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# Consistent Investment of Sophisticated Rank-Dependent Utility Agents in Continuous Time

Ying Hu\*      Hanqing Jin<sup>†</sup>      Xun Yu Zhou<sup>‡</sup>

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## Abstract

We study portfolio selection in a complete continuous-time market where the preference is dictated by the rank-dependent utility. As such a model is inherently time inconsistent due to the underlying probability weighting, we study the investment behavior of sophisticated consistent planners who seek (subgame perfect) intra-personal equilibrium strategies. We provide sufficient conditions under which an equilibrium strategy is a replicating portfolio of a final wealth. We derive this final wealth profile explicitly, which turns out to be in the same form as in the classical Merton model with the market price of risk process properly scaled by a deterministic function in time. We present this scaling function explicitly through the solution to a highly nonlinear and singular ordinary differential equation, whose existence of solutions is established. Finally, we give a necessary and sufficient condition for the scaling function to be smaller than 1 corresponding to an effective reduction in risk premium due to probability weighting.

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## 1 Introduction

The classical expected utility theory (EUT) is unable to explain many puzzling phenomena and paradoxes, such as, just to name a few, the Allais paradox, the co-existence of risk-averse and risk-seeking behavior of a same individual, and the disposition effect in financial investment. Rank-dependent utility theory (RDUT), first proposed by Quiggin (1982), has been developed to address some of these puzzles and has thus far been widely considered as one of the most prominent alternative theories on preferences and choices. In addition to a concave outcome utility function as in EUT, RDUT features a probability weighting (or distortion) function whose slopes give uneven weights on random outcomes when calculating the mean. The presence of probability weighting is supported by a large amount of experimental and empirical studies, many of which find that such a weighting function typically displays an “inverse S-shape” (namely, it is first concave and then convex in its domain); see, e.g. Tversky and Kahneman (1992), Wu and Gonzalez (1996) and Tanaka et al (2010). This particular shape captures individuals’ tendency to exaggerate the tiny probabilities of both improbable large gains (such as winning a lottery) and improbable large losses (such as encountering a plane crash). In particular, Tversky and Kahneman (1992) propose a specific parametric class of inverse S-shaped weighting functions, which will be used as a baseline example to test the assumptions in our paper.

In this paper we study a continuous-time financial portfolio selection model in which an agent pursues the highest rank-dependent utility (RDU) value. In contrast to classical continuous-time portfolio models such as Merton’s (Merton 1969), a dynamic RDU model is intrinsically *time inconsistent*; namely, any “optimal” strategy for today will generally not be optimal for tomorrow. As a result, there is no notion of a *dynamically optimal strategy* for a time-inconsistent model because any such a strategy, once devised for this moment, will *have to* be abandoned immediately (and indeed infinitesimally) at the next moment. The time inconsistency of the RDU model emanates from the probability weighting which weights the random outcomes *unevenly* according to their probabilities of occurrence. Consider as an example a

10-period binomial lattice model with equal probabilities of moving up and down at any given state. Standing at  $t = 0$  the probability of reaching the top most state (TMS) at  $t = 10$  is extremely small ( $2^{-10}$ ). If the agent has an inverse S-shaped probability weighting then she will greatly inflate this probability. As time goes by and the agent moves along the lattice the probabilities of this same event – eventually reaching the TMS – keep changing, and so do the *degrees* of probability weighting. Indeed, once at  $t = 9$  the probability of finally reaching the TMS is either  $1/2$  or  $0$  – which any reasonably intelligent individual is able to tell – and hence there is no probability weighting at all. We hereby see an inconsistency in the *strength* of probability weighting over time which is the key reason behind the time inconsistency of an RDU model.

Time inconsistency changes fundamentally the way we deal with dynamic optimization in general. Optimization is intimately associated with decision making, and finding an optimal solution is therefore to advise on the best decisions. Now, there is *no* optimal solutions under time inconsistency: so what is the purpose of studying a time-inconsistent problem?

In his seminal paper, Strotz (1955) describes three types of agents when facing time inconsistency. Type 1, a “spendthrift” (or a naiveté as in the more recent literature), does not recognize the time inconsistency and at any given time seeks an optimal solution for that moment only. As a result, his strategies are always myopic and change all the times. The next two types are aware of time inconsistency but act differently. Type 2 is a “precommmitter” who solves the optimization problem only *once* at time 0 and then commits to the resulting strategy throughout, even though she knows that the original solution may no longer be optimal at later times. Type 3 is a “thrift” (or a sophisticated agent) who is unable to precommit and realizes that her future selves will disobey whatever plans she makes now. Her resolution is to compromise and choose *consistent planning* in the sense that she optimizes taking the future disobedience as a *constraint*. The Strotzian approach to time inconsistency is, therefore, *descriptive*, namely to describe what people actually do – the different reactions and behaviors in front of the inconsistency, as opposed to being *normative*, namely to tell people what to do.

It is both interesting and challenging to formulate mathematical models for *each* of the three types and solve them. It is interesting because these models are very different from the classical stochastic control based ones and different from each other, and it is challenging because the most powerful tools for tackling dynamic optimization such as dynamic programming and

martingale analysis are based on time consistency and hence fail for time-inconsistent problems. Recently, there is an upsurge of research interest and effort in the fields of stochastic control and mathematical finance/insurance in studying time-inconsistent models, mostly focusing on three different problems: mean–variance portfolio selection, and those involving non-exponential discounting or probability weighting. Earlier works focused on Type 2, precommitted agents (see, e.g., Li and Ng 2000, Zhou and Li 2000, He and Zhou 2011), and later ones gradually shifted to Type 3, consistent planners or sophisticated agents (Ekeland and Lazrak 2006, Bjork and Murgoci 2010, Hu et al 2012, Hu et al 2017, Bjork et al 2014).

The Type 3 problem can be mathematically formulated as a game in the following way. The sophisticated agent, anticipating the disagreement between her current and future selves, searches for a dynamic strategy that all the future selves have no incentive to deviate from. The resulting strategy is a (subgame perfect) *intra-personal equilibrium* that will be carried through. In the continuous-time setting, Ekeland and Lazrak (2006) are the first to give a formal definition of such an equilibrium (albeit for a deterministic Ramsey model with non-exponential discounting), based on a first-order condition of a “spike variation” of the equilibrium strategy.

This paper derives the consistent investment strategies of a Type 3 RDU agent in a continuous-time market in which the asset prices are described by stochastic differential equations (SDEs). We make several contributions, both methodologically and economically. First of all, to our best knowledge, this paper is the first to formulate and attack an RDU investment problem in the continuous-time setting. There are substantial difficulties in approaching the problem. Most notably, in deriving an equilibrium strategy, one needs to analyze the effect of the aforementioned spike variation on the RDU objective functional. In the absence of probability weighting, there is a well-developed approach at disposal to do this, based on calculus of variations and *sample-path* dependence of SDEs on parameters.<sup>1</sup> With probability weighting, however, we need to study the *distributional* dependence of the SDE solutions on parameters, which is actually a largely unexplored topic to our best knowledge. Starting from scratch, we carry out a delicate analysis to solve the problem thoroughly. The final solution requires the existence of solutions to a highly nonlinear, singular ordinary differential equation

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<sup>1</sup>The same approach is used to derive the stochastic maximum principle for optimal stochastic controls; see, e.g., Yong and Zhou (1999).

