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► **To cite this version:**

Roberta Bianchini, Charlotte Perrin. Soft congestion approximation to the one-dimensional constrained Euler equations. *Nonlinearity*, In press, 34 (10), pp.6901-6929. 10.1088/1361-6544/ac1e33 . hal-02624318v2

**HAL Id: hal-02624318**

**<https://hal.science/hal-02624318v2>**

Submitted on 30 Sep 2021

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# Soft congestion approximation to the one-dimensional constrained Euler equations

Roberta Bianchini\* and Charlotte Perrin†

September 30, 2021

## Abstract

This article is concerned with the analysis of the one-dimensional compressible Euler equations with a singular pressure law, the so-called *hard sphere equation of state*. We provide a detailed description of the effect of the singular pressure on the breakdown of the smooth solutions. Moreover, we rigorously justify the singular limit for smooth solutions towards the free-congested Euler equations, where the compressible (free) dynamics is coupled with the incompressible one in the constrained (i.e. congested) domain.

**Keywords:** Compressible Euler equations, maximal packing constraint, singularity formation, singular limit, free boundary problem.

**MSC:** 35Q35, 35L87, 35L81.

## 1 Introduction and main results

The topic of this work is the analysis of the following one-dimensional compressible Euler equations

$$\begin{cases} \partial_t \rho + \partial_x m = 0, & (1a) \\ \partial_t m + \partial_x \left( \frac{m^2}{\rho} \right) + \partial_x p_\varepsilon(\rho) = 0, & (1b) \end{cases}$$

where  $\rho$  stands for the density and  $m = \rho u$  for the momentum of the fluid, with  $u$  the velocity of the fluid. The novelty of the model that we shall consider in this paper lies in the choice the pressure law  $p_\varepsilon$ , which is supposed to satisfy the so-called *hard-sphere equation of state* introduced by Carnahan and Starling in [6]. The latter is identified by means of the following conditions at  $\varepsilon > 0$  fixed:

$$p_\varepsilon \in \mathcal{C}^1([0, 1)), \quad p_\varepsilon(0) = 0, \quad p'_\varepsilon(\rho) > 0 \quad \text{on } (0, 1), \quad \lim_{\rho \rightarrow 1^-} p_\varepsilon(\rho) = +\infty, \quad (2)$$

where the physical meaning of the parameter  $\varepsilon > 0$  is discussed below. The class of equations in (1)-(2) gained the interest of the mathematical community for the modeling of collective

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motions (see for instance [25] and [14]) and of dispersed mixtures like bubbly fluids or granular suspensions (see for instance [19], [16], [26]). In the collective motion models,  $\rho$  in (1) is the density of the crowd, while the pressure  $p_\varepsilon(\rho)$  is the cumulative response of short-range repulsive social forces preventing contacts among individuals. From the macroscopic viewpoint, the singularity of the pressure plays the role of a barrier, formally ruling out the creation of congested regions where  $\rho = 1$ . The parameter  $\varepsilon$  models the strength of the repulsive forces.

In the rest of the paper, the expression of the pressure term is explicitly chosen as follows

$$p_\varepsilon(\rho) = \varepsilon \left( \frac{\rho}{1-\rho} \right)^\gamma + \kappa \rho^{\tilde{\gamma}} = p_{1,\varepsilon}(\rho) + p_2(\rho), \quad (3)$$

where  $\varepsilon > 0$  is small and fixed, while the specific ranges of  $\gamma, \tilde{\gamma} > 1$  and  $\kappa \geq 0$  will be discussed later on. The pressure is thus split into two parts: the first one  $p_{1,\varepsilon}$  dictates the singular behavior close to the maximal density constraint, while  $p_2$  is the classical equation of state for isentropic gases and models additional non-singular effects. For instance, shallow water flows can be described by system (1) (the so-called *shallow water* or *Saint-Venant equations*), where the variable  $\rho$  is replaced by the height of the flow  $h$ , and  $p_2$  is the hydrostatic part of the pressure due to gravity, namely  $p_2(h) = gh^2/2$ .

A heuristic reasoning shows that the solutions  $(\rho_\varepsilon, m_\varepsilon)$  to system (1) coupled with the equation of state (3) converge as  $\varepsilon \rightarrow 0$  towards the solutions  $(\rho, m)$  to the *free-congested Euler equations*

$$\begin{cases} \partial_t \rho + \partial_x m = 0, & (4a) \end{cases}$$

$$\begin{cases} \partial_t m + \partial_x \left( \frac{m^2}{\rho} \right) + \partial_x p + \kappa \partial_x \rho^{\tilde{\gamma}} = 0, & (4b) \end{cases}$$

$$\begin{cases} 0 \leq \rho \leq 1, (1-\rho)p = 0, p \geq 0, & (4c) \end{cases}$$

where the pressure  $p$  is the limit (in a sense that will be clarified later on) of  $p_{1,\varepsilon}(\rho_\varepsilon)$ . The above system is a hybrid model describing both regions where the density is “free”, in the sense that  $\rho < 1$  and  $p = 0$ , and constrained regions where the density is saturated  $\rho = 1$  and  $p$  activates itself. From the mathematical viewpoint, the pressure  $p$  can be seen as a Lagrange multiplier associated to the incompressibility constraint

$$\partial_x u = 0 \quad \text{in} \quad \{\rho = 1\}. \quad (5)$$

Following the terminology introduced by Maury in [25], compressible systems with singular constitutive laws like (1)-(2) are called *soft congestion models*, whereas free-congested systems of type (4) are called *hard congestion models*. It is worth pointing out that, unlike the standard formulation of free-boundary problems, in (4) there is no explicit equation for the evolution of the interface between the free domain and the congested one, which is indeed implicitly encoded in the exclusion relation (4c).

The limit system (4), with  $\kappa = 0$ , has been heuristically introduced by Bouchut et al. in [3] as an asymptotic model for two-phase (gas-liquid or solid-liquid) flows when the ratio between the characteristic densities of the two phases is very small (or conversely very large). The existence of global weak solutions to system (4) with  $\kappa = 0$  has been established by Berthelin in [1] (see also [28] for a closely related model and [2] for an extension to the multi-D case) and numerical approaches based on optimal transport are proposed in [30].

Let us also mention the connections between (4) and models for wave-structure interactions developed in the very recent years by Lannes [21] and Godlewski et al. [17]-[18], on the

basis of the Shallow Water Equations with  $\kappa = g/2 > 0$ ,  $\tilde{\gamma} = 2$  in (4). In wave-structure interaction models, a constraint similar to (4c) can be indeed introduced to model the two possible states of the flow, which is *pressurized* in the “interior” domain where the fluid is in contact with the above structure (the height being then constrained by the structure, a floating object or a roof), and a *free* surface flow in the “exterior” domain. As in (4), the momentum equation involves a pressure  $p$ , which is different from the hydrostatic pressure  $gh^2/2$  and it is associated to the constraint on the height through the relation  $(h_{structure} - h)(p - p_{atm}) = 0$ , which implies that  $p$  differs from the atmospheric pressure only in the pressurized domain where  $h = h_{structure}$ . In the floating body problem, it is important to notice that, in contrast to (4), the constraint can be “heterogenous”, *i.e.*  $h_{structure}$  is not constant (in space but also possibly in time), and it is localized, *i.e.*  $h_{structure} = +\infty$  beyond the lateral boundaries of the solid body. Two approaches have been developed in this context. The first one, followed by Iguchi and Lannes [20], is based on the theory of initial boundary value hyperbolic problems and transmission problems. Transmission conditions adapted to the geometry of the floating structure are explicitly prescribed at the contact points (*i.e.* the interface) between the interior region and the exterior domain. It enables to study the system separately in each region and the dynamics of the interface is then implicitly encoded in the transmission condition. Local well-posedness results are proved in [20] for different cases: fixed structure, prescribed motion and free motion. In [17], the authors adopt another point of view with a *pseudo-compressibility method* (or *relaxation method*). It consists in replacing the pressure  $p$  by a “compressible” pressure  $p_\lambda(h) = \frac{(h - h_{structure})_+}{\lambda}$  and take  $\lambda \rightarrow 0$ . As for the  $\varepsilon$ -approximation (3) analysed in the present paper, one circumvents the difficulties associated with the interface (transmission conditions and dynamics of the free boundary) but here the variable  $h_\lambda$  is allowed to pass above the maximal constraint for fixed  $\lambda > 0$ . The study [17] contains numerical simulations based on the approximation by  $p_\lambda$ .

To our knowledge, the rigorous proof of the convergence as  $\varepsilon \rightarrow 0$  or similarly the limit  $\lambda \rightarrow 0$  as introduced in [17] of solutions to (1)-(3) towards solutions of (4) is a largely open question, since the existing literature establishes the link between the two models only at the formal level. For instance, Degond et al. in [14] take advantage of this formal limit to provide a new numerical scheme for the free boundary problem (4). Interestingly, the analysis of the asymptotic behavior as  $\varepsilon \rightarrow 0$  of the solutions of the Riemann problem associated to (1) is also carried out in [14]. In [4] Bresch and Renardy analyse the shock formation at the interface between the congested region where  $\rho = 1$  and the free region where  $\rho < 1$ . In that case, the heuristic connection between (1) and (4) plays again a crucial role in identifying numerically the formation of these shocks when the congestion constraint is reached. More precisely, assuming compression in the initial velocity at a given point  $x^*$  and also assuming that a shock wave arises around  $(t^*, x^*)$  with  $\rho(t^*, x^*) = 1$ , the authors deduce suitable scalings (in terms of  $\varepsilon$ ) of  $t, x, \rho, u$  in the neighborhood of  $(t^*, x^*, 1, 0)$  from the Rankine-Hugoniot conditions. After rewriting the approximate system (1) in the scaled variables and only taking into account the main order terms, the authors exhibit a numerical solution which develops shock for an initial velocity for  $u'_0(x^*) = -q < 0$ , so confirming the initial ansatz.

As a matter of facts, the asymptotic limit for  $\varepsilon \rightarrow 0$  is better understood in the viscous case, that is the Navier-Stokes equations, where the viscosity term  $-\nu\partial_{xx}u$  is added to the momentum equation (1b). The interested reader is referred to [29], where the behavior as  $\varepsilon \rightarrow 0$  for the multi-dimensional Navier-Stokes equations with a hard-sphere potential is investigated, and to the survey paper [27], which provides a precise picture on the related

state of the art. Finally, we remark that the asymptotics  $\varepsilon \rightarrow 0$  also shares some features with other kinds of singular limits for the compressible Euler equations, as the vanishing pressure limit [9] and the low Mach number limit [12].

The aim of this work is twofold. Our first goal is to provide a quantitative description of the role played by the (singular) hard-sphere potential on the breakdown of the smooth ( $C^1$ ) solutions. In a second step, we aim at rigorously justifying the convergence  $\varepsilon \rightarrow 0$  of smooth solutions to (1) towards (weak) solutions to system (4). Finally, an appendix section is dedicated to the existence of global weak solutions at  $\varepsilon$  fixed to (1) by means of the compensated compactness method.

The analysis of one-dimensional gas dynamics in the smooth framework has a long history, which started with Lax [22] in the 60's and was further developed by Chen and his coauthors in a series of recent papers (see for instance [10], [11], and references therein). In his original paper [22] on  $2 \times 2$  strictly hyperbolic systems, Lax considers initial data which are small perturbations of a constant state and shows that if these initial data contain some ‘‘compression’’ (in a sense precised below) then the corresponding smooth solutions develop singularities (i.e. blow-up of the gradient of the solution) in finite time; otherwise the solutions are global in time. This result applies in particular to the compressible Euler equations, more precisely to its reformulation in Lagrangian coordinates, the so-called *p-system* (see system (6) below), in the context of *small initial data*. The appearance of singularities for *large initial data* was instead an open question until the recent work of Chen et al. [10]. They show that singularity formations occur in the p-system with the pressure law  $p(\rho) = \kappa\rho^\gamma$  and  $\gamma > 1$ , if the initial datum (whose size is arbitrarily chosen) contains some compression (in the sense of Definition 1.1). Otherwise, if the initial datum is *everywhere rarefactive* (see again Definition 1.1), the smooth solution is global in time. One key point of the proof of Chen et al. is the derivation of upper and, more importantly, lower bounds (in the case of a polytropic gas  $p(\rho) = \kappa\rho^\gamma$  with  $\gamma \in (1, 3)$ ) for the density  $\rho$ . The upper bound is easily obtained by using the *Riemann invariants* of the system. The (time-dependent) lower bound is more subtle and relies on the control of the gradients of the Riemann invariants.

The analysis of the singular system is more delicate in our case, where the tracking of the small parameter  $\varepsilon$  is a fundamental issue for dealing with the singular pressure  $p_\varepsilon(\rho)$  in (3). Taking inspiration from [10], in this context the control of the Riemann invariants of the system allows us to provide a detailed description of the life-span of the solution, highlighting and making a distinction between the gradients blows up and the vanishing parameter  $\varepsilon$  as responsible for the breakdown of the smooth solutions. This last point confirms the above-mentioned numerical study of Bresch and Renardy [4]. Moreover, we perform the limit as  $\varepsilon \rightarrow 0$ , so rigorously justifying the connection between (1) and (4) for ‘‘well-prepared’’ initial data. This convergence result is the main novelty of the present paper, where, to the best of our knowledge, the limit from the soft congestion model to the free-congested Euler equations is proven for the first time.

