$\lim_{N \rightarrow +\infty} I_T(\chi_N e^{-z \cdot}) = (-1)^{n_s} \langle S_T, R_{n_s}(z) \iota_V(U_{-T}) \rangle_{\mathcal{H}_{n_s}^{-m_{N_0, N_1}} \times \mathcal{H}_{n_s}^{m_{N_0, N_1}}}.$$

In particular, it can be rewritten in terms of the distribution appearing in Lemma 4.8:

$$\lim_{N \rightarrow +\infty} I_T(\chi_N e^{-z \cdot}) = (-1)^{n_s + n(n-n_s)} \langle R_{n_s}(z) \wedge (S_T \otimes \iota_V(U_{-T})), 1 \rangle_{\mathcal{D}'^{2n}(M \times M), \Omega^0(M \times M)}.$$

¹⁷This follows from the hyperbolicity of the flow and from the transversality property (14).

Moreover, as the number of points lying in $(\varphi^{-T}(\Sigma_1) \cap \varphi^{T+t}(\Sigma_2))_{t \geq T}$ has at most exponential growth according to Lemma 3.11, we also know from (31) that, for $\operatorname{Re}(z)$ large enough,

$$\lim_{N \rightarrow +\infty} I_T(\chi_N e^{-z \cdot}) = \sum_{t \geq -t_0: \varphi^{-T}(\Sigma_1) \cap \varphi^{T+t}(\Sigma_2) \neq \emptyset} \left(\sum_{x \in \varphi^{-T}(\Sigma_1) \cap \varphi^{T+t}(\Sigma_2)} \epsilon_t(x) e^{-zt} e^{-\int_{-t}^0 W \circ \varphi^s(x) |ds|} \right).$$

This concludes the proof of Theorem 4.3 and we have in fact proved the slightly more precise statement (involving the weight function W):

Theorem 4.15. *Let $W \in \mathcal{C}^\infty(M, \mathbb{C})$. There exist $T_0, C_0 > 0$ such that, for every $T \geq T_0$, the function*

$$z \in \{\operatorname{Re}(z) > C_0\} \mapsto \sum_{t \geq -t_0: \varphi^{-T}(\Sigma_1) \cap \varphi^{T+t}(\Sigma_2) \neq \emptyset} \left(\sum_{x \in \varphi^{-T}(\Sigma_1) \cap \varphi^{T+t}(\Sigma_2)} \epsilon_t(x) e^{-zt} e^{-\int_{-t}^0 W \circ \varphi^s(x) |ds|} \right) \in \mathbb{C}$$

is holomorphic and coincides with the function

$$z \mapsto (-1)^{n(n-n_s)+n_s} \langle R_{n_s}(z) \wedge (S_T \otimes \iota_V(U_{-T})), 1 \rangle_{\mathcal{D}^{2n}(M \times M), \Omega^0(M \times M)}.$$

In particular, it has a meromorphic continuation to \mathbb{C} .

This Theorem implies Theorem 1.1 from the introduction by taking $W = 0$ and by recalling that $\epsilon_t(x) = \varepsilon(\mathbf{c}_2)$ for every (t, x) in that case.

In fact, the result is slightly more precise :

Theorem 4.16. *Suppose that $M = S^*X$ where (X, g) is a negatively curved Riemannian surface and that V is the geodesic vector field. Let \mathbf{c}_1 and \mathbf{c}_2 be two elements in $\pi_1(X)$ and let c_1 and c_2 be two of their geodesic representatives.*

Then, the function

$$z \in \{\operatorname{Re}(z) > h_{\text{top}}\} \mapsto \sum_{t > 0: \Sigma(c_1) \cap \varphi^t(\Sigma(c_2)) \neq \emptyset} \sum_{x \in \Sigma(c_1) \cap \varphi^t(\Sigma(c_2))} e^{-zt} \in \mathbb{C}$$

is holomorphic and, there exists $T_1 > 0$ such that, for every $0 < T_0 < T_1$, it coincides with the function

$$z \mapsto -\varepsilon(\mathbf{c}_2) e^{-zT_0} \langle R_1(z) \wedge ([\Sigma(c_1)] \otimes \iota_V(\varphi^{-T_0*}[\Sigma(c_2)])), 1 \rangle_{\mathcal{D}^6(M \times M), \Omega^0(M \times M)},$$

where

$$\varepsilon(\mathbf{c}_2) := 1, \text{ if } \mathbf{c}_2 \text{ is trivial,}$$

and

$$\varepsilon(\mathbf{c}_2) := -1, \text{ otherwise.}$$

In particular, it has a meromorphic continuation to \mathbb{C} .

The proof is exactly the same as the proof of Theorem 4.15 recalling that Lemmas 3.9 and 4.9 and Proposition 4.12 hold with any nonnegative T_0 such that $\varphi^{-T_0}(\Sigma(c_1)) \cap \Sigma(c_2) = \emptyset$. Finally, for $\Sigma(c_1) \cap \Sigma(c_2) = \emptyset$, we observe that the quantity

$$z \mapsto e^{-zT_0} \langle R_1(z) \wedge ([\Sigma(c_1)] \otimes \iota_V(\varphi^{-T_0*}[\Sigma(c_2)])), 1 \rangle_{\mathcal{D}^6(M \times M), \Omega^0(M \times M)}$$

is well defined for $T_0 = 0$ (see Lemma 4.9) and is continuous at $T_0 = 0$ for every value of z where the resolvent converges.

Remark 4.17. As was pointed to us by Y. Guedes-Bonthonneau, these two theorems can be thought of as analogues in the context of Pollicott-Ruelle resonances of the Kuznetsov trace formulas for the Laplace-Beltrami operator [90]. In that context, Zelditch considered the spectral projector of the Laplacian on the eigenvalues $\leq \lambda$ and he integrated the kernel of the operator against singular distributions carried by smooth submanifolds. Here, we did the exact same thing with the resolvent of our operator. Yet, compared with that reference, we need to restrict ourselves to certain families of submanifolds verifying our transversality assumptions (14) and (15), to ensure that they satisfy the appropriate wavefront set conditions so that they can be integrated against the Schwartz kernel of our resolvent.

5. BEHAVIOUR AT 0 FOR 3-DIMENSIONAL CONTACT FLOWS

In all this section, we make the assumption that $\dim(M) = 3$ and that the smooth Anosov vector field V preserves a smooth *contact form* α , i.e. $\mathcal{L}_V \alpha = 0$. This is for instance the case when V is the geodesic vector field on the unit cotangent bundle of a negatively curved surface. In particular, the currents $[\Sigma_1]$ and $[\Sigma_2]$ are elements of $\mathcal{D}^2(M)$. Given the fact that they are currents of integration over a smooth closed curve, we also note that, for $i = 1, 2$,

$$(33) \quad d[\Sigma_i] = 0.$$

Compared with Section 4, we also suppose that $W = 0$ and that α is normalized in the following sense

$$(34) \quad \int_M \alpha \wedge d\alpha = 1.$$

5.1. Description of the spectral projector at 0. In order to describe the behaviour of our zeta function at 0, we need to describe the spectral projector $\pi_0^{(1)}$ at $z = 0$. This can be achieved following the recent results of Dyatlov and Zworski [27] on the behaviour of the Ruelle zeta function at 0. In particular, this will crucially use the contact structure – see [45] for extensions to manifolds with boundary and [14] for extensions to the volume preserving case. For $0 \leq k \leq 3$, we set

$$C^k := \text{Ran}(\pi_0^{(k)}) \quad \text{and} \quad C_0^k := C^k \cap \text{Ker}(\iota_V).$$

According to [17, Lemma 7.1], one has

$$\forall 0 \leq k \leq 3, \quad C^k = C_0^k \oplus (\alpha \wedge C_0^{k-1}),$$

with the convention that $C_0^{-1} = \{0\}$. In [27, Lemma 3.2], it is shown that

$$C^0 := \mathbb{C}1, \quad \text{and} \quad C_0^2 := \mathbb{C}d\alpha.$$

Still in [27, Lemma 3.4], the authors proved that

$$(35) \quad C_0^1 = C^1 \cap \text{Ker}(d) \simeq H^1(M, \mathbb{C}).$$

In particular, the spectral projector can be written as

$$\forall \psi \in \Omega^1(M), \quad \pi_0^{(1)}(\psi) = \left(\int_M \tilde{\mathbf{S}}_0 \wedge \psi \right) \alpha + \sum_{j=1}^{b_1(M)} \left(\int_M \tilde{\mathbf{S}}_j \wedge \psi \right) \mathbf{U}_j,$$

with $d\mathbf{U}_j = \iota_V(\mathbf{U}_j) = 0$ for every $1 \leq j \leq b_1(M) := \dim H^1(M, \mathbb{C})$. Note from Proposition 4.7 that \mathbf{U}_j (resp. $\tilde{\mathbf{S}}_j$) belongs to $\mathcal{D}_{\tilde{E}_u^*}^1(M)$ (resp. $\mathcal{D}_{\tilde{E}_s^*}^2(M)$) for every $1 \leq j \leq b_1(M)$.

By Poincaré duality [19, §4.6], we can observe that $(\tilde{\mathbf{S}}_0, \tilde{\mathbf{S}}_1, \dots, \tilde{\mathbf{S}}_{b_1(M)})$ forms a basis for the dual operator to $-\mathcal{L}_V$ which is $-\mathcal{L}_{-V}$ acting on some dual anisotropic space of 2-forms. Thanks to (34) and to the fact that \mathbf{U}_j is closed, one already knows that $\tilde{\mathbf{S}}_0 = d\alpha$. Observing now that

$$C^2 = C_0^2 \oplus \left(\oplus_{j=1}^{b_1(M)} \mathbb{C}(\alpha \wedge \mathbf{U}_j) \right),$$

and applying (35) to $-V$ instead of V , we find that there exists a family¹⁸ $(\mathbf{S}_j)_{j=1, \dots, b_1(M)}$ in $\mathcal{D}_{\tilde{E}_s^*}^1(M)$ such that, for every $1 \leq j \leq b_1(M)$, $\iota_V(\mathbf{S}_j) = 0$, $d\mathbf{S}_j = 0$ and $\tilde{\mathbf{S}}_j = \alpha \wedge \mathbf{S}_j$. Hence, to summarize, one has

$$(36) \quad \forall \psi \in \Omega^1(M), \quad \pi_0^{(1)}(\psi) = \left(\int_M d\alpha \wedge \psi \right) \alpha + \sum_{j=1}^{b_1(M)} \left(\int_M \alpha \wedge \mathbf{S}_j \wedge \psi \right) \mathbf{U}_j,$$

where, for every $1 \leq j \leq b_1(M)$,

- (1) $\mathbf{U}_j \in \mathcal{D}_{\tilde{E}_u^*}^1(M)$, $\mathbf{S}_j \in \mathcal{D}_{\tilde{E}_s^*}^1(M)$,
- (2) $d\mathbf{U}_j = d\mathbf{S}_j = 0$,
- (3) $\iota_V(\mathbf{U}_j) = \iota_V(\mathbf{S}_j) = 0$.

Remark 5.1. In a similar way, we can write

$$\forall \psi \in \Omega^2(M), \quad \pi_0^{(2)}(\psi) = \left(\int_M \alpha \wedge \psi \right) d\alpha - \sum_{j=1}^{b_1(M)} \left(\int_M \mathbf{S}_j \wedge \psi \right) \alpha \wedge \mathbf{U}_j.$$

5.2. Behaviour at 0 of the Poincaré series. We are now in position to study the behaviour at 0 of the zeta function $\zeta_{\Sigma_1, \Sigma_2}$ appearing in Theorem 4.3, *i.e. taking into account the orientation indices but with no weight function*. Recall from [27, Lemma 3.5] that there is no Jordan blocks in the kernel of the operator. In particular, according to Proposition 4.7 and Theorem 4.15, one has, near $z = 0$,

$$\zeta_{\Sigma_1, \Sigma_2}(z) = - \frac{\left\langle \varphi^{T*}([\Sigma_1]), \pi_0^{(1)}(\iota_V \varphi^{-T*}([\Sigma_2])) \right\rangle}{z} + h(z),$$

¹⁸It is given by the family of “dual” eigenvectors.

where $h(z)$ is a holomorphic function. Using now the explicit expression given by (36) and (33), one finds

$$\begin{aligned}
z(h(z) - \zeta_{\Sigma_1, \Sigma_2}(z)) &= \left(\int_M d\alpha \wedge \iota_V \varphi^{-T*}([\Sigma_2]) \right) \left(\int_M \varphi^{T*}([\Sigma_1]) \wedge \alpha \right) \\
&+ \sum_{j=1}^{b_1(M)} \left(\int_M \alpha \wedge \mathbf{S}_j \wedge \iota_V \varphi^{-T*}([\Sigma_2]) \right) \left(\int_M \varphi^{T*}([\Sigma_1]) \wedge \mathbf{U}_j \right) \\
&= - \sum_{j=1}^{b_1(M)} \left(\int_M \mathbf{S}_j \wedge \varphi^{-T*}([\Sigma_2]) \right) \left(\int_M \varphi^{T*}([\Sigma_1]) \wedge \mathbf{U}_j \right) \\
&= - \sum_{j=1}^{b_1(M)} \left(\int_M \mathbf{S}_j \wedge [\Sigma_2] \right) \left(\int_M [\Sigma_1] \wedge \mathbf{U}_j \right)
\end{aligned}$$

To summarize, we have shown:

Proposition 5.2. *Suppose that $\dim M = 3$ and that V preserves a smooth contact form α . Then, there exist a holomorphic function h (in a neighborhood of 0) and two families of linearly independent closed currents $(\mathbf{U}_j)_{j=1, \dots, b_1(M)}$ in $\mathcal{D}_{E_u^*}^1(M)$ and $(\mathbf{S}_j)_{j=1, \dots, b_1(M)}$ in $\mathcal{D}_{E_s^*}^1(M)$ such that*

$$\forall 1 \leq i, j \leq b_1(M), \quad \int_M \alpha \wedge \mathbf{S}_i \wedge \mathbf{U}_j = \delta_{ij},$$

and, near $z = 0$,

$$\zeta_{\Sigma_1, \Sigma_2}(z) = \frac{1}{z} \sum_{j=1}^{b_1(M)} \left(\int_M \mathbf{S}_j \wedge [\Sigma_2] \right) \left(\int_M [\Sigma_1] \wedge \mathbf{U}_j \right) + h(z).$$

Recall that Hodge-De Rham theory shows that the ellipticity of d implies that the cohomology is independent of the choice of the spaces we are working with – see e.g. [27, Lemma 2.1]. In particular, the currents $(\mathbf{U}_j)_{j=1, \dots, b_1(M)}$ in $\mathcal{D}_{E_u^*}^1(M)$ and $(\mathbf{S}_j)_{j=1, \dots, b_1(M)}$ form a basis of $H^1(M, \mathbb{C})$. As a direct Corollary of this Proposition, we find that

Corollary 5.3. *Suppose that $\dim M = 3$ and that V preserves a smooth contact form α . If either $[\Sigma_1]$ or $[\Sigma_2]$ is exact, then $\zeta_{\Sigma_1, \Sigma_2}(z)$ is holomorphic in a neighborhood of $z = 0$.*

6. GEODESIC ARCS IN DIMENSION 2

All along this section, we suppose that $M = S^*X$ where (X, g) is a negatively curved surface and that φ^t is the geodesic flow. In particular, we are in the framework of contact Anosov flows in dimension 3 as in Proposition 5.2. Our goal is now to prove Theorem 1.2.

We fix two homotopy classes \mathbf{c}_1 and \mathbf{c}_2 inside X and two of their geodesic representatives c_1 and c_2 . Without loss of generality, we suppose that \mathbf{c}_i are both primitive elements of $\pi_1(X)$ in the sense that

$$(37) \quad \forall i = 1, 2, \quad \mathbf{c}_i = \mathbf{c}^p \text{ for some } p > 1 \Rightarrow \mathbf{c}_i \text{ is trivial in } \pi_1(X).$$

We saw in paragraph 3.3.1 how to associate to each c_i a Legendrian curve $\Sigma(c_i)$ by taking a connected component of their unit conormal bundle. We also explained that these two curves can be naturally oriented using the vertical and horizontal bundles induced by g .

We already observed in Theorem 4.16 that we can take any positive T_0 in the definition of the zeta function $\zeta_{\Sigma(c_1), \Sigma(c_2)}(z)$. In particular, taking T_0 small enough, we can define

$$\zeta_{\Sigma(c_1), \Sigma(c_2)}(z) := \varepsilon(\mathbf{c}_2) \sum_{t \in \mathcal{P}_{\Sigma(c_1), \Sigma(c_2)}} m_{\Sigma(c_1), \Sigma(c_2)}(t) e^{-zt},$$

where

$$\mathcal{P}_{\Sigma(c_1), \Sigma(c_2)} := \{t > 0 : \varphi^{-t}(\Sigma(c_1)) \cap \Sigma(c_2) \neq \emptyset\},$$

and

$$m_{\Sigma(c_1), \Sigma(c_2)}(t) := |\{x \in \varphi^{-t}(\Sigma(c_1)) \cap \Sigma(c_2)\}|.$$

Note that this function slightly differs from the function $\mathcal{N}_\infty(c_1, c_2, z)$ from Theorem 1.2, namely

$$\zeta_{\Sigma(c_1), \Sigma(c_2)}(z) = \varepsilon(\mathbf{c}_2) \mathcal{N}_\infty(c_2, c_1, z)$$

Using these conventions and recalling that we denote by $[\Sigma(c_i)]$ the current of integration over $\Sigma(c_i)$, one has:

Proposition 6.1. *Suppose that $[\Sigma(c_2)]$ is exact. Then one can find some $T_1 > 0$ such that, for $0 < T_0 < T_1$,*

$$\zeta_{\Sigma(c_1), \Sigma(c_2)}(0) = - \int_M \varphi^{T_0*} [\Sigma(c_1)] \wedge R_2,$$

where $[\Sigma(c_2)] = dR_2$ and $R_2 \in \mathcal{D}_{N^*(\Sigma(c_2))}^1(S^*X)$. Moreover, T_0 can be chosen equal to 0 if $\Sigma(c_1) \cap \Sigma(c_2) = \emptyset$.