(ODE), the validity of an inequality, and the existence of the Lagrange multiplier to a budget constraint. We then provide sufficient conditions on the model primitives to ensure that these three requirements are met.

The equilibrium strategy is a replicating portfolio to a terminal wealth profile we derive explicitly, assuming that the market is complete. Curiously, the terminal wealth is of the same form as that with the classical Merton optimal strategy under EUT, except that the market price of risk or risk premium process needs to be multiplied by a scaling factor, the latter in turn determined by a solution to the aforementioned nonlinear ODE. To put this in a different way, the sophisticated RDU behaves as if she was an EUT agent, albeit in a *revised* market where the risk premium is properly scaled. The scaling factor depends on the original investment opportunity set and the agent's probability weighting function, but *not* her outcome utility function. This suggests that the additional constraint arising from the consistent planning due to time inconsistency can be transferred to the market opportunity set. This observation could be a key leading to identifying market equilibria where all the agents are RDU consistent planners.

When the scaling factor is less than 1, the risk premium is reduced and the agent is more risk averse than her EUT counterpart. In this case the probability weighting and consistent planning make the agent to take less risky exposure. We present a necessary and sufficient condition for this to happen.

It should be noted that, in order to derive our main results, we make several assumptions on the model primitives. Some of them may look quite technical especially those on the weighting function. However, we do not impose those assumptions for mathematical convenience; instead we make sure that they are mathematically mild and economically reasonable. In particular, all the assumptions on the weighting function are satisfied by the baseline function proposed by Tversky and Kahneman (1992).

The rest of the paper is organized as follows. In Section 2, we state our problem and define the intra-personal equilibrium in the same spirit as in our previous work Hu et al. (2012,2017). In Section 3, we present sufficient conditions under which the equilibrium terminal wealth can be explicitly derived. In Section 4, we examine these sufficient conditions closely on our model primitives. Section 5 is devoted to a concrete example demonstrating the results of the paper. In Section 6, we present an equivalent condition for an effectively reduced risk

premium. Finally, Section 7 concludes. In Appendices, we state related general results on a class of singular ODEs, and verify our assumptions on a family of time-varying Tversky–Kahneman’s weighting functions.

## 2 Problem Formulation

### 2.1 The market

We consider a continuous-time market in a finite time horizon  $[0, T]$ , where there are a risk-free asset and  $n$  risky assets being traded frictionlessly with price processes  $S_0(\cdot)$  and  $S_i(\cdot)$ ,  $i = 1, \dots, n$ , respectively. The risk-free interest rate, without loss of generality, is set to be  $r(\cdot) \equiv 0$ , or equivalently,  $S_0(t) \equiv S_0(0)$ . The dynamics of  $S_i(\cdot)$ ,  $i = 1, \dots, n$ , of the risky assets follow a multi-dimensional geometric Brownian motion

$$dS_i(t) = S_i(t) \left[ \mu_i(t)dt + \sum_{j=1}^n \sigma_{i,j}(t)dW_j(t) \right], \quad i = 1, \dots, n,$$

where  $\mu_i(\cdot)$  and  $\sigma_{i,j}(\cdot)$  are all deterministic functions of time  $t$ ,  $W(\cdot) = (W_1(\cdot), \dots, W_n(\cdot))^\top$  is an  $n$ -dimensional standard Brownian motion in a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$  with  $\mathcal{F}_t = \sigma(W_s : s \leq t) \vee \mathcal{N}(\mathbb{P})$ . Here  $^\top$  denotes the matrix transpose.

Denote  $\mu(t) = (\mu_1(t), \dots, \mu_n(t))^\top$  and  $\sigma(t) = (\sigma_{i,j}(t))_{n \times n}$ . We assume that  $\sigma(\cdot)$  is invertible and that the market price of risk is  $\theta(t) := \sigma(t)^{-1}\mu(t)$ ; so the market is arbitrage free and complete.

A trading strategy is a self-financing portfolio described by an  $\mathcal{F}_t$ -adapted process  $\pi(\cdot) = (\pi_1(\cdot), \dots, \pi_n(\cdot))^\top$ , where  $\pi_i(t)$  is the dollar amount allocated to asset  $i$  at time  $t$ . Under such a portfolio  $\pi(\cdot)$ , the dynamics of the corresponding wealth process  $X(\cdot)$  evolve according to the wealth equation

$$dX(t) = \pi(t)^\top \mu(t)dt + \pi(t)^\top \sigma(t)dW(t). \quad (2.1)$$

For any  $t \in [0, T)$ , we say a trading strategy  $\pi(\cdot)$  is admissible on  $[t, T]$  if  $\mathbb{E} \int_t^T |\pi(s)^\top \mu(s)| ds < +\infty$  and  $\mathbb{E} \int_t^T |\pi(s)^\top \sigma(s)|^2 ds < +\infty$ .

## 2.2 Rank-dependent utility

An agent, with an initial endowment  $x_0$  at time  $t = 0$ , pursues the highest possible rank-dependent utility (RDU) of her wealth at  $t = T$  by dynamically trading in the market. At any given time  $t$  with the wealth state  $X(t) = x$  she takes an admissible portfolio  $\pi(\cdot)$  on  $[t, T]$  leading to the terminal wealth  $X(T)$ . The RDU value of this terminal wealth is

$$J(X(T); t, x) = \int_0^{+\infty} w(t, \mathbb{P}_t(u(X(T)) > y)) dy + \int_{-\infty}^0 [w(t, \mathbb{P}_t(u(X(T)) > y)) - 1] dy,$$

where  $w(t, \cdot)$  is the probability weighting applied at time  $t$ ,  $u(\cdot)$  is the (outcome) utility function, and  $\mathbb{P}_t := \mathbb{P}(\cdot | \mathcal{F}_t)$  denotes the conditional probability given  $\mathcal{F}_t$ , which includes the information  $X(t) = x$ . The agent's original objective is to maximize  $J(X(T); t, x)$  by choosing a proper admissible investment strategy.

Here we allow the weighting function  $w$  to depend explicitly on time  $t$ . This is not just for mathematical generality; the time variation of probability weighting is supported empirically (Dierkes 2013, Cui et al 2020) and argued for on a psychology ground (Cui et al 2020).

## 2.3 Equilibrium strategies

As discussed in Introduction, probability weighting in general causes time-inconsistency. This means an optimal strategy with respect to  $J(X(T); 0, X(0))$  is not necessarily still optimal with respect to  $J(X(T); t, X(t))$  for a future time  $t > 0$ , where  $X(\cdot)$  is the ‘‘optimal’’ wealth process projected at  $t = 0$  for the objective  $J(X(T); 0, X(0))$ . In this paper, we study the behaviors of a sophisticated agent who is aware of the time-inconsistency but lacks commitment, and who instead seeks a consistent investment among all the different selves  $t \in [0, T]$  by finding intra-personal equilibrium strategies.