Finally, an appendix section of this paper is dedicated to the proof of existence of global weak solutions at  $\varepsilon > 0$  fixed to (1), where we noticed an interesting connection with the smooth framework, which we would like to underline. We already mentioned that the Riemann invariants play a key role for our results in the smooth setting, as their control gives indeed a refined estimate for the density of system (1) with the singular pressure law  $p_\varepsilon(\rho)$  in (3). Somehow similarly, in the context of weak solutions at  $\varepsilon$  fixed, an  $\varepsilon$ -uniform bound in  $L^\infty$  of the (singular) internal energy  $H_\varepsilon$  satisfying  $H'_\varepsilon(\rho)\rho - H_\varepsilon(\rho) = p_\varepsilon(\rho)$  follows as an application

of the invariant region method, as detailed in the Appendix. The nice connection to notice is that both the uniform lower bound on the density in the smooth setting and the uniform upper bound of the internal energy in the weak framework hold under the same scaling assumptions on the initial data. Lastly, it could be interesting to mention that the proof of existence of weak solutions at  $\varepsilon$  fixed does not rely on the use of the whole family of (infinite and implicit) entropies generated by the entropy kernel, as done for instance in [8], but rather on the manipulation of four explicit entropy-entropy flux pairs in the spirit of Lu's work, [Chapter 8, [24]] and allows us to keep track of the singular parameter  $\varepsilon$  throughout all the computations.

We conclude this introduction recalling that other kinds of solutions in the existing literature which study the compressible Euler system are relevant for our problem and they will be investigated in future works. We just mention the finite-energy solutions studied by LeFloch and Westdickenberg in [23] and by Chen and Perepelitsa in [7]. In that context, the bound on  $H_\varepsilon(\rho)$  would hold in  $L_t^\infty L_x^1$  rather than  $L_{t,x}^\infty$ . Lastly, the case of  $BV$  solutions, displaying a quite vast literature, see for instance the books of Bressan [5], Dafermos [Chapter 15, [13]] and references therein, will be addressed in a forthcoming paper.

## Notations and conventions

- Given a Banach space  $\mathcal{B}$ , we indistinctly use both  $\mathcal{B}([a, b] \times \Omega)$  and  $\mathcal{B}([a, b]; \mathcal{B}(\Omega))$ , where  $[a, b] \subset \mathbb{R}^+$  and  $\Omega \subset \mathbb{R}$  (thus in the second part on the smooth setting we often shortly denote  $C_{t,x}^1 = C^1([0, T] \times \mathbb{R})$ ). In the case where the time and space functional spaces  $\mathcal{B}_1, \mathcal{B}_2$  are different one from another, we use the standard notation  $\mathcal{B}_1([a, b]; \mathcal{B}_2(\Omega))$ .
- We use the notation  $f_1 \lesssim f_2$  if there exists a constant  $C$ , independent of  $\varepsilon$ , such that  $f_1 \leq C f_2$ . We also employ the notation  $f(t, x) = \mathcal{O}(\varepsilon^\alpha)$ , with the constant  $\alpha \in \mathbb{R}$ , which means that  $f(t, x) = \varepsilon^\alpha \tilde{f}(t, x)$ , where  $\tilde{f}$  is a bounded continuous function in time and space.

## Main results: smooth solutions and their asymptotic behavior as $\varepsilon \rightarrow 0$ .

In the context of smooth ( $C_{t,x}^1$ ) solutions to system (1), the use of Lagrangian coordinates allows us to provide a refined description of the solutions. In this setting, we are able to analyse and exactly quantify the influence of the singular component of the pressure ( $p_{1,\varepsilon}$  in (3)) on the breakdown of the smooth solutions. After obtaining an existence theory at  $\varepsilon$  fixed, we are allowed to justify the asymptotics  $\varepsilon \rightarrow 0$  under additional assumptions on the values of the initial data which are close to the congestion constraint.

*Lagrangian coordinates.* The previous system (1) is written in the so-called Eulerian coordinates  $(t, x)$ . If, instead of  $x$ , we choose as space variable the material coordinate  $\tilde{x}$  such that

$$dx = udt + v d\tilde{x} \quad \text{where} \quad v := \frac{1}{\rho},$$

then the system can be rewritten as

$$\begin{cases} \partial_t v - \partial_{\tilde{x}} u = 0, \\ \partial_t u + \partial_{\tilde{x}} \tilde{p}_\varepsilon(v) + \kappa \partial_{\tilde{x}} (v^{-\tilde{\gamma}}) = 0, \end{cases} \quad \text{for } (t, \tilde{x}) \in \mathbb{R}_+ \times \mathbb{R}, \quad (6)$$

with the pressure law  $\tilde{p}_\varepsilon(v) := p_{1,\varepsilon}(v^{-1})$ . For sake of simplicity, when it is clear that we are in the Lagrangian setting, we shall drop hereafter the notation  $\tilde{\cdot}$ . In the context of gas motion,

the variable  $v$  denotes the *specific volume* (the reciprocal of the gas density) and system (6) is called *p-system*. The change of variable can be justified not only for smooth solutions but also in the framework of weak bounded solutions, as shown by Wagner in [33]. Nevertheless, in the latter setting, the definition of weak solutions for the Lagrangian equations must be adapted in the regions where vacuum occurs. This discussion is detailed in [33] and [Section 1.2, [31]].

As  $\varepsilon \rightarrow 0$ , we expect that the sequence of solutions  $(v_\varepsilon, u_\varepsilon)_\varepsilon$  to (6) converges to a solution  $(v, u)$  of the following free-congested p-system (namely the Lagrangian version of (4)):

$$\begin{cases} \partial_t v - \partial_x u = 0 & (7a) \\ \partial_t u + \partial_x p + \kappa \partial_x v^{-\tilde{\gamma}} = 0 & (7b) \\ v \geq 1, (v-1)p = 0, p \geq 0 & (7c) \end{cases}$$

*Statement of the results.* This part provides two main results. The first one concerns the existence of smooth solutions at  $\varepsilon > 0$  fixed, and makes a distinction between two cases which depend on the initial data and are defined below.

**Definition 1.1.** Let us introduce the function  $\theta_\varepsilon$ , defined as

$$\theta_\varepsilon(v) := \int_v^{+\infty} \sqrt{-p'_\varepsilon(\tau)} d\tau$$

and the Riemann invariants

$$w_\varepsilon^0 := u_\varepsilon^0 + \theta_\varepsilon(v_\varepsilon^0), \quad z_\varepsilon^0 := u_\varepsilon^0 - \theta_\varepsilon(v_\varepsilon^0).$$

At a point  $x \in \mathbb{R}$ , the initial datum  $(v_\varepsilon^0, u_\varepsilon^0)$  is said to be *rarefactive* if it is such that

$$\partial_x w_\varepsilon^0(x) \geq 0 \quad \text{and} \quad \partial_x z_\varepsilon^0(x) \geq 0 \quad (8)$$

and *compressive* otherwise.

### Initial data setup

**Assumption 1.2.** For any  $\varepsilon > 0$ ,  $(v_\varepsilon^0, u_\varepsilon^0)$  are  $\mathcal{C}^1$  functions and that there exist  $M_1, M_2 > 0$ , independent of  $\varepsilon$ , such that

$$(v_\varepsilon^0 - 1)^{\gamma-1} \geq M_1^{-1} \varepsilon, \quad \|(v_\varepsilon^0, u_\varepsilon^0)\|_{L^\infty(\mathbb{R})} + \|(\partial_x v_\varepsilon^0, \partial_x u_\varepsilon^0)\|_{L^\infty(\mathbb{R})} \leq M_2. \quad (9)$$

The first inequality is quite natural and it is not so restrictive at it could seem at a first glance. In fact it allows to consider a large setting of initial data. If the initial specific volume  $(v_0 - 1) = M_1^{-\frac{1}{\gamma-1}} \varepsilon^{\frac{1}{\gamma-1}}$ , this corresponds to a very congested state (in the sense of the approximated system at  $\varepsilon$  fixed), as the singular pressure would be

$$p_\varepsilon(v_0) = \frac{\varepsilon}{M_1^{-\frac{\gamma}{\gamma-1}} \varepsilon^{\frac{\gamma}{\gamma-1}}} = \varepsilon^{-\frac{1}{\gamma-1}} M_1^{\frac{\gamma}{\gamma-1}},$$

which blows up as  $\varepsilon \rightarrow 0$ . It is therefore the minimal requirement that allows us to pass to the limit in the classical setting: the pressure can blow up, but it guarantees that we have a control of the Riemann invariants (the internal energy  $H_\varepsilon$ , see Remark 1.10). Further conditions on the Riemann invariants at the initial time are given below.

**Assumption 1.3.** *There exist two constant  $Y^0, Q^0$  independent of  $\varepsilon$  such that*

$$\sqrt{c_\varepsilon^0} \partial_x w_\varepsilon^0 \leq Y^0, \quad \sqrt{c_\varepsilon^0} \partial_x z_\varepsilon^0 \leq Q^0. \quad (10)$$

We state our existence result in the smooth setting.

**Theorem 1.4** (Existence and life-span of  $(v_\varepsilon, u_\varepsilon)$ ). *Let*

$$p_\varepsilon(v) = \frac{\varepsilon}{(v-1)^\gamma} + \frac{\kappa}{v^{\tilde{\gamma}}} \quad \text{with } \kappa > 0, \gamma > 1, \tilde{\gamma} \in (1, 3) \text{ and } \varepsilon \leq \varepsilon_0 \text{ small enough.}$$

*Assume that the initial data satisfy Assumptions 1.2-1.3 We have two cases.*

1. *If the initial datum is everywhere rarefactive in the sense of Definition 1.1, then there exists a unique global-in-time  $\mathcal{C}_{t,x}^1$  solution  $(v_\varepsilon, u_\varepsilon)$  to (6), whose  $\mathcal{C}_{t,x}^1$ -norm is independent of  $\varepsilon$ .*
2. *Otherwise (i.e. if there exists  $x^* \in \mathbb{R}$  such that  $\partial_x w_\varepsilon^0(x^*) < 0$  or  $\partial_x z_\varepsilon^0(x^*) < 0$ ), there exists a unique local  $\mathcal{C}_{t,x}^1$  solution  $(v_\varepsilon, u_\varepsilon)$  to (6) which breaks down in finite time.*

*Moreover, in case 2 where a blowup in finite time occurs, we have the following lower bounds on the maximal time of existence  $T_\varepsilon^* < +\infty$*

$$T_\varepsilon^* \geq \begin{cases} \inf_{x^* \in \mathbb{R}} \frac{\varepsilon^{\frac{1}{2(\gamma-1)}}}{C \sqrt{c_\varepsilon^0(x^*)} \max\{-\partial_x w_\varepsilon^0(x^*), -\partial_x z_\varepsilon^0(x^*)\}} & \text{if } \gamma \in (1, 3), \\ \inf_{x^* \in \mathbb{R}} \frac{\varepsilon^{\frac{1}{4}}}{C \sqrt{c_\varepsilon^0(x^*)} \max\{-\partial_x w_\varepsilon^0(x^*), -\partial_x z_\varepsilon^0(x^*)\}} & \text{if } \gamma = 3, \\ \inf_{x^* \in \mathbb{R}} \frac{\varepsilon^{\frac{1}{\gamma+1}}}{C \sqrt{c_\varepsilon^0(x^*)} \max\{-\partial_x w_\varepsilon^0(x^*), -\partial_x z_\varepsilon^0(x^*)\}} & \text{if } \gamma > 3, \end{cases} \quad (11)$$

*where  $C > 0$  is a suitable constant independent of  $\varepsilon$ .*

Notice that, in full generality, the maximal existence time  $T_\varepsilon^*$  depends on  $\varepsilon$  and may a priori degenerate to 0 if no additional assumption is satisfied by the initial data  $(\partial_x w_\varepsilon^0, \partial_x z_\varepsilon^0)$ . The specific hypotheses that ensure an  $\varepsilon$ -uniform lower bound on  $T_\varepsilon^*$  are given below in Assumption 1.6. The derivation of this lower bound is the preliminary step for the analysis of the singular limit  $\varepsilon \rightarrow 0$ . Before stating our convergence result, let us recall the notion of solutions for the target limit system (7).

**Definition 1.5** (Weak solutions to the free-congested p-system). Let  $(v^0, u^0) \in \mathcal{C}^1(\mathbb{R})$  satisfying

$$v^0(x) \geq 1 \quad \forall x \in \mathbb{R},$$

and let  $T > 0$  be fixed. We say that  $(v, u, p)$  is a *weak solution* to (7) on the time interval  $[0, T]$  if the following hold:

- the mass equation is satisfied a.e.

$$\partial_t v - \partial_x u = 0, \quad v|_{t=0} = v^0.$$

- the momentum equation is satisfied in the sense of distributions

$$\begin{aligned} \int_{\mathbb{R}_+} \int_{\mathbb{R}} u \partial_t \varphi \, dx dt + \kappa \int_{\mathbb{R}_+} \int_{\mathbb{R}} v^{\tilde{\gamma}} \partial_x \varphi \, dx dt + \int_{\mathbb{R}_+} \int_{\mathbb{R}} \partial_x \varphi \, dp(t, x) \\ = - \int_{\mathbb{R}} u^0(x) \varphi(0, x) dx \quad \forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}); \end{aligned}$$

- the congestion and exclusion constraints are satisfied in the following sense

$$v(t, x) \geq 1 \quad \forall (t, x) \quad \text{and} \quad p \geq 0, \quad (v - 1)p = 0 \quad \text{in } \mathcal{D}'.$$

We now state the following two assumptions which will be used to pass to the limit  $\varepsilon \rightarrow 0$ .