As we shall see in Theorem 6.3, our assumption on $[\Sigma(c)]$ is automatically satisfied when \mathbf{c} is homologically trivial. It could also be viewed as a consequence of Remark 6.4 below and of the de Rham Theorem showing the isomorphism between the de Rham cohomology and the non torsion part of the homology groups [36, §5.4]. In the case where \mathbf{c} is trivial in $\pi_1(X)$, the situation is even simpler:

Lemma 6.2. *If c is a point in X , then $[\Sigma(c)] = [S_c^*X]$ is exact.*

Proof. In order to prove this Lemma (see also Remark 6.10 below for a more constructive proof), we can use Poincaré duality. In other words, it is sufficient to check that, for any closed form $\tilde{\theta} \in \Omega^1(S^*X)$, $\int_{S^*X} [\Sigma(c)] \wedge \tilde{\theta} = 0$. Now, as we are in dimension 2 and as $\chi(X) \neq 0$, any closed form $\tilde{\theta}$ is (up to some coboundary) the pullback of a smooth closed form θ on X – see for instance [27, Lemma 2.4]. Hence, it is sufficient to evaluate, for any closed form $\theta \in \Omega^1(X)$,

$$\int_{S^*X} [\Sigma(c)] \wedge \Pi^* \theta,$$

where $\Pi : (q, p \in S^*X \rightarrow q \in X)$ is the canonical projection. In particular, $\Pi^* \theta = \theta \circ d\Pi$ and $\Pi^* \theta(x)$ belongs to $\mathbb{R}e_0^*(x) \oplus \mathbb{R}e_1^*(x)$ with the conventions of Section 2. We now use the fact that \mathbf{c} is trivial, i.e. $\Sigma(c) = S_c^*X$. In that case, it follows from Paragraph 3.3.1 that

$[\Sigma(c)]$ belongs to $\mathcal{D}'(S^*X, \mathbb{R}e_0^* \oplus \mathbb{R}e_1^*)$. Hence, one can verify that $[\Sigma(c)] \wedge \Pi^*\theta = 0$, which allows to conclude. \square

In order to determine the value at 0 (and to conclude the proof of Theorem 1.2), we are left with two problems

- verify that $[\Sigma(c)]$ is exact when \mathbf{c} is homologically trivial;
- determine the value of

$$\mathbf{L}(c_1, c_2) := \int_M \varphi^{T_0*}[\Sigma(c_1)] \wedge R_2,$$

for some $0 < T_0 < T_1$.

In other words, we are reduced to a purely topological question formulated in terms of De Rham currents. The number $\mathbf{L}(c_1, c_2)$ can be understood as the linking number between the two Legendrian knots $\varphi^{-T_0}(\Sigma(c_1))$ and $\Sigma(c_2)$. In fact, even if $[\Sigma(c_i)]$ is trivial in De Rham cohomology, the homology class of the curve $\Sigma(c_2)$ may have a non trivial torsion component in homology. Hence, it is rather the quantity $\chi(X)\mathbf{L}(c_1, c_2)$ which can be viewed as the linking between the two Legendrian knots $\varphi^{T_0*}(\Sigma(c_1))$ and $\chi(X)\Sigma(c_2)$ if we take the convention that a knot¹⁹ has to be homologically trivial.

The main result of this section is which is more or less a reformulation of Theorem 1.2 thanks to (6.1):

Theorem 6.3. *Suppose that \mathbf{c}_1 and \mathbf{c}_2 are homologically trivial. Then, both $[\Sigma(c_i)]$ are exact and one has, for any of their geodesic representatives c_1 and c_2 ,*

$$\chi(X)\mathbf{L}(c_1, c_2) \in \mathbb{Z}.$$

Moreover, if c_1 and c_2 are embedded and if $X(c_i)$ is the compact surface²⁰ whose oriented boundary is given by c_i , then one has

$$(38) \quad \mathbf{L}(c_1, c_2) = -\frac{\chi(X(c_1))\chi(X(c_2))}{\chi(X)} + \chi(X(c_2) \cap X(c_1)) - \frac{1}{2}\chi(c_1 \cap c_2),$$

in the following cases:

- \mathbf{c}_1 and \mathbf{c}_2 are distinct nontrivial homotopy classes,
- at least one \mathbf{c}_i is trivial and $c_1 \cap c_2 = \emptyset$.

Finally, for general c_1 and c_2 , formula (38) remains true if we replace each c_i on the right-hand side by $\tilde{c}_i = \Pi \circ \varphi^{t_i}(\Sigma(c_i))$ for some small enough $0 \leq t_1, t_2 < T_0$.

For the last part of the Theorem, we need to define precisely what $X(\tilde{c}_i)$ is and the meaning of its Euler characteristic. See § 6.4.3 for details. As we shall see, the fact that we take some small Hamiltonian deformation of $\Sigma(c_i)$ in the last part of the Theorem ensures that the two curves \tilde{c}_1 and \tilde{c}_2 intersect each other transversally and that they have only simple self-intersection. This is a standard procedure when one considers the projection

¹⁹This is not always part of the definition. In the following, a smooth closed curve in S^*X will be referred as a knot if the induced current is De Rham exact.

²⁰Recall that, in the case of points, $X(c_i) = c_i$.

of knots on a two-dimensional space. This simplification makes the combinatorics slightly less involved and it is plausible that this assumption can be relaxed a little bit but not completely (for instance if $\mathbf{c}_1 = \mathbf{c}_2$ is a nontrivial homotopy class).

The first part of the proof consists in showing that, if \mathbf{c} is homologically trivial²¹, then $[\Sigma(c)]$ is exact in the sense of currents on S^*X . We already saw in Lemma 6.2 that this is true when \mathbf{c} is trivial in $\pi_1(X)$. In fact, when proving the second part of the Theorem, we will need to construct *explicitly* some current S such that $dS = [\Sigma(c)]$. Hence, we will actually verify both parts at the same time.

Remark 6.4. A more conceptual (but not explicit enough for our means) manner to verify the first part would be as follows. For a negatively curved Riemannian surface (X, g) , one has the following short exact sequence between the fundamental groups:

$$\pi_1(\mathbb{S}^1) \simeq \mathbb{Z} \xrightarrow{i} \pi_1(S^*X, x_0) \xrightarrow{\Pi_*} \pi_1(X, q_0),$$

where Π_* is the map induced by the canonical projection. The morphism i associates to $1 \in \pi_1(\mathbb{S}^1)$ the homotopy class of $S^*_{x_0}M$. Moreover, each homotopy class in $\pi_1(X, q_0)$ is represented by a unique closed geodesic [55, Th. 3.8.14]. Hence, the exact sequence admits a splitting : there is a natural section $s : \pi_1(X, q_0) \rightarrow \pi_1(S^*X, x_0)$ obtained by associating to the homotopy class of every closed geodesic c_i , the homotopy class of its conormal bundle $\Sigma(c_i)$ (as we defined it in § 3.3.1). Hence, $\pi_1(S^*X, x_0)$ is isomorphic to a semidirect product $\mathbb{Z} \rtimes \pi_1(X, q_0)$. From this observation, we can verify that $H_1(S^*X, \mathbb{Z})$ (which is defined as the abelianization of $\pi_1(S^*X, x_0)$) is given by

$$H_1(S^*X, \mathbb{Z}) \simeq \mathbb{Z}/(p(X)\mathbb{Z}) \oplus \mathbb{Z}^{2-\chi(X)} \simeq \mathbb{Z}/(p(X)\mathbb{Z}) \oplus H^1(X)$$

for some $p(X) \geq 1$. In fact $p(X) = |\chi(X)|$ but we will not use this explicitly. Hence, the (oriented) conormal $\Sigma(c_i)$ to any closed geodesic c_i which is homologically trivial in X , may have a nontrivial torsion component. Yet, it remains trivial in De Rham cohomology [36, Th.1, p.575 and Th.8, p.620] since tensoring with \mathbb{R} kills all the information contained in the torsion part.

This quite long section is organized as follows. First, we prove Proposition 6.1 in §6.1 using the results of Section 5. This identifies the value at 0 of Poincaré series as a certain linking number that we compute in the next paragraphs. In §6.2, we compute this linking number in the case of trivial homotopy classes. Then, in §6.3, we compute this quantity in two particular examples that will illustrate our upcoming strategy. In §6.4, we prove Theorem 6.3 in its full generality. Finally, in §6.5 and in the case of nontrivial homotopy classes, we relate this linking number with the linking number of closed geodesics in S^*X and, in §6.6, we reinterpret the results of Margulis and Parkkonen–Paulin as a property on the asymptotic linking of two Legendrian knots.

²¹Recall that $H_1(X, \mathbb{Z}) \simeq \mathbb{Z}^{2-\chi(X)}$ has no torsion. Hence, by [36, Th.1, p.575 and Th.8, p.620], it is equivalent to be trivial in homology and in De Rham cohomology in the case of negatively curved surfaces. In particular, $[c] = d\theta$ for some $\theta \in \mathcal{D}^0(X)$.

6.1. Proof of Proposition 6.1. In order to alleviate notations, we will write $\Sigma_1 = \Sigma(c_1)$ and $\Sigma_2 = \Sigma(c_2)$ all along this proof. According to Theorem 4.16, one can find some $T_0 > 0$ (with $T_0 = 0$ in the case specified in Proposition 6.1) such that

$$\zeta_{\Sigma_1, \Sigma_2}(z) = -e^{-zT_0} \int_{S^*X} [\Sigma_1] \wedge (\mathcal{L}_V + z)^{-1} (\iota_V(\varphi^{-T_0*}[\Sigma_2])) ,$$

which can be rewritten, using Corollary 5.3 and the conventions of Proposition 4.7, as

$$\zeta_{\Sigma_1, \Sigma_2}(z) = -e^{-zT_0} \int_{S^*X} [\Sigma_1] \wedge R_1^H(z) (\iota_V(\varphi^{-T_0*}[\Sigma_2])) .$$

Under that form, we see that

$$(39) \quad \lim_{z \rightarrow 0} \zeta_{\Sigma_1, \Sigma_2}(z) = - \int_{S^*X} [\Sigma_1] \wedge R_1^H(0) (\iota_V(\varphi^{-t_0*}[\Sigma_2])) .$$

Moreover, we can rewrite

$$[\Sigma_2] = \pi_0^{(2)}([\Sigma_2]) + (\text{Id} - \pi_0^{(2)})([\Sigma_2]) .$$

Using Remark 5.1 and the facts that $[\Sigma_2]$ is a coboundary and that Σ_2 is a Legendrian curve, this can be simplified as

$$(40) \quad [\Sigma_2] = \left(\int_{S^*X} [\Sigma_2] \wedge \alpha \right) d\alpha + (\text{Id} - \pi_0^{(2)})([\Sigma_2]) = (\text{Id} - \pi_0^{(2)})([\Sigma_2]) .$$

Recall now that $[\Sigma_2]$ belongs to some anisotropic space $\mathcal{H}_2^{-\tilde{m}_{N_0, N_1}}(S^*X)$. Hence, as in [19, §4.2], we can write

$$\begin{aligned} [\Sigma_2] &= \mathcal{L}_V \mathcal{L}_V^{-1} (\text{Id} - \pi_0^{(2)})([\Sigma_2]) \\ &= d\iota_V \mathcal{L}_V^{-1} (\text{Id} - \pi_0^{(2)})([\Sigma_2]) + \iota_V d\mathcal{L}_V^{-1} (\text{Id} - \pi_0^{(2)})([\Sigma_2]) \\ &= d \left(\mathcal{L}_V^{-1} (\text{Id} - \pi_0^{(1)}) (\iota_V([\Sigma_2])) \right) , \end{aligned}$$

where the last line follows from the fact that d and ι_V commutes with \mathcal{L}_V^{-1} and π_0 and that $[\Sigma_2]$ is a cocycle. Using the notations of Proposition 4.7, one has

$$[\Sigma_2] = d \left(R_1^H(0) (\text{Id} - \pi_0^{(1)}) (\iota_V([\Sigma_2])) \right) .$$

Applying ι_V to (40) and using that $\iota_V \pi_0^{(2)} = \pi_0^{(1)} \iota_V$ (as \mathcal{L}_V and ι_V commute), we find that

$$[\Sigma_2] = d \left(R_1^H(0) (\iota_V([\Sigma_2])) \right) .$$

Hence, combining this with (39), one finds that

$$\lim_{z \rightarrow 0} \zeta_{\Sigma_1, \Sigma_2}(z) = - \int_{S^*X} [\Sigma_1] \wedge \varphi^{-T_0*}(\tilde{R}_2) = - \int_{S^*X} \varphi^{T_0*}[\Sigma_1] \wedge \tilde{R}_2 ,$$

where $[\Sigma_2] = d\tilde{R}_2$.

It now remains to show that we can replace \tilde{R}_2 with R_2 having the expected regularity. To that aim, we combine Proposition 4.7 and the product properties from Appendix A.2

to find out that the wavefront set of $R_1^H(0)(x, y, dx, dy)\iota_V([\Sigma_2])(y, dy)$ is contained in $\Gamma_1 \cup \Gamma_2 \cup \Gamma_1 + \Gamma_2$ where $\Gamma_1 = \underline{0}_{S^*X} \times N^*(\Sigma_2)$ and $\Gamma_2 := E_u^* \times E_s^* \cup \Omega_+ \cup \Delta(T^*S^*X)$. Now, we can use the explicit description of the unstable bundle on Riemannian surface (see paragraph 3.2) and apply the pushforward properties of Appendix A.4 to deduce that \tilde{R}_2 belongs to $\mathcal{D}_{\tilde{\Gamma}}^1(S^*X)$, with

$$\tilde{\Gamma} \subset E_u^* \cup N^*(\Sigma_2) \cup (\cup_{t \geq 0} \Phi^t(N^*(\Sigma_2) \cap \text{Ker}(\iota_V)).$$

We now observe that $d\tilde{R}_2 = [\Sigma_2] \in \mathcal{D}_{N^*(\Sigma_2)}^2$. We are in position to conclude by using elliptic regularity. To that aim, we follow the lines of [27, Lemma 2.1]. Recall that, for $0 \leq k \leq 3$, the Laplace Beltrami operator²² $\Delta_{\tilde{g}}^{(k)} = dd^* + d^*d$ acting on $\Omega^k(S^*X)$ is a pseudodifferential operator of order 2 whose principal symbol is $\|\xi\|_x^2 \text{Id}_{\Lambda^k(T^*S^*X)}$. In particular, one can find some pseudodifferential operator $Q^{(k)}$ of order -2 such that

$$\Delta_{\tilde{g}}^{(k)} Q^{(k)} - \text{Id} = Q^{(k)} \Delta_{\tilde{g}}^{(k)} - \text{Id} : \mathcal{D}'^k(S^*X) \rightarrow \Omega^k(S^*X).$$

Set now $R_2 = d^*dQ^{(1)}\tilde{R}_2$. One has $\tilde{R}_2 - R_2 - dd^*Q^{(1)}\tilde{R}_2 \in \Omega^1(S^*X)$ by construction of $Q^{(1)}$. Thus, one finds that

$$[\Sigma_2] - dR_2 = [\Sigma_2] - d\Delta_{\tilde{g}}^{(1)}Q^{(1)}\tilde{R}_2 \in \Omega^1(S^*X),$$

from which we infer $\Delta_{\tilde{g}}^{(2)}dQ^{(1)}\tilde{R}_2 \in \mathcal{D}_{N^*(\Sigma_2)}^2$. By elliptic regularity, we can deduce that $dQ^{(1)}\tilde{R}_2$ belongs to $\mathcal{D}_{N^*(\Sigma_2)}^2$. Hence, $R_2 = d^*dQ^{(1)}\tilde{R}_2 \in \mathcal{D}_{N^*(\Sigma_2)}^1$. Recalling that $\tilde{R}_2 - R_2 - dd^*Q^{(1)}\tilde{R}_2 \in \Omega^1(S^*X)$, we can conclude (up to modifying R_2 by a smooth function) that there exist $R_2 \in \mathcal{D}_{N^*(\Sigma_2)}^1(S^*X)$ and some $\theta_2 \in \mathcal{D}_{\tilde{\Gamma}}^0(S^*X)$ such that $\tilde{R}_2 = R_2 + d\theta_2$. Hence, the value at 0 can be rewritten as

$$\zeta_{\Sigma_1, \Sigma_2}(0) = - \int_{S^*X} \varphi^{T_0^*}[\Sigma_1] \wedge R_2,$$

with $R_2 \in \mathcal{D}_{N^*(\Sigma_2)}^1(S^*X)$ verifying $dR_2 = [\Sigma_2]$.

6.2. Proof of Theorem 6.3: the case of trivial homotopy classes. The end of the article is devoted to the proof of Theorem 6.3 and it will be in some sense a constructive argument. One of the difficulty is that we are aiming at an explicit formula for this value at 0 in terms of the two geodesic representatives c_1 and c_2 . We begin with the case where \mathbf{c}_1 and \mathbf{c}_2 are trivial homotopy classes:

Lemma 6.5. *Suppose that \mathbf{c}_1 and \mathbf{c}_2 are trivial homotopy classes. Then, for any of their geodesic representatives c_1 and c_2 , one has*

$$\chi(X)\mathbf{L}(c_1, c_2) = -1 \text{ if } c_1 \neq c_2,$$

and

$$\chi(X)\mathbf{L}(c_1, c_1) = \chi(X) - 1 \text{ otherwise.}$$

²²Here \tilde{g} is the Sasaki metric induced by g on S^*X – see Section 2.

Remark 6.6. B. Chantraine explained to us that the linking between $S_{c_1}^* X$ and $S_{c_2}^* X$ was equal to the inverse of the Euler characteristic and the Morse theoretic proof given below was shown to us by J.Y. Welschinger.

In order to prove this Lemma, we begin with the following observation:

Lemma 6.7. *Let Y be a smooth vector field on X which has finitely many zeroes. Suppose that all its zeroes are real hyperbolic. Set*

$$S := \left\{ \left(q, \frac{Y(q)^b}{\|Y(q)^b\|} \right) : q \in X \setminus \text{Crit}(Y) \right\}.$$

Then the current of integration $[S]$ on $S^(X \setminus \text{Crit}(Y))$ extends as a current on S^*X and it satisfies the equation:*

$$(41) \quad d[S] = - \sum_{a \in \text{Crit}(Y)} (-1)^{\text{Ind}(a)} [S_a^* X].$$

Remark 6.8. Let us recall the meaning of our assumptions on the vector field Y . Here, by a real hyperbolic zero q of Y , we mean that the linearization of the vector field Y at q is a matrix all of whose eigenvalues are *real* and different from 0. The index $\text{Ind}(q)$ of a critical point is the number of negative eigenvalues. The main example is given by the gradient vector field of a Morse function which is the case we will mostly use in the following.

We postpone the proof of this geometric Lemma and we first show how it implies Lemma 6.5.

Proof. As \mathbf{c}_i is trivial, we know that c_i is reduced to a point $q_i \in X$ for $i = 1, 2$. Hence, one has $\Sigma(c_i) = S_{q_i}^* X$. Recall from Lemma 6.2 that $[\Sigma(c_i)]$ is exact in that case and that thanks to Proposition 6.1, $[\Sigma(c_i)] = dR_i$ for some $R_i \in \mathcal{D}_{N^*(\Sigma(c_i))}^1(S^*X)$. We shall write $R_i = R_{q_i}$ all along this proof in order to emphasize the dependence on the point $q_i \in X$. We now fix some $0 \leq T_0 < T_1$ such that $S_{q_2}^* X \cap \varphi^{-T_0}(S_{q_1}^* X) = \emptyset$, and we want to compute

$$\mathbf{L}(q_1, q_2) = \int_{S^*X} \varphi^{T_0*}([S_{q_1}^* X]) \wedge R_{q_2}.$$

We begin with the case where $q_1 \neq q_2$ for which one can take $T_0 = 0$ in the previous integral. We take f to be a smooth Morse function which has no critical point at q_1 .

We denote its set of critical points by $\text{Crit}(f)$, and we define

$$S := \left\{ \left(q, \frac{d_q f}{\|d_q f\|} \right) : q \notin \text{Crit}(f) \right\},$$

which is oriented by the orientation on X .

Remark 6.9. Given a smooth vector field $q \mapsto Y(q)$ with finitely many critical points, we need to compute the tangent space to the surface

$$\tilde{S} := \left\{ \left(q, \frac{Y(q)^b}{\|Y(q)^b\|} \right) : q \notin Y^{-1}(0) \right\} \subset S^*X,$$

in the horizontal/vertical bundles of Section 2. Given a curve $x : t \mapsto (q(t), p(t)) \in S$ such that $x'(0) \neq 0$, we find that $d_{q(0), p(0)} \Pi(x'(0)) = q'(0) \neq 0$. As this is valid for any curve, this shows that the tangent space to S is transversal to the vertical bundle.