Precisely, given an admissible trading strategy  $\pi(\cdot)$  with the corresponding wealth process  $X(\cdot)$  starting from  $X(0) = x_0$ , a time  $t \in [0, T)$  and a small number  $\varepsilon \in (0, T - t)$ , we define a slightly perturbed strategy which adds  $\$k$  on top of  $\pi(\cdot)$  over the small time interval  $[t, t + \varepsilon)$  while keeping  $\pi(\cdot)$  unchanged outside of this interval. Here  $k$  is an  $\mathcal{F}_t$ -measurable random *vector*. This technique is called a *spike variation*. It follows from the wealth equation (2.1) that the perturbed terminal wealth is  $X(T) + k^\top \Delta(t, \varepsilon)$ , where  $\Delta(t, \varepsilon) = \int_t^{t+\varepsilon} \mu(s) ds + \int_t^{t+\varepsilon} \sigma(s) dW(s)$ .



**Definition 2.1** *An admissible strategy  $\pi(\cdot)$  with the wealth process  $X(\cdot)$  starting from  $X(0) = x_0$  is called an equilibrium strategy, if for any  $t \in [0, T)$  and any  $\mathcal{F}_t$ -measurable random vector  $k$ , one has*

$$\limsup_{\varepsilon \downarrow 0} \frac{J(X(T) + k^\top \Delta(t, \varepsilon); t, X(t)) - J(X(T); t, X(t))}{\varepsilon} \leq 0. \quad (2.2)$$

This definition follows most existing works on time-inconsistent optimal controls in continuous time; see e.g. Ekeland and Lazrak (2006), Bjork and Murgoci (2010), and Hu et al (2012, 2017). Theoretically, an equilibrium strategy is an “infinitesimally” sub-game perfect equilibrium among all the selves  $t \in [0, T)$ .<sup>2</sup>

The objectives of this paper are to find conditions under which there exists an equilibrium strategy, to derive the terminal wealth under such a strategy, and to draw economic implications and interpretations of the results. Since the market is complete, once the desired terminal wealth profile is identified its replicating portfolio is then the corresponding equilibrium strategy for consistent investment.

## 2.4 Assumptions

In this subsection we collect all the assumptions needed on the market parameters as well as on the preference functions.<sup>3</sup> These assumptions are henceforth in force, without necessarily being mentioned again in all the subsequent statements of results.

**Assumption 2.2** *The functions  $\mu(\cdot)$ ,  $\sigma(\cdot)$  and  $\theta(\cdot)$  are all right continuous and uniformly bounded, and  $\theta(t) \neq 0 \forall t \in [0, T]$ .*

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<sup>2</sup>Definition 2.1 is also in line with the original Strotz’s vision of intrapersonal equilibrium (Strotz 1955), adjusted for the continuous-time setting. More specifically, a Strotzian equilibrium strategy stipulates that for any given self  $t$ , all the future selves will commit to the strategy, because any deviation from the strategy will make the deviating self  $t$  worse off. In a continuous-time setting, however, any fixed  $t$  alone has no influence on the terminal wealth because it has a measure of zero. Therefore, one considers instead a small “alliance” of self  $t$ : the interval  $[t, t + \varepsilon)$ . Definition 2.1 posits that a “deviation-in-alliance” from equilibrium fares worse in a first-order sense.

<sup>3</sup>Throughout this paper, by an “increasing” function we mean a “non-decreasing” function, namely  $f$  is increasing if  $f(x) \geq f(y)$  whenever  $x > y$ . We say  $f$  is “strictly increasing” if  $f(x) > f(y)$  whenever  $x > y$ . Similar conventions are used for “decreasing” and “strictly decreasing” functions.

**Assumption 2.3** (i)  $w(\cdot, \cdot) : [0, T] \times [0, 1] \mapsto [0, 1]$  is measurable. Moreover, for each  $t \in [0, T]$ ,  $w(t, \cdot) : [0, 1] \mapsto [0, 1]$  is strictly increasing,  $C^2$ , and  $w(t, 0) = 0, w(t, 1) = 1 \forall t \in [0, T]$ .

(ii)  $u(\cdot) : \mathbb{R} \mapsto \mathbb{R}$  is strictly increasing, strictly concave,  $C^3$ , and  $u(-\infty) = -\infty, u'(-\infty) = +\infty, u'(+\infty) = 0$ .

These two are very weak assumptions, representing some “minimum requirements” for the model primitives.

Define  $I(x) := (u')^{-1}(x)$ ,  $x > 0$ , and  $l(x) := -\ln u'(x)$ ,  $x \in \mathbb{R}$ .

**Assumption 2.4** (i) There exists  $\alpha > 0$  such that  $\limsup_{x \rightarrow +\infty} \frac{x^{-\alpha}}{-I'(x)} < +\infty$ , or equivalently,  $\limsup_{x \rightarrow -\infty} \frac{-u''(x)}{u'(x)^\alpha} < +\infty$ .

(ii) There exist  $a > 0, b > 0$  such that  $l'(x) \leq e^{a|l(x)|+b}, l''(x) \leq e^{a|l(x)|+b} \forall x \in \mathbb{R}$ .

(iii)  $u'''(x) \geq 0 \forall x \in \mathbb{R}$ .

(iv) For any  $t \in [0, T]$ , there exist  $c > 0$  and  $m \in (-1, 0)$ , both possibly depending on  $t$ , such that

$$w'_p(t, p) \leq c[p^m + (1 - p)^m] \quad \forall p \in (0, 1).$$

Assumption 2.4-(i) and -(ii) are very mild conditions satisfied by most commonly used utility functions defined on  $\mathbb{R}$  (e.g. the exponential utility). Assumption 2.4-(iii) implies the *risk prudence* of the agent, capturing her tendency to take precautions against future risk. Many common utility functions are prudent.<sup>4</sup> Assumption 2.4-(iv) is to control the level of probability weighting on very small and very large probabilities. It is satisfied by some well-known weighting functions, e.g. that of Tversky and Kahneman (1992); see Appendix B.

**Assumption 2.5** There exist constants  $\nu > 0, \zeta > 0$ , such that

$$\int_{-\infty}^0 u'(x)^{-\nu} dx < +\infty, \quad \int_0^{\infty} u'(x)^\nu dx < +\infty, \quad \int_{-\infty}^{+\infty} u'(x)e^{-\zeta x^2} dx < +\infty.$$

This is also a mild assumption, which is satisfied by, say, the exponential utility function.

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<sup>4</sup>Through an experiment with a large number of subjects, Noussair et al (2014) observe that the majority of individuals' decisions are consistent with prudence.

## 2.5 A crucial function

The following real-valued function, generated from the weighting function, will play a central role throughout this paper:

$$h(t, x) := \mathbb{E} [w'_p(t, N(\xi))e^{x\xi}] > 0, \quad t \in [0, T], \quad x \in \mathbb{R},$$

where (and henceforth)  $w'_p(t, p) := \frac{\partial}{\partial p} w(t, p)$ ,  $\xi$  is a standard normal random variable with  $N$  being its probability distribution function. Similarly, we denote by  $h'_x(t, x)$  and  $h''_x(t, x)$  the first- and second-order partial derivatives of  $h$  in  $x$  respectively, and in general  $h_x^{(n)}(t, x)$  the  $n$ -th order partial derivative of  $h$  in  $x$ .