**Assumption 1.6.** *Assume initially that*

$$\left( \frac{\varepsilon}{(v_\varepsilon^0 - 1)^{\gamma+1}} \right)^{\frac{1}{4}} \left( [\partial_x w_\varepsilon^0]_- + [\partial_x z_\varepsilon^0]_- \right) = \begin{cases} \mathcal{O}(\varepsilon^{\frac{1}{2(\gamma-1)}}) & \text{if } \gamma \in (1, 3), \\ \mathcal{O}(\varepsilon^{\frac{1}{4}}) & \text{if } \gamma = 3, \\ \mathcal{O}(\varepsilon^{\frac{1}{\gamma+1}}) & \text{if } \gamma > 3. \end{cases} \quad (12)$$

where  $[f]_- = \max(-f, 0)$ .

This assumption will be crucial to prove that the whole sequence  $(v_\varepsilon, u_\varepsilon)_\varepsilon$  exists on a time interval  $[0, T]$  independent of  $\varepsilon$ , provided that one has a control (in terms of  $\varepsilon$ ) of  $\partial_x u_\varepsilon^0, \partial_x v_\varepsilon^0$  in the regions initially close to the congestion constraint. In other words, Assumption 1.6 is designed so that the maximal time of existence of smooth solutions to (17) can be  $\varepsilon$ -uniformly bounded from below.

**Assumption 1.7.** *There exist  $\ell^*, \underline{v} > 0$ , both independent of  $\varepsilon$ , such that*

$$\frac{1}{2\ell} \int_{-\ell}^{\ell} v_\varepsilon^0(x) \, dx \geq \underline{v} > 1 \quad \text{for all } \ell \geq \ell^*. \quad (13)$$

Roughly speaking, this hypothesis guarantees that the initial specific volume  $v^0$  is not congested (i.e. equal to 1) in the whole domain and it is fundamental to provide an  $L^1$  local (in time and space) control of the pressure  $p_\varepsilon(v_\varepsilon)$  in Subsection 3.2 (see analogous conditions in [29] for instance).

The result below establishes the validity of the soft congestion approximation to the free-congested Euler equations.

**Theorem 1.8** (Singular limit). *Under the hypotheses of the previous Theorem and the additional Assumptions 1.6-1.7 on the initial data  $(v_\varepsilon^0, u_\varepsilon^0) \in \mathcal{C}^1(\mathbb{R})$ , there exist a time interval  $[0, T]$ , where  $T > 0$  is independent of  $\varepsilon$ , a sequence of unique (at  $\varepsilon$  fixed) smooth solutions  $(v_\varepsilon, u_\varepsilon) \in \mathcal{C}^1([0, T] \times \mathbb{R})$  to system (6), a limit initial datum  $(v^0, u^0)$  and a triplet  $(v, u, p)$  such that the following convergences hold (up to the extraction of a subsequence):*

$$\begin{aligned} v_\varepsilon^0 &\rightarrow v^0 \quad \text{strongly in } \mathcal{C}([-L, L]) \text{ and weakly-}^* \text{ in } W^{1,\infty}(\mathbb{R}) \\ u_\varepsilon^0 &\rightarrow u^0 \quad \text{strongly in } \mathcal{C}([-L, L]) \text{ and weakly-}^* \text{ in } W^{1,\infty}(\mathbb{R}) \\ v_\varepsilon &\rightarrow v \quad \text{strongly in } \mathcal{C}([0, T] \times [-L, L]) \text{ and weakly-}^* \text{ in } L^\infty((0, T); W^{1,\infty}(\mathbb{R})), \\ u_\varepsilon &\rightarrow u \quad \text{strongly in } L^q((0, T); \mathcal{C}([-L, L])), \quad \forall q \in [1, +\infty), \quad L > 0, \\ &\quad \text{and weakly-}^* \text{ in } L^\infty((0, T); W^{1,\infty}(\mathbb{R})) \\ p_{\varepsilon,1}(v_\varepsilon) &\rightharpoonup p \quad \text{in } \mathcal{M}_+((0, T) \times (-L, L)) \quad \forall L > 0. \end{aligned}$$

Moreover, the limit  $(v, u, p)$  is a weak solution of the free-congested  $p$ -system associated to the initial datum  $(v^0, u^0)$  in the sense of Definition 1.5. Finally, the couple  $(v, u)$  satisfies the incompressibility constraint in the congested domain, i.e.

$$\partial_x u = 0 \quad \text{a.e. on} \quad \{v = 1\}.$$

### Additional remarks.

- Remark 1.9* (Assumptions on the pressure). • We assumed in Theorems 1.4 and 1.8 that the exponent  $\tilde{\gamma}$  of the non-singular component of the pressure  $p_2$  lies in the interval  $(1, 3)$ . This assumption is mainly used when deriving a lower bound on the sequence of the maximal times  $(T_\varepsilon^*)_\varepsilon$  (see Proposition 3.1). However it is actually not necessary to guarantee the first part of Theorem 1.4, that is the global existence or the blow-up in finite time depending on the presence or not of a compression in the initial datum.
- The specific form of the pressure (3) (which blows up close to 1 like a power law) is used in the paper to exhibit the small scales associated to the singular limit  $\varepsilon \rightarrow 0$  (see in particular estimate (11) and Assumption 1.6). Nevertheless, we expect similar results for more general hard-sphere potentials. All the estimates will then depend on the specific balance between the parameter  $\varepsilon$  and the type of the singularity close to 1 encoded in the pressure law.
  - The assumption  $\kappa > 0$  is crucial in our analysis, since it allows the derivation of a uniform upper bound on  $v_\varepsilon$  (see Lemma 2.2). With a more general perspective, if  $\kappa = 0$ , then in the (limit) free domain where  $v > 1$  we would have a pressureless dynamics, where vacuum states are known to occur, see [9].

*Remark 1.10* (Control of the Riemann invariants and link with the internal energy). Assumption 1.2 bounds from below the distance between the initial specific volume  $v_\varepsilon^0$  and the minimal threshold  $v^* = 1$ , and allows a control of the Riemann invariants,  $w_\varepsilon$  and  $z_\varepsilon$  (see Section 2.1). From another perspective, the initial Assumption 1.2 guarantees that the internal energy at time 0 (and consequently for all times) is bounded uniformly with respect to  $\varepsilon$ . Indeed, defining the internal energy as

$$H_\varepsilon(\rho) = \frac{\varepsilon}{\gamma - 1} \frac{\rho^\gamma}{(1 - \rho)^{\gamma-1}} \quad (14)$$

which is such that

$$\rho H'_\varepsilon(\rho) - H_\varepsilon(\rho) = p_\varepsilon(\rho),$$

we ensure, thanks to (58), that

$$H_\varepsilon(\rho_\varepsilon^0) = \frac{\varepsilon}{\gamma - 1} \frac{(\rho_\varepsilon^0)^\gamma}{(1 - \rho_\varepsilon^0)^{\gamma-1}} \leq \frac{1}{(\gamma - 1)C_0^{\gamma-1}} =: H_0. \quad (15)$$

This provides a connection between the framework of smooth solutions and the setting of the weak ( $L^\infty$ ) solutions, which is summarized in the Appendix.

*Remark 1.11* (Consequences of the assumptions on the initial data). As a direct consequence of (12) and the relation  $\partial_x w_\varepsilon^0 + \partial_x z_\varepsilon^0 = \partial_x u_\varepsilon^0$ , we have

$$\left( \frac{\varepsilon}{(v_\varepsilon^0 - 1)^{\gamma+1}} \right)^{\frac{1}{4}} [\partial_x u_\varepsilon^0]_- \lesssim \begin{cases} \varepsilon^{\frac{1}{2(\gamma-1)}} & \text{if } \gamma \in (1, 3), \\ \varepsilon^{\frac{1}{4}} & \text{if } \gamma = 3, \\ \varepsilon^{\frac{1}{\gamma+1}} & \text{if } \gamma > 3. \end{cases} \quad (16)$$

Hence, in the regions close to the congestion constraint, i.e. where  $v_\varepsilon^0(x) - 1 = \mathcal{O}(\varepsilon^\alpha)$  with  $\alpha > 0$ , Assumption 1.6 implies

$$[\partial_x u_\varepsilon^0]_- \lesssim \begin{cases} \varepsilon^{\frac{3-\gamma}{4(\gamma-1)} + \frac{\alpha}{4}(\gamma+1)} & \text{if } \gamma \in (1, 3), \\ \varepsilon^{\frac{\alpha}{4}(\gamma+1)} & \text{if } \gamma = 3, \\ \varepsilon^{\frac{3-\gamma}{4(\gamma+1)} + \frac{\alpha}{4}(\gamma+1)} & \text{if } \gamma > 3, \end{cases}$$

so that  $[\partial_x u_\varepsilon^0]_-$  is small w.r.t.  $\varepsilon$  (except the case  $\gamma > 3$ ). The previous bounds (16) for  $\gamma \leq 3$  lead then to the following condition on the limit initial datum

$$\partial_x u^0 \geq 0 \quad \text{a.e. on } \{v^0 = 1\}.$$

We recall the less restrictive Assumption 1.3 constraining the positive part  $[\partial_x u^0]_+$  in the domain where  $c_\varepsilon(v_\varepsilon^0)$  blows up. We recover therefore a kind of compatibility condition  $\operatorname{div} u = 0$  in the congested/saturated regions. In the end, admissible initial data are such that  $\partial_x v_\varepsilon^0, \partial_x u_\varepsilon^0$  vary very slowly close to the congestion constraint.

Finally, let us mention that the present results are not in conflict with the numerical simulations by Bresch and Renardy [4] already discussed in the introduction. Indeed, the latter consider a non-negligible compression in the initial velocity:  $\partial_x u^0(x^*) = -1$  (independent of  $\varepsilon$ , which is not compatible with our initial assumption 1.6) leading to the development of shocks at the finite time  $t^*$  when  $v(t^*, x^*) = 1$ .

## Organization of the paper

In Section 2 we provide a detailed analysis of the singularity formation in the framework of smooth solutions to system (6) at  $\varepsilon$  fixed. Section 3 is dedicated to the study of the limit  $\varepsilon \rightarrow 0$  which provides convergence to the free-congested system (4). Finally, the existence of global weak solutions to (6) at  $\varepsilon$  fixed is postponed to the Appendix.

## 2 Analysis of the smooth solutions at $\varepsilon$ fixed

The one-dimensional compressible Euler equations in Lagrangian coordinates read as follows

$$\begin{cases} \partial_t v_\varepsilon - \partial_x u_\varepsilon = 0, & (17a) \\ \partial_t u_\varepsilon + \partial_x p_\varepsilon(v_\varepsilon) = 0, & (17b) \end{cases}$$

where  $v_\varepsilon = 1/\rho_\varepsilon$  is the specific volume and the pressure  $p_\varepsilon$  (which is in this section a function of  $v$ ) is given by

$$p_\varepsilon(v) = \frac{\varepsilon}{(v-1)^\gamma} + \frac{\kappa}{v^\gamma} =: p_{\varepsilon,1}(v) + p_2(v), \quad (18)$$

where  $\kappa > 0$ ,  $\gamma > 1$ ,  $\tilde{\gamma} \in (1, 3)$  and  $\varepsilon \leq \varepsilon_0$  is a positive small parameter. The characteristic speeds of system (17) are

$$\pm c_\varepsilon = \pm \sqrt{-p'_\varepsilon(v_\varepsilon)} = \pm \sqrt{\frac{\varepsilon\gamma}{(v_\varepsilon - 1)^{\gamma+1}} + \frac{\kappa\tilde{\gamma}}{v_\varepsilon^{\tilde{\gamma}+1}}} \quad (19)$$

and, introducing the quantity

$$\theta_\varepsilon(v) := \int_v^\infty c_\varepsilon(\tau) d\tau, \quad (20)$$

the Riemann invariants of system (17) read

$$w_\varepsilon = u_\varepsilon + \theta_\varepsilon(v_\varepsilon), \quad z_\varepsilon = u_\varepsilon - \theta_\varepsilon(v_\varepsilon). \quad (21)$$

## 2.1 Invariant regions: lower and upper bounds

Aiming at obtaining uniform bounds, the next step is to rearrange system (17) in terms of the Riemann invariants (21), so that

$$\begin{cases} \partial_t w_\varepsilon + c_\varepsilon \partial_x w_\varepsilon = 0, & (22a) \\ \partial_t z_\varepsilon - c_\varepsilon \partial_x z_\varepsilon = 0. & (22b) \end{cases}$$

It is now an easy task to get an a priori lower bound for the specific volume  $v_\varepsilon$  and an upper bound for the velocity  $u_\varepsilon$  as follows.