By lemma 6.7, one finds that

$$d[S] = - \sum_{a \in \text{Crit}(f)} (-1)^{\text{ind}(a)} [S_a^* X] = - \sum_{a \in \text{Crit}(f)} (-1)^{\text{ind}(a)} dR_a.$$

Remark 6.10. We remark that, as X is path-connected, one could verify that $[S_a^* X] = [S_b^* X] + d\theta_{ab}$ for some $\theta_{ab} \in \mathcal{D}^1(S^* X)$. In that manner, if we recall that $\sum_{a \in \text{Crit}(f)} (-1)^{\text{ind}(a)} = \chi(X) \neq 0$, we would recover the conclusion of Lemma 6.2, i.e. that $[S_a^* X]$ is De Rham exact for every $a \in X$.

As the intersection of $S_{q_1}^* X$ and Σ is reduced to one point and taking into account the orientation of S and $[S_{q_1}^* X]$ (see also Remark 6.9), we find that

$$1 = \int_{S^* X} [S_{q_1}^* X] \wedge [S] = - \sum_{a \in \text{Crit}(f)} (-1)^{\text{ind}(a)} \int_{S^* X} R_{q_1} \wedge [S_a^* X] = - \sum_{a \in \text{Crit}(f)} (-1)^{\text{ind}(a)} \mathbf{L}(q_1, a).$$

Now, if we fix $a \in \text{Crit}(f)$ and if we modify the Morse function f inside a small neighborhood of a , we can observe that the map $q \mapsto \mathbf{L}(q_1, q)$ is locally constant on $X \setminus \{q_1\}$ which is connected. Hence, $\mathbf{L}(q_1, a)$ is independent of the choice of a . Thanks to the case of equality in the Morse inequalities, this yields

$$-1 = \sum_{a \in \text{Crit}(f)} (-1)^{\text{ind}(a)} \mathbf{L}(q_1, a) = \chi(X) \mathbf{L}(q_1, q_2),$$

which concludes the proof when $q_1 \neq q_2$.

Suppose now that $q_1 = q_2$. In that case, we fix f which has a local minimum at q_1 and no other critical points inside the disk bounded by $\Pi(\varphi^{-T_0}(S_{q_1}^* X))$. We also suppose that f is constant on $\Pi(\varphi^{-T_0}(S_{q_1}^* X))$. Then, we define

$$S := \left\{ \left(q, \frac{df(q)}{\|df(q)\|} \right) : q \notin \text{Crit}(f) \right\}$$

which does not intersect $\varphi^{-T_0}(S_{q_1}^* X)$. Reproducing the above arguments, this yields

$$0 = \int_{S^* X} \varphi^{T_0*}([S_{q_1}^* X]) \wedge [S] = - \sum_{a \in \text{Crit}(f)} (-1)^{\text{ind}(a)} \int_{S^* X} \varphi^{T_0*}(R_{q_1}) \wedge [S_a^* X] = - \sum_{a \in \text{Crit}(f)} (-1)^{\text{ind}(a)} \mathbf{L}(q_1, a).$$

Thanks to the case where $q_1 \neq q_2$ and to the case of equality in Morse inequalities, we finally obtain

$$0 = -\mathbf{L}(q_1, q_1) + \frac{1}{\chi(X)} \sum_{a \in \text{Crit}(f) \setminus q_1} (-1)^{\text{ind}(a)} = -\mathbf{L}(q_1, q_1) + 1 - \frac{1}{\chi(X)},$$

which concludes the proof, except for the proof of Lemma 6.7. \square

Proof of Lemma 6.7. We only need to prove this formula near a fixed critical point a of Y . The argument is just a variation of the proof of Stokes formula except that we do not know a priori that \bar{S} is a *smooth* manifold with boundary. We let $\kappa : U \rightarrow \mathbb{R}^2$ be a local chart centered at a (i.e. $\kappa(a) = 0$). Using the symplectic lift of κ , this chart lifts into a chart $\tilde{\kappa} : S^*U \rightarrow \mathbb{R}^2 \times \mathbb{S}^1$. In this chart, the boundary ∂S of S is exactly given by $\{0\} \times \mathbb{S}^1$. Similarly, S reads $\{(\tilde{q}, \phi(\tilde{q})) : \tilde{q} \neq 0\}$ where $\phi : \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{S}^1$ is a smooth map obtained via the local chart and the initial vector field Y . Without loss of generality, we may assume that the image of the submanifold S in \mathbb{R}^2 is oriented by the canonical orientation of \mathbb{R}^2 . As the critical points are of real hyperbolic type, we can always choose the local chart centered at a in such a way that, in the (induced) local coordinates

$$Y(\tilde{q}_1, \tilde{q}_2) = (\chi_1 \tilde{q}_1 + h_1(\tilde{q})) \partial_{\tilde{q}_1} + (\chi_2 \tilde{q}_2 + h_2(\tilde{q})) \partial_{\tilde{q}_2},$$

with $\chi_1 \chi_2 \neq 0$ and with $h_1, h_2 = \mathcal{O}(\|\tilde{q}\|^2)$ which are smooth functions defined near 0. As in the examples of paragraph 3.3.2, we define the submanifold of $\mathbb{R}^2 \times \mathbb{S}^1$:

$$S := \left\{ (\tilde{q}, \phi) : \tilde{q} \neq 0, \cos \phi = \frac{\chi_1 \tilde{q}_1 + h_1(\tilde{q})}{\|Y(\tilde{q})\|}, \sin \phi = \frac{\chi_2 \tilde{q}_2 + h_2(\tilde{q})}{\|Y(\tilde{q})\|} \right\},$$

which is oriented with $d\tilde{q}_1 \wedge d\tilde{q}_2$. We set

$$F(\tilde{q}, \phi) := \chi_2 \tilde{q}_2 + h_2(\tilde{q}) - \tan \phi (\chi_1 \tilde{q}_1 + h_1(\tilde{q})).$$

Then, as in this example, one can define, in $\mathbb{R}^2 \times (-\pi/2, \pi/2)$

$$[S] = -\frac{\chi_1}{|\chi_1|} \mathbf{1}_{\mathbb{R}_+}(\tilde{q}_1) \delta_0(F(\tilde{q}, \phi)) dF,$$

which extends the current of integration $[S]$ defined on $\mathbb{R}^2 \setminus \{0\} \times (-\pi/2, \pi/2)$. Then, taking a partition of unity on \mathbb{S}^1 (associated with each half plane of \mathbb{R}^2), one can verify that $[S]$ is well defined on $\mathbb{R}^2 \times \mathbb{S}^1$. Finally, if we differentiate this expression, we find

$$d[S] = -\frac{\chi_1 \chi_2}{|\chi_1 \chi_2|} \delta_0(\tilde{q}_1, \tilde{q}_2) d\tilde{q}_1 \wedge d\tilde{q}_2.$$

Recalling that $[S_a^* X] = \delta_0(\tilde{q}_1, \tilde{q}_2) d\tilde{q}_1 \wedge d\tilde{q}_2$ was oriented by $d\phi$, we obtain the expected result. \square

6.3. Proof of Theorem 6.3: two warm up examples. We now consider the case where at least one of the two homotopy classes \mathbf{c}_i is nontrivial (say \mathbf{c}_1). Before handling the general case, we begin with the case where the geodesic representative c_1 and c_2 do not intersect each other and have no self-intersection points. In that case, one can take $T_0 = 0$ in the definition of $\mathbf{L}(c_1, c_2)$.

Remark 6.11. Eventhough the general case will be technically more involved, it will essentially consist in splitting the problem into small elementary pieces where we can apply the strategy used in these two examples.

6.3.1. *Classical definition of the Euler characteristic.* Before starting with these two examples, let us recall the definition of the Euler characteristic of a CW-complex [46, App. A]. Let \mathbf{X} be a space which can be written as a disjoint union of open cells, i.e. $\mathbf{X} = \bigsqcup_{j \in J} \mathbf{X}_j$, each cell being homeomorphic to some $\mathbb{R}^{\dim \mathbf{X}_j}$. Then, the Euler characteristic of \mathbf{X} is given by [15, p. 3]

$$\chi(\mathbf{X}) = \sum_{j \in J} (-1)^{\dim \mathbf{X}_j},$$

which extends the classical formula for polyhedra. In particular, any continuous closed curve (without selfintersection points) on our closed surface X has Euler characteristic equal to 0. Similarly, any closed domain $X_1 \subset X$ with piecewise smooth boundary ∂X_1 can be triangulated and it can be decomposed as above. As we are in dimension 2, one has

$$(42) \quad \chi(X_1) = \chi(X_1 \setminus \partial X_1) + \chi(\partial X_1) = \chi(X_1 \setminus \partial X_1), \quad \text{and} \quad \chi(X \setminus X_1) + \chi(X_1) = \chi(X).$$

6.3.2. *The case where c_2 is trivial.* As c_1 is homologically trivial, its geodesic representative c_1 splits the surface X into two connected components X_1 and $X \setminus X_1$. We suppose that $d[X_1] = -[c_1]$ in terms of De Rham currents on X .

Remark 6.12. In the following, we always take the conventions that the surfaces X_i (or $X_{i,j}$, $X_{i,j}^m$) are closed, i.e. they contain their topological boundary.

We fix f to be a smooth Morse function which is constant on c_1 and which has no critical point at q_2 . We define

$$S := \left\{ \left(q, \frac{d_q f}{\|d_q f\|} \right) : q \in X_1, q \notin \text{Crit}(f) \right\}.$$

Hence, up to changing f into $-f$, we have, thanks to Lemma 6.7 and to the Stokes formula near the boundary of X_1 (see e.g. § 3.3.2),

$$d[S] = [\Sigma(c_1)] - \sum_{a \in \text{Crit}(f) \cap X_1} (-1)^{\text{ind}(a)} [S_a^* X].$$

Using Lemma 6.2, we deduce that $[\Sigma(c_1)]$ is exact which is the first part of Theorem 6.3. If q_2 belongs to X_1 , one finds that

$$1 = \int_{S^* X} [S] \wedge [S_{q_2}^* X] = \int_{S^* X} [\Sigma(c_1)] \wedge R_{q_2} - \sum_{a \in \text{Crit}(f) \cap X_1} (-1)^{\text{ind}(a)} \int_{S^* X} [S_a^* X] \wedge R_{q_2}.$$

From Lemma 6.5, we get

$$\int_{S^* X} [\Sigma(c_1)] \wedge R_{q_2} = 1 - \frac{1}{\chi(X)} \sum_{a \in \text{Crit}(f) \cap X_1} (-1)^{\text{ind}(a)}.$$

From the case of equality in Morse inequalities for manifolds with boundary [65, Th. A_0] and according to (42), we find that, for $q_2 \in X_1$

$$\int_{S^* X} [\Sigma(c_1)] \wedge R_{q_2} = 1 - \frac{\chi(X_1)}{\chi(X)} = \frac{\chi(X \setminus X_1)}{\chi(X)}.$$

If $q_2 \notin X_1$, $\int_{S^*X} [S] \wedge [S_{q_2}^* X] = 0$ and the same argument would give

$$\int_{S^*X} [\Sigma(c_1)] \wedge R_{q_2} = -\frac{\chi(X_1)}{\chi(X)} = \frac{\chi(X \setminus X_1)}{\chi(X)} - 1.$$

6.3.3. *The case where c_2 is nontrivial.* Recall that we want to compute

$$\mathbf{L}(c_1, c_2) = \int_M [\Sigma(c_1)] \wedge R_2,$$

where $[\Sigma(c_2)] = dR_2$. Thanks to the previous example, we already know that both $[\Sigma(c_i)]$ are De Rham exact. For $i = 1, 2$, the set $X \setminus c_i$ has either one or two connected components. As c_1 and c_2 are both homologically trivial, we deduce that we have two connected components in both cases. Hence, for $i = 1, 2$, c_i (with the orientation induced by the homotopy class \mathbf{c}_i) is the oriented boundary of a surface X_i .

As before, we let f be a smooth Morse function on X which has no critical points on c_1 and on c_2 . We also suppose f to be constant on c_1 and that c_1 is the oriented boundary of X_1 . Suppose first that $c_2 \subset X_1$ and set

$$S := \left\{ \left(q, \frac{d_q f}{\|d_q f\|} \right) : q \in X \setminus X_1, q \notin \text{Crit}(f) \right\}.$$

Up to changing f into $-f$, one has one more time by an application of Lemma 6.7 and of §3.3.2:

$$d[S] = -[\Sigma(c_1)] - \sum_{a \in \text{Crit}(f) \setminus X_1} [S_a^* X] = -[\Sigma(c_1)] - \sum_{a \in \text{Crit}(f) \setminus X_1} (-1)^{\text{ind}(a)} dR_a.$$

This yields

$$\mathbf{L}(c_1, c_2) = - \int_{S^*X} d[S] \wedge R_2 - \sum_{a \in \text{Crit}(f) \setminus X_1} (-1)^{\text{ind}(a)} \int_{S^*X} dR_a \wedge R_2.$$

Hence, one finds

$$\mathbf{L}(c_1, c_2) = - \int_{S^*X} [S] \wedge [\Sigma(c_2)] - \sum_{a \in \text{Crit}(f) \setminus X_1} (-1)^{\text{ind}(a)} \int_{S^*X} R_a \wedge [\Sigma(c_2)].$$

As S does not intersect $[\Sigma(c_2)]$, we are left with

$$\mathbf{L}(c_1, c_2) = - \sum_{a \in \text{Crit}(f) \setminus X_1} (-1)^{\text{ind}(a)} \int_{S^*X} [\Sigma(c_2)] \wedge R_a.$$

We already computed the quantities appearing in this sum and we obtained

$$\int_{S^*X} [\Sigma(c_2)] \wedge R_a = -\frac{\chi(X_2)}{\chi(X)} \text{ if } a \in X \setminus X_2,$$

and

$$\int_{S^*X} [\Sigma(c_2)] \wedge R_a = 1 - \frac{\chi(X_2)}{\chi(X)} \text{ if } a \in X_2.$$

Applying the case of equality in Morse inequalities for manifolds with boundary [65, Th. A_0] together with (42), we find that

$$\mathbf{L}(c_1, c_2) = -\chi(X_1^c \cap X_2) + \frac{\chi(X_2)\chi(X_1^c)}{\chi(X)} = -\frac{\chi(X_2)\chi(X_1)}{\chi(X)} + \chi(X_1 \cap X_2).$$

If $c_2 \subset X \setminus X_1$, then the same argument would also show

$$\mathbf{L}(c_1, c_2) = -\frac{\chi(X_2)\chi(X_1)}{\chi(X)} + \chi(X_1 \cap X_2).$$

6.4. Proof of Theorem 6.3: the general case. We can now consider the general case still supposing that both \mathbf{c}_i are homologically trivial and primitive in the sense of (37). Yet, we do not suppose anymore that both geodesic representatives have no selfintersection nor intersection between each other. Recall that we want to compute, for $T_0 + T'_0 > 0$ small enough,

$$\mathbf{L}(c_1, c_2) = \int_M \varphi^{(T_0+T'_0)*}([\Sigma(c_1)]) \wedge R_2,$$

where $[\Sigma(c_2)] = dR_2$, which is the same as

$$\mathbf{L}(c_1, c_2) = \int_M \varphi^{T_0*}([\Sigma(c_1)]) \wedge R_2^{-T'_0},$$

where $\varphi^{-T'_0*}([\Sigma(c_2)]) = dR_2^{-T'_0}$. We note that we first need to justify that $[\Sigma(c_2)]$ is exact when \mathbf{c}_2 is homologically trivial and this will be part of our argument.

In the following, we shall write things a little bit more compactly by letting $[\Sigma_1^{T_0}] = \varphi^{T_0*}([\Sigma(c_1)])$ which is the current of integration over the smooth submanifold $\varphi^{-T_0}(\Sigma(c_1))$. Similarly, $[\Sigma_2^{-T'_0}] = dR_2^{-T'_0}$ will denote the current of integration over the submanifold $\varphi^{T'_0}(\Sigma(c_2))$. In both cases, we denote by \tilde{c}_i , the projection (via the canonical projection) on X of these two curves of S^*X .

The proof can be decomposed as follows. First, in §6.4.1, we make small homotopic deformations of our geodesic representatives c_1 and c_2 . This gives two new curves \tilde{c}_1 and \tilde{c}_2 with nice intersection properties. In §6.4.2, we decompose these curves into a family of *embedded* and piecewise smooth curves using the notion of constructible functions. Then, in §6.4.3, we recall the definitions of Euler characteristics for these functions. In §6.4.4, we lift these curves to S^*X in a manner which is consistent with the currents $[\Sigma_1^{T_0}]$ and $[\Sigma_2^{-T'_0}]$. In §6.4.5, this allows us to reduce our problem to computing the linking number of the piecewise smooth curves appearing in this decomposition. In §6.4.6 and §6.4.7, we show that these curves can be smoothed out without affecting the topological quantities we are interested in. Finally, in §6.4.8–6.4.10, we use the strategy explained in our warm up examples (based on Poincaré-Hopf formula) to compute the linking number of this family of smooth curves.

6.4.1. *First properties of the perturbed curves \tilde{c}_1 and \tilde{c}_2 .* We start by collecting some properties of these curves which are obtained by pushing $\Sigma(c_i)$ along the geodesic flow:

Proposition 6.13. *There exist $T_0 > 0$ and $T'_0 > 0$ small enough (with T'_0 depending on T_0), such that the following properties hold:*

- for $i = 1, 2$, one can find some smooth map $\tilde{c}_i : \mathbb{R}/\ell_i\mathbb{Z} \rightarrow X$ (with $\ell_i > 0$) representing the projected curve \tilde{c}_i and such that $\tilde{c}'_i(t) \neq 0$ for every $t \in \mathbb{R}/\ell_i\mathbb{Z}$;
- for $i = 1, 2$, for every $t \in [0, \ell_i)$,

$$(43) \quad \{s \in [0, \ell_i) : s \neq t \text{ and } \tilde{c}_i(t) = \tilde{c}_i(s)\} \leq 1.$$

In other words, the selfintersections of each curve \tilde{c}_i is made of double points²³;

- if $q_0 = \tilde{c}_1(t) = \tilde{c}_2(s)$ for some $(t, s) \in [0, \ell_1) \times [0, \ell_2)$, then q_0 is neither a double point of \tilde{c}_1 , nor of \tilde{c}_2 .

Finally, if c_2 (resp. c_1) is trivial in $\pi_1(X)$, T'_0 (resp. T_0) can be taken equal to 0 and²⁴ $T_0 > 0$ (resp. $T'_0 > 0$) such that $\tilde{c}_1 \cap c_2 = \emptyset$ (resp. $\tilde{c}_2 \cap c_1 = \emptyset$).