**Lemma 2.6**  $h(t, x) < +\infty \quad \forall (t, x) \in [0, T] \times \mathbb{R}$ .

*Proof.* Fix  $(t, x) \in [0, T] \times \mathbb{R}$ . It follows from Assumption 2.4-(iv) that  $w'_p(t, p) \leq c(p^m + (1 - p)^m) \quad \forall p \in (0, 1)$  for some  $c > 0$  and  $m \in (-1, 0)$ . Denote  $\hat{m} := \frac{m-1}{2} \in (-1, m)$ ,  $p := \frac{\hat{m}}{m} > 1$ ,  $q := \frac{p}{p-1} > 1$ . Applying Young's inequality  $a^m b \leq \frac{a^{\hat{m}}}{p} + \frac{b^q}{q} \quad \forall a \geq 0, b \geq 0$ , we obtain

$$\mathbb{E}[(N(\xi))^m e^{\pm x\xi}] \leq \mathbb{E} \left[ \frac{1}{p} (N(\xi))^{\hat{m}} + \frac{1}{q} e^{\pm xq\xi} \right] = \frac{1}{(\hat{m} + 1)p} + \frac{1}{q} e^{(qx)^2/2}.$$

So

$$\begin{aligned} h(t, x) &\leq c \left( \mathbb{E}[N(\xi)^m e^{x\xi}] + \mathbb{E}[N(-\xi)^m e^{x\xi}] \right) \\ &= c \left( \mathbb{E}[N(\xi)^m e^{x\xi}] + \mathbb{E}[N(\xi)^m e^{-x\xi}] \right) \\ &\leq 2c \left[ \frac{1}{(\hat{m} + 1)p} + \frac{1}{q} e^{(qx)^2/2} \right] < +\infty. \end{aligned}$$

The proof is complete. □

The following lemma collects some basic properties of  $h$  which will be useful in the sequel.

**Lemma 2.7** *For any  $t \in [0, T]$ ,  $h(t, \cdot)$  has the following properties:*

- (i)  $h(t, 0) = 1$ .
- (ii)  $h(t, x)$  is  $C^\infty$  in  $x \geq 0$ , with  $h_x^{(n)}(t, x) = \mathbb{E} [w'_p(t, N(\xi)) \xi^n e^{x\xi}]$ ,  $x \geq 0, \forall n \geq 0$ .
- (iii)  $h_x^{(n)}(t, x)$  is convex in  $x \geq 0$  for all even  $n \geq 0$ , and increasing in  $x \geq 0$  for all odd  $n \geq 1$ .
- (iv)  $\ln h(t, x)$  is convex in  $x \geq 0$ .

*Proof.* (i) We have

$$h(t, 0) = \mathbb{E}[w'_p(t, N(\xi))] = \int_{-\infty}^{\infty} w'_p(t, N(x)) dN(x) = \int_0^1 w'_p(t, p) dp = w(t, 1) - w(t, 0) = 1.$$

(ii) For  $\varepsilon \neq 0$ , write

$$\frac{h(t, x + \varepsilon) - h(t, x)}{\varepsilon} = \mathbb{E} \left[ w'_p(t, N(\xi)) \xi \int_0^1 e^{(x+\theta\varepsilon)\xi} d\theta \right].$$

For  $x \geq 0$ , we have

$$0 < \int_0^1 e^{(x+\theta\varepsilon)\xi} d\theta \leq e^{(x+|\varepsilon|)\xi} + e^{-|\varepsilon|\xi}.$$

Fix  $a < 0$ . For any integer  $n \geq 0$ , it follows from  $h(t, a) < +\infty$  that there exists a constant  $C_n > 0$  such that

$$|\xi^n| \leq C_n (e^{a\xi/2} + e^\xi).$$

Now, for any  $\varepsilon$  with  $|\varepsilon| < \min(-a/2, 1)$ , we have

$$\begin{aligned} & |w'_p(t, N(\xi)) \xi \int_0^1 e^{(x+\theta\varepsilon)\xi} d\theta| \\ & \leq w'_p(t, N(\xi)) |\xi| (e^{(x+|\varepsilon|)\xi} + e^{-|\varepsilon|\xi}) \\ & \leq C_1 w'_p(t, N(\xi)) (e^{(x+|\varepsilon|+1)\xi} + e^{(1-|\varepsilon|)\xi} + e^{(x+|\varepsilon|+a/2)\xi} + e^{(a/2-|\varepsilon|)\xi}) \\ & < C_1 w'_p(t, N(\xi)) (e^{(x-a/2+1)\xi} + e^{(x+1)\xi} + e^{(1+a/2)\xi} + e^\xi + e^{(x+a/2)\xi} + e^{x\xi} + e^{a\xi} + e^{a\xi/2}), \end{aligned}$$

where to deduce the last inequality we have repeatedly used the fact that  $e^y < e^{y_1} + e^{y_2}$  for any  $y_2 > y_1$  and  $y \in [y_1, y_2]$ . From the assumption of the lemma, it follows that the last term of the above is a random variable with a finite mean; hence Lebesgue's dominated convergence theorem yields

$$h'_x(t, x) = \mathbb{E}[w'_p(t, N(\xi)) \xi e^{x\xi}].$$

Similarly, we can derive the desired expressions of higher-order derivatives.

(iii) The result is straightforward by (ii).

(iv) It follows from (ii) and the Cauchy–Schwarz inequality that, for  $x \geq 0$ ,

$$\begin{aligned} h'_x(t, x) &= \mathbb{E} [w'_p(t, N(\xi)) \xi e^{x\xi}] \\ &= \mathbb{E} \left[ \sqrt{w'_p(t, N(\xi)) \xi e^{x\xi/2}} \cdot \sqrt{w'_p(t, N(\xi)) \xi e^{x\xi/2}} \right] \\ &\leq \sqrt{\mathbb{E} [w'_p(t, N(\xi)) \xi^2 e^{x\xi}]} \cdot \sqrt{\mathbb{E} [w'_p(t, N(\xi)) \xi e^{x\xi}]} \\ &= \sqrt{h''_x(t, x)} \sqrt{h(t, x)}. \end{aligned}$$

Thus

$$\frac{\partial^2}{\partial x^2} \ln h(t, x) = \frac{h''_x(t, x)h(t, x) - (h'_x(t, x))^2}{h^2(t, x)} \geq 0,$$

establishing the desired convexity.  $\square$

## 3 Equilibrium Strategies

### 3.1 An Ansatz and an ODE

Our approach to deriving the equilibrium strategies is inspired by an *Ansatz* we now make. If the agent was an expected utility maximizer, then her optimal strategy would be to dynamically replicate the terminal wealth  $I(\kappa\rho(T))$  where  $\rho(\cdot)$  is the state-price density process defined as

$$\rho(t) := \exp\left(-\frac{1}{2}\int_0^t |\theta(s)|^2 ds - \int_0^t \theta(s)^\top dW(s)\right), \quad (3.1)$$

and  $\kappa$  is the Lagrange multiplier for the budget constraint. We conjecture that in the current RDU setting the terminal wealth from an equilibrium strategy is still of the form  $I(\kappa\bar{\rho}(T))$  with a *revised* state-price density process determined by multiplying the market price of risk function  $\theta(\cdot)$  by a scaling function  $\lambda(\cdot)$ :

$$\bar{\rho}(t) = \exp\left(-\frac{1}{2}\int_0^t |\lambda(s)\theta(s)|^2 ds - \int_0^t \lambda(s)\theta(s)^\top dW(s)\right). \quad (3.2)$$

Since now we have conjectured a *specific* form of the desired terminal wealth profile  $X(T)$ , we will be able to calculate its RDU value along with that of a slightly perturbed wealth process in the spirit of Definition 2.1. Then, the equilibrium condition (2.2) will lead to an equation that can be used to identify  $\lambda(\cdot)$  as well as to other conditions.