**Lemma 2.1.** *Under Assumption 1.2, there exists two positive constants  $C_1, C_2 > 0$  independent of  $\varepsilon$ , such that*

$$v_\varepsilon \geq 1 + C_1 \varepsilon^{\frac{1}{\gamma-1}}. \quad (23)$$

and

$$\|u_\varepsilon\|_{L_{t,x}^\infty} \leq C_2. \quad (24)$$

*Proof.* From the definition of the Riemann invariants (21),

$$w_\varepsilon^0 = u_\varepsilon^0 + \theta_\varepsilon(v_\varepsilon^0),$$

and from Assumption 1.2,

$$u_\varepsilon^0 \leq M_2, \quad \theta_\varepsilon(v_\varepsilon^0) \leq C(M_1).$$

Hence  $\|w_\varepsilon^0\|_{L^\infty} \leq M$  where  $M$  is independent of  $\varepsilon$ . Observing that  $w_\varepsilon$  and  $z_\varepsilon$  are constant along the characteristics, it is now classical to show that the domain is invariant

$$\Sigma := \{(v_\varepsilon, u_\varepsilon), w_\varepsilon \leq M, z_\varepsilon \geq -M\}. \quad (25)$$

This implies that

$$\theta_\varepsilon(v_\varepsilon) \leq 2M, \quad (26)$$

which directly yields the lower bound on  $v_\varepsilon$  in (23). One also has the control of the velocity

$$\|u_\varepsilon\|_{L^\infty} \leq \|w_\varepsilon\|_{L^\infty} + \|\theta_\varepsilon(v_\varepsilon)\|_{L^\infty} \leq 3M, \quad (27)$$

which concludes the proof.  $\square$

## 2.2 A uniform upper bound on the specific volume

Now we introduce the following change of variables, due to [10],

$$y_\varepsilon := \sqrt{c_\varepsilon} \partial_x w_\varepsilon, \quad q_\varepsilon := \sqrt{c_\varepsilon} \partial_x z_\varepsilon. \quad (28)$$

In terms of the new variables  $(y_\varepsilon, q_\varepsilon)$ , system (17) read

$$\begin{cases} \partial_t y_\varepsilon + c_\varepsilon \partial_x y_\varepsilon = -a_\varepsilon y_\varepsilon^2, \\ \partial_t q_\varepsilon - c_\varepsilon \partial_x q_\varepsilon = -a_\varepsilon q_\varepsilon^2, \end{cases} \quad (29a)$$

$$(29b)$$

where

$$a_\varepsilon = a_\varepsilon(v_\varepsilon) = -\frac{c'_\varepsilon(v_\varepsilon)}{2\sqrt{c_\varepsilon(v_\varepsilon)}c_\varepsilon(v_\varepsilon)} = \frac{p''_\varepsilon(v_\varepsilon)}{2(-p'_\varepsilon(v_\varepsilon))^{5/4}}. \quad (30)$$

We provide an  $\varepsilon$ -uniform upper bound on the specific volume  $v_\varepsilon$ .

**Lemma 2.2** (Upper bound on  $v_\varepsilon$ ). *Let  $\kappa > 0$ ,  $\tilde{\gamma} \in (1, 3)$ . Let  $(v_\varepsilon, u_\varepsilon)$  belonging to  $C^1_{t,x} = C^1([0, T] \times \mathbb{R})$  be a solution to system (17) on the time interval  $[0, T]$ , with initial data satisfying Assumptions 1.2 1.3. Then, there exists  $K = K(\kappa, M_1, M_2, Y^0, Q^0) > 0$ , independent of  $\varepsilon$ , such that*

$$v_\varepsilon(t, x) \leq K(1+t)^{\frac{4}{3-\tilde{\gamma}}} \quad \forall t \in [0, T]. \quad (31)$$

*Proof.* By comparison principle for ODEs, we ensure thanks to Assumption 1.2-1.3 that

$$y_\varepsilon(t, x) \leq \bar{Y} = \max\{0, Y^0\}, \quad q_\varepsilon(t, x) \leq \bar{Q} = \max\{0, Q^0\}, \quad (32)$$

which implies

$$\int_0^t y_\varepsilon(\tau, x) + q_\varepsilon(\tau, x) d\tau \leq (\bar{Y} + \bar{Q})t.$$

We recall that

$$y_\varepsilon + q_\varepsilon = 2\sqrt{c_\varepsilon} \partial_x u_\varepsilon = 2\sqrt{c_\varepsilon} \partial_t v_\varepsilon, \quad c_\varepsilon = \sqrt{-p'_\varepsilon(v_\varepsilon)} = \sqrt{\frac{\varepsilon\gamma}{(v_\varepsilon - 1)^{\gamma+1}} + \frac{\kappa\tilde{\gamma}}{v_\varepsilon^{\tilde{\gamma}+1}}}.$$

This way

$$\begin{aligned} \int_0^t y_\varepsilon(\tau, x) + q_\varepsilon(\tau, x) d\tau &= 2 \int_0^t \sqrt{c_\varepsilon} \partial_\tau v_\varepsilon d\tau = 2 \int_{v_\varepsilon^0}^{v_\varepsilon} \sqrt{c_\varepsilon(w)} dw \\ &\geq (\varepsilon\gamma)^{\frac{1}{4}} \int_{v_\varepsilon^0}^{v_\varepsilon} (w-1)^{-\frac{(\gamma+1)}{4}} dw + (\kappa\tilde{\gamma})^{\frac{1}{4}} \int_{v_\varepsilon^0}^{v_\varepsilon} w^{-\frac{(\tilde{\gamma}+1)}{4}} dw. \end{aligned}$$

Therefore we obtain

$$\Theta_\varepsilon(v_\varepsilon) \leq \Theta_\varepsilon(v_\varepsilon^0) + (\bar{Y} + \bar{Q})t \quad (33)$$

where

$$\Theta_\varepsilon(v) = \frac{4(\kappa\tilde{\gamma})^{1/4}}{(3-\tilde{\gamma})} v^{\frac{3-\tilde{\gamma}}{4}} + \begin{cases} \frac{4(\varepsilon\gamma)^{1/4}}{(3-\gamma)} (v-1)^{\frac{3-\gamma}{4}} & \text{if } \gamma \neq 3, \\ (\varepsilon\gamma)^{1/4} \ln(v-1) & \text{if } \gamma = 3. \end{cases}$$

Then, recalling that  $\tilde{\gamma} \in (1, 3)$ , we have if  $\gamma \in (1, 3)$

$$\Theta_\varepsilon(v_\varepsilon^0) = \frac{4(\kappa\tilde{\gamma})^{1/4}}{(3-\tilde{\gamma})} (v_\varepsilon^0)^{\frac{3-\tilde{\gamma}}{4}} + \frac{4(\varepsilon\gamma)^{1/4}}{(3-\gamma)} (v_\varepsilon^0 - 1)^{\frac{3-\gamma}{4}} \leq K \|v_\varepsilon^0\|_{L^\infty}^{\frac{3-\min(\gamma,\tilde{\gamma})}{4}},$$

where  $K$  denotes any  $K(\kappa, \tilde{\gamma})$ , and

$$\Theta_\varepsilon(v_\varepsilon) \geq \frac{4(\kappa\tilde{\gamma})^{1/4}}{(3-\tilde{\gamma})} (v_\varepsilon)^{\frac{3-\tilde{\gamma}}{4}}. \quad (34)$$

In the complementing case  $\gamma \geq 3$ , using (23) we have that for all  $\varepsilon < \varepsilon_0$

$$\begin{aligned} \frac{4(\varepsilon\gamma)^{\frac{1}{4}}}{(3-\gamma)} (v_\varepsilon - 1)^{\frac{3-\gamma}{4}} &\geq -\frac{4(\varepsilon\gamma)^{\frac{1}{4}}}{(\gamma-3)} (M_1^{-1}\varepsilon)^{\frac{\gamma-3}{4(\gamma-1)}} \geq -C(\gamma, M_1)\varepsilon^{\frac{1}{2(\gamma-1)}}, \text{ if } \gamma > 3, \\ (\varepsilon\gamma)^{1/4} \ln(v_\varepsilon - 1) &\geq -C(M_1)\varepsilon^{\frac{1}{4}}, \text{ if } \gamma = 3. \end{aligned}$$

Hence, for  $\varepsilon < \varepsilon_0$  small enough,  $\gamma \geq 3$ , it holds that

$$\begin{aligned} \Theta_\varepsilon(v_\varepsilon) &= \frac{4(\kappa\tilde{\gamma})^{1/4}}{(3-\tilde{\gamma})} v^{\frac{3-\tilde{\gamma}}{4}} + \begin{cases} \frac{4(\varepsilon\gamma)^{1/4}}{(3-\gamma)} (v-1)^{\frac{3-\gamma}{4}} & \text{if } \gamma \neq 3, \\ (\varepsilon\gamma)^{1/4} \ln(v-1) & \text{if } \gamma = 3. \end{cases} \\ &\geq \frac{(\kappa\tilde{\gamma})^{1/4}}{(3-\tilde{\gamma})} v^{\frac{3-\tilde{\gamma}}{4}} \end{aligned}$$

and on the other hand

$$\Theta_\varepsilon(v_\varepsilon^0) \leq K \|v_\varepsilon^0\|_{L^\infty}^{\frac{3-\tilde{\gamma}}{4}}.$$

Using again (34) and the bound  $\|v_\varepsilon^0\|_{L^\infty} \leq M_2$  from Assumption 1.2, we obtain

$$\frac{4(\kappa\tilde{\gamma})^{1/4}}{(3-\tilde{\gamma})} v_\varepsilon(t, x)^{\frac{3-\tilde{\gamma}}{4}} \leq K(1+t),$$

for a constant  $K = K(\kappa, \tilde{\gamma}, C_1, Y^0, Q^0)$  independent of  $\varepsilon$ . The proof is concluded.  $\square$

*Remark 2.3.* Notice that the assumption  $\kappa > 0$  is crucial for the derivation of a  $\varepsilon$ -uniform upper bound on  $v_\varepsilon$ . From the previous argument one has indeed that  $K \rightarrow +\infty$  as  $\kappa \rightarrow 0$ .

### 2.3 Existence of smooth solutions: non-compressive and compressive case

In this section, we provide an analysis of the smooth solutions to system (17). Two different situations are identified. We rely on Definition 1.1 presented at the beginning.

**Theorem 2.4.** *Under Assumption 1.2-1.3, we obtain the following dichotomy result.*

- If the initial datum is everywhere rarefactive in the sense of Definition 1.1, then there exists a unique global-in-time  $\mathcal{C}_{t,x}^1$  solution  $(v_\varepsilon, u_\varepsilon)$ , whose  $\mathcal{C}_{t,x}^1$ -norm is independent of  $\varepsilon$ .
- Otherwise, there exists a unique local  $\mathcal{C}_{t,x}^1$  solution  $(v_\varepsilon, u_\varepsilon)$  which breaks down in finite time  $T^* = T^*(\varepsilon) < +\infty$ .

The proof of this theorem relies on the following Lemma.

**Lemma 2.5.** *Let  $\kappa > 0$  be fixed,  $\varepsilon \leq \varepsilon_0$  small enough, and consider for  $v_\varepsilon > 1$ :*

$$a_\varepsilon(v_\varepsilon) = \frac{p_\varepsilon''(v_\varepsilon)}{2(-p_\varepsilon'(v_\varepsilon))^{5/4}} \quad \text{where} \quad p_\varepsilon(v_\varepsilon) = \frac{\varepsilon}{(v_\varepsilon - 1)^\gamma} + \frac{\kappa}{v_\varepsilon^{\tilde{\gamma}}} = p_{\varepsilon,1}(v_\varepsilon) + p_2(v_\varepsilon),$$

with  $\gamma > 1$ ,  $\tilde{\gamma} \in (1, 3)$ . We distinguish three main cases:

1. Case where  $v_\varepsilon - 1 = \mathcal{O}(\varepsilon^\alpha)$  with  $\frac{1}{\gamma+1} \leq \alpha \leq \frac{1}{\gamma-1}$ . There exist two positive constants  $K_1, K_2$ , independent of  $\varepsilon$ , such that

$$K_1 \varepsilon^{-\frac{1}{\gamma+1}} \leq a_\varepsilon(v_\varepsilon) \leq K_2 \varepsilon^{-\frac{1}{2(\gamma-1)}} \quad \text{if } \gamma \in (1, 3), \quad (35)$$

$$K_1 \varepsilon^{-\frac{1}{4}} \leq a_\varepsilon(v_\varepsilon) \leq K_2 \varepsilon^{-\frac{1}{4}} \quad \text{if } \gamma = 3, \quad (36)$$

$$K_1 \varepsilon^{-\frac{1}{2(\gamma-1)}} \leq a_\varepsilon(v_\varepsilon) \leq K_2 \varepsilon^{-\frac{1}{\gamma+1}} \quad \text{if } \gamma > 3. \quad (37)$$

2. Case where  $v - 1 = \mathcal{O}(\varepsilon^\alpha)$  with  $\frac{1}{\gamma+2} < \alpha < \frac{1}{\gamma+1}$ . There exist two positive constants  $K_1, K_2$ , independent of  $\varepsilon$ , such that

$$K_1 < K_1 \varepsilon^{1-\alpha(\gamma+2)} \leq a_\varepsilon(v_\varepsilon) \leq K_2 \varepsilon^{1-\alpha(\gamma+2)} < K_2 \varepsilon^{-\frac{1}{\gamma+1}}; \quad (38)$$

3. Case where  $\varepsilon^{\frac{1}{\gamma+2}} \lesssim v_\varepsilon - 1 \leq v_{max} - 1$ . There exist two positive constants  $K_1, K_2$ , independent of  $\varepsilon$ , such that

$$K_1 v_{max}^{\frac{\tilde{\gamma}-3}{4}} \leq a_\varepsilon(v_\varepsilon) \leq K_2.$$

*Proof.* The bounds on  $a_\varepsilon$  are directly derived from the expression of the pressure, which simplifies according to the considered regime, Case 1, 2 or 3. In each case, one of the two components of the pressure law is indeed negligible.