Proof. Let us explain how to find T_0 and T'_0 with the above properties. The first point is clear and one only needs to discuss the two other items. We start by acting on c_1 (i.e. we will fix the range of T_0). We would like to remove all the selfintersections of the curve c_1 that correspond to points with multiplicity > 2 . We note that any such point q_0 of the curve c_1 is isolated in the sense that one can find some $r > 0$ such that $B(q_0, r)$ contains no other selfintersection point of c_1 . We would like to show that, applying the geodesic flow allow to remove any such point q_0 with multiplicity > 2 . We suppose by contradiction that, for every $n \geq 1$, one can find $0 < T_n \leq \frac{1}{n}$ such that one can find three distinct points $(q_i^n, p_i^n)_{1 \leq i \leq 3}$ on $\Sigma(c_1)$ with the property that $\Pi \circ \varphi^{-T_n}(q_i^n, p_i^n) = \tilde{q}_n$. One has $\tilde{q}_n \rightarrow q_0$. Without loss of generality, we can suppose that, for every $n \geq 1$, q_i^n always belongs to the same geodesic branch of c_1 passing through q_0 . As the three points are distinct, these correspond to three geodesic distinct branches γ_i . In particular, by construction, $d_g(q_i^n, \tilde{q}_n) = T_n$ is equal to the distance to each geodesic branch. Moreover, the point \tilde{q}_n lies on the curve of points passing through q_0 which are at equal distance from γ_1 and γ_2 . Similarly, it also belongs to the curve of points at equal distance from γ_2 and γ_3 . As the three branches are distinct, this leads to a contradiction. Hence, picking $T_0 > 0$ small enough, we can transform any point of multiplicity > 2 in a family of double points. By compactness, this allows to find some $\tilde{T}_0 > 0$ such that, for any $0 < T_0, T'_0 \leq \tilde{T}_0$, the second property holds for \tilde{c}_1 and \tilde{c}_2 .

It now remains to find some small enough $T'_0 > 0$ in order to verify the last property. We fix some $T_0 > 0$ so that the curve $\tilde{c}_1(t)$ has only simple selfintersection points. Taking $T'_0 > 0$ small enough, we saw that the curve \tilde{c}_2 has also only simple selfintersection points that correspond to the perturbation of selfintersection points of the initial curve c_2 . Then, one can verify that, by eventually taking $T'_0 > 0$ slightly smaller in a way that depends

²³This will be referred as a simple selfintersection point.

²⁴When c_1 and c_2 are distinct points, we can take $T_0 = T'_0 = 0$ but we already treated this case in Lemma 6.5.

on \tilde{c}_1 (thus on T_0), none of these new self intersection points belong to the curve \tilde{c}_1 and \tilde{c}_2 does not intersect the selfintersection points of \tilde{c}_1 . We note that, when \mathbf{c}_2 is a trivial homotopy class, one can in fact take $T'_0 = 0$ (but not $T_0 = 0$ in general) as we can choose $T_0 > 0$ such that $c_2 \notin \tilde{c}_1$. \square

Now that we have slightly simplified the representatives of our homotopy classes \mathbf{c}_1 and \mathbf{c}_2 , we are in position to begin our proof whose general strategy is reminiscent of the Seifert algorithm in classical knot theory. In the rest of the proof, we shall always assume (without loss of generality) that \mathbf{c}_1 is a nontrivial homotopy class.

6.4.2. Decomposing the curve \tilde{c}_i into elementary pieces. Our goal in this paragraph is to provide some algorithm which allows to decompose each curve \tilde{c}_1, \tilde{c}_2 as a union of simple, closed, piecewise smooth curves. Moreover, we shall verify in the next paragraphs that this decomposition leads to a decomposition of the Legendrian curves $\Sigma_1^{T_0}, \Sigma_2^{-T'_0}$ (lifting \tilde{c}_1, \tilde{c}_2) to S^*X into a sum of conormals.

Let $i \in \{1, 2\}$. Our algorithm is based on the construction of a nice function $f_i : X \mapsto \mathbb{Z}$ which takes constant value in each connected component of $X \setminus \tilde{c}_i$. Even if the next definition may slightly differ from what can be found in the literature [15, p. 5], such a function is often referred as a *constructible function*:

Definition 6.14. Let $i \in \{1, 2\}$ and suppose that \mathbf{c}_i is homologically trivial. Write

$$X \setminus \tilde{c}_i = \bigsqcup_{j \in J} \Omega_j,$$

where each Ω_j is an open connected subset of X and where \tilde{c}_i is the curve from Proposition 6.13. We say that $f_i : X \setminus \tilde{c}_i \rightarrow \mathbb{Z}_+$ is a constructible function associated to \tilde{c}_i if

- $f_i^{-1}(0) = \Omega_{j_0}$ for some $j_0 \in J$;
- there exists $x_0 \in \Omega_{j_0}$ such that, for every $j \in J$, for every $y \in \Omega_j$,

$$f_i(y) = \int_X [\tilde{c}_i] \wedge [\gamma],$$

where γ is any smooth path going from x_0 to y which is transverse to \tilde{c}_i .

The next Proposition shows the existence of such a constructible function:

Proposition 6.15. *Let $i \in \{1, 2\}$ and suppose that \mathbf{c}_i is homologically trivial. Then, there exists a constructible function associated to \tilde{c}_i in the sense of definition 6.14.*

Proof. We start by fixing some open connected component Ω_{j_1} of $X \setminus \tilde{c}_i$ and some point x_1 in Ω_{j_1} . Then, given some $j \in J$ and some $y \in \Omega_j$, we define

$$\tilde{f}_i(y) = \int_X [\tilde{c}_i] \wedge [\gamma],$$

where γ is any smooth path going from x_1 to y which is transverse to \tilde{c}_i . Since $[\tilde{c}_i]$ is an exact current²⁵, it is immediate that the value of \tilde{f}_i in each Ω_j does not depend on the choice of paths γ used to construct \tilde{f}_i . Moreover, \tilde{f}_i takes value in \mathbb{Z} since it is defined by intersecting integral currents in transverse positions. We now fix some $j_0 \in J$ such that $\tilde{f}_i|_{\Omega_{j_0}} = \text{essinf}(\tilde{f}_i)$ and we set

$$f_i := \tilde{f}_i - \text{essinf}(\tilde{f}_i) : X \rightarrow \mathbb{Z}_+.$$

This function now verifies that $f_i \geq 0$ a.e. on X and f_i is identically 0 on Ω_{j_0} . From the definition of \tilde{f}_i and the exactness of $[c_i]$, we can finally verify that f_i is a constructible in the sense of definition 6.14. \square

Now let us define an algorithm which extracts surfaces from the constructible function f_i . These surfaces are going to bound the decomposition of the curve \tilde{c}_i we are looking for. We also note that f_i is defined on $X \setminus \tilde{c}_i$ for the moment. In particular, the sets $U_{i,j} := \{f_i \geq j\}$ are open in X and they have piecewise smooth boundaries. The following construction comes from Euler integration and motion sensing as in [3, 15]. Observe first that on $X \setminus \tilde{c}_i$, we have the identity [15, Prop 4.2],[3, Prop 4.1 p. 833]:

$$f_i = \sum_{j=0}^{\infty} j \mathbf{1}_{\{f_i=j\}} = \sum_{j=1}^{\infty} \mathbf{1}_{\{f_i \geq j\}}$$

where both sums are finite since f_i takes finitely many values²⁶. If we set $X_{i,j} = \overline{U_{i,j}}$, we may extend f_i to the whole manifold X by the formula

$$f_i = \sum_{j=1}^{\infty} \mathbf{1}_{X_{i,j}}.$$

Each $X_{i,j}$ is a smooth manifold with piecewise smooth boundary $\tilde{c}_{i,j} := \partial X_{i,j}$. Note that the singularities of the boundary only occur at the selfintersection points of the curve \tilde{c}_i . We have the following chains of inclusions

$$X_{i,\sup(f_i)} \subset \cdots \subset X_{i,0} = X.$$

Note that each $\tilde{c}_{i,j}$ is not necessarily connected since our surfaces $X_{i,j}$ may have several boundary components. We shall need the following important observation:

Lemma 6.16. *Let $i \in \{1, 2\}$. Let q be some element in \tilde{c}_i . If q is not a selfintersection point of \tilde{c}_i , then $q \in \tilde{c}_{i,j}$ for exactly one index j . Moreover, in a neighborhood of such a point, one has $d[X_{i,j}] = -[\tilde{c}_{i,j}]$ in the sense of De Rham currents.*

Otherwise, there exists $j \geq 0$ such that $q \in \tilde{c}_{i,j} \cap \tilde{c}_{i,j+1}$ and $q \notin \tilde{c}_{i,j'}$ if $j' \notin \{j, j+1\}$.

²⁵Recall that $H_1(X, \mathbb{Z})$ has no torsion. Thus a closed curve is homologically trivial if and only if it is De Rham exact [36, Ch. 5].

²⁶The level sets $\{f_i \geq j\}$ are related to the excursion sets of [3]

Proof. We begin with the case where q is not a selfintersection point. Consider some small open neighborhood Ω of q (diffeomorphic to some open ball). The intersection $\Omega \cap \tilde{c}_i$ is just some open connected interval containing q and having no self-intersection point. Let us consider the restriction of f_i on Ω . The open subset $\Omega \setminus \tilde{c}_i$ is divided into two connected components $\Omega \setminus \tilde{c}_i = \tilde{\Omega}_{j-1} \cup \tilde{\Omega}_j$ where $f_i|_{\tilde{\Omega}_{j-1}} = j-1$, $f_i|_{\tilde{\Omega}_j} = j$ and $j \geq 1$. We note that f_i takes different values since we can choose to cross \tilde{c}_i exactly one time to go from one component $\tilde{\Omega}_{j-1}$ to the other component $\tilde{\Omega}_j$. By construction of the surfaces $X_{i,0}, \dots, X_{i,\sup(f)}$, we have $\tilde{\Omega}_{j-1} \subset X_{i,j-1} \subset \dots \subset X_{i,0}$ but $\tilde{\Omega}_{j-1} \cap X_{i,j} = \emptyset$. On the other hand, $\tilde{\Omega}_j \subset X_{i,j} \subset \dots \subset X_{i,0}$. This implies that $\Omega \cap \tilde{c}_i$ is a subset of a smooth part of $\tilde{c}_{i,j}$. Moreover, one has $d[X_{i,j}] = -[\tilde{c}_{i,j}]$ near this point where the boundary of $X_{i,j}$ is smooth. To see this, it is sufficient to check the formula in the following toy model (which is equivalent to ours in a local chart $(\tilde{q}_1, \tilde{q}_2)$):

$$X_{i,j} := \{(\tilde{q}_1, \tilde{q}_2) : \tilde{q}_2 > 0\} \quad \text{and} \quad X_{i,j-1} := \{(\tilde{q}_1, \tilde{q}_2) : \tilde{q}_2 < 0\},$$

where $\tilde{c}_{i,j} := \{(\tilde{q}_1, 0)\}$ is oriented by $d\tilde{q}_1$, i.e. $[\tilde{c}_{i,j}] = -\delta_0(\tilde{q}_2)d\tilde{q}_2$. In fact, taking $[\gamma] = \delta_0(\tilde{q}_1)d\tilde{q}_1$ (which is oriented by $d\tilde{q}_2$), one finds that the value in the upper half-plane is larger than the value in the lower half plane. Hence, by a direct calculation, one finds that $d[X_{i,j}] = d\mathbf{1}_{\mathbb{R}_+}(\tilde{q}_2) = \delta_0(\tilde{q}_2)d\tilde{q}_2 = -[\tilde{c}_{i,j}]$.

Suppose now that q is a selfintersection point of the curve \tilde{c}_i . In that case, the function f_i takes exactly three values on the four connected components of $\Omega \setminus \tilde{c}_i$. By construction of the function f_i , these three values are given by $j-1$ (one time), j (two times) and $j+1$ (one time) for some $j \geq 1$ and one can verify that $q \in \tilde{c}_{i,j} \cap \tilde{c}_{i,j+1}$. \square

By construction, we obtain the expected decomposition of the curve \tilde{c}_i :

Proposition 6.17. *Let $i \in \{1, 2\}$. We have the following decomposition of the current $[\tilde{c}_i]$:*

$$[\tilde{c}_i] = \sum_{j=1}^{\infty} [\tilde{c}_{i,j}]$$

where each $\tilde{c}_{i,j} = \partial X_{i,j}$ is a finite union of closed, simple and piecewise smooth curves with $X_{i,j} := \{f_i \geq j\}$. Moreover, for every $1 \leq j \leq N_i := \sup f_i$, one has

$$d\mathbf{1}_{X_{i,j}} = d[X_{i,j}] = -[\tilde{c}_{i,j}],$$

in the sense of De Rham currents.

Note that using the dual operator $\partial T = -(-1)^{\deg(T)} dT$ on \mathcal{D}' , this would read equivalently $\partial[X_{i,j}] = [\tilde{c}_{i,j}]$. As a consequence, the orientation induced by $X_{i,j}$ on its boundary $\tilde{c}_{i,j}$ is the same as the orientation induced by \tilde{c}_i and each $\tilde{c}_{i,j}$ is cohomologically trivial. Hence, \tilde{c}_i is in this sense the (oriented) boundary of the system of surfaces $X_i = (X_{i,1}, \dots, X_{i,N_i})$. More precisely, in the sense of De Rham currents, one has

$$(44) \quad [\tilde{c}_i] = - \sum_{j=1}^{N_i} d[X_{i,j}] = -df_i.$$

Proof. The first part is direct consequence of our construction and of Lemma 6.16. For the second part, we already know that it holds true away from the singularities of $\tilde{c}_{i,j}$. Near a singular point, we can work in a local chart $(\tilde{q}_1, \tilde{q}_2)$. It is then sufficient to write the relation $d\mathbf{1}_{\mathbb{R}_+^2} = \mathbf{1}_{\mathbb{R}_+}(\tilde{q}_2)\delta_0(\tilde{q}_1)d\tilde{q}_1 + \mathbf{1}_{\mathbb{R}_+}(\tilde{q}_1)\delta_0(\tilde{q}_2)d\tilde{q}_2$ and to verify as in the proof of Lemma 6.16 that it is the expected current. \square

6.4.3. Euler characteristics of surfaces and constructible functions. As we have just seen, it is equivalent to think of the constructible functions f_i associated to \tilde{c}_i with $i \in \{1, 2\}$ as the system of surfaces $X_i = (X_{i,1}, \dots, X_{i,N_i})$, $X_{i,j} = \{f_i \geq j\}$. Note that this system of surfaces with piecewise smooth boundary generates an abstract CW-complex that we denote by $X(\tilde{c}_i)$ and whose “(oriented) boundary” is given by \tilde{c}_i . This was already express more precisely in terms of De Rham currents by equality (44). In the case where the initial curve has no selfintersection, one has $X(\tilde{c}_i) = X(c_i) := X_{i,1}$ which is a surface with smooth boundary. Hence, in that case, this CW-complex is exactly what we meant in Theorems 1.2 and 6.3 by the compact surface $X(c_i)$ whose oriented boundary is c_i . For more general geodesic representatives c_i not entering the assumptions of these Theorems, our main formula can be extended if we introduce these curves \tilde{c}_i and if we define the appropriate notion of Euler characteristic for the system of surfaces $X_i = (X_{i,1}, \dots, X_{i,N_i})$ (or equivalently for the CW-complex $X(\tilde{c}_i)$).

Thus we would like to assign a natural notion of Euler characteristic to the constructible function f_i or equivalently to the system of surfaces $X_i = (X_{i,1}, \dots, X_{i,N_i})$. Our definition follows the presentation of Euler integration due to Viro [88] and Schapira [79, 80, 81]:

Definition 6.18. [Euler characteristic of constructible functions] We define the Euler characteristic of f_i as

$$(45) \quad \chi(f_i) := \sum_{j=0}^{N_i} j \chi(\{f_i = j\}) = \sum_{j=1}^{N_i} \chi(\{f_i \geq j\}) = \sum_{j=1}^{N_i} \chi(X_{i,j}).$$

Note that the advantage of the second formulation for $\chi(f_i)$ is that the excursion sets $\{f_i \geq j\}$ are compact whereas $\{f_i = j\}$ is only relatively compact [3, Prop. 4.1].

Remark 6.19. We can relate this definition with the classical one for CW-complex as follows: $\chi(f_i) = \chi(X(\tilde{c}_i))$.

We emphasize that the Euler integral is in fact defined for much more general bounded and constructible functions, $f : X \rightarrow \mathbb{Z}$ whose level sets are tame sets [15, §4]. In that context, one can define

$$\chi(f) := \int_X f d\chi = \sum_{j=-\infty}^{+\infty} j \chi(f = j).$$

For instance, we can define the Euler characteristic $\chi(f_1 f_2)$ of the product $f_1 f_2$ as:

$$(46) \quad \chi(f_1 f_2) = \int_X f_1 f_2 d\chi = \sum_{1 \leq j_1 \leq N_1} \sum_{1 \leq j_2 \leq N_2} \int_X \mathbf{1}_{X_{1,j_1}} \mathbf{1}_{X_{2,j_2}} d\chi = \sum_{1 \leq j_1 \leq N_1} \sum_{1 \leq j_2 \leq N_2} \chi(X_{1,j_1} \cap X_{2,j_2}),$$

or the Euler characteristic of $\mathbf{1}_{\tilde{c}_1 \cap \tilde{c}_2}$ as

$$\chi(\mathbf{1}_{\tilde{c}_1 \cap \tilde{c}_2}) = \sum_{1 \leq j_1 \leq N_1} \sum_{1 \leq j_2 \leq N_2} \chi(\partial X_{1,j_1} \cap \partial X_{2,j_2}).$$

6.4.4. *Lifting everything to S^*X .* We would now like to turn this decomposition of the curve \tilde{c}_i into a proper decomposition of $\Sigma(\tilde{c}_i)$. The convenient way to do that is to introduce the (unit) conormal bundle of $X_{i,j}$ – see [1, Def. 2.4.1 p. 442] for the case of more general polyhedra. Recall that, for every vector $v \in T_x X$, we defined in Section 2 the covector $v^\flat \in T_x^* X$ as the image of v by the isomorphism induced by the metric g on X . In order to define the unit conormal bundle above $X_{i,j}$, we have three kind of points to distinguish:

- The points in the interior of $X_{i,j}$. Above such points, the (unit) conormal bundle is obviously empty.
- The regular points of $\tilde{c}_{i,j}$. Here, we take the same convention as for $\Sigma(\tilde{c}_i)$, i.e. the points in the unit conormal bundle above some regular point $\tilde{c}_{i,j}(t_0)$ are given by the point

$$(\tilde{c}_{i,j}(t_0), (\tilde{c}'_{i,j}(t_0)^\flat)^\perp),$$

where $\tilde{c}_{i,j}(t)$ is parametrized by arc length.

- The singular points of $\tilde{c}_{i,j}$. Again, we take an arc-length (away from the singularities) parametrization $t \mapsto \tilde{c}_{i,j}(t)$ of the curve $\tilde{c}_{i,j}$. Above such a point $\tilde{c}_{i,j}(t_0)$, the derivative $\tilde{c}'_{i,j}(t_0)$ is not well defined. Yet, we have the existence of the two following limits:

$$\tilde{c}'_{i,j}(t_0+) = \lim_{\tau \rightarrow 0, \tau > 0} \tilde{c}'_{i,j}(t_0 + \tau) \quad \text{and} \quad \tilde{c}'_{i,j}(t_0-) = \lim_{\tau \rightarrow 0, \tau > 0} \tilde{c}'_{i,j}(t_0 - \tau).$$

Then, the conormal bundle above such a point is defined as the *connected* set of unit covectors lying in $S_{\tilde{c}_{i,j}(t_0)}^* X$ and in the cone of cotangent vectors between $(\tilde{c}'_{i,j}(t_0-)^\flat)^\perp$ and $(\tilde{c}'_{i,j}(t_0+)^\flat)^\perp$ that are all pointing inward $X_{i,j}$. Here, a covector p is pointing inward $X_{i,j}$ if, for any curve γ passing through $\tilde{c}_{i,j}(t_0)$ and cotangent to p at $t = 0$, one has $\gamma(t) \in X_{i,j}$ for every $t \geq 0$ small enough.