It turns out that the equation to derive  $\lambda(\cdot)$  is an ODE, explicitly expressed in the following form:

$$\begin{cases} \Lambda'(t) = -|\theta(t)|^2 \left(\frac{h(t, \sqrt{\Lambda(t)})}{h'_x(t, \sqrt{\Lambda(t)})}\right)^2 \Lambda(t), & t \in [0, T), \\ \Lambda(T) = 0. \end{cases} \quad (3.3)$$

This is a highly nonlinear ODE that is singular at  $T$ . The existence of its *positive* solutions will be established in Subsection 4.2.

The scaling function  $\lambda(\cdot)$  in determining (3.2) is then given by

$$\lambda(t) := \sqrt{-\Lambda'(t)/|\theta(t)|^2} > 0, \quad t \in [0, T), \quad (3.4)$$

where  $\Lambda(\cdot)$  is a positive solution of (3.3).

### 3.2 Terminal wealth of an equilibrium strategy

The following result gives a complete solution to our problem by presenting the explicit terminal wealth profile of an equilibrium strategy.

**Theorem 3.1** *Assume that equation (3.3) admits a solution  $\Lambda(\cdot) \in C[0, T] \cap C^1[0, T]$  with  $\Lambda(t) > 0 \forall t \in [0, T]$ , and that the following inequality holds for any  $c \in \mathbb{R}$ :*

$$\int_{-\infty}^{+\infty} w'_p \left( t, N \left( \frac{c - g(x)}{\sqrt{\Lambda(t)}} \right) \right) N' \left( \frac{c - g(x)}{\sqrt{\Lambda(t)}} \right) \left( g''(x) + \frac{c - g(x)}{\Lambda(t)} g'(x)^2 \right) du(x) \geq 0, \quad a.e.t \in [0, T]. \quad (3.5)$$

Moreover, assume there is  $\kappa > 0$  such that the following holds

$$\mathbb{E} [\rho(T) I(\kappa \bar{\rho}(T))] = x_0 \quad (3.6)$$

where  $x_0 > 0$  is the initial endowment of the agent at  $t = 0$ . Then the portfolio replicating the terminal wealth

$$X(T) := I(\kappa \bar{\rho}(T)) \quad (3.7)$$

where  $\bar{\rho}(T)$  is determined through (3.2) – (3.4), is an equilibrium strategy.

*Proof.* Denote

$$\mathcal{E}_{s,t} := \int_s^t \lambda(v) \theta(v)^\top dW(v) \quad \forall 0 \leq s \leq t \leq T, \quad f(x) := I(\kappa e^{-\frac{1}{2}\Lambda(0)} e^{-x}) =: g^{-1}(x), \quad v(x) := u^{-1}(x), \quad x \in \mathbb{R}.$$

It is easy to see that, for any  $s \in [0, T]$  and conditional on  $\mathcal{F}_s$ ,  $\mathcal{E}_{s,t}$  is normal, i.e.,  $\mathcal{E}_{s,T} | \mathcal{F}_s \sim N(0, \Lambda(s))$ , and (3.7) can be rewritten as  $X(T) = f(\mathcal{E}_{0,T})$ .

Let  $\pi(\cdot)$  be the replicating strategy of  $X(T)$ , which exists by the market completeness. Moreover, the budget constraint (3.6) ensures that  $\pi(\cdot)$  is an admissible portfolio starting from the initial wealth  $x_0$ . The goal is to prove that  $\pi(\cdot)$  is an equilibrium strategy.

Fix  $t \in [0, T)$ . Consider the perturbed strategy described in Subsection 2.3 with the perturbed final wealth  $X(T) + k^\top \Delta(t, \varepsilon)$ , where  $\varepsilon \in (0, T - t)$ . To compute the RDU value of this perturbed strategy, we first calculate, for any  $y \in \mathbb{R}$ :

$$\begin{aligned} \mathbb{P}_t (u(X(T) + k^\top \Delta(t, \varepsilon)) > y) &= \mathbb{P}_t (X(T) > v(y) - k^\top \Delta(t, \varepsilon)) \\ &= \mathbb{P}_t (\mathcal{E}_{t+\varepsilon, T} > g(v(y) - k^\top \Delta(t, \varepsilon)) - \mathcal{E}_{0, t+\varepsilon}) \\ &= \mathbb{E}_t [\mathbb{P}_{t+\varepsilon} (\mathcal{E}_{t+\varepsilon, T} > g(v(y) - k^\top \Delta(t, \varepsilon)) - \mathcal{E}_{0, t+\varepsilon})] \\ &= \mathbb{E}_t \left[ N \left( \frac{\mathcal{E}_{0, t+\varepsilon} - g(v(y) - k^\top \Delta(t, \varepsilon))}{\sqrt{\Lambda(t + \varepsilon)}} \right) \right], \end{aligned}$$

where  $\mathbb{P}_t = \mathbb{P}(\cdot | \mathcal{F}_t)$  and  $\mathbb{E}_t = \mathbb{E}[\cdot | \mathcal{F}_t]$ .

Denote  $m(s, y) = v(y) - k^\top \Delta(t, s)$  and  $Y(s, y) = \frac{\varepsilon_{0,t+s} - g(m(s, y))}{\sqrt{\Lambda(t+s)}}$ ,  $s \in [0, \varepsilon]$ ,  $y \in \mathbb{R}$ . Applying Itô's formula and noting that  $\Lambda'(t+s) = -\lambda(t+s)^2 |\theta(t+s)|^2$ , we derive<sup>5</sup>

$$dY(s) = \left[ \frac{Y(s)}{2\Lambda(t+s)} \lambda(t+s)^2 |\theta(t+s)|^2 + \frac{g'(m(s)) k^\top \mu(t+s)}{\sqrt{\Lambda(t+s)}} - \frac{g''(m(s)) |k^\top \sigma(t+s)|^2}{2\sqrt{\Lambda(t+s)}} \right] ds + \frac{\lambda(t+s) \theta(t+s)^\top + g'(m(s)) k^\top \sigma(t+s)}{\sqrt{\Lambda(t+s)}} dW(t+s), \quad s \in [0, \varepsilon].$$