1. In the first regime, the singular component  $p_{\varepsilon,1}(v_\varepsilon)$  is dominant both in  $p_\varepsilon''$ ,  $p_\varepsilon'$ :

$$p_\varepsilon''(v_\varepsilon) \underset{\varepsilon \rightarrow 0}{\sim} \gamma(\gamma+1) \frac{\varepsilon}{(v_\varepsilon - 1)^{\gamma+2}}, \quad p_\varepsilon'(v_\varepsilon) \underset{\varepsilon \rightarrow 0}{\sim} -\gamma \frac{\varepsilon}{(v_\varepsilon - 1)^{\gamma+1}};$$

Thus,

$$a_\varepsilon(v_\varepsilon) \underset{\varepsilon \rightarrow 0}{\sim} \varepsilon^{-\frac{1}{4}(1+\alpha(3-\gamma))},$$

which directly provides the bounds of Case 1 using that  $\gamma \in (1, 3)$ .

2. In the intermediate regime, since for  $\frac{1}{\gamma+2} < \alpha < \frac{1}{\gamma+1}$

$$p_\varepsilon'(v_\varepsilon) = -\gamma \varepsilon^{1-\alpha(\gamma+1)} - \kappa \tilde{\gamma} v_\varepsilon^{-(\tilde{\gamma}+1)} \underset{\varepsilon \rightarrow 0}{\sim} -\tilde{\gamma} \frac{\kappa}{v_\varepsilon^{\tilde{\gamma}+1}} = p_2'(v_\varepsilon),$$

then the singular component  $p_{\varepsilon,1}(v_\varepsilon)$  is dominant only in  $p_\varepsilon''$ ,

$$p_\varepsilon''(v_\varepsilon) \underset{\varepsilon \rightarrow 0}{\sim} \gamma(\gamma+1) \frac{\varepsilon}{(v_\varepsilon - 1)^{\gamma+2}}.$$

3. In the last regime, since  $v_\varepsilon$  is “far” from 1, the component  $p_{\varepsilon,1}(v_\varepsilon)$  is negligible in both  $p_\varepsilon''$  and  $p_\varepsilon'$  as  $\varepsilon \rightarrow 0$ . The bounds on  $a_\varepsilon$  are directly derived from the upper and lower bounds on  $v_\varepsilon$  in  $p_2(v_\varepsilon)$ , using the fact that  $\tilde{\gamma} \in (1, 3)$ .

For sake of brevity, we omit further details.  $\square$

*Proof of Theorem 2.4.* Under Assumption 1.2 on the initial data, one can prove by classical arguments (see for instance [Section 7.8, [13]] the local existence of a unique  $C^1$  solution  $(v_\varepsilon, u_\varepsilon)$ .

*The rarefactive case.* We want to extend the previous local solution  $(v_\varepsilon, u_\varepsilon)$  by a continuity argument. For that purpose, we need to show a priori controls of the  $L^\infty$  and Lipschitz norms of  $(v_\varepsilon, u_\varepsilon)$ . Let us recall the result of Lemma 2.1: from the bounds on  $w_\varepsilon$  and  $z_\varepsilon$  (see (25)), we infer a control in  $L^\infty$  (uniform in  $\varepsilon$ ) on  $u_\varepsilon$  as well as a lower bound (23) on  $v_\varepsilon$ . Hence, it remains to show the control on  $(\partial_x v_\varepsilon, \partial_x u_\varepsilon)$ .

From hypothesis (8), we have  $(y_\varepsilon)|_{t=0} \geq 0$ ,  $(q_\varepsilon)|_{t=0} \geq 0$ . We ensure then  $y_\varepsilon(t, x) \geq 0$ ,  $q_\varepsilon(t, x) \geq 0$  for all times  $t \geq 0$  and, recalling (32), we deduce that

$$\|y_\varepsilon(t, \cdot)\|_{L^\infty} \leq \bar{Y}, \quad \|q_\varepsilon(t, \cdot)\|_{L^\infty} \leq \bar{Q}.$$

Thanks to Lemma 2.2, we have the following upper bound on  $v_\varepsilon$  (recall that  $\tilde{\gamma} \in (1, 3)$ ):

$$v_\varepsilon(t, x) \leq \left( v_\varepsilon^0(x)^{\frac{3-\tilde{\gamma}}{4}} + K(\bar{Y} + \bar{Q})t \right)^{\frac{4}{3-\tilde{\gamma}}}$$

and therefore

$$c_\varepsilon(t, x) \geq \sqrt{\frac{\kappa \tilde{\gamma}}{v_\varepsilon(t, x)^{\tilde{\gamma}+1}}} \geq \underline{c}(T) \quad \forall t \leq T.$$

This lower bound on  $c_\varepsilon$  allows us to control  $\partial_x w_\varepsilon$ :

$$\|\partial_x w_\varepsilon(t, \cdot)\|_{L^\infty} \leq \left\| \frac{y_\varepsilon(t, \cdot)}{\sqrt{c_\varepsilon(t, \cdot)}} \right\|_{L^\infty} \leq \bar{K}(t) \bar{Y} \leq \bar{K}(t),$$

where, hereafter,  $\bar{K}$  denotes a generic function of time which is bounded uniformly w.r.t  $\varepsilon$  for any finite time  $t$ . Similarly,

$$\|\partial_x z_\varepsilon(t, \cdot)\|_{L^\infty} \leq \left\| \frac{q_\varepsilon(t, \cdot)}{\sqrt{c_\varepsilon(t, \cdot)}} \right\|_{L^\infty} \leq \bar{K}(t).$$

Combining these two bounds then leads to the control of  $\partial_x u_\varepsilon$ :

$$\|\partial_x u_\varepsilon(t, \cdot)\|_{L^\infty} = \|\partial_x(w_\varepsilon + z_\varepsilon)(t, \cdot)\|_{L^\infty} \leq \bar{K}(t). \quad (39)$$

Finally, since

$$\theta'_\varepsilon(v_\varepsilon) \partial_x v_\varepsilon = \partial_x w_\varepsilon - \partial_x u_\varepsilon \quad \text{with} \quad \theta'_\varepsilon(v_\varepsilon) = -c_\varepsilon(v_\varepsilon)$$

we also control  $\partial_x v_\varepsilon$

$$\|\partial_x v_\varepsilon(t, \cdot)\|_{L^\infty} \leq \bar{K}(t). \quad (40)$$

Now, let us assume that the local solution  $(v_\varepsilon, u_\varepsilon)$  admits a finite maximal existence time  $T^* < +\infty$ . Since  $T^*$  is finite, then  $\underline{c}(T^*) > 0$ . As a consequence  $\overline{K}(T^*) < +\infty$  and the spatial  $\mathcal{C}_x^1$ -norm of  $(v_\varepsilon(T^*, \cdot), u_\varepsilon(T^*, \cdot))$  is controlled, Assumption 1.2-1.3 are satisfied at time  $T^*$ . Applying once again the local existence result starting at time  $T^*$ , we deduce that there exists  $t^* > 0$  such that the solution  $(v_\varepsilon, u_\varepsilon)$  can be extended on the time interval  $[0, T^* + t^*]$ . This is in contradiction with the fact that  $T^*$  is the maximal time of existence. Hence, the solution  $(v_\varepsilon, u_\varepsilon)$  exists globally in time and the previous estimates show that its  $\mathcal{C}_{t,x}^1$ -norm is independent of  $\varepsilon$ .

*The compressive case.* This is the case where (8) is not satisfied, i.e. there exists a point  $x^* \in \mathbb{R}$  such that

$$y_\varepsilon(0, x^*) = \sqrt{c_\varepsilon^0(x^*)} \partial_x w_\varepsilon^0(x^*) < 0 \quad \text{or} \quad q_\varepsilon(0, x^*) = \sqrt{c_\varepsilon^0(x^*)} \partial_x z_\varepsilon^0(x^*) < 0.$$

Let us consider the case where  $y_\varepsilon(0, x^*) < 0$  ( $q_\varepsilon(0, x^*) < 0$  is analogous). As a consequence of the Riccati equation (29a) one has, as long as the solution exists,

$$\frac{1}{y_\varepsilon(t, x_\varepsilon^+(t))} = \frac{1}{y_\varepsilon(0, x^*)} + \int_0^t a_\varepsilon(\tau, x_\varepsilon^+(\tau)) d\tau \quad (41)$$

where  $x_\varepsilon^+$  is the forward characteristic emanating from  $x^*$ , i.e.

$$\frac{dx_\varepsilon^+}{dt}(t) = c_\varepsilon(t, x_\varepsilon^+), \quad x_\varepsilon^+(0) = x^*.$$

This way, the appearance of a singularity in  $y_\varepsilon$  essentially depends on the function  $a_\varepsilon$ , whose asymptotic is detailed in Lemma 2.5. Since by Lemma 2.2,

$$v_\varepsilon(t, x) \leq K(1+t)^{\frac{4}{3-\bar{\gamma}}} =: v_{max}(t),$$

then we observe that, in all cases,

$$\int_0^T a_\varepsilon(t, x_\varepsilon^+(t)) dt \longrightarrow +\infty \quad \text{as} \quad T \rightarrow +\infty. \quad (42)$$

As a consequence, if there exists a point  $x^* \in \mathbb{R}$  such that  $y_\varepsilon(0, x^*) < 0$ , then there exists a finite time  $T_\varepsilon^*$  such that

$$\int_0^{T_\varepsilon^*} a_\varepsilon(t, x_\varepsilon^+(t)) dt = -\frac{1}{y_\varepsilon(0, x^*)} > 0, \quad (43)$$

whence  $y_\varepsilon(t, x_\varepsilon^+(t)) \rightarrow -\infty$  as  $t \rightarrow T_\varepsilon^*$ .

Using the same arguments, if there exists  $x^* \in \mathbb{R}$  such that  $q_\varepsilon(0, x^*) < 0$ , one can show the existence of a finite time  $T_\varepsilon^*$  such that

$$\int_0^{T_\varepsilon^*} a_\varepsilon(t, x_\varepsilon^-(t)) dt = -\frac{1}{q_\varepsilon(0, x^*)}$$

where  $x_\varepsilon^-$  is the backward characteristic emanating from  $x^*$ , i.e. s.t.  $(x_\varepsilon^-)'(t) = -c_\varepsilon$  and  $x_\varepsilon^-(0) = x^*$ . This achieves the proof of the second part of Theorem 2.4.  $\square$

### 3 Singular limit in the smooth setting

In this section, we aim at justifying the limit in the vanishing  $\varepsilon$  parameter of the singular p-system (17). To this end, since we know from Theorem 2.4 that the maximal time of existence of the smooth solution  $(u_\varepsilon, v_\varepsilon)$  can be finite at  $\varepsilon$  fixed, we need to make sure that it is  $\varepsilon$ -uniformly bounded from below, and does not shrink to zero as  $\varepsilon$  vanishes. This is indeed proved in Proposition 3.1. Later, we employ Assumption 1.7 (together with Assumption 1.2) to obtain a uniform control of the singular pressure. Putting all these results together, we finally pass to the limit in Subsection 3.3.

#### 3.1 Lower bound on the maximal existence time

**Proposition 3.1.** *Let Assumptions 1.2-1.3-1.6 hold. Then there exists  $T > 0$ , independent of  $\varepsilon$ , such that, for any  $\varepsilon < \varepsilon_0$ , the smooth solution  $(v_\varepsilon, u_\varepsilon)$  provided by Theorem 2.4 exists on the whole interval  $[0, T]$ .*

*Proof.* In the case where the initial datum is *everywhere rarefactive*, we know from Theorem 2.4 that the smooth solution to (17) exists for all times. Then we need to handle the *compressive case*. More precisely, we need to show that we can bound from below, uniformly in  $\varepsilon$ , the maximal time of existence  $T_\varepsilon^*$  when there is some compression in the initial data. Let us assume for instance that  $y_\varepsilon(0, x^*) < 0$ . Then by (41) we have at the explosion time  $T_\varepsilon^*$  that

$$\int_0^{T_\varepsilon^*} a_\varepsilon(t, x_\varepsilon^+(t)) dt = -\frac{1}{y_\varepsilon(0, x^*)}. \quad (44)$$

To derive a lower bound on  $T_\varepsilon^*$ , we need an estimate of  $a_\varepsilon(v_\varepsilon)$ , which is in fact provided by Lemma 2.5 and depends on the distance between  $v_\varepsilon$  and 1. In this regard, Lemma 2.5 distinguishes three main cases, which we analyse here.