The union of all these covectors will be referred as the (unit) conormal bundle to $X_{i,j}$ and we will denote it by $N_1^*(X_{i,j})$. This defines a closed, piecewise smooth and embedded curve in S^*X . This curve is naturally oriented by the orientation of $\tilde{c}_{i,j}$ which itself comes from the orientation of \tilde{c}_i . In particular, we can define the integration current $[N_1^*(X_{i,j})]$ along this curve and one has $d[N_1^*(X_{i,j})] = 0$. We can also note that we still have a Legendrian curve, i.e. $[N_1^*(X_{i,j})] \wedge \alpha = 0$, where α is the Liouville one-form.

Remark 6.20. We remark that, in this construction, we implicitly supposed that \tilde{c}_i was not reduced to a point (i.e. $T_0, T'_0 \neq 0$ if \mathbf{c}_i is trivial in $\pi_1(X)$). In the case of a point, the conormal bundle of a point and its orientation were already defined in §3.3.

Finally, we observe that, as soon as one curve $\tilde{c}_{i,j}$ has singular points, the union $\cup_{j=1}^{N_i} N_1^*(X_{i,j})$ is larger than the set $\Sigma(\tilde{c}_i)$ (as it contains more cotangent vectors above each selfintersection point of \tilde{c}_i). Yet, in terms of currents, we can verify that the following holds:

Theorem 6.21. *With the above conventions, one has, in the sense of De Rham currents,*

$$(47) \quad [\Sigma_1^{T_0}] = \sum_{j=1}^{N_1} [\Sigma_{1,j}] \quad \text{and} \quad [\Sigma_2^{-T'_0}] = \sum_{j=1}^{N_2} [\Sigma_{2,j}]$$

where $[\Sigma_{i,j}] = [N_1^* X_{i,j}]$.

Note that the currents $\Sigma_{i,j}$ depend implicitly on T_0 and T'_0 but we dropped this dependence to alleviate the notations. As a corollary of this result, we will be able to compute the linking between Σ_1 and Σ_2 by evaluating the linking numbers between each elementary piece $\Sigma_{1,j}$ and $\Sigma_{2,j'}$ which are simple closed curves as in our warm-up examples. See next paragraph for more details.

Proof. We only treat the case $i = 1$. The other case is similar. Recall first that $[\Sigma_1^{T_0}]$ and $([N_1^*(X_{1,j})])_{1 \leq j \leq N_1}$ are currents of integration over closed simple and piecewise smooth curves in S^*X . For every $1 \leq j \leq N_1$, the oriented curve $N_1^*(X_{1,j})$ coincide with $\Sigma_1^{T_0}$ away from the singularities of $\tilde{c}_{1,j}$. In particular, thanks to Lemma 6.16, the expected equality holds away from these singularities. Hence, we only need to understand what happen in a neighborhood of such a singularity q . Thanks to Lemma 6.16, the point q belongs to exactly two curves $\tilde{c}_{1,j}$ and $\tilde{c}_{1,j+1}$ for some $j \geq 1$. In a neighborhood of this point, the current $[\Sigma_1^{T_0}]$ is the current of integration over the two curves

$$\{(\tilde{c}_1(t_1 + t), (\tilde{c}'_1(t_1 + t)^b)^\perp) : t \in (-\epsilon, \epsilon)\},$$

and

$$\{(\tilde{c}_1(t_2 + t), (\tilde{c}'_1(t_2 + t)^b)^\perp) : t \in (-\epsilon, \epsilon)\},$$

where $t_1 \neq t_2$ are such that $\tilde{c}_1(t_1) = \tilde{c}_1(t_2) = q$. Equivalently, it is the current of integration over the curves

$$\{(\tilde{c}_{1,j}(t), (\tilde{c}'_{1,j}(t)^b)^\perp) : t \in (-\epsilon, \epsilon) \setminus \{0\}\},$$

and

$$\{(\tilde{c}_{1,j+1}(t), (\tilde{c}'_{1,j+1}(t)^b)^\perp) : t \in (-\epsilon, \epsilon) \setminus \{0\}\},$$

oriented by the orientation induced by the one of \tilde{c}_1 . Note that we took the convention that $\tilde{c}_{1,j}(0) = \tilde{c}_{1,j+1}(0) = q$ and that there are four piece of curves.

Now, if we consider the current of integration over $N_1^*(X_{1,j})$ near q , it is given by the current of integration over the curve

$$\{(\tilde{c}_{1,j}(t), (\tilde{c}'_{1,j}(t)^b)^\perp) : t \in (-\epsilon, \epsilon) \setminus \{0\}\},$$

oriented by \tilde{c}_1 and over the curve above q which joins $(\tilde{c}'_{1,j}(0-)^b)^\perp$ and $(\tilde{c}'_{1,j}(0+)^b)^\perp$ as explained above. Equivalently, it is given by the current of integration along the following three pieces of curves:

$$\{(\tilde{c}_1(t_1 + t), (\tilde{c}'_1(t_1 + t)^b)^\perp) : t \in (-\epsilon, 0)\},$$

$$\{(\tilde{c}_1(t_2 + t), (\tilde{c}'_1(t_2 + t)^b)^\perp) : t \in (0, \epsilon)\},$$

and the curve above q which joins $(\tilde{c}'_1(t_1)^b)^\perp$ and $(\tilde{c}'_1(t_2)^b)^\perp$ as before. Similarly, for $j + 1$, we find that the current of integration over $N_1^*(X_{1,j+1})$ near q is given by the current of integration over the following three distinct pieces of curves:

$$\begin{aligned} & \{(\tilde{c}_1(t_1 + t), (\tilde{c}'_1(t_1 + t)^b)^\perp) : t \in (0, \epsilon)\}, \\ & \{(\tilde{c}_1(t_2 + t), (\tilde{c}'_1(t_2 + t)^b)^\perp) : t \in (-\epsilon, 0)\}, \end{aligned}$$

and the curve above q which joins $(\tilde{c}'_1(t_2)^b)^\perp$ and $(\tilde{c}'_1(t_1)^b)^\perp$ as before. Hence, the contributions coming from the regular points of $N_1^*(X_{1,j}) \cup N_1^*(X_{1,j+1})$ and the points of $\Sigma_1^{T_0}$ coincide. While above the singular point q , the contributions of $N_1^*(X_{1,j})$ and $N_1^*(X_{1,j+1})$ compensate each other. Summing all these contributions, we find that, in a neighborhood of q , one has

$$[\Sigma_1^{T_0}] = [N_1^*(X_{1,j})] + [N_1^*(X_{1,j+1})],$$

which allows us to conclude. \square

6.4.5. Consequence of the decomposition for the linking numbers. Let us summarize the situation so far and fix some notations for the sequel. We started from our two geodesic representatives c_1 and c_2 and we applied the geodesic flow to their Legendrian lifts $\Sigma(c_1)$ and $\Sigma(c_2)$. This gives rise to two curves \tilde{c}_1 and \tilde{c}_2 that are homotopic to c_1 and c_2 and to a new pair of Legendrian knots $\Sigma_1^{T_0}$ and $\Sigma_2^{-T'_0}$. Then, we decomposed the curves \tilde{c}_1 and \tilde{c}_2 as an union of embedded, closed curves which are only piecewise smooth. In terms of De Rham currents, it reads

$$\forall i \in \{1, 2\}, \quad [\tilde{c}_i] = \sum_{j=1}^{N_i} [\tilde{c}_{i,j}],$$

where each $\tilde{c}_{i,j}$ is the oriented boundary of some surface $X_{i,j}$ with piecewise smooth boundary. In terms of currents, we have $[\tilde{c}_{i,j}] = -d[X_{i,j}] = \partial[X_{i,j}]$. Then, we defined the (unit) conormal bundle $N_1^*(X_{i,j})$ to each surface $X_{i,j}$. This conormal bundle is in fact a Legendrian knot in S^*X (again piecewise smooth) and we denote it by $\Sigma_{i,j}$. This yields the following decompositions of our initial Legendrian knots:

$$(48) \quad [\Sigma_1^{T_0}] = \sum_{j=1}^{N_1} [\Sigma_{1,j}] \quad \text{and} \quad [\Sigma_2^{-T'_0}] = \sum_{j=1}^{N_2} [\Sigma_{2,j}],$$

We can rewrite the quantity we are interested in as

$$\mathbf{L}(c_1, c_2) = \sum_{j=1}^{N_1} \int_{S^*X} [\Sigma_{1,j}] \wedge R_2^{-T'_0}.$$

Hence, it remains to evaluate the “linking number” associated with every elementary piece $\Sigma_{1,j}$. One more time, we still note that we need to justify that $[\Sigma(c_2)]$ is exact as soon as c_2 is homologically trivial. To that aim, it is sufficient to justify that $\Sigma_{2,j}$ is exact for every $1 \leq j \leq N_2$.

Before continuing the proof, let us state a microlocal statement reformulating part of the Theorem 6.3 we are aiming at and summarizing what we already did. We reformulate

it in terms of constructible functions in order to clarify the relation of our constructions with the microlocal index Theorems discussed in the introduction:

Theorem 6.22. *Suppose that \mathbf{c}_1 and \mathbf{c}_2 are homologically trivial. Let c_1 and c_2 be two of their geodesic representatives and let \tilde{c}_1 and \tilde{c}_2 be the two (small) homotopic deformations given by Proposition 6.13.*

Then there exists a pair (f_1, f_2) of constructible functions associated to $(\tilde{c}_1, \tilde{c}_2)$ such that that

$$(49) \quad \sum_{j=1}^{\infty} [N_1^* (\{f_i \geq j\})] = [\Sigma(\tilde{c}_i)],$$

where the equality holds in the sense of De Rham currents. Moreover, the linking of the Legendrians is given by the formula

$$(50) \quad \mathbf{L}(c_1, c_2) = -\frac{\chi(f_1)\chi(f_2)}{\chi(X)} + \chi(f_1 f_2) - \frac{1}{2}\chi(\mathbf{1}_{\tilde{c}_1 \cap \tilde{c}_2}).$$

We note that, in the case where one of the \mathbf{c}_i is trivial in $\pi_1(X)$, our curves are chosen so that $\tilde{c}_1 \cap \tilde{c}_2 = \emptyset$. In the terminology of the introduction, we say that the constructible function f_i is quantizing the Legendrian knot $\Sigma(\tilde{c}_i)$, where $\Sigma(\tilde{c}_1)$ (resp. $\Sigma(\tilde{c}_2)$) is the Legendrian knot $\Sigma_1^{T_0}$ (resp. $\Sigma_2^{-T'_0}$) with the conventions of paragraph 3.3.1.

6.4.6. Smoothing each elementary piece. When we removed the selfintersections of our initial (smooth) curves \tilde{c}_1 and \tilde{c}_2 , we introduced some families of embedded curves that are only piecewise smooth. We would now like to regularize these new curves without affecting the linking number we want to compute. Let $i = 1, 2$ and let $1 \leq j \leq N_i$. Recall that $\tilde{c}_{i,j}$ are finite union of simple, closed, piecewise smooth curves and that they are not necessarily connected. Before smoothing the curves, let us begin with a few observations on the wavefront sets of the currents $[\Sigma_{i,j}]$. First, by construction of $\Sigma_{i,j} = N_1^*(X_{i,j})$ (see § 6.4.4), we have

$$\bigcup_{j=1}^{N_1} \text{supp}([\Sigma_{1,j}]) \subset \Sigma_1^{T_0} \cup \left(\bigcup_{a \in \text{Cross}(\tilde{c}_1)} S_a^* X \right) \text{ and } \bigcup_{j=1}^{N_2} \text{supp}([\Sigma_{2,j}]) \subset \Sigma_2^{-T'_0} \cup \left(\bigcup_{a \in \text{Cross}(\tilde{c}_2)} S_a^* X \right)$$

since we added some subset of the cotangent fibers over the selfintersection points $\text{Cross}(\tilde{c}_i)$ of \tilde{c}_i . Still, by construction of $\Sigma_{i,j} = N_1^*(X_{i,j})$ (see § 6.4.4), the following holds:

Lemma 6.23. *If $\tilde{c}_{i,j}$ has k singular points, then $\Sigma_{i,j}$ is itself a piecewise smooth curve with exactly $2k$ singular points which are isolated. Over each singular point a of $\tilde{c}_{i,j}$, there are exactly two singular points of $\Sigma_{i,j}$.*

We denote by $\text{Sing}(\Sigma_{i,j})$ this finite subset of singular points. In terms of wavefront sets, this allows to give the simple upper bound:

$$\forall (i, j), \text{WF}([\Sigma_{i,j}]) \subset \bigcup_{j=0}^{N_i} N^* \Sigma_{i,j} \subset N^* \Sigma_i \cup \left(\bigcup_{a \in \text{Cross}(\tilde{c}_i)} N^*(S_a^* X) \right) \cup \bigcup_{b \in \text{Sing}(\Sigma_{i,j})} T_b^*(S^* X) \setminus 0,$$

where we dropped the dependence in T_0 and T'_0 for $\Sigma_1 = \Sigma_1^{T_0}$ and $\Sigma_2 = \Sigma_2^{-T'_0}$. In order to smooth the curves $\tilde{c}_{i,j}$ near their singularities, we fix some conic neighborhood Γ_i of

$$N^*\Sigma_i \cup \left(\bigcup_{a \in \text{Cross}(\tilde{c}_i)} N^*(S_a^*X) \right) \cup \bigcup_{b \in \text{Sing}(\Sigma_{i,j})} T_b^*(S^*X) \setminus 0.$$

We begin by observing that

Lemma 6.24. *There exists Γ_1, Γ_2 some closed conic subsets of $T^*(S^*X) \setminus \underline{0}$ s.t. Γ_i is a conic neighborhood of*

$$N^*\Sigma_i \cup \left(\bigcup_{a \in \text{Cross}(\tilde{c}_i)} N^*(S_a^*X) \right) \cup \bigcup_{b \in \text{Sing}(\Sigma_{i,j})} T_b^*(S^*X) \setminus 0$$

and

$$(51) \quad \Gamma_1 \cap \Gamma_2 = \emptyset.$$

Proof. Thanks to the hypothesis following (43), the selfintersection points of \tilde{c}_1 do not meet \tilde{c}_2 and conversely the selfintersection points of \tilde{c}_2 do not meet \tilde{c}_1 . Moreover the intersection of both curves are transverse. It means that the following intersection is disjoint

$$\left(\Sigma_1 \cup \left(\bigcup_{a \in \text{Cross}(\tilde{c}_1)} S_a^*X \right) \right) \cap \left(\Sigma_2 \cup \left(\bigcup_{a \in \text{Cross}(\tilde{c}_2)} S_a^*X \right) \right) = \emptyset.$$

Thus the union of supports $\bigcup_{j=1}^{N_1} \text{supp}([\Sigma_{1,j}])$, $\bigcup_{j=1}^{N_2} \text{supp}([\Sigma_{2,j}])$ are disjoint. Since the projection on S^*X of the wave front set of a current in $\mathcal{D}'(S^*X)$ is contained in the support of the current, this implies that one can choose Γ_1, Γ_2 with the expected properties. \square

We now turn to the smoothing of our curves:

Lemma 6.25. *One can construct a family of smooth curves $(\tilde{c}_{i,j}^m)_{m \geq 1, i \in \{1,2\}, 1 \leq j \leq N_i}$ on X with the following properties:*

- for every t , $\|(\tilde{c}_{i,j}^m)'(t)\| = 1$,
- $[\tilde{c}_{i,j}^m]$ converges weakly to $[\tilde{c}_{i,j}]$ in $\mathcal{D}^1(X)$,
- $[\tilde{c}_{i,j}^m] = [\tilde{c}_{i,j}]$ outside of some neighborhood (depending on m) of the singularities of $\tilde{c}_{i,j}$;
- one can attach above each point $\tilde{c}_{i,j}^m(t)$ of the curve, its normalized conormal vector $((\tilde{c}_{i,j}^m)'(t)^b)^\perp$ so that the closed curve $t \mapsto (\tilde{c}_{i,j}^m(t), ((\tilde{c}_{i,j}^m)'(t)^b)^\perp)$ is smooth and if we denote by $\Sigma_{i,j,m}$ the image of this curve in S^*X , then one has $[\Sigma_{i,j,m}] = [\Sigma_{i,j}]$ away from the singularities and, as $m \rightarrow +\infty$,

$$[\Sigma_{i,j,m}] \rightarrow [\Sigma_{i,j}],$$

in $\mathcal{D}_{\Gamma_i}^2(S^*X)$.

We refer to the Appendix A.1 for a reminder on the topology on $\mathcal{D}'_{\Gamma_i}(S^*X)$

Proof. Up to the fact that we may have to reparametrize the curve (and up to using a partition of unity), we note that we only need to modify the curve $\tilde{c}_{i,j}$ in a small neighborhood of its singularities. The point is that we will round the “corners” of the bounding surface $X_{i,j}$ to make the curve $\tilde{c}_{i,j}^m$ smooth.

Thanks to assumption (43), we note that, in a small chart $(\tilde{q}_1, \tilde{q}_2)$ near a point q_0 at some corner of $X_{i,j}$, $X_{i,j}$ is given by $\{\tilde{q}_1 \leq 0, \tilde{q}_2 \geq 0\}$ or $\{\tilde{q}_1 \leq 0\} \cup \{\tilde{q}_2 \geq 0\}$ (with the usual orientations on \mathbb{R}^2). We only discuss the first case and the second case is treated in a similar way. In this local chart, the boundary of $X_{i,j}$ near the singular point has a local piecewise smooth parametrization which reads

$$t \in [-1, 1] \mapsto \gamma(t) = (-t\mathbf{1}_{[-1,0]}(t), 0) + (0, t\mathbf{1}_{[0,1]}(t)) \in \mathbb{R}^2.$$

Observe that

$$\begin{aligned} \tilde{\gamma}_m(t) &:= \left((t\mathbf{1}_{[-1, -\frac{1}{m}]}(t), 0) + \left(0, t\mathbf{1}_{[\frac{1}{m}, 1]}(t) \right) \right) \\ &+ \left(\frac{1}{m} \cos\left(\frac{m\pi t}{4} - \frac{\pi}{4}\right) - \frac{1}{m}, \frac{1}{m} \sin\left(\frac{m\pi t}{4} - \frac{\pi}{4}\right) + \frac{1}{m} \right) \mathbf{1}_{[-\frac{1}{m}, \frac{1}{m}]}(t) \end{aligned}$$

is a \mathcal{C}^1 -path, which is only piecewise \mathcal{C}^∞ , and lies in some $\frac{1}{m}$ -neighborhood of γ . Hence $\tilde{\gamma}_m$ bounds the domain

$$\left\{ \tilde{q}_1 \leq -\frac{1}{m}, \tilde{q}_2 \geq 0 \right\} \cup \left\{ \tilde{q}_2 \geq \frac{1}{m}, \tilde{q}_1 \leq 0 \right\} \cup \left\{ \left(\tilde{q}_1 + \frac{1}{m} \right)^2 + \left(\tilde{q}_2 - \frac{1}{m} \right)^2 \leq \frac{1}{m^2} \right\}$$

which has \mathcal{C}^1 boundary. Hence, instead of the corner point $\{(0, 0)\}$, we obtained a quarter circle. Now we fix $\chi \in C^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} \chi = 1, \chi \geq 0, \text{supp}(\chi) \subset [-\frac{1}{2}, \frac{1}{2}]$ and we define $\chi_m(\tilde{q}_1, \tilde{q}_2) = \frac{1}{m^2} \chi(m\tilde{q}_1, m\tilde{q}_2)$. We can define the new parametrization $\gamma_m = \tilde{\gamma}_m * \chi_m \in \mathbb{R}^2$ obtained by convolution. This new curve γ_m converges to $\tilde{\gamma}_m$ in the \mathcal{C}^1 -topology and the image of both curves coincide outside some $\frac{4}{m}$ -neighborhood of the corner point $(0, 0)$.