Applying Itô's formula again yields

$$\begin{aligned} dN(Y(s)) &= N'(Y(s)) dY(s) + \frac{1}{2\Lambda(t+s)} N''(Y(s)) |\lambda(t+s) \theta(t+s)^\top + g'(m(s)) k^\top \sigma(t+s)|^2 ds \\ &= N'(Y(s)) dY(s) - \frac{1}{2\Lambda(t+s)} N'(Y(s)) Y(s) |\lambda(t+s) \theta(t+s)^\top + g'(m(s)) k^\top \sigma(t+s)|^2 ds \\ &= A(s) ds + B(s) dW(t+s), \quad s \in [0, \varepsilon], \end{aligned}$$

where

$$\begin{aligned} A(s) &:= \frac{N'(Y(s))}{\sqrt{\Lambda(t+s)}} \left[ g'(m(s)) k^\top \mu(t+s) - \frac{g''(m(s))}{2} |\sigma(t+s)^\top k|^2 \right. \\ &\quad \left. - \frac{Y(s) g'(m(s))^2}{2\sqrt{\Lambda(t+s)}} |\sigma(t+s)^\top k|^2 - \frac{Y(s) g'(m(s))}{\sqrt{\Lambda(t+s)}} k^\top \sigma(t+s) \lambda(t+s) \theta(t+s) \right] \\ &= \frac{N'(Y(s))}{\sqrt{\Lambda(t+s)}} \left[ -\frac{1}{2} \left( g''(m(s)) + g'(m(s))^2 \frac{Y(s)}{\sqrt{\Lambda(t+s)}} \right) |\sigma(t+s)^\top k|^2 \right. \\ &\quad \left. + g'(m(s)) \left( 1 - \frac{Y(s) \lambda(t+s)}{\sqrt{\Lambda(t+s)}} \right) \theta(t+s)^\top \sigma(t+s)^\top k \right], \quad (3.8) \\ B(s) &:= \frac{N'(Y(s))}{\sqrt{\Lambda(t+s)}} [\lambda(t+s) \theta(t+s)^\top + g'(m(s)) k^\top \sigma(t+s)]. \end{aligned}$$

Integrating from  $s = 0$  to  $s = \delta$ , where  $\delta \in (0, \varepsilon]$ , and then taking conditional expectations on the above, we obtain

$$\mathbb{E}_t[N(Y(\delta))] = N(Y(0)) + \int_0^\delta \mathbb{E}_t[A(s)] ds, \quad (3.9)$$

where we have used the fact that  $\int_0^\cdot B(s) dW(t+s)$  is a martingale on  $[0, \varepsilon]$  for sufficiently small  $\varepsilon > 0$  and hence

$$\mathbb{E}_t \int_0^\delta B(s) dW(t+s) = 0, \quad \delta \in (0, \varepsilon]. \quad (3.10)$$

---

<sup>5</sup>To save space, we will omit to write out the dependence of  $m$  and  $Y$  in  $y$  from this point of the proof, except when it is important to spell out this dependence. However, the reader is urged to bear in mind this dependence while reading the proof.

A proof of this martingality will be delayed to Subsection 3.3. Now, we have

$$\begin{aligned}
& \limsup_{\varepsilon \downarrow 0} \frac{w(t, \mathbb{P}_t(u(X(T) + k^\top \Delta(t, \varepsilon)) > y)) - w(t, \mathbb{P}_t(u(X(T)) > y))}{\varepsilon} \\
&= \limsup_{\varepsilon \downarrow 0} \frac{w(t, \mathbb{E}_t[N(Y(\varepsilon))]) - w(t, N(Y(0)))}{\varepsilon} \\
&= \limsup_{\varepsilon \downarrow 0} \frac{w(t, N(Y(0)) + \int_0^\varepsilon \mathbb{E}_t[A(s)] ds) - w(t, N(Y(0)))}{\varepsilon} \\
&= w'_p(t, N(Y(0))) \cdot \limsup_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^\varepsilon \mathbb{E}_t[A(s)] ds \\
&= w'_p(t, N(Y(0))) A(0).
\end{aligned}$$

Hence (from this point on we will write back the variable  $y$ )

$$\begin{aligned}
& \limsup_{\varepsilon \downarrow 0} \frac{J(X(T) + k^\top \Delta(t, \varepsilon); t, X(t)) - J(X(T); t, X(t))}{\varepsilon} \\
&= \limsup_{\varepsilon \downarrow 0} \int_{-\infty}^{+\infty} \frac{w(t, \mathbb{E}_t[N(Y(\varepsilon, y))]) - w(t, N(Y(0, y)))}{\varepsilon} dy \\
&\leq \int_{-\infty}^{+\infty} \limsup_{\varepsilon \downarrow 0} \frac{w(t, \mathbb{E}_t[N(Y(\varepsilon, y))]) - w(t, N(Y(0, y)))}{\varepsilon} dy \\
&= \int_{-\infty}^{+\infty} w'_p(t, N(Y(0, y))) A(0, y) dy \\
&= -|\sigma(t)^\top k|^2 \frac{1}{2\sqrt{\Lambda(t)}} \int_{-\infty}^{+\infty} w'_p(t, N(Y(0, y))) N'(Y(0, y)) \left( g''(m(0, y)) + g'(m(0, y))^2 \frac{Y(0, y)}{\sqrt{\Lambda(t)}} \right) dy \\
&\quad + \theta(t)^\top \sigma(t)^\top k \frac{1}{\sqrt{\Lambda(t)}} \int_{-\infty}^{+\infty} w'_p(t, N(Y(0, y))) N'(Y(0, y)) g'(m(0, y)) \left( 1 - \frac{Y(0, y)}{\sqrt{\Lambda(t)}} \lambda(t) \right) dy.
\end{aligned} \tag{3.11}$$

In the above, the first inequality

$$\begin{aligned}
& \limsup_{\varepsilon \downarrow 0} \int_{-\infty}^{+\infty} \frac{w(t, \mathbb{E}_t[N(Y(\varepsilon, y))]) - w(t, N(Y(0, y)))}{\varepsilon} dy \\
&\leq \int_{-\infty}^{+\infty} \limsup_{\varepsilon \downarrow 0} \frac{w(t, \mathbb{E}_t[N(Y(\varepsilon, y))]) - w(t, N(Y(0, y)))}{\varepsilon} dy
\end{aligned} \tag{3.12}$$

will be proved in Subsection 3.4.

By Definition 2.2, the underlying strategy is an equilibrium if the right hand side of (3.11) is non-positive for *any*  $k \in \mathbb{R}^n$ .<sup>6</sup> However, the right hand side, being quadratic in  $\sigma(t)^\top k$  while  $\sigma(t)$  is invertible, is non-positive for any  $k$  if and only if (noting  $\theta(t) \neq 0$ )

$$\int_{-\infty}^{+\infty} w'_p(t, N(Y(0, y))) N'(Y(0, y)) g'(m(0, y)) \left( 1 - \frac{Y(0, y)}{\sqrt{\Lambda(t)}} \lambda(t) \right) dy = 0, \tag{3.13}$$

and

$$\int_{-\infty}^{+\infty} w'_p(t, N(Y(0, y))) N'(Y(0, y)) \left( g''(m(0, y)) + g'(m(0, y))^2 \frac{Y(0, y)}{\sqrt{\Lambda(t)}} \right) dy \geq 0. \tag{3.14}$$

---

<sup>6</sup>Recall that  $t$  is fixed; so any  $\mathcal{F}_t$ -measurable random vector  $k$  is almost surely deterministic conditional on  $\mathcal{F}_t$ .