- *Case 3 of Lemma 2.5.* We begin with the case where  $v_\varepsilon^0(x^*) \gtrsim 1 + \varepsilon^{\frac{1}{\gamma+2}}$ , namely the initial specific volume evaluated at the point  $x^*$  is “far” from 1. Then, Lemma 2.5 ensures the existence a constant  $K_2 > 0$ , independent of  $\varepsilon$  such that

$$0 < a_\varepsilon(0, x^*) \leq K_2.$$

On the one hand, thanks to Assumption 1.2, we also have that

$$y_\varepsilon(0, x^*) = \sqrt{c_\varepsilon^0(x^*)} \left( \partial_x u_\varepsilon^0(x^*) + \theta'_\varepsilon(v_\varepsilon^0(x^*)) \partial_x v_\varepsilon^0(x^*) \right) \geq -K_3$$

for some constant  $K_3 > 0$  which is independent of  $\varepsilon$ . We have then two possibilities. The first option is that  $v_\varepsilon$  remains “far” from the congestion constraint (i.e. in Case 3) until the singularity occurs at  $T_\varepsilon^*$ . In this case,  $a_\varepsilon$  remains bounded from above uniformly with respect to  $\varepsilon > 0$  on the whole time interval  $[0, T_\varepsilon^*)$ , i.e.

$$0 < a_\varepsilon(v_\varepsilon(t, x)) \leq K_4, \quad t \in [0, T_\varepsilon^*),$$

with  $K_4 > 0$  independent of  $\varepsilon$ . Hence, using again (44), we infer the desired  $\varepsilon$ -uniform lower bound on  $T_\varepsilon^*$

$$T_\varepsilon^* \geq \frac{1}{\sup_{t \in [0, T_\varepsilon^*)} a_\varepsilon(t, x_\varepsilon^+(t))} \int_0^{T_\varepsilon^*} a_\varepsilon(t, x_\varepsilon^+(t)) dt \geq -\frac{1}{K_4} \frac{1}{y_\varepsilon(0, x^*)} \geq \frac{1}{K_4} \frac{1}{K_3}. \quad (45)$$

The alternative scenario is that, at some time  $t^* < T_\varepsilon^*$ ,  $v_\varepsilon$  gets closer to the congestion threshold, passing through the intermediate regime of Case 2. This would imply that  $v_\varepsilon(t^*, x_\varepsilon^+(t^*)) < 1 + \varepsilon^\alpha$  with  $\alpha > \frac{1}{\gamma+2}$ . In this case, by continuity of the solution  $(v_\varepsilon, u_\varepsilon)$ , we can find a positive time  $\bar{t} < t^*$  such that

$$0 < a_\varepsilon(t, x_\varepsilon^+(t)) \leq 2K_2, \quad \forall t \in [0, \bar{t}].$$

Replacing  $T_\varepsilon^*$  by  $\bar{t}$  and  $K_4$  by  $2K_2$  in (45), we deduce that

$$\bar{t} \geq \frac{1}{2K_2K_3},$$

namely  $\bar{t}$  is bounded from below uniformly with respect to  $\varepsilon$ , and  $T_\varepsilon^* > t^* > \bar{t}$  as well.

- *Case 1 and 2 of Lemma 2.5.* Let us now deal with the worst scenario: the case where  $v_\varepsilon^0(x^*)$  is close to the congestion constraint, namely

$$v_\varepsilon^0(x^*) - 1 = \mathcal{O}(\varepsilon^\alpha), \quad \frac{1}{\gamma+2} \leq \alpha \leq \frac{1}{\gamma-1}.$$

If at some time  $\bar{t} < T_\varepsilon^*$ ,  $v_\varepsilon(\bar{t}, x_\varepsilon^+(\bar{t}))$  escapes from this domain, i.e. the specific volume gets away from the congestion constraint, then we are back to the previous case and we bound from below  $T_\varepsilon^*$ . So, we assume that

$$v_\varepsilon(t, x_\varepsilon^+(t)) - 1 = \mathcal{O}(\varepsilon^\alpha), \quad \frac{1}{\gamma+2} \leq \alpha \leq \frac{1}{\gamma-1}, \quad \forall t \in [0, T_\varepsilon^*].$$

Let us recall that from Lemma 2.5, we have

$$\begin{aligned} a_\varepsilon(t, x_\varepsilon^+(t)) &\leq K_2 \varepsilon^{-\frac{1}{2(\gamma-1)}} && \text{if } \gamma \in (1, 3), \\ a_\varepsilon(t, x_\varepsilon^+(t)) &\leq K_2 \varepsilon^{-\frac{1}{4}} && \text{if } \gamma = 3, \\ a_\varepsilon(t, x_\varepsilon^+(t)) &\leq K_2 \varepsilon^{-\frac{1}{\gamma+1}} && \text{if } \gamma > 3; \end{aligned}$$

and thus

$$T_\varepsilon^* \geq -\frac{1}{\sup_{t \in [0, T_\varepsilon^*]} a_\varepsilon(t, x_\varepsilon^+(t))} \frac{1}{y_\varepsilon(0, x^*)} \geq \begin{cases} -\frac{\varepsilon^{\frac{1}{2(\gamma-1)}}}{K_2 y_\varepsilon(0, x^*)} & \text{if } \gamma \in (1, 3), \\ -\frac{\varepsilon^{\frac{1}{4}}}{K_2 y_\varepsilon(0, x^*)} & \text{if } \gamma = 3, \\ -\frac{\varepsilon^{\frac{1}{\gamma+1}}}{K_2 y_\varepsilon(0, x^*)} & \text{if } \gamma > 3. \end{cases}$$

Thanks to Assumption 1.3, we guarantee in the three cases that  $y_\varepsilon(0, x^*)$  will be small enough to compensate the blow up of  $a_\varepsilon$  as  $\varepsilon \rightarrow 0$ , namely

$$|y_\varepsilon(0, x^*)| = \sqrt{c_\varepsilon^0(x^*)} |\partial_x w_\varepsilon^0(x^*)| = \begin{cases} \mathcal{O}(\varepsilon^{\frac{1}{2(\gamma-1)}}) & \text{if } \gamma \in (1, 3), \\ \mathcal{O}(\varepsilon^{\frac{1}{4}}) & \text{if } \gamma = 3, \\ \mathcal{O}(\varepsilon^{\frac{1}{\gamma+1}}) & \text{if } \gamma > 3. \end{cases}$$

Hence, we obtain a lower bound on  $T_\varepsilon^*$  which is uniform with respect to  $\varepsilon$ . Notice that this is the point where Assumption 1.6 plays its key role in providing an  $\varepsilon$ -uniform lower bound on the maximal existence time.

□

From now on, we shall consider the time interval  $[0, T]$  on which the whole sequence of solutions  $(v_\varepsilon, u_\varepsilon)_\varepsilon$  exists,  $T$  being independent of  $\varepsilon$ .

### 3.2 Control of the pressure

We have previously proved in Lemma 2.1 that  $v_\varepsilon$  was bounded from below (cf (23)):

$$v_\varepsilon \geq 1 + C_1 \varepsilon^{\frac{1}{\gamma-1}}.$$

Unfortunately, this bound does not provide any control on the pressure  $p_\varepsilon(v_\varepsilon)$  as  $\varepsilon \rightarrow 0$  since it only yields the inequality

$$p_\varepsilon(v_\varepsilon) \leq \frac{\varepsilon}{(1 + C_1 \varepsilon^{\frac{1}{\gamma-1}} - 1)^\gamma} \lesssim \varepsilon^{-\frac{1}{\gamma-1}}.$$

The goal of this section is to prove a uniform control of  $\|p_\varepsilon(v_\varepsilon)\|_{L^1_{loc}}$ .

**Proposition 3.2.** *Let Assumption 1.2-1.7 hold. Then there exists a positive constant  $C$ , independent of  $\varepsilon$ , such that*

$$\|p_\varepsilon(v_\varepsilon)\|_{L^1((0,T) \times (-L,L))} \leq C \quad \forall L > 0. \quad (46)$$

*Proof.* Thanks to Assumption 1.7, there exists two positive constants  $\ell^* > 0$  and  $\underline{v} > 1$ , independent of  $\varepsilon$ , such that

$$\langle v_\varepsilon^0 \rangle := \frac{1}{2\ell} \int_{-\ell}^{\ell} v_\varepsilon^0(x) dx \geq \underline{v} > 1 \quad \forall \ell \geq \ell^*. \quad (47)$$

From the first equation of system (17), we infer that

$$\langle v_\varepsilon(t) \rangle := \frac{1}{2\ell} \int_{-\ell}^{\ell} v_\varepsilon(t, x) dx = \langle v_\varepsilon^0 \rangle + \frac{1}{2\ell} \int_0^t (u_\varepsilon(s, \ell) - u_\varepsilon(s, -\ell)) ds. \quad (48)$$

From the  $L_x^\infty$  bound on  $u_\varepsilon$  provided by (24) of Lemma 2.1, we have

$$u_\varepsilon(t, \ell) - u_\varepsilon(t, -\ell) \geq -2\|u_\varepsilon\|_{L^\infty} = -2C_2 \quad \forall t \in [0, T],$$

so that

$$\begin{aligned} \langle v_\varepsilon(t) \rangle &= \langle v_\varepsilon^0 \rangle + \frac{1}{2\ell} \int_0^t u_\varepsilon(s, \ell) - u_\varepsilon(s, -\ell) ds \\ &\geq \langle v_\varepsilon^0 \rangle - \frac{C_2 t}{\ell} \\ &\geq \underline{v} - \frac{C_2 t}{\ell}. \end{aligned} \quad (49)$$

Choosing now

$$\ell \geq \max \left\{ \ell^*, \frac{2C_2T}{\underline{v}-1} \right\} := L^*, \quad (50)$$

for all  $t \in [0, T]$  one has that

$$\langle v_\varepsilon(t) \rangle \geq \underline{v} - \frac{C_2T}{\ell} \geq \underline{v} + \frac{1-\underline{v}}{2} \geq \frac{\underline{v}+1}{2} > 1. \quad (51)$$

In order to control of the pressure, let us define the function

$$\phi(t, x) = \begin{cases} \frac{(x+L)}{2L} \int_{-L}^L v_\varepsilon(t, z) dz - \int_{-L}^x v_\varepsilon(t, z) dz & \text{if } x \in [-L, L], \\ 0 & \text{otherwise,} \end{cases}$$

for some fixed  $L \geq L^*$  where  $L^*$  has been introduced in (50) and is independent of  $\varepsilon$ . Since  $v_\varepsilon$  is smooth, then  $\phi \in \mathcal{C}_c^1([0, T] \times \mathbb{R})$ . Now, we multiply the momentum equation in (17) (which holds point-wisely) by  $\phi$  and we integrate in space and time. This yields

$$\int_0^T \int_{-L}^L p_\varepsilon(v_\varepsilon) \partial_x \phi \, dx dt = - \int_0^T \int_{-L}^L u_\varepsilon \partial_t \phi \, dx dt + \int_{-L}^L u_\varepsilon^0(x) \phi(0, x) \, dx.$$

In the right-hand side first we have

$$\begin{aligned} & \int_0^T \int_{-L}^L u_\varepsilon(t, x) \partial_t \phi(t, x) \, dx dt \\ &= \int_0^T \int_{-L}^L u_\varepsilon(t, x) \left[ \frac{(x+L)}{2L} \int_{-L}^L \partial_t v_\varepsilon(t, z) \, dz - \int_{-L}^x \partial_t v_\varepsilon(t, z) \, dz \right] \, dx dt \\ &= \int_0^T \int_{-L}^L u_\varepsilon(t, x) \left[ \frac{(x+L)}{2L} \int_{-L}^L \partial_x u_\varepsilon(t, z) \, dz - \int_{-L}^x \partial_x u_\varepsilon(t, z) \, dz \right] \, dx dt \\ &= \int_0^T \int_{-L}^L u_\varepsilon(t, x) \left[ \frac{(x+L)}{2L} (u_\varepsilon(t, L) - u_\varepsilon(t, -L)) - (u_\varepsilon(t, x) - u_\varepsilon(t, -L)) \right] \, dx dt. \end{aligned}$$

Hence

$$\left| \int_0^T \int_{-L}^L u_\varepsilon(t, x) \partial_t \phi(t, x) \, dx dt \right| \leq C(T, L) \|u_\varepsilon\|_{L^\infty}.$$

The next term is

$$\int_{-L}^L u_\varepsilon^0(x) \phi(0, x) \, dx = \int_{-L}^L u_\varepsilon^0(x) \left[ \frac{(x+L)}{2L} \int_{-L}^L v_\varepsilon^0(z) \, dz - \int_{-L}^x v_\varepsilon^0(z) \, dz \right] \, dx,$$

which is controlled as follows

$$\left| \int_{-L}^L u_\varepsilon^0(x) \phi(0, x) \, dx \right| \leq C(L) \|u_\varepsilon^0\|_{L^\infty} \|v_\varepsilon^0\|_{L^\infty}.$$

Now, we split in two parts the integral involving the pressure as

$$\begin{aligned} \int_0^T \int_{-L}^L p_\varepsilon(v_\varepsilon) \partial_x \phi \, dx dt &= \int_0^T \int_{-L}^L p_\varepsilon(v_\varepsilon) \partial_x \phi \mathbf{1}_{\{v_\varepsilon > \frac{3+\underline{v}}{4}\}} \, dx dt \\ &\quad + \int_0^T \int_{-L}^L p_\varepsilon(v_\varepsilon) \partial_x \phi \mathbf{1}_{\{v_\varepsilon \leq \frac{3+\underline{v}}{4}\}} \, dx dt. \end{aligned}$$

Since  $\underline{v} > 1$  uniformly in  $\varepsilon$  and

$$1 < \frac{3+\underline{v}}{4} = \frac{1+\frac{\underline{v}+1}{2}}{2} < \frac{\underline{v}+1}{2} < \underline{v},$$

then the pressure  $p_\varepsilon(v_\varepsilon)$  remains bounded on the set  $\{v_\varepsilon > \frac{3+\underline{v}}{4}\}$ , so providing a positive constant  $C$ , independent of  $\varepsilon$ , such that

$$\left| \int_0^T \int_{-L}^L p_\varepsilon(v_\varepsilon) \partial_x \phi \mathbf{1}_{\{v_\varepsilon > \frac{3+\underline{v}}{4}\}} \, dx dt \right| \leq C.$$