Define $X_{i,j}^m$ to be the new surface obtained from the above smoothing procedure at every corner point, this is a manifold with smooth boundary which is homotopic to $\partial X_{i,j}$ by construction. Proceeding like this, one can verify that the first three properties are satisfied locally near the singular point (and thus globally via a partition of unity).

Regarding now the last property, we are in fact taking the (oriented and normalized) conormal to the curve $t \mapsto \tilde{c}_{i,j}^m(t)$. By construction, it has the expected properties away from the singularities of the initial curve. Near the singularity, one can write the above expression in local coordinates in \mathbb{R}^2 as above and verify that the current of integration along the curve

$$t \in [-1, 1] \mapsto \left(\gamma_m(t), \frac{(\gamma'_m(t)^b)^\perp}{\|(\gamma'_m(t)^b)^\perp\|} \right)$$

converges to the current of integration along

$$\begin{aligned} N^*(\{\tilde{q}_1 \leq 0, \tilde{q}_1 \geq 0\}) &= \{(t, 0; 0, 1); t \in (-1, 0]\} \cup \{(0, t; -1, 0); t \in [0, 1)\} \\ &\cup \left\{ (0, 0; \cos(\theta), \sin(\theta)); \theta \in \left[\frac{\pi}{2}, \pi\right] \right\}. \end{aligned}$$

The only discussion is near the corner point. By construction, we see that the conormal lift of the \mathcal{C}^1 -curve $\tilde{\gamma}_m$ which is the map

$$t \in [-1, 1] \mapsto \left(\tilde{\gamma}_m(t), \frac{(\tilde{\gamma}'_m(t)^b)^\perp}{\|(\tilde{\gamma}'_m(t)^b)^\perp\|} \right)$$

converges in the sense of currents to the conormal $N^*(\{\tilde{q}_1 \leq 0, \tilde{q}_2 \geq 0\})$. Since γ_m is \mathcal{C}^1 close to $\tilde{\gamma}_m$ for m large enough, we are done. \square

By construction, we remark that, for every m large enough, the curve $\tilde{c}_{i,j}^m$ bounds a compact surface $X_{i,j}^m$ with smooth boundary which has the same topology as $X_{i,j}$. In particular, for m large enough, for $i \in \{1, 2\}$ and for $1 \leq j \leq N_i$

$$(52) \quad \chi(X_{i,j}) = \chi(X_{i,j}^m).$$

Similarly, recall that the singularities of $\partial X_{1,j}$ and $\partial X_{2,j'}$ are away from each other by construction of \tilde{c}_1 and \tilde{c}_2 . Thus, one finds that, for m, m' large enough, for $1 \leq j \leq N_1$ and for $1 \leq j' \leq N_2$,

$$(53) \quad \chi(X_{1,j} \cap X_{2,j'}) = \chi(X_{1,j}^m \cap X_{2,j'}^{m'}) \quad \text{and} \quad \chi(\partial X_{1,j} \cap \partial X_{2,j'}) = \chi(\partial X_{1,j}^m \cap \partial X_{2,j'}^{m'}).$$

6.4.7. Back to linking numbers. We now come back to our computation of the linking number $\mathbf{L}(c_1, c_2)$ and we recall that we still have to prove that $[\Sigma_{2,j}]$ (and also $[\Sigma_{1,j}]$) is exact. To that aim, we have the following:

Lemma 6.26. *Suppose that $[\Sigma_{2,j,m}]$ is exact for m large enough. Then, for m large enough, $[\Sigma_{2,j,m}] = dR_{2,j,m}$ with $R_{2,j,m} \in \mathcal{D}'_{\Gamma_2}(S^*X)$ and $[\Sigma_{2,j}]$ is exact.*

Hence, to show the exactness of $[\Sigma_{2,j}]$ for every $0 \leq j \leq N_2$ (thus of $[\Sigma(c_2)]$), it is sufficient to show that, for $m \geq 1$ large enough, $[\Sigma_{2,j,m}]$ is exact. Even if this property more or less follows from the proof in §6.3 (as $\tilde{c}_{2,j}^m$ is smooth), we will revisit this argument below in a way that will be convenient for us in the end of our proof.

Proof. Let $1 \leq j \leq N_2$ and let $m \geq 1$ be large enough to ensure that $[\Sigma_{2,j,m}]$ is exact. From the elliptic properties of the Hodge-De Rham Laplacian $\Delta_{\tilde{g}^1} = dd^* + d^*d$, we know that $(\Delta_{\tilde{g}^1} + z)^{-1} : \mathcal{D}'_{\Gamma_2}(S^*X) \rightarrow \mathcal{D}'_{\Gamma_2}(S^*X)$ has a meromorphic continuation from $\{\operatorname{Re}(z) > 0\}$ to the whole complex plane with the poles given by the eigenvalues of $-\Delta_{\tilde{g}^{(1)}}$. In particular, in a neighborhood of 0, the resolvent reads

$$(\Delta_{\tilde{g}^1} + z)^{-1} = \frac{\mathbf{1}_0(\Delta_{\tilde{g}^1})}{z} + R(z),$$

where $R(z)$ is some holomorphic function and where $\mathbf{1}_0(\Delta_{\tilde{g}^1})$ is the spectral projector on the eigenvalue 0 (whose range generates the De Rham cohomology). We now proceed as in

the proof of Proposition 6.1. In particular, if $[\Sigma_{2,j,m}]$ is De Rham exact, then we can write

$$\begin{aligned} [\Sigma_{2,j,m}] &= \mathbf{1}_0(\Delta_{\tilde{g}^1})[\Sigma_{2,j,m}] + (\text{Id} - \mathbf{1}_0(\Delta_{\tilde{g}^1}))[\Sigma_{2,j,m}] \\ &= (\text{Id} - \mathbf{1}_0(\Delta_{\tilde{g}^1}))[\Sigma_{2,j,m}] \\ &= \lim_{z \rightarrow 0} (dd^* + d^*d + z)R(z) (\text{Id} - \mathbf{1}_0(\Delta_{\tilde{g}^1}))[\Sigma_{2,j,m}] \\ &= \lim_{z \rightarrow 0} dd^*R(z) (\text{Id} - \mathbf{1}_0(\Delta_{\tilde{g}^1}))[\Sigma_{2,j,m}], \end{aligned}$$

where we used that d commutes with $\Delta_{\tilde{g}^1}$ (thus with its spectral projectors and $R(z)$). As $R(z)$ is a holomorphic family of pseudodifferential operators of order -2 , by composition we find that $d^*R(z) (\text{Id} - \mathbf{1}_0(\Delta_{\tilde{g}^1}))$ is a holomorphic family of pseudodifferential operators of order -1 and we can write $[\Sigma_{2,j,m}] = dR_{2,j,m}$ with $R_{2,j,m} \in \mathcal{D}_{\Gamma_2}^1(S^*X)$. Moreover, $d^*R(z) (\text{Id} - \mathbf{1}_0(\Delta_{\tilde{g}^1}))$ is continuous on currents $\mathcal{D}_{\Gamma_2}^1(S^*X) \rightarrow \mathcal{D}_{\Gamma_2}^1(S^*X)$, uniformly near $z = 0 \in \mathbb{C}$. Thus, as $[\Sigma_{2,j,m}]$ converges to $[\Sigma_{2,j}]$ in $\mathcal{D}_{\Gamma_2}^1(S^*X)$, we find by passing to the limit in the above equation that $[\Sigma_{2,j}] = dR_{2,j}$ with $R_{2,j} \in \mathcal{D}_{\Gamma_2}^1(S^*X)$. \square

Using now that the wavefront set of $R_2^{-T'_0}$ is disjoint from Γ_1 by Proposition 6.1 and Lemma 6.24, we obtain

$$\mathbf{L}(c_1, c_2) = \sum_{j=1}^{N_1} \lim_{m \rightarrow +\infty} \int_{S^*X} [\Sigma_{1,j,m}] \wedge R_2^{-T'_0},$$

where we use the continuity property of the wedge product on $\mathcal{D}'_{\Gamma}(M)$ – see Appendix A.2. Similarly, if we suppose that $[\Sigma_{2,j',m'}]$ is exact for m' large enough, then we find thanks to Lemmas 6.26 and 6.24:

$$(54) \quad \mathbf{L}(c_1, c_2) = \sum_{j=1}^{N_1} \sum_{j'=1}^{N_2} \lim_{m \rightarrow +\infty} \lim_{m' \rightarrow +\infty} \int_{S^*X} [\Sigma_{1,j,m}] \wedge R_{2,j',m'}.$$

Hence, modulo the fact, that we still have to prove the exactness of $[\Sigma_{2,j',m'}]$ for every (j', m') , we have reduced our problem to the computation of the linking number in the general case to the case of two smooth embedded curves $\tilde{c}_{1,j}^m$ and $\tilde{c}_{2,j'}^{m'}$ as in our two warm-up examples. In the sequel, we shall thus compute the value of

$$\mathbf{L}(\tilde{c}_{1,j}^m, \tilde{c}_{2,j'}^{m'}) := \int_{S^*X} [\Sigma_{1,j,m}] \wedge R_{2,j',m'},$$

for every (j, j') and for every $m, m' \geq 1$ large enough. A notable difference with our warm up examples is that the two curves $\tilde{c}_{1,j}^m$ and $\tilde{c}_{2,j'}^{m'}$ may intersect each other (transversally).

6.4.8. Exactness of $[\Sigma_{i,j,m}]$. We now fix $1 \leq j \leq N_1$, $1 \leq j' \leq N_2$ and $m, m' \geq 1$ large enough to ensure that (52) and (53) hold. We want to prove that $[\Sigma_{i,j,m}]$ is exact for $m \geq 1$ large enough and to compute

$$\mathbf{L}(\tilde{c}_{1,j}^m, \tilde{c}_{2,j'}^{m'}) = \int_{S^*X} [\Sigma_{1,j,m}] \wedge R_{2,j',m'}.$$

We let $Y_{j,m} \in \Gamma^\infty(TX)$ be a smooth vector field which is positively colinear to the normalized normal vector $(\tilde{c}_{1,j}^m)'(t)^\perp$ above each point of the smooth curve $\tilde{c}_{1,j}^m(t)$. It does not matter if the close curve $\tilde{c}_{1,j}^m$ has several connected components, the only thing which is needed here is the smoothness of the curve. Without loss of generality, we can suppose that this vector field has finitely many zeroes and that they are all of real hyperbolic type. We denote these critical points by $\text{Crit}(Y_{j,m})$.

Remark 6.27. A way to construct this vector field goes as follows. Take \tilde{f} to be a smooth function which is constant on $\tilde{c}_{1,j}^m$ and whose gradient vector field is positively colinear to $(\tilde{c}_{1,j}^m)'(t)^\perp$. This implies that $\nabla_g \tilde{f}$ has no critical points in some neighborhood of $\tilde{c}_{1,j}^m$. By density of Morse functions in the \mathcal{C}^∞ -topology, we can find arbitrarily close to \tilde{f} a smooth Morse function f . In particular, its gradient vector field has now finitely many critical points which are all of real-hyperbolic type and which are away from $\tilde{c}_{1,j}^m = \partial X_{1,j}^m$. The gradient vector field $\nabla_g f$ may not be normal to $\tilde{c}_{1,j}^m$ anymore. Take some \mathcal{C}^∞ cut-off function such that $\chi = 1$ near $\tilde{c}_{1,j}^m$ and such that χ is supported in some small tubular neighborhood of $\tilde{c}_{1,j}^m$. Then $h = \chi \tilde{f} + (1 - \chi)f$ is arbitrarily \mathcal{C}^1 close to both f and \tilde{f} . The function h is \mathcal{C}^1 close to \tilde{f} hence we can choose f and χ in such a way that h has no critical points in the support of χ . So all the critical points of h are in the region where $\chi = 0$ which is the region where $h = f$ hence h is Morse. Finally $Y_{j,m} = \nabla_g h$ does the job.

One can in fact ensure that all the fixed points of $Y_{j,m}|_{X_{1,j}^m}$ are away from the curve \tilde{c}_2 :

Lemma 6.28. *Let $1 \leq j \leq N_1$ and let $m \geq 1$ be large enough. One can smoothly deform $Y_{j,m}$ in the interior of $X_{1,j}^m$ so that*

- *the new vector field has the same number of critical points in $X_{1,j}^m$,*
- *all its critical points in $X_{1,j}^m$ are still of real hyperbolic type,*
- *they do not lie in a small neighborhood of $X_{1,j}^m \cap \tilde{c}_2$.*

In particular, the critical points of $Y_{j,m}$ are away from $\tilde{c}_{2,j'}^{m'} = \partial X_{2,j'}^{m'}$ for m' large enough.

Proof. Let us modify $Y_{j,m}$ into a vector field Y with the expected properties. We only need to discuss the critical points that belong to \tilde{c}_2 . For any such point a , one can associate $b(a) \in X_{1,j}^m \setminus \tilde{c}_2$. We make the assumption that for every critical points $a \neq a'$ lying on \tilde{c}_2 , we choose $b(a) \neq b(a')$. To every pair of points $(a, b(a))$, we associate a smooth curve (with no selfintersection points) γ_a joining a to $b(a)$ and such that γ_a and $\gamma_{a'}$ do not intersect each other if $a \neq a'$. One can then find tubular neighborhoods O_a of these curves which are diffeomorphic to $\mathbb{R} \times (0, 1)$, which lie inside the interior of $X_{1,j}^m$ and which do not intersect each other. On each of these neighborhoods, one can build a diffeomorphism κ_a which sends a to $b(a)$ which is equal to the identity near the boundary of O_a . Gluing these “local” diffeomorphisms together by taking the identity outside the O_a yields a global diffeomorphism κ . Taking Y to be the pullback of $Y_{j,m}$ under κ , we find a vector field with the expected properties. \square

Remark 6.29. Let us collect a few useful properties that we can impose to our vector field by modifying it in a small neighborhood $\partial X_{2,j'}^{m'} \cap X_{1,j}^m$.

- (1) We note that, thanks to Proposition 6.13 and to Lemma 6.24, the piece of surface

$$q \mapsto \left(q, \frac{Y_{j,m}(q)^b}{\|Y_{j,m}(q)^b\|} \right)$$

does not intersect $\Sigma_2^{-T'_0}$ in a small neighborhood of $\partial X_{1,j}^m$. If \mathbf{c}_2 is trivial, then we know that we can choose $T'_0 = 0$ and this piece of surface intersects $\Sigma_2^{-T'_0}$ at one single point (if $c_2 \in X_{1,j}$). In that case, the intersection is transversal – see Remark 6.9. If \mathbf{c}_1 is nontrivial, then, outside this small neighborhood, we can slightly modify the vector field $Y_{j,m}$ (in a small neighborhood of $\partial X_{2,j'}^{m'} \cap X_{1,j}^m$) so that it intersects the normal of $\partial X_{2,j'}^{m'} \cap X_{1,j}^m$ at finitely many points. In that case, the new vector field may depend on (j', m') . We will denote it by $Y_{j,j',m,m'}$ and the corresponding surface by $S_{j,j',m,m'}$.

- (2) Now, if we consider an intersection point between the piece of surface $S_{j,j',m,m'}$ and $\Sigma_{2,j',m'}$, we would like to have a transverse intersection at these intersection points. This would ensure that the product between the currents of integration is well-defined. If \mathbf{c}_2 is trivial, we saw that it is automatically satisfied. When \mathbf{c}_2 is nontrivial, let us briefly explain how it can be ensured up to modifying slightly the vector field. Fix a point $x_0 = (q_0, p_0)$ where the piece of surface intersects $\Sigma_{2,j',m'}$. Let $(U_0 \subset X, \kappa_0)$ be a small chart near q_0 . We can choose local coordinates $(\tilde{q}_1, \tilde{q}_2)$ such that $X_{2,j'}^{m'}$ is given in this local chart by the local coordinates of § 3.3.2, i.e. $X_{2,j'}^{m'} := \{(\tilde{q}_1, \tilde{q}_2) : \tilde{q}_2 \geq 0\}$. In this local chart, we know that $Y_{j,j',m,m'}(0,0)$ is proportional to $\partial_{\tilde{q}_2}$ and that it points inside $X_{2,j'}^{m'}$. Hence locally, up to multiplying the vector field by a positive constant near 0, it reads $Y_{j,j',m,m'}(\tilde{q}) = \partial_{\tilde{q}_2} + f_1(\tilde{q})\partial_{\tilde{q}_1} + f_2(\tilde{q})\partial_{\tilde{q}_2}$ with $f_1(0) = f_2(0) = 0$. If we perturb the vector field in such a way that $\partial_{\tilde{q}_1}f_1(0) \neq 0$, then, we get a transversal intersection at 0. By partition of unity, we can use this procedure to make the intersection transversal at every intersection point.
- (3) Suppose that \mathbf{c}_2 is nontrivial. We can lift the local chart as a map $S^*U_0 \rightarrow \mathbb{R}^2 \times \mathbb{S}^1$. We saw that the transversality at 0 means that $\partial_{\tilde{q}_1}f_1(0) \neq 0$. In the following, we aim at intersecting $[S_{j,j',m,m'}]$ and $[\Sigma_{2,j'}^{m'}]$ which is well defined thanks to our transversality assumption. In particular, $\int_{S^*U_0} [S_{j,j',m,m'}] \wedge [\Sigma_{2,j'}^{m'}] = \pm 1$. In order to determine the sign of this intersection, recall from § 3.3.2 that

$$[\Sigma_{2,j'}^{m'}] = \delta_0(\tilde{q}_2)\delta_0\left(\phi - \frac{\pi}{2}\right) d\tilde{q}_2 \wedge d\phi.$$

Similarly, one has locally

$$S_{j,j',m,m'} := \left\{ \left(\tilde{q}_1, \tilde{q}_2, \phi(\tilde{q}) := \arccos \left(\frac{f_1(\tilde{q})}{\sqrt{f_1(\tilde{q})^2 + (1 + f_2(\tilde{q}))^2}} \right) \right) \right\}.$$

Hence, $[S_{j,j',m,m'}] = \delta_0(\phi - \phi(\tilde{q}))(d\phi - \partial_{\tilde{q}_1}\phi(\tilde{q})d\tilde{q}_1 - \partial_{\tilde{q}_2}\phi(\tilde{q})d\tilde{q}_2)$. Taking the product of these two currents of integration, we find that

$$\begin{aligned} \int_{S^*U_0} [S_{j,j',m,m'}] \wedge [\Sigma_{2,j'}^{m'}] &= - \int_{\mathbb{R}^2 \times S^1} \delta_0(\tilde{q}_2) \delta_0\left(\phi - \frac{\pi}{2}\right) \delta_0(\phi - \phi(\tilde{q}_1)) \partial_{\tilde{q}_1}\phi(\tilde{q}) d\tilde{q}_1 d\tilde{q}_2 d\phi \\ &= - \frac{\partial_{\tilde{q}_1}\phi(0)}{|\partial_{\tilde{q}_1}\phi(0)|}. \end{aligned}$$

Hence, it is equal to 1 if $\partial_{\tilde{q}_1}f_1(0) > 0$ and to -1 otherwise.