Recall that  $m(0, y) = v(y)$  and  $Y(0, y) = \frac{\mathcal{E}_{0,t}-g(v(y))}{\sqrt{\Lambda(t)}}$ . By changing variables  $x = v(y)$  and  $z = \frac{\mathcal{E}_{0,t}-g(x)}{\sqrt{\Lambda(t)}}$ , the equation (3.13) becomes

$$\begin{aligned}
0 &= \int_{-\infty}^{+\infty} w'_p \left( t, N \left( \frac{\mathcal{E}_{0,t}-g(x)}{\sqrt{\Lambda(t)}} \right) \right) N' \left( \frac{\mathcal{E}_{0,t}-g(x)}{\sqrt{\Lambda(t)}} \right) g'(x) \left( 1 - \frac{\mathcal{E}_{0,t}-g(x)}{\Lambda(t)} \lambda(t) \right) u'(x) dx \\
&= - \int_{-\infty}^{+\infty} w'_p \left( t, N \left( \frac{\mathcal{E}_{0,t}-g(x)}{\sqrt{\Lambda(t)}} \right) \right) N' \left( \frac{\mathcal{E}_{0,t}-g(x)}{\sqrt{\Lambda(t)}} \right) \left( 1 - \frac{\mathcal{E}_{0,t}-g(x)}{\Lambda(t)} \lambda(t) \right) du'(x) \\
&= - \int_{-\infty}^{+\infty} w'_p(t, N(z)) N'(z) \left( 1 - \frac{z}{\sqrt{\Lambda(t)}} \lambda(t) \right) de^{\sqrt{\Lambda(t)}z - \mathcal{E}_{0,t}} \\
&= -e^{-\mathcal{E}_{0,t}} \int_{-\infty}^{+\infty} w'_p(t, N(z)) N'(z) (\sqrt{\Lambda(t)} - z\lambda(t)) e^{\sqrt{\Lambda(t)}z} dz \\
&= -e^{-\mathcal{E}_{0,t}} \mathbb{E} \left[ w'_p(t, N(\xi)) \left( \sqrt{\Lambda(t)} - \xi\lambda(t) \right) e^{\sqrt{\Lambda(t)}\xi} \right],
\end{aligned}$$

where  $\xi \sim N(0, 1)$ . This equation is further equivalent to

$$\sqrt{\Lambda(t)} \mathbb{E} \left[ w'_p(t, N(\xi)) e^{\sqrt{\Lambda(t)}\xi} \right] = \lambda(t) \mathbb{E} \left[ w'_p(t, N(\xi)) \xi e^{\sqrt{\Lambda(t)}\xi} \right],$$

or

$$\lambda(t) = \frac{h(t, \sqrt{\Lambda(t)})}{h'(t, \sqrt{\Lambda(t)})} \sqrt{\Lambda(t)},$$

which holds true by the facts that  $\Lambda$  solves equation (3.3) and  $\Lambda'(t) = -\lambda(t)^2 |\theta(t)|^2$ .

Similarly, we can rewrite the inequality (3.14) as

$$\int_{-\infty}^{+\infty} w'_p \left( t, N \left( \frac{\mathcal{E}_{0,t}-g(x)}{\sqrt{\Lambda(t)}} \right) \right) N' \left( \frac{\mathcal{E}_{0,t}-g(x)}{\sqrt{\Lambda(t)}} \right) \left( g''(x) + g'(x)^2 \frac{\mathcal{E}_{0,t}-g(x)}{\Lambda(t)} \right) du(x) \geq 0, \quad (3.15)$$

which is satisfied under (3.5).  $\square$

In the above proof, there are two technical results, the martingale condition (3.10) and the inequality (3.12), left unproved. We provide proofs in the next two subsections.

### 3.3 The martingale condition (3.10)

**Proposition 3.2** *The martingale condition (3.10) holds for sufficiently small  $\varepsilon > 0$ .*

*Proof.* As before the variable  $y$  is suppressed. It suffices to prove that  $B(s)$  is locally square integrable at  $s = 0+$ . Note that  $\lambda(\cdot), \theta(\cdot), \sigma(\cdot)$  are all bounded, and  $\frac{1}{\sqrt{\Lambda(t+s)}}$  is locally bounded

at  $s = 0+$ . It thus follows from the expression of  $B(\cdot)$  that we only need to estimate a bound of  $N'(Y(s))g'(m(s))$ .

Because  $\limsup_{y \rightarrow +\infty} \frac{y^{-\alpha}}{-I'(y)} < +\infty$  (Assumption 2.4-(i)) and  $\frac{1}{-I'(y)} \equiv -u''(I(y))$  is increasing in  $y$  (Assumption 2.4-(iii)), we have that there exists  $K > 0$  such that

$$\frac{1}{-I'(y)} \leq K(y^\alpha + 1) \quad \forall y > 0. \quad (3.16)$$

Recalling that  $m(s) = f(\mathcal{E}_{0,t+s} - \sqrt{\Lambda(t+s)}Y(s))$  and  $f'(x) = -\tilde{\kappa}I'(\tilde{\kappa}e^{-x})e^{-x}$  where  $\tilde{\kappa} := \kappa e^{-\frac{1}{2}\Lambda(0)}$ , we have

$$\begin{aligned} g'(m(s)) &= g'(f(\mathcal{E}_{0,t+s} - \sqrt{\Lambda(t+s)}Y(s))) \\ &= \frac{1}{f'(\mathcal{E}_{0,t+s} - \sqrt{\Lambda(t+s)}Y(s))} \\ &= \frac{1}{-\tilde{\kappa}I'(\tilde{\kappa}e^{\mathcal{E}_{0,t+s} - \sqrt{\Lambda(t+s)}Y(s)})} \\ &\leq \frac{K}{\tilde{\kappa}} \left( \tilde{\kappa}^\alpha e^{(\alpha-1)\sqrt{\Lambda(t+s)}Y(s) - (\alpha-1)\mathcal{E}_{0,t+s}} + e^{\mathcal{E}_{0,t+s} - \sqrt{\Lambda(t+s)}Y(s)} \right), \end{aligned}$$

where the last inequality is due to (3.16). However,  $N'(x)e^{\gamma x}$  is bounded in  $x \in \mathbb{R}$  for any given  $\gamma \in \mathbb{R}$ ; hence

$$0 < N'(Y(s))g'(m(s)) \leq c_1 e^{-(\alpha-1)\mathcal{E}_{0,t+s}} + c_2 e^{\mathcal{E}_{0,t+s}}.$$

The right hand side above is locally square integrable as a process in  $s$ . The proof is complete.  $\square$

### 3.4 The inequality (3.12)

We now prove the inequality (3.12). In view of Fatou's lemma, it suffices to show that the integrand on its left hand side is dominated by an integrable function. Throughout this subsection, we keep  $t \in [0, T)$  fixed.