Therefore, using (51) and the fact that  $L \geq L^*$  in (50), we have

$$\begin{aligned} C &\geq \left| \int_0^T \int_{-L}^L p_\varepsilon(v_\varepsilon) \partial_x \phi \mathbf{1}_{\{v_\varepsilon \leq \frac{3+\underline{v}}{4}\}} \, dx dt \right| \\ &= \left| \int_0^T \int_{-L}^L p_\varepsilon(v_\varepsilon) \left( \frac{1}{2L} \int_{-L}^L v_\varepsilon(t, z) \, dz - v_\varepsilon(t, x) \right) \mathbf{1}_{\{v_\varepsilon \leq \frac{3+\underline{v}}{4}\}} \, dx dt \right| \\ &\geq \int_0^T \int_{-L}^L p_\varepsilon(v_\varepsilon) \left( \frac{\underline{v}+1}{2} - \frac{3+\underline{v}}{4} \right) \mathbf{1}_{\{v_\varepsilon \leq \frac{3+\underline{v}}{4}\}} \, dx dt \\ &\geq \frac{\underline{v}-1}{4} \int_0^T \int_{-L}^L p_\varepsilon(v_\varepsilon) \mathbf{1}_{\{v_\varepsilon \leq \frac{3+\underline{v}}{4}\}} \, dx dt. \end{aligned}$$

Thus we also obtain the control of the integral of the singular pressure in the region close to the singularity. In the end, we proved that

$$(p_\varepsilon(v_\varepsilon))_\varepsilon \text{ is bounded in } L^1((0, T) \times (-L, L)) \text{ for all } L \geq L^*,$$

and thus

$$(p_\varepsilon(v_\varepsilon))_\varepsilon \text{ is bounded in } L^1((0, T) \times (-L, L)) \text{ for all } L > 0.$$

□

### 3.3 Passing to the limit as $\varepsilon \rightarrow 0$

In this section, we prove Theorem 1.8 under Assumptions 1.2-1.7. On the time interval  $[0, T]$ , the solution  $(v_\varepsilon, u_\varepsilon)$  exists and is regular. More precisely, the previous sections have shown that there exists a constant  $C$ , independent of  $\varepsilon$  such that

$$\begin{aligned} \|u_\varepsilon\|_{L^\infty((0, T) \times \mathbb{R})} &\leq C, \\ \|v_\varepsilon\|_{L^\infty((0, T) \times \mathbb{R})} &\leq C, \\ \|\partial_x u_\varepsilon\|_{L^\infty((0, T) \times \mathbb{R})} &\leq C, \\ \|\partial_x v_\varepsilon\|_{L^\infty((0, T) \times \mathbb{R})} &\leq C, \\ \|p_\varepsilon(v_\varepsilon)\|_{L^1((0, T) \times (-L, L))} &\leq C, \quad \forall L > 0. \end{aligned}$$

From these bounds, we infer the existence of a pair  $(v, u)$  such that

$$\begin{aligned} v_\varepsilon &\rightharpoonup v, \quad \text{weakly-}^* \text{ in } L^\infty((0, T); W^{1, \infty}(\mathbb{R})), \\ u_\varepsilon &\rightharpoonup u \quad \text{weakly-}^* \text{ in } L^\infty((0, T); W^{1, \infty}(\mathbb{R})). \end{aligned}$$

Since the lower bound (23) holds at  $\varepsilon$  fixed thanks to Lemma 2.1, then the congestion constraint involving the specific volume is satisfied in the limit, i.e.

$$v(t, x) \geq 1 \quad \text{a.e. } (t, x) \in (0, T) \times \mathbb{R}. \quad (52)$$

Employing the bound on  $\partial_x u_\varepsilon$  in (40) in the mass equation, (17a), we uniformly control the time derivative of  $v_\varepsilon$  as follows

$$\|\partial_t v_\varepsilon\|_{L^\infty((0, T) \times \mathbb{R})} = \|\partial_x u_\varepsilon\|_{L^\infty((0, T) \times \mathbb{R})} \leq C.$$

This way, it is now an easy task to apply the classical Aubin-Lions Lemma to get

$$v_\varepsilon \rightarrow v \quad \text{in } C([0, T] \times [-L, L]), \quad \forall L > 0. \quad (53)$$

Applying the same reasoning to the velocity  $u_\varepsilon$ , and using in particular the control of the  $L^1_{t,x}$ -norm of the pressure, we obtain

$$\|\partial_t u_\varepsilon\|_{L^1((0, T); W_{loc}^{-1, 1}(\mathbb{R}))} \leq C.$$

A generalization of the Aubin-Lions Lemma due to Simon, see [32], provides the convergence

$$u_\varepsilon \rightarrow u \quad \text{in } L^q((0, T); \mathcal{C}([-L, L])), \quad \forall q \in [1, +\infty), \quad L > 0. \quad (54)$$

As the pressure is made of two parts

$$p_\varepsilon(v_\varepsilon) = \frac{\varepsilon}{(v_\varepsilon - 1)^\gamma} + \frac{\kappa}{v_\varepsilon^\gamma} = p_{\varepsilon, 1}(v_\varepsilon) + p_{\varepsilon, 2}(v_\varepsilon),$$

the strong convergence of the non-singular component  $p_{\varepsilon, 2}(v_\varepsilon)$  directly follows from (53), so that

$$p_{\varepsilon, 2}(v_\varepsilon) \rightarrow p_2(v) = \frac{\kappa}{v^\gamma} \quad \text{in } C([0, T], W_{loc}^{s, \infty}(\mathbb{R})), \quad \forall 0 < s < 1.$$

About the singular component  $p_{\varepsilon, 1}(v_\varepsilon)$ , we use the  $L^1$  uniform bound to infer that

$$p_{\varepsilon, 1}(v_\varepsilon) \rightharpoonup p \quad \text{in } \mathcal{M}_+((0, T) \times (-L, L)) \quad \forall L > 0. \quad (55)$$

Finally, to recover the exclusion constraint, we pass to the limit in the equality

$$(v_\varepsilon - 1)p_{\varepsilon, 1}(v_\varepsilon) = \frac{\varepsilon}{(v_\varepsilon - 1)^{\gamma-1}}.$$

On the one hand  $(v_\varepsilon - 1)$  converges in  $\mathcal{C}([0, T] \times [-L, L])$  to  $v - 1$ , so that the left-hand side of the above equality converges in the sense of distribution towards  $(v - 1)p$ . While the right-hand side converges strongly to 0 in  $L^{\frac{\gamma}{\gamma-1}}((0, T) \times (-L, L))$  thanks to the following inequality (and the uniform  $L^1$  bound on the pressure),

$$\frac{\varepsilon}{(v_\varepsilon - 1)^{\gamma-1}} \leq \varepsilon^{\frac{1}{\gamma}} \left( \frac{\varepsilon}{(v_\varepsilon - 1)^\gamma} \right)^{\frac{\gamma-1}{\gamma}} = \varepsilon^{\frac{1}{\gamma}} (p_\varepsilon^1(v_\varepsilon))^{\frac{\gamma-1}{\gamma}}.$$

Hence, we get the desired exclusion constraint

$$(v - 1) p = 0, \quad (56)$$

which finally allows to show that  $(v, u, p)$  is a weak solution of the free-congested Euler equations.

As a final remark, note that we are able to show that the incompressibility constraint is satisfied by the limit velocity  $u$  in the congested domain where  $v = 1$ . We have indeed the following lemma.

**Lemma 3.3.** *Let  $T > 0$ ,  $v \in W^{1,\infty}((0, T) \times \mathbb{R})$  and  $u \in L^\infty((0, T); W^{1,\infty}(\mathbb{R}))$  satisfying*

$$\partial_t v = \partial_x u, \quad v|_{t=0} = v^0 \quad \text{a.e.}$$

*The following two assertions are equivalent:*

1.  $v(t, x) \geq 1$  for all  $(t, x) \in [0, T] \times \mathbb{R}$ ;
2.  $\partial_x u = 0$  a.e. on  $\{v \leq 1\}$  and  $v^0 \geq 1$ .

For sake of brevity, we omit the proof of this lemma. The interested reader is referred to [29] where a similar result is demonstrated.

## 4 Appendix: Existence of global weak solutions at $\varepsilon$ fixed

To our knowledge, the existence of global weak solutions to (1) at  $\varepsilon > 0$  has not been demonstrated in the literature. An familiar reader can presume that the general result of Chen and LeFloch [8] applies here although this study does not explicitly deal with hard-sphere potentials. For sake of completeness, and with the purpose of tracking explicitly the  $\varepsilon$ -dependency in the pressure law, we sketch in this appendix the main lines of the proof of existence of weak bounded solutions for the pressure

$$p_\varepsilon(\rho) = \varepsilon \left( \frac{\rho}{1 - \rho} \right)^\gamma \quad \text{with } \gamma \in (1, 3]. \quad (57)$$

**Definition 4.1** (Weak entropy solutions to (1)). Let  $(\rho_\varepsilon^0(x), m_\varepsilon^0(x)) \in (L^\infty(\mathbb{R}))^2$  satisfying for some  $C_0 > 0$  (independent of  $\varepsilon$ )

$$0 \leq \rho_\varepsilon^0 \leq 1 - C_0 \varepsilon^{\frac{1}{\gamma-1}} =: A_\varepsilon^0 \quad \text{a.e. on } \mathbb{R}, \quad m_\varepsilon^0(x) \leq A_\varepsilon^0 \rho_\varepsilon^0(x) \quad \text{a.e. on } \mathbb{R}. \quad (58)$$

We call  $(\rho_\varepsilon, m_\varepsilon)$  *global weak entropy solution* to (1) if the following hold:

- $(\rho_\varepsilon, m_\varepsilon) \in (L^\infty(\mathbb{R}_+ \times \mathbb{R}))^2$  and there exists  $A_\varepsilon > 0$  such that

$$0 \leq \rho_\varepsilon \leq A_\varepsilon < 1 \quad \text{and} \quad |m_\varepsilon| \leq A_\varepsilon \rho_\varepsilon \quad \text{a.e.};$$

- the mass and momentum equations are satisfied in the weak sense

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} \rho_\varepsilon \partial_t \varphi \, dx dt + \int_{\mathbb{R}_+} \int_{\mathbb{R}} m_\varepsilon \partial_x \varphi \, dx dt = - \int_{\mathbb{R}} \rho_\varepsilon^0(x) \varphi(0, x) dx \quad \forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R});$$

$$\begin{aligned} & \int_{\mathbb{R}_+} \int_{\mathbb{R}} m_\varepsilon \partial_t \varphi \, dx dt + \int_{\mathbb{R}_+} \int_{\mathbb{R}} \left( \frac{m_\varepsilon^2}{\rho_\varepsilon} + p_\varepsilon(\rho_\varepsilon) \right) \partial_x \varphi \, dx dt \\ &= - \int_{\mathbb{R}} m_\varepsilon^0(x) \varphi(0, x) dx \quad \forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R}), \end{aligned}$$

- the entropy inequality is satisfied, i.e. for any pair  $(\eta, q)$  of entropy-entropy flux,  $\eta$  convex, and  $\phi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \times \mathbb{R})$ ,  $\phi \geq 0$ ,

$$- \int_{\mathbb{R}_+} \int_{\mathbb{R}} \eta(\rho_\varepsilon, m_\varepsilon) \partial_t \phi \, dx dt - \int_{\mathbb{R}_+} \int_{\mathbb{R}} q(\rho_\varepsilon, m_\varepsilon) \partial_x \phi \, dx dt \leq 0.$$

**Theorem 4.2** (Existence of global weak solutions). *Consider the pressure law (57) with  $\gamma \in (1, 3]$ . Let  $(\rho_\varepsilon^0, m_\varepsilon^0) \in (L^\infty(\mathbb{R}))^2$  satisfy conditions (58). Then there exists a global weak entropy solution  $(\rho_\varepsilon, m_\varepsilon)$  to (1) in the sense of Definition 4.1. Moreover, the following inequality holds*

$$0 \leq \rho_\varepsilon \leq 1 - C\varepsilon^{\frac{1}{\gamma-1}} \quad \text{a.e.} \quad (59)$$

for some generic constant  $C$  independent of  $\varepsilon$ .

As usual for this kind of problem, the general strategy of proof consists of three steps. First we need to find a sequence of approximate solutions  $(\rho_\mu, m_\mu)_{\mu>0}$ . Next, we have to prove that the sequence is relatively compact in a suitable functional space. We can then extract a subsequence which converges (for the corresponding topology) to a limit  $(\rho, m)$  and the last step is to show that  $(\rho, m)$  is actually a solution to the original equations.

As starting system, we choose the following parabolic regularization (other regularizations could be studied, like the Navier-Stokes equations in the context of weak finite energy solutions, see [7] or [34]):

$$\begin{cases} \partial_t \rho_\mu + \partial_x m_\mu = \mu \partial_{xx} \rho_\mu, & (60a) \\ \partial_t m_\mu + \partial_x \left( \frac{m_\mu^2}{\rho_\mu} \right) + \partial_x p_\varepsilon(\rho_\mu) = \mu \partial_{xx} m_\mu & (60b) \end{cases}$$

To this system we associate the regularized initial data  $(\rho_{\varepsilon,\mu}^0(x), m_{\varepsilon,\mu}^0(x))$  constructed by convolution of  $(\rho_\varepsilon^0, m_\varepsilon^0)$  with a standard mollifier and which is such that

$$0 < a_\mu^0 \leq \rho_{\varepsilon,\mu}^0 \leq A_\varepsilon^0 < 1, \quad m_{\varepsilon,\mu}^0(x) \leq A_\varepsilon^0 \rho_{\varepsilon,\mu}^0(x) \quad \text{a.e. on } \mathbb{R}.$$

Weak-\* convergence of the sequence of approximate solutions is ensured by the *invariant region method* which provides  $L^\infty$  bounds (uniform with respect to the viscosity parameter) on the sequence as presented in the following lemma.