- (4) We note that all these modifications of the vector field are performed in a small neighborhood $\tilde{c}_{2,j'}^{m'} \cap X_{1,j}^m$ where the vector field has no critical points. In particular, they do not affect the critical points of the vector field which remains the same for every $m' \geq 1$ and every $1 \leq j' \leq N_2$.

As in our preliminary examples, we form the following surface in S^*X :

$$S_{j,j',m,m'} := \left\{ \left(q, \frac{Y_{j,j',m,m'}(q)^b}{\|Y_{j,j',m,m'}(q)^b\|} \right) : q \in X_{1,j}^m \setminus \text{Crit}(Y_{j,j',m,m'}) \right\}.$$

Using Lemma 6.7, this surface gives rise to a well-defined current on S^*X and, combining with §3.3.2, one has in the sense of currents:

$$d[S_{j,j',m,m'}] = [\Sigma_{1,j,m}] - \sum_{a \in X_{1,m} \cap \text{Crit}(Y_{j,j',m,m'})} (-1)^{\text{ind}(a)} [S_a^*X].$$

Since each current $[S_a^*X]$ is De Rham exact with primitive R_a by Lemmas 6.4 and 6.5, we immediately deduce that $[\Sigma_{1,j,m}]$ is exact for every $1 \leq j \leq N_1$ and every large enough m . The same argument applied to $i = 2$ shows that, for every $m' \geq 1$ and for every $1 \leq j' \leq N_2$, the current $[\Sigma_{2,j',m'}]$ is exact. Thanks to Lemma 6.26, this implies that both $[\Sigma(c_1)]$ and $[\Sigma(c_2)]$ are exact and this concludes the first part of Theorem 6.3.

Thanks to Remark 6.29, the intersection between $\Sigma_{2,j',m'}$ and $S_{j,j',m,m'}$ is transverse. Hence, one finds that, for $1 \leq j \leq N_1$, for $1 \leq j' \leq N_2$, for $m, m' \geq 1$ large enough,

$$(55) \quad \mathbf{L}\left(\tilde{c}_{1,j}^m, \tilde{c}_{2,j'}^{m'}\right) = \int_{S^*X} [S_{j,j',m,m'}] \wedge [\Sigma_{2,j',m'}] + \sum_{a \in X_{1,j}^m \cap \text{Crit}(Y_{j,j',m,m'})} (-1)^{\text{ind}(a)} \int_{S^*X} [\Sigma_{2,j',m'}] \wedge R_a,$$

where $dR_a = [S_a^*X]$. We note from Remark 6.29 that the terms coming from the first term on the right-hand side defines an integer by construction (that may depend on $m, m' \geq 1$). It remains to compute the two terms on the right-hand side and we will in fact show that they are independent of m and m' (at least if they are large enough). As in our two warm-up examples, we begin with the case where c_2 is a point and then use this case to handle the general case.

6.4.9. Conclusion when c_1 has self-intersections but c_2 is a point. Let us first treat the case where \mathbf{c}_2 is a trivial homotopy class in which case the representative c_2 is a point and the Legendrian knot $\Sigma(c_2)$ is just the cotangent fiber $S_{c_2}^*X$. Hence, one can take $N_2 = 1$, $\Sigma_2^{-T'_0} = \Sigma_{2,1} = \Sigma_{2,1,m'}$, $S_{j,m} = S_{j,1,m,m'}$ and $Y_{j,m} = Y_{j,1,m,m'}$ for every $m' \geq 1$. Recall also

from Proposition 6.13 that T'_0 can be chosen equal to 0. Hence, $\Sigma_2^{-T'_0} = \Sigma(c_2)$. From Lemma 6.5, one finds that, for every $a \in \text{Crit}(Y_{j,m})$:

$$\int_{S^*X} [\Sigma_2^{-T'_0}] \wedge R_a = -\frac{1}{\chi(X)}.$$

Hence, we have a smooth vector field $Y_{j,m}$ defined in some neighborhood of some surface with boundary $X_{1,j}^m$ and pointing inward the boundary of $X_{1,j}^m$. The Poincaré-Hopf formula for surfaces with boundary [65, Th. A₀] yields:

$$\chi(X_{1,j}^m) = \sum_{a \in X_{1,j}^m \cap \text{Crit}(Y_{j,m})} (-1)^{\text{ind}(a)}.$$

Therefore, using (52), one finds that:

$$\mathbf{L}(\tilde{c}_{1,j}^m, \tilde{c}_{2,1}^{m'}) = \int_{S^*X} [S_{j,m}] \wedge [\Sigma_2^{-T'_0}] - \frac{\chi(X_{1,j})}{\chi(X)}$$

where $\chi(X(\tilde{c}_1))$ is the Euler characteristic of a surface with oriented boundary \tilde{c}_1 . Hence, it remains to discuss the value of

$$\int_{S^*X} [S_{j,m}] \wedge [\Sigma_2^{-T'_0}] = \int_{S^*X} [S_{j,m}] \wedge \varphi^{-T'_0*}([S_{c_2}^*X]).$$

Recalling the calculation of our first warm-up example in § 6.3 (which only used the fact that c_1 was smooth without selfintersection points), we find, for every m, m' large enough and for every $1 \leq j \leq N_1$,

$$\mathbf{L}(\tilde{c}_{1,j}^m, \tilde{c}_{2,1}^{m'}) = 1 - \frac{\chi(X_{1,j})}{\chi(X)} \quad \text{if } c_2 \in X_{1,j},$$

and

$$\mathbf{L}(\tilde{c}_{1,j}^m, \tilde{c}_{2,1}^{m'}) = -\frac{\chi(X_{1,j})}{\chi(X)} \quad \text{if } c_2 \notin X_{1,j}.$$

Hence, in both case, this reads as $\chi(c_2 \cap X_{1,j}) - \frac{\chi(X_{1,j})}{\chi(X)}$ and, if we sum, over $1 \leq j \leq N_1$, we obtain

$$\mathbf{L}(c_1, c_2) = \chi(X(c_2) \cap X(\tilde{c}_1)) - \frac{\chi(X(\tilde{c}_1))}{\chi(X)},$$

where in the present case, we understand $X(c_2)$ as a point and the intersection $X(c_2) \cap X(\tilde{c}_1)$ is a point with some multiplicity since $X(\tilde{c}_1)$ is a sum of oriented surfaces. This means $\chi(X(c_2) \cap X(\tilde{c}_1)) = |\{j : c_2 \in X_{1,j}\}|$ counts the number of surfaces in $(X_{1,j})_{j=1}^{N_1}$ which contain the point c_2 . In terms of constructible functions, recall from Theorem 6.21 that there is a constructible function $f_1 : X \mapsto \mathbb{Z}$ which quantizes the Legendrian $\Sigma(\tilde{c}_1)$, $df_1 = [\tilde{c}_1]$ and this integer reads $\chi(X(c_2) \cap X(\tilde{c}_1)) = f_1(c_2)$. The other term can be rewritten $\chi(X(\tilde{c}_1)) = \chi(f_1)$. Hence, we have

$$\mathbf{L}(c_1, c_2) = f_1(c_2) - \frac{\chi(f_1)}{\chi(X)}.$$

We remark that in that case $\mathbf{1}_{\tilde{c}_1 \cap c_2} = 0$ and we have indeed obtained Theorem 6.22 and thus Theorem 6.3 in that case.

6.4.10. *The conclusion for arbitrary c_2 .* We can use the result for trivial classes to conclude in the general case. We use it to deal with the term

$$\int_{S^*X} [\Sigma_{2,j',m'}] \wedge R_a$$

appearing on the right-hand side of (55). We suppose now that \mathbf{c}_2 is a nontrivial homotopy class. This quantity is exactly the linking between $\Sigma_{2,j',m'}$ and S_a^*X . We have verified in §6.4.9 that, for m' large enough,

$$\int_{S^*X} [\Sigma_{2,j',m'}] \wedge R_a = -\frac{\chi(X_{2,j'})}{\chi(X)} + \chi(X_{2,j'} \cap \{a\}),$$

where we used that no critical point a of $Y_{j,j',m,m'}$ lies on a neighborhood of the curve \tilde{c}_2 . Hence, for $1 \leq j \leq N_1$ and $m \geq 1$ large enough,

$$\begin{aligned} \sum_{a \in X_{1,j}^m \cap \text{Crit}(Y_{j,j',m,m'})} (-1)^{\text{ind}(a)} \int_{S^*X} [\Sigma_{2,j',m'}] \wedge R_a &= - \left(\sum_{a \in X_{1,j}^m \cap \text{Crit}(Y_{j,j',m,m'})} (-1)^{\text{ind}(a)} \right) \frac{\chi(X_{2,j'})}{\chi(X)} \\ &\quad + \sum_{a \in X_{1,j}^m \cap \text{Crit}(Y_{j,j',m,m'})} (-1)^{\text{ind}(a)} \chi(X_{2,j'} \cap \{a\}). \end{aligned}$$

Applying Poincaré-Hopf formula for manifolds with boundary one more time, exactly as in §6.4.9, to the first term yields

$$\sum_{a \in X_{1,j}^m \cap \text{Crit}(Y_{j,j',m,m'})} (-1)^{\text{ind}(a)} = \chi(X_{1,j}^m).$$

Hence, thanks to (52), we obtain

$$\begin{aligned} \sum_{a \in X_{1,j}^m \cap \text{Crit}(Y_{j,j',m,m'})} (-1)^{\text{ind}(a)} \int_{S^*X} [\Sigma_{2,j',m'}] \wedge R_a &= -\frac{\chi(X_{1,j})\chi(X_{2,j'})}{\chi(X)} \\ &\quad + \sum_{a \in X_{1,j}^m \cap \text{Crit}(Y_{j,j',m,m'})} (-1)^{\text{ind}(a)} \chi(X_{2,j'} \cap \{a\}). \end{aligned}$$

As the critical points of $Y_{j,j',m,m'}|_{X_{1,j}^m}$ are independent of (j', m') and away from $\tilde{c}_{2,j'} = \partial X_{2,j'}$, one has $\chi(X_{2,j'} \cap \{a\}) = 1$ if $a \in X_{2,j'}^{m'}$ and $\chi(X_{2,j'} \cap \{a\}) = 0$ otherwise (for m' large enough). Thus, we find that

$$\sum_{a \in X_{1,j}^m \cap \text{Crit}(Y_{j,j',m,m'})} (-1)^{\text{ind}(a)} \chi(X_{2,j'} \cap \{a\}) = \sum_{a \in X_{1,j}^m \cap \text{Crit}(Y_{j,j',m,m'}) \cap X_{2,j'}^{m'}} (-1)^{\text{ind}(a)}.$$

Hence,

$$\begin{aligned} \sum_{a \in X_{1,j}^m \cap \text{Crit}(Y_{j,j',m,m'})} (-1)^{\text{ind}(a)} \int_{S^*X} [\Sigma_{2,j',m'}] \wedge R_a &= -\frac{\chi(X_{1,j})\chi(X_{2,j'})}{\chi(X)} \\ &+ \sum_{a \in \text{Crit}(Y_{j,j',m,m'}) \cap X_{1,j}^m \cap X_{2,j'}^{m'}} (-1)^{\text{ind}(a)}. \end{aligned}$$

For the moment, we have shown that for $1 \leq j \leq N_1$, $1 \leq j' \leq N_2$ and m, m' large enough

$$\begin{aligned} \mathbf{L}(\tilde{c}_{1,j}^m, \tilde{c}_{2,j'}^{m'}) &= \int_{S^*X} [S_{j,j',m,m'}] \wedge [\Sigma_{2,j',m'}] + \sum_{a \in \text{Crit}(Y_{j,j',m,m'}) \cap X_{1,j}^m \cap X_{2,j'}^{m'}} (-1)^{\text{ind}(a)} \\ &- \frac{\chi(X_{1,j})\chi(X_{2,j'})}{\chi(X)}. \end{aligned}$$

Hence, we are left with the computation of

$$(56) \quad \int_{S^*X} [S_{j,j',m,m'}] \wedge [\Sigma_{2,j'}^{m'}] + \sum_{a \in X_{1,j}^m \cap X_{2,j'}^{m'} \cap \text{Crit}(Y_{j,j',m,m'})} (-1)^{\text{ind}(a)},$$

for every $1 \leq j \leq N_1$, $1 \leq j' \leq N_2$ and m, m' large enough. In order to finish the proof, let us now interpret properly (56) in terms of Poincaré-Hopf theory in order to compute it terms of Euler characteristics. Thanks to Remark 6.29, each integral

$$\int_{S^*X} [S_{j,j',m,m'}] \wedge [\Sigma_{2,j'}^{m'}]$$

is an alternate sum of $+1$ and -1 which correspond to each point of the curve $\tilde{c}_{2,j'}^{m'} \cap X_{1,j}^m$ where the vector field $Y_{j,j',m,m'}$ points inward $X_{2,j'}^{m'}$ (here we are using the fact that \mathbf{c}_2 is nontrivial) and normally to the curve. Equivalently, the normalized vector field $Y_{j,j',m,m'}/\|Y_{j,j',m,m'}\|$ induces a vector field $\tilde{Y}_{j,j',m,m'}$ tangent to the curve $\tilde{c}_{2,j'}^{m'} \cap X_{1,j}^m$ and each contribution to the integral will come from the points where the induced vector field $\tilde{Y}_{j,j',m,m'}$ vanishes and where the vector field $Y_{j,j',m,m'}$ is pointing inward $X_{2,j'}^{m'}$. When the point is attracting (resp. repulsing), it will give a contribution -1 (resp. 1) to the integral – see Remark 6.29. Hence, one finds

$$(57) \quad \int_{S^*X} [S_{j,j',m,m'}] \wedge [\Sigma_{2,j'}^{m'}] = \sum_{a \in \text{Crit}_{\text{in}}(\tilde{Y}_{j,j',m,m'}) \cap (X_{1,j}^m \cap \partial X_{2,j'}^{m'})} (-1)^{\text{ind}(a)},$$

where $\text{Crit}_{\text{in}}(\tilde{Y}_{j,j',m,m'})$ is the set of critical points of $\tilde{Y}_{j,j',m,m'}$ where $Y_{j,j',m,m'}$ points inward $X_{2,j'}^{m'}$. We will now apply the Poincaré-Hopf formula to the compact one-dimensional submanifold $X_{1,j}^m \cap \partial X_{2,j'}^{m'}$. This is just a union of smooth curves with boundary. The boundary $\partial X_{1,j}^m \cap \partial X_{2,j'}^{m'}$ may be non empty and $\tilde{Y}_{j,j',m,m'}$ is pointing inward on $\partial X_{1,j}^m \cap \partial X_{2,j'}^{m'}$ since

the initial vector field $\tilde{Y}_{j,j',m,m'}$ coincides with the inward normal of $X_{1,j}^m$ on $\partial X_{1,j}^m$ and the intersection $\partial X_{1,j}^m \cap \partial X_{2,j'}^{m'}$ is transverse. We have still according to [65, Th. 1]

$$(58) \quad \sum_{a \in \text{Crit}(\tilde{Y}_{j,j',m,m'}) \cap (X_{1,j}^m \cap \partial X_{2,j'}^{m'})} (-1)^{\text{ind}(a)} = -\frac{1}{2} \chi \left(\partial X_{1,j}^m \cap \partial X_{2,j'}^{m'} \right).$$

Note that the right-hand side is an integer. Indeed, $\partial X_{1,j}^m$ and $\partial X_{2,j'}^{m'}$ are both homologically trivial by construction. In particular, as the curves are transverse to each other, $\int_X [\partial X_{1,j}^m] \wedge [\partial X_{2,j'}^{m'}] = 0$ and the two curves intersect each other an even number of times. We can also remark that, thanks to Proposition 6.13 and to Lemma 6.16, the orientation of the boundary of $X_{1,j}^m \cap X_{2,j'}^{m'}$ (induced by X) is the same as the one on $\partial X_{1,j}^m$ and $\partial X_{2,j'}^{m'}$. Equation (58) applied to (57) yields

$$\begin{aligned} \int_{S^*X} [S_{j,j',m,m'}] \wedge [\Sigma_{2,j'}^{m'}] &= - \sum_{a \in \text{Crit}_{\text{out}}(\tilde{Y}_{j,j',m,m'}) \cap (X_{1,j}^m \cap \partial X_{2,j'}^{m'})} (-1)^{\text{ind}(a)} \\ &\quad - \frac{1}{2} \chi \left(\partial X_{1,j}^m \cap \partial X_{2,j'}^{m'} \right). \end{aligned}$$

We would now like to apply the Poincaré-Hopf formula [65, Th. A₀] to $X_{1,j}^m \cap X_{2,j'}^{m'}$ in (56) except that this surface has only piecewise smooth boundary. To solve this problem, we can smooth the singular points of $\partial(X_{1,j}^m \cap X_{2,j'}^{m'})$ as in the proof of Lemma 6.25 in order to obtain a compact surface $X_{j,j',m,m'}$ with smooth boundary. As in (52), we can ensure that

$$\chi(X_{j,j',m,m'}) = \chi \left(X_{1,j}^m \cap X_{2,j'}^{m'} \right).$$

Moreover, as $Y_{j,j',m,m'}$ is pointing inward on $\partial X_{1,j}^m$, we find

$$(59) \quad \int_{S^*X} [S_{j,j',m,m'}] \wedge [\Sigma_{2,j'}^{m'}] = - \sum_{a \in \text{Crit}_{\text{out}}(\tilde{Y}_{j,j',m,m'})} (-1)^{\text{ind}(a)} - \frac{1}{2} \chi \left(\partial X_{1,j}^m \cap \partial X_{2,j'}^{m'} \right).$$

Then, if we apply the Poincaré-Hopf formula [65, Th. A₀], we get

$$\chi(X_{j,j',m,m'}) = \sum_{a \in X_{j,j',m,m'} \cap \text{Crit}(Y_{j,j',m,m'})} (-1)^{\text{ind}(a)} - \sum_{a \in \text{Crit}_{\text{out}}(\tilde{Y}_{j,j',m,m'})} (-1)^{\text{ind}(a)}.$$

By construction of the vector field $Y_{j,j',m,m'}$ and using that the critical points of the vector field are away from $\partial(X_{1,j}^m \cap X_{2,j'}^{m'})$ (see Remark 6.29), we find

$$(60) \quad \chi \left(X_{1,j}^m \cap X_{2,j'}^{m'} \right) = \sum_{a \in X_{1,j}^m \cap X_{2,j'}^{m'} \cap \text{Crit}(Y_{j,j',m,m'})} (-1)^{\text{ind}(a)} - \sum_{a \in \text{Crit}_{\text{out}}(\tilde{Y}_{j,j',m,m'})} (-1)^{\text{ind}(a)}.$$

Using (53), we finally obtain by combining (59) and (60) that, for m, m' large enough,

$$\sum_{a \in X_{1,j}^m \cap X_{2,j'}^{m'} \cap \text{Crit}(Y_{j,m})} (-1)^{\text{ind}(a)} + \int_{S^*X} [S_{j,m}] \wedge [\Sigma_{2,j'}^{m'}] = \chi(X_{1,j} \cap X_{2,j'}) - \frac{1}{2} \chi(\partial X_{1,j} \cap \partial X_{2,j'}),$$

which belongs to \mathbb{Z} . Finally, we will be done by summing over $1 \leq j \leq N_1$ and $1 \leq j' \leq N_2$. In fact, this last formula together with Definition 6.18, equation (54) and the equation preceding (56) yield

$$\mathbf{L}(c_1, c_2) = -\frac{\chi(X(\tilde{c}_1))\chi(X(\tilde{c}_2))}{\chi(X)} + \sum_{j=1}^{N_1} \sum_{j'=1}^{N_2} \left(\chi(X_{1,j} \cap X_{2,j'}) - \frac{1}{2} \chi(\partial X_{1,j} \cap \partial X_{2,j'}) \right).$$

Using (46), this gives in terms of constructible functions

$$\mathbf{L}(c_1, c_2) = -\frac{\chi(f_1)\chi(f_2)}{\chi(X)} + \chi(f_1 f_2) - \frac{1}{2} \chi(\mathbf{1}_{\tilde{c}_1 \cap \tilde{c}_2}).$$

Remark 6.30. This shows that $\chi(X)\mathbf{L}(c_1, c_2)$ belongs to \mathbb{Z} but also provides us with an explicit formula in terms of constructible functions associated with \tilde{c}_1 and \tilde{c}_2 . Recall that \tilde{c}_i was obtained by pushing the initial curve c_i transversally by the geodesic flow. This was done to remove the nonsimple selfintersection points of the curve but also to ensure that $\Sigma(\tilde{c}_1) \cap \Sigma(\tilde{c}_2) = \emptyset$ and that \tilde{c}_1 and \tilde{c}_2 intersect away from their selfintersection points. In particular, one can replace \tilde{c}_i by c_i in the following situation:

- $\Sigma(c_1) \cap \Sigma(c_2) = \emptyset$;
- c_1 and c_2 have only simple selfintersection points;
- $c_1 \cap c_2$ does not contain selfintersection points of c_1 or c_2 .