**Lemma 4.3.** *Define the Riemann invariants of system (1) as*

$$w(\rho, m) = \frac{m}{\rho} + \int_0^\rho \frac{\sqrt{p'_\varepsilon(s)}}{s} ds, \quad z(\rho, m) = \frac{m}{\rho} - \int_0^\rho \frac{\sqrt{p'_\varepsilon(s)}}{s} ds. \quad (61)$$

*Assuming (58) for  $(v_\varepsilon^0, u_\varepsilon^0) \in (L^\infty(\mathbb{R}))^2$ , the quantity  $M = \|w(\rho_\mu^0, m_\mu^0)\|_{L_x^\infty}$  is bounded uniformly with respect to  $\varepsilon$  and  $\mu$ , and the domain*

$$\Sigma := \{(\rho_\mu, m_\mu), w(\rho_\mu, m_\mu) \leq M\} \cap \{(\rho_\mu, m_\mu), z(\rho_\mu, m_\mu) \geq -M\} \quad (62)$$

is an invariant region for (60). As a consequence, the unique global smooth solution  $(\rho_\mu, m_\mu)$  to (60) satisfies:

$$0 < a_\mu \leq \rho_\mu \leq 1 - C\varepsilon^{\frac{1}{\gamma-1}} < 1, \quad |m_\mu| \leq B\rho_\mu \quad a.e., \quad (63)$$

for some constant  $a_\mu > 0$ , while  $C > 0$  is independent of  $\mu, \varepsilon$ , and  $B$  is independent of  $\mu$  and  $\varepsilon$ .

However, this weak convergence is not enough to be allowed to pass to limit in the non-linear terms of the equations, namely the convective term  $m^2/\rho$  and the pressure, because of potential high-frequency oscillations of the approximate solutions. The main core of the compensated compactness method (initiated by DiPerna [15]) consists exactly in showing that the mechanism of entropy dissipation actually quenches the high-frequency oscillations, so enforcing strong convergence of the approximate solutions.

The proof sketched below makes use of four explicit entropy-entropy flux pairs in the spirit of Lu's work, [Chapter 8, [24]] and allows us to keep track of the singular parameter  $\varepsilon$  throughout all the computations.

The latter result heavily relies on the following classical compensated compactness theorem due to Murat and Tartar (see for instance Dafermos book [Section 16.2, [13]]).

**Theorem 4.4** (Div-Curl Lemma). *Let  $\Omega$  be an open subset of  $\mathbb{R}^m$ ,  $m \geq 2$ ,  $(G_j)_j, (H_j)_j$  be sequences of vector fields belonging to  $(L^2(\Omega))^m$  such that*

$$G_j \rightharpoonup \bar{G}, \quad H_j \rightharpoonup \bar{H} \quad \text{weakly in } (L^2(\Omega))^m.$$

*If  $(\operatorname{div} G_j)_j, (\operatorname{curl} H_j)_j \subset$  compact sets of  $H_{loc}^{-1}(\Omega)$ , then*

$$G_j \cdot H_j \rightharpoonup \bar{G} \cdot \bar{H} \quad \text{weakly in } \mathcal{D}'.$$

### Entropy-entropy flux pairs, compactness in $H^{-1}$

One can check that the pairs  $(\eta_i, q_i) = (\eta_i(\rho, m), q_i(\rho, m))$ ,  $i = 1, \dots, 4$ , defined as follows

$$\begin{aligned} (\eta_1, q_1) &= (\rho, m), \\ (\eta_2, q_2) &= \left( m, \frac{m^2}{\rho} + p_\varepsilon(\rho) \right), \\ (\eta_3, q_3) &= \left( \frac{m^2}{2\rho} + \rho \int^\rho \frac{p_\varepsilon(s)}{s^2} ds, \frac{m^3}{2\rho^2} + m \left[ \frac{p_\varepsilon(\rho)}{\rho} + \int^\rho \frac{p_\varepsilon(s)}{s^2} ds \right] \right), \\ (\eta_4, q_4) &= \left( \frac{m^3}{\rho^2} + 6m \int^\rho \frac{p_\varepsilon(s)}{s^2} ds, \right. \\ &\quad \left. \frac{m^4}{\rho^3} + 3m^2 \left[ \frac{p_\varepsilon(\rho)}{\rho^2} + \frac{2}{\rho} \int^\rho \frac{p_\varepsilon(s)}{s^2} ds \right] + 6 \left[ p_\varepsilon(\rho) \int^\rho \frac{p_\varepsilon(s)}{s^2} ds - \int^\rho \frac{(p_\varepsilon(s))^2}{s^2} ds \right] \right), \end{aligned}$$

are entropy-entropy flux pairs for system (1). Namely, by definition (see again [13]), given a smooth solution  $U = (\rho, m)$  to system (1), one has that

$$\partial_t \eta_i(U) + \partial_x q_i(U) = 0, \quad i = 1, \dots, 4.$$

The pairs  $(\eta_1, q_1)$  and  $(\eta_2, q_2)$  are associated with the mass and momentum equation respectively, while  $(\eta_3, q_3)$  corresponds to the energy equality.

Before applying the Div-Curl lemma we need the following result which can be proved by following [Lemma 16.2.2, [13]].

**Proposition 4.5.** *The following property holds for  $i = 1, \dots, 4$ :*

$$\partial_t \eta_i(U_\mu) + \partial_x q_i(U_\mu) \subset \text{compact set of } H_{loc}^{-1}(\mathbb{R}^+ \times \mathbb{R}). \quad (64)$$

### Strong convergence of $U_\mu = (\rho_\mu, m_\mu)$ , reduction of the Young measure

Thanks to the bounds sated in Lemma 4.3, we obtain the weak-\* convergence of a subsequence of  $(\rho_\mu, m_\mu)_\mu$ , to which is associated a family of Young measures  $(\nu_{(t,x)})_{t,x}$  such that, for any continuous function  $h$ ,

$$h(U_\mu) \rightharpoonup \bar{h} \quad \text{with} \quad \bar{h}(t, x) = \langle \nu_{(t,x)}, h(U) \rangle = \int_{\mathbb{R}^2} h(U) \, d\nu_{(t,x)}(U).$$

Passing to the limit  $\mu \rightarrow 0$ , we deduce that system (1) is satisfied in the following sense:

$$\partial_t \langle \nu_{(t,x)}, U \rangle + \partial_x \langle \nu_{(t,x)}, f_\varepsilon(U) \rangle = 0 \quad \text{in } \mathcal{D}'. \quad (65)$$

In addition, we notice that the maximal bound satisfied by  $\rho_\mu$  implies that the weak limit

$$\bar{\rho}(t, x) := \langle \nu_{(t,x)}, \rho \rangle$$

also satisfies

$$0 \leq \bar{\rho}(t, x) \leq A^\varepsilon < 1 \quad \text{a.e.} \quad (66)$$

We recover a weak solution to (1) as  $\mu \rightarrow 0$  in the sense of Definition 4.1 provided that we are able to prove that  $\nu_{(t,x)}$  reduces to a Dirac mass, i.e.

$$\nu_{(t,x)} = \delta_{U(t,x)} \quad \text{a.e.}$$

In what follows, for the sake of brevity, we shall omit the dependency on  $(t, x)$  and we replace  $\nu_{(t,x)}$  with  $\nu$ . We also denote

$$\bar{U} = (\bar{\rho}, \bar{m}) := (\langle \nu, \rho \rangle, \langle \nu, m \rangle).$$

The first step to prove the reduction of the Young measure is to introduce four explicit *relative* entropy-entropy flux pairs  $(\tilde{\eta}_i(U, \bar{U}), \tilde{q}_i(U, \bar{U}))$ , which are related to the four pairs  $(\eta_i, q_i)$  introduced above as follows

$$\begin{cases} \tilde{\eta}_i(U, \bar{U}) = \eta_i(U) - \eta_i(\bar{U}) \\ \tilde{q}_i(U, \bar{U}) = q_i(U) - q_i(\bar{U}) \end{cases} \quad \text{for } i = 1, 2,$$

$$\begin{cases} \tilde{\eta}_i(U, \bar{U}) = \eta_i(U) - \eta_i(\bar{U}) - \nabla \eta_i(\bar{U}) \cdot (U - \bar{U}) \\ \tilde{q}_i(U, \bar{U}) = q_i(U) - q_i(\bar{U}) - \nabla \eta_i(\bar{U}) \cdot (f_\varepsilon(U) - f_\varepsilon(\bar{U})) \end{cases} \quad \text{for } i = 3, 4.$$

Using Proposition 4.5, it is easy to check that for  $i = 1, 2, 3, 4$ ,

$$\partial_t \tilde{\eta}_i(U_\mu, \bar{U}) + \partial_x \tilde{q}_i(U_\mu, \bar{U}) \subset \text{compact set of } H_{loc}^{-1}(\mathbb{R}^+ \times \mathbb{R}). \quad (67)$$

Now, applying the Div-Curl Lemma and smartly combining its resulting identities, one gets the following relations.

**Lemma 4.6.** *Let the pressure*

$$p_\varepsilon(\rho) = \varepsilon \left( \frac{\rho}{1-\rho} \right)^\gamma \quad \text{with } \gamma \in (1, 3].$$

*The following equality holds true*

$$\begin{aligned} & \varepsilon \frac{3-\gamma}{2(\gamma+1)} \frac{\bar{\rho}^{\gamma+1}}{(1-\bar{\rho})^\gamma} \langle \nu, (u-\bar{u})^4 \rangle + \varepsilon^3 \frac{\gamma^2(5\gamma+1)}{2(\gamma+1)} \frac{\bar{\rho}^{3\gamma-5}}{(1-\bar{\rho})^{3\gamma+2}} \langle \nu, (\rho-\bar{\rho})^4 \rangle \\ & + \varepsilon^2 6\gamma \frac{\bar{\rho}^{2(\gamma-1)}}{(1-\bar{\rho})^{2\gamma+1}} \left( \langle \nu, (u-\bar{u})(\rho-\bar{\rho}) \rangle \right)^2 + \text{Error} = 0, \end{aligned} \quad (68)$$

where Error denotes “an error term”, whose  $L^\infty$  norm is negligible with respect to the norm of the other terms.

The proof, inspired by [24], relies on long and technical calculations, we omit it for sake of brevity. The main core of this part is represented by the result below.

**Proposition 4.7** (Reduction of the Young measure). *Assume that  $\gamma \in (1, 3]$ . The support of  $\nu$  is either confined in  $\{\bar{\rho} = 0\}$  or is reduced to the point  $(\bar{\rho}, \bar{m})$ .*

*Proof.* From Lemma 4.6, we observe that (68) implies, if  $\gamma \in (1, 3)$ , that

$$C_1(\bar{\rho}) \langle \nu, (u-\bar{u})^4 \rangle + C_2(\bar{\rho}) \langle \nu, (\rho-\bar{\rho})^4 \rangle + \text{Error} \leq 0,$$

where the coefficients  $C_1(\bar{\rho})$  and  $C_2(\bar{\rho})$  are positive on the set  $\{\bar{\rho} > 0\}$ . Therefore, we have

$$\begin{aligned} \langle \nu_{(t,x)}, (\rho-\bar{\rho})^4 \rangle &= \int_{\mathbb{R}^2} (\rho-\bar{\rho})^4 d\nu_{(t,x)}(\rho, m) = 0 \quad \text{a.e. on } \{(t,x) \mid \bar{\rho}(t,x) > 0\}, \\ \langle \nu_{(t,x)}, (u-\bar{u})^4 \rangle &= \int_{\mathbb{R}^2} \left( \frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}} \right)^4 d\nu_{(t,x)}(\rho, m) = 0 \quad \text{a.e. on } \{(t,x) \mid \bar{\rho}(t,x) > 0\}, \end{aligned}$$

from which we deduce that

$$\nu_{(t,x)} = \delta_{(\bar{\rho}(t,x), \bar{m}(t,x))} \quad \text{a.e. on } \{(t,x) \mid \bar{\rho}(t,x) > 0\}, \quad (69)$$

so that the strong convergence of  $(\rho_\mu, m_\mu)$  towards  $(\bar{\rho}, \bar{m})$  is proven.

If  $\gamma = 3$ , then in (68) the term

$$\varepsilon \frac{3-\gamma}{2(\gamma+1)} \frac{\bar{\rho}^{\gamma+1}}{(1-\bar{\rho})^\gamma} \langle \nu, (u-\bar{u})^4 \rangle$$

vanishes. However, the strong convergence of  $u$  can be recovered in that case from the following equality

$$\langle \nu, (u-\bar{u})^2 \rangle = \frac{p'_\varepsilon(\bar{\rho})}{\bar{\rho}^2} \langle \nu, (\rho-\bar{\rho})^2 \rangle + \text{Error},$$

which can be obtained by a careful analysis and combination of the identities provided by the Div-Curl Lemma. We refer to [24] for a detailed explanation of this point. The proof is over.  $\square$

*Remark 4.8.* Note that the previous estimates, and in particular (68), degenerate as  $\varepsilon \rightarrow 0$ . As a consequence, a similar compactness argument would not work as  $\varepsilon \rightarrow 0$ .

## Acknowledgement

This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation program Grant agreement No 637653, project BLOC “Mathematical Study of Boundary Layers in Oceanic Motion”. This work was supported by the SingFlows project, grant ANR-18-CE40-0027 of the French National Research Agency (ANR). RB was partially supported by the GNAMPA group of INdAM (GNAMPA project 2019). The authors thank Roberto Natalini for an introduction on compensated compactness.

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