6.5. Linking of closed geodesics. When \mathbf{c}_1 and \mathbf{c}_2 are both nontrivial in $\pi_1(X)$, there are other natural curves in S^*X that one may associate to $c_i : \mathbb{R}/\ell_i\mathbb{Z} \rightarrow X$:

$$\Sigma_{\text{geod}}(c_i) := \{(c_i(t), c'_i(t)^b) : t \in \mathbb{R}/\ell_i\mathbb{Z}\}.$$

This is just the closed geodesic lifting c_i in S^*X . It is then natural to ask whether the linking number $\mathbf{L}(c_1, c_2)$ is related to the linking number of $\Sigma_{\text{geod}}(c_1)$ and $\Sigma_{\text{geod}}(c_2)$ and this is indeed the case as we will establish. First of all, we can define using the conventions of Section 2 the following diffeomorphism of S^*X :

$$\mathcal{R} : x = (q, p) \in S^*X \mapsto (q, p^\perp) \in S^*X.$$

This map is orientation-preserving as it is isotopic to the identity and, for $i = 1, 2$, one has

$$[\Sigma(c_i)] = \mathcal{R}^{-1*}[\Sigma_{\text{geod}}(c_i)].$$

In particular, $[\Sigma_{\text{geod}}(c_i)]$ is de Rham exact when \mathbf{c}_i is homologically trivial thanks to Theorem 6.3. Using the conventions of Proposition 6.13, one has

$$\mathbf{L}(c_1, c_2) = \int_{S^*X} [\Sigma(\tilde{c}_1)] \wedge R_2^{-T'_0},$$

where $[\Sigma(\tilde{c}_2)] = dR_2^{-T'_0}$. Hence, using that \mathcal{R} is orientation preserving and the continuity of the wedge product of currents whose wave front sets are transverse (see appendix A), we can deduce that

$$\mathbf{L}(c_1, c_2) = \int_{S^*X} \mathcal{R}^*[\Sigma(\tilde{c}_1)] \wedge \mathcal{R}^*R_2^{-T'_0} = \int_{S^*X} [\Sigma_{\text{geod}}(\tilde{c}_1)] \wedge \mathcal{R}^*R_2^{-T'_0},$$

where $[\Sigma_{\text{geod}}(\tilde{c}_2)] = d \left(\mathcal{R}^* R_2^{-T'_0} \right)$ (as d commutes with \mathcal{R}^*). Recall from the remark following Theorem 4.16 that one can take $T_0 = T'_0 = 0$ whenever $\Sigma(c_1) \cap \Sigma(c_2) = \emptyset$, e.g. if $c_1 \neq c_2$. In other words, for nontrivial homotopy classes, the linking number $\mathbf{L}(c_1, c_2)$ we have been computing in this section is equal to the linking number of the geodesic curves lifting c_1 and c_2 . Hence, we can reformulate Theorem 1.2 as follows:

Theorem 6.31. *Suppose that c_i are both nontrivial homotopy classes which are homologically trivial.*

*If $c_1 \neq c_2$, then $\mathcal{N}_\infty(c_1, c_2, 0) = \mathbf{L}(c_1, c_2)$ is the linking number of the two closed geodesics $\Sigma_{\text{geod}}(c_1)$ and $\Sigma_{\text{geod}}(c_2)$ which lift c_1 and c_2 to S^*X .*

Moreover, if $c_1 = c_2$, this remains true if we replace c_1 by the (small) homotopic deformation \tilde{c}_1 given in Proposition 6.13.

In particular, this establishes a direct connection with the works of Duke–Imamoğlu–Tóth who expressed the linking number of two closed geodesics on the modular surface as the special value of a certain Dirichlet series [23]. Similarly, for suspension of toral automorphisms, the linking of closed orbits was identified with the special value of certain L -functions by Bergeron–Charollois–Garcia–Venkatesh [5].

6.6. Margulis asymptotic formula (1) revisited. We already explained that Margulis asymptotic formula,

$$\mathcal{N}_T(c_1, c_2) \sim A_{c_1, c_2} e^{Th_{\text{top}}}, \quad \text{as } T \rightarrow +\infty,$$

could be recovered from our analysis of the meromorphic continuation of $s \mapsto \mathcal{N}_\infty(c_1, c_2, s)$ and the spectral results from [37]. Let us now reinterpret this result when c_1 and c_2 are homologically trivial. In that case, one has

$$\begin{aligned} \varphi^{-T*}[\Sigma(c_2)] &= [\Sigma(c_2)] + \int_0^T \mathcal{L}_V \varphi^{-t*}([\Sigma(c_2)]) dt \\ &= [\Sigma(c_2)] + d \left(\int_0^T \iota_V \varphi^{-t*}([\Sigma(c_2)]) dt \right) + \int_0^T \iota_V \varphi^{-t*}(d[\Sigma(c_2)]) dt. \end{aligned}$$

Using now Proposition 6.1 and Theorem 6.3, we find that

$$\varphi^{-T*}[\Sigma(c_2)] = d \left(R_2 + \int_0^T \iota_V \varphi^{-t*}([\Sigma(c_2)]) dt \right),$$

where $R_2 \in \mathcal{D}_{N^*(\Sigma(c_2))}^1(S^*X)$ is such that $[\Sigma(c_2)] = dR_2$. Equivalently, we have written a primitive of $\varphi^{-T*}[\Sigma(c_2)]$. Hence, up to applying the flow in a fixed (small) forward time $T_0 > 0$ to $\Sigma(c_1)$, we can make the wedge product between $\varphi^{T_0*}([\Sigma(c_1)]) = [\Sigma(\tilde{c}_1)]$ and this primitive of $\varphi^{-T*}[\Sigma(c_2)]$ – see Appendix A.2. In particular, the linking number

$$\int_{S^*X} \varphi^{-T*}([\Sigma(c_2)]) \wedge R_1$$

between $\varphi^{-T_0}(\Sigma(c_1))$ and $\varphi^T(\Sigma(c_2))$ is given by

$$\int_{S^*X} [\Sigma(c_1)] \wedge R_2 + \int_{S^*X} \int_0^T [\Sigma(c_1)] \wedge \iota_V \varphi^{-t*}([\Sigma(c_2)]) dt.$$

The first term is the quantity we have computed in this section – see Theorem 6.22 for the precise formula. The second term can be expressed thanks to Lemma 4.11 and it is equal to $\varepsilon(\mathbf{c}_1) \mathcal{N}_T(c_1, c_2)$. Hence, Margulis–Parkkonen–Paulin Theorem can be rewritten as a Theorem on the asymptotic linking between two Legendrian knots:

Theorem 6.32. *Let (X, g) be a smooth (\mathcal{C}^∞) , compact, oriented, connected, Riemannian surface which has no boundary and which has negative curvature. Then, for every \mathbf{c}_1 and \mathbf{c}_2 in $\pi_1(X)$ which are homologically trivial and for any of their geodesic representatives c_1 and c_2 , there exists $A_{\mathbf{c}_1, \mathbf{c}_2} > 0$ such that*

$$\int_{S^*X} \varphi^{-T*}([\Sigma(c_2)]) \wedge R_1 \sim \varepsilon(\mathbf{c}_2) A_{\mathbf{c}_1, \mathbf{c}_2} e^{Th_{\text{top}}}, \quad \text{as } T \rightarrow +\infty,$$

where $[\Sigma(c_1)] = dR_1$.

APPENDIX A. A BRIEF REMINDER ON THE WAVEFRONT SET OF A DISTRIBUTION

In this appendix, we briefly recall the notion of the wavefront set of a distribution and collect some classical properties that were used all along this article. The presentation is close to [26, 9, 20] to which we refer for more informations and references.

The space $\mathcal{D}'_k(M)$ denotes the currents of degree $0 \leq k \leq n = \dim(M)$ whose wavefront set is contained in a fixed closed conic set $\Gamma \subset T^*M \setminus \underline{0}$, with $\underline{0}$ denoting the zero section. Recall first that an element in $\mathcal{D}'_k(M)$ is a current u of degree k such that, for every $N \geq 1$, for every open set U , for every closed cone C such that $(\text{supp } \chi \times C) \cap \Gamma = \emptyset$, one has

$$(61) \quad \|u\|_{N, C, \chi, \alpha, U} := \|(1 + \|\xi\|)^N \mathcal{F}(u_\alpha \chi)(\xi)\|_{L^\infty(C)} < +\infty,$$

where χ is supported on the chart U , where $u = \sum_{|\alpha|=k} u_\alpha dx^\alpha$ where α is a multi-index and where \mathcal{F} is the Fourier transform computed in the local chart U . Given a smooth, closed, embedded, oriented submanifold Σ of dimension $n - k$ inside M , one can verify that the current of integration $[\Sigma]$ over Σ , defined as

$$\forall \psi \in \Omega^{n-k}(M), \quad \langle [\Sigma], \psi \rangle = \int_\Sigma \psi,$$

is an element in $\mathcal{D}'_{N^*(\Sigma)}(M)$, where

$$N^*(\Sigma) := \{(x, \xi) \in T^*M \setminus \underline{0} : x \in \Sigma \text{ and } \forall v \in T_x \Sigma, \xi(v) = 0\}.$$

Remark A.1. For a current u of degree k , the wavefront set of u , denoted by $\text{WF}(u)$, is the smallest conic cone Γ such that $u \in \mathcal{D}'_k(M)$.

Remark A.2. Following [26, App. C.1], we also define the wavefront set of an operator $B : \Omega^k(M) \rightarrow \mathcal{D}'^k(M)$ by considering its Schwartz kernel $K_B(x, y, dx, dy)$ that we view as an element in $\mathcal{D}'^{k, n-k}(M \times M)$. The wavefront set $\text{WF}'(B) \subset T^*(M \times M) \setminus \underline{0}$ is then

defined as the set of points $(y, \eta, x, -\xi)$ such that (y, η, x, ξ) belongs to the wavefront set of K_B .

A.1. Topology on the space $\mathcal{D}'_\Gamma(M)$. Let us first recall the notion of bounded subsets in $\mathcal{D}'^k(M)$ following [82, Ch. 3, p. 72]:

Definition A.3. A subset B of currents is bounded if, for every test form $\varphi \in \Omega^{n-k}(M)$, $\sup_{t \in B} |\langle t, \varphi \rangle| < +\infty$.

This definition is often referred as weak boundedness and it is equivalent to the notion of boundedness induced by the strong topology on $\mathcal{D}'^k(M)$ [82, Ch. 3]. We note that this is equivalent to B being bounded in some Sobolev space $H^s(M, \Lambda^k(T^*M))$ of currents by suitable application of the uniform boundedness principle [16, § 5, Lemma 23]. We can now define the normal topology in the space of currents essentially following [9, Sect. 3]:

Definition A.4 (Normal topology on the space of currents). For every closed conic subset $\Gamma \subset T^*M \setminus \underline{0}$, the topology of $\mathcal{D}'_\Gamma^k(M)$ is defined as the weakest topology which makes continuous the seminorms of the strong topology of $\mathcal{D}'^k(M)$ and the seminorms:

$$(62) \quad \|u\|_{N,C,\chi,\alpha,U} = \|(1 + \|\xi\|)^N \mathcal{F}(u_\alpha \chi)(\xi)\|_{L^\infty(C)}$$

where χ is supported on some chart U , where $u = \sum_{|\alpha|=k} u_\alpha dx^\alpha$ where α is a multi-index, where \mathcal{F} is the Fourier transform computed in the local chart and C is a closed cone such that $(\text{supp } \chi \times C) \cap \Gamma = \emptyset$. A subset $B \subset \mathcal{D}'_\Gamma^k$ is called bounded in \mathcal{D}'_Γ^k if it is bounded in \mathcal{D}'^k and if all seminorms $\|\cdot\|_{N,C,\chi,\alpha,U}$ are bounded on B .

We emphasize that this definition is given purely in terms of local charts without loss of generality. The above topology is in fact *intrinsic as a consequence of the continuity of the pull-back* [9, Prop 5.1 p. 211] as emphasized by Hörmander [47, p. 265] (see below for a brief reminder). Note that it is the same to consider currents or distributions when we define the relevant topologies since currents are just elements of the form $\sum u_{i_1, \dots, i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$ in local coordinates (x^1, \dots, x^n) where the coefficients u_{i_1, \dots, i_k} are distributions. We note that the above seminorms involve the L^∞ norm while the anisotropic spaces we deal with in this article are built from L^2 norms. This problem is handled by the following Lemma [20, App. B]:

Lemma A.5 (L^2 vs L^∞). Let N, \tilde{N} be some positive integers and let $\tilde{\Gamma}_0$ be a closed cone in \mathbb{R}^{n*} . Then, for every closed conic neighborhood $\tilde{\Gamma}$ of $\tilde{\Gamma}_0$, one can find a constant $C = C(N, \tilde{N}, \tilde{\Gamma}) > 0$ such that, for every u in $\mathcal{C}_c^\infty(B_{\mathbb{R}^n}(0, 1))$, one has

$$\sup_{\xi \in \tilde{\Gamma}_0} (1 + |\xi|)^N |\hat{u}(\xi)| \leq C \left(\|(1 + |\xi|)^N \hat{u}(\xi)\|_{L^2(\tilde{\Gamma})} + \|u\|_{H^{-\tilde{N}}} \right).$$

Remark A.6. Regarding the definition of anisotropic Sobolev spaces via pseudodifferential operators [48, §18], it is also convenient to define the wavefront set using pseudodifferential operators as in [26, App. C.1]. Recall that the wavefront set $\text{WF}(A)$ of a pseudodifferential operator $A \in \Psi^k(M, \Lambda^k(T^*M))$ is a closed conic set such that the full symbol of A decays as $\mathcal{O}(|\xi|^{-\infty})$ in a conic neighborhood of each point of the complementary of $\text{WF}(A)$. Then,

one can verify that a point (x, ξ) does not lie in the wavefront of a current u if there exists a conic neighborhood U of (x, ξ) such that, for any pseudodifferential $A \in \Psi^0(M, \Lambda^k(T^*M))$ with $\text{WF}(A) \subset U$, one has $Au \in \Omega^k(M)$.

Let us now discuss some of the properties of the space $\mathcal{D}'^k(M)$ under standard operations: product, pullback, pushforward.

A.2. Product of currents. Given two closed conic sets (Γ_1, Γ_2) which have empty intersection, the usual wedge product of smooth forms

$$\wedge : (\varphi_1, \varphi_2) \in \Omega^k(M) \times \Omega^l(M) \longmapsto \varphi_1 \wedge \varphi_2 \in \Omega^{k+l}(M)$$

extends uniquely as a hypocontinuous map for the normal topology [9, Th. 6.1]

$$\wedge : (\varphi_1, \varphi_2) \in \mathcal{D}'^k_{\Gamma_1}(M) \times \mathcal{D}'^l_{\Gamma_2}(M) \longmapsto \varphi_1 \wedge \varphi_2 \in \mathcal{D}'^{k+l}_{s(\Gamma_1, \Gamma_2)}(M),$$

with $s(\Gamma_1, \Gamma_2) = \Gamma_1 \cup \Gamma_2 \cup (\Gamma_1 + \Gamma_2)$. The notion of hypocontinuity is a strong notion of continuity adapted to bilinear maps from $E \times F \mapsto G$ where E, F, G are locally convex spaces [9, p. 204-205]. It is weaker than joint continuity but implies that the bilinear map is separately continuous in each factor uniformly in the other factor in a bounded subset²⁷.

A.3. Pullback of currents. Let Γ be a closed conic set and let f be a smooth *diffeomorphism* on M . The usual pullback operation on smooth forms,

$$f^* : \Omega^k(M) \rightarrow \Omega^k(M)$$

extends uniquely as a continuous map [9, Prop. 5.1] from $\mathcal{D}'^k_{\Gamma}(M)$ to $\mathcal{D}'^k_{f^*\Gamma}(M)$ for the normal topology, with $f^*\Gamma$ defined as

$$f^*\Gamma := \{ (f^{-1}(x), (df^{-1}(x)^T)^{-1}\xi) \in T^*M \setminus \underline{0} : (x, \xi) \in \Gamma \}.$$

A.4. Pushforward of currents. Let Γ be a closed conic set and let $f : M \rightarrow N$ be a smooth map between the smooth, compact, boundaryless manifolds M and N . The usual pushforward operation on smooth forms,

$$f_* : \Omega^k(M) \rightarrow \Omega^k(N)$$

extends uniquely as a continuous map [9, Th. 6.3] from $\mathcal{D}'^k_{\Gamma}(M)$ to $\mathcal{D}'^k_{f_*\Gamma}(N)$ for the normal topology, with $f_*\Gamma$ defined as

$$f_*\Gamma := \{ (y, \eta) \in T^*N \setminus \underline{0} : \exists (x, \xi) \in \Gamma \cup \underline{0} \text{ s.t. } f(x) = y \text{ and } \xi = df(x)^T \eta \}.$$

²⁷The tensor product of distributions for the strong topology is hypocontinuous but not continuous [9, p. 205]

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