**Lemma 3.3** *For any constant  $\gamma \in (0, 1]$  and  $\varepsilon \in (0, T - t)$ , there exists a constant  $c_1(\gamma) > 0$  such that*

$$\frac{|A(\delta, y)|}{N'(Y(\delta, y))^{1-\gamma}} = \frac{|A(\delta, y)|}{N'(-Y(\delta, y))^{1-\gamma}} \leq c_1(\gamma) e^{\frac{\gamma}{4\Lambda(t+\delta)}\mathcal{E}_{0,t+\delta}^2} \quad \forall \delta \in (0, \varepsilon], y \in \mathbb{R}.$$

*Proof.* As before, we drop the  $y$  variable to save space in this proof. Since  $g(x) = l(x) + \ln \kappa - \frac{1}{2}\Lambda(0)$ , Assumption 2.4-(ii) is satisfied with  $l(\cdot)$  replaced by  $g(\cdot)$  (and the constants  $a$  and  $b$

properly modified). Hence,

$$\begin{aligned}
\frac{|A(\delta)|}{N'(Y(\delta))^{1-\gamma}} &\leq c_2(\gamma)N'(Y(\delta))^\gamma(e^{a|g(m(\delta))|+b} + |Y(\delta)|e^{2a|g(m(\delta))|+2b}) \\
&= \frac{c_2(\gamma)}{\sqrt{2\pi}}e^{-\frac{\gamma}{2}Y(\delta)^2}(e^{a|g(m(\delta))|+b} + |Y(\delta)|e^{2a|g(m(\delta))|+2b}) \\
&\leq c_3(\gamma)e^{-\frac{\gamma}{4}Y(\delta)^2}e^{2a|g(m(\delta))|+2b} \\
&= c_3(\gamma)e^{-\frac{\gamma}{4\Lambda(t+\delta)}(\mathcal{E}_{0,t+\delta}-g(m(\delta)))^2}e^{2a|g(m(\delta))|+2b},
\end{aligned}$$

where  $c_2(\gamma) > 0$  and  $c_3(\gamma) > 0$  are suitable constants. Making use of the general inequality  $(x + y)^2 \geq \frac{1}{2}x^2 - y^2$ , we deduce

$$\begin{aligned}
\frac{|A(\delta)|}{N'(Y(\delta))^{1-\gamma}} &\leq c_3(\gamma)e^{-\frac{\gamma}{4\Lambda(t+\delta)}(\mathcal{E}_{0,t+\delta}-g(m(\delta)))^2}e^{2a|g(m(\delta))|+2b} \\
&\leq c_3(\gamma)e^{\frac{\gamma}{4\Lambda(t+\delta)}\mathcal{E}_{0,t+\delta}^2}e^{-\frac{\gamma}{8\Lambda(t+\delta)}g(m(t+\delta))^2+2a|g(m(t+\delta))|+2b} \\
&\leq c_3(\gamma)e^{n_1(\gamma)}e^{\frac{\gamma}{4\Lambda(t+\delta)}\mathcal{E}_{0,t+\delta}^2}, \tag{3.17}
\end{aligned}$$

where  $n_1(\gamma) := \max_{x \in \mathbb{R}} \{-\frac{\gamma}{8\Lambda t+\delta}x^2 + 2a|x| + 2b\} < +\infty$ . This completes the proof.  $\square$

**Lemma 3.4** *For any  $\gamma \in (0, 1)$ , there exists  $\varepsilon \in (0, T - t)$  and a constant  $M_1(\gamma) < +\infty$  such that*

$$\frac{\mathbb{E}_t|A(\delta, y)|}{[\mathbb{E}_tN(Y(\delta, y))]^\gamma} \leq M_1(\gamma) \quad \forall \delta \in (0, \varepsilon], y \in \mathbb{R}.$$

*Proof.* Again we omit to write out  $y$ . Denote  $m = 1/\gamma$  and  $n = \frac{m}{m-1}$ . Then Cauchy–Schwarz’s inequality yields

$$\begin{aligned}
\frac{\mathbb{E}_t|A(\delta)|}{[\mathbb{E}_tN(Y(\delta))]^\gamma} &\equiv \frac{\mathbb{E}_t|A(\delta)|}{[\mathbb{E}_tN(Y(\delta))]^{1/m}} \\
&\leq \left(\mathbb{E}_t[|A(\delta)|^n N(Y(\delta))^{-n/m}]\right)^{1/n}.
\end{aligned}$$

Write

$$\begin{aligned}
|A(\delta)|^n N(Y(\delta))^{-n/m} &= \left(\frac{N'(Y(\delta))}{N(Y(\delta))}\right)^{n/m} \left(\frac{|A(\delta)|}{N'(Y(\delta))^{1/m}}\right)^n \\
&= \left(\frac{N'(Y(\delta))}{N(Y(\delta))}\right)^{n/m} \left(\frac{|A(\delta)|}{N'(Y(\delta))^{(\gamma+1)/2}}\right)^n N'(Y(\delta))^{n(1-\gamma)/2}.
\end{aligned}$$

Noting  $\frac{N'(x)}{N(x)} \leq C(|x| + 1) \forall x \in \mathbb{R}$  for some  $C > 0$ , we conclude that the last term above converges to 0 faster than the first term going to  $+\infty$  when  $Y(\delta)$  goes to  $+\infty$ ; hence we can

find a bound  $M_2(\gamma) > 0$  such that

$$\begin{aligned} |A(\delta)|^n N(Y(\delta))^{-n/m} &\leq M_2(\gamma) \left( \frac{|A(\delta)|}{N'(Y(\delta))^{(\gamma+1)/2}} \right)^n \\ &\leq M_2(\gamma) c_1 ((1-\gamma)/2)^n e^{\frac{1}{8\Lambda(t+\delta)} \mathcal{E}_{0,t+\delta}^2}, \end{aligned}$$

where the last inequality is due to Lemma 3.3. When  $\varepsilon > 0$  is small enough, there exists  $M_3(\gamma) < \infty$  such that  $\mathbb{E}_t[e^{\frac{1}{8\Lambda(t+\delta)} \mathcal{E}_{0,t+\delta}^2}] < M_3(\gamma)$  for any  $\delta \in (0, \varepsilon]$ . This leads to the desired inequality.  $\square$

**Proposition 3.5** *We have the following conclusions:*

- (i) *For any  $\gamma > 0$ , there are a sufficiently small  $\varepsilon \in (0, T-t)$  and a function  $H_1(\cdot; \gamma)$  with  $\int_{-\infty}^{\underline{y}(\gamma)} H_1(y; \gamma) dy < +\infty$  for some  $\underline{y}(\gamma) < 0$ , such that when  $\delta \in (0, \varepsilon)$ , we have*

$$(\mathbb{E}_t |A(\delta)|)^\gamma \leq H_1(y; \gamma) \quad \forall y \in (-\infty, \underline{y}(\gamma)].$$

- (ii) *For any  $\gamma > 0$ , there are a sufficiently small  $\varepsilon \in (0, T-t)$  and a function  $H_2(\cdot; \gamma)$  with  $\int_{\bar{y}(\gamma)}^{+\infty} H_2(y; \gamma) dy < +\infty$  for some  $\bar{y}(\gamma) > 0$ , such that when  $\delta \in (0, \varepsilon)$ , we have*

$$(\mathbb{E}_t |A(\delta)|)^\gamma \leq H_2(y; \gamma) \quad \forall y \in [\bar{y}(\gamma), +\infty).$$

*Proof.* We prove only (i), while (ii) being similar. Applying  $\gamma = 1$  to the second inequality of (3.17), we get

$$|A(\delta)| \leq c_3(1) e^{\frac{1}{4\Lambda(t+\delta)} \mathcal{E}_{0,t+\delta}^2} e^{-\frac{1}{8\Lambda(t+\delta)} g(m(\delta))^2 + 2a|g(m(\delta))| + 2b}.$$

Define  $n_2 := \max_{x \in \mathbb{R}} \{-\frac{1}{16\Lambda(t+\delta)} x^2 + 2a|x| + 2b\} < +\infty$ . Then

$$|A(\delta)| \leq c_3(1) e^{n_2} e^{\frac{1}{4\Lambda(t+\delta)} \mathcal{E}_{0,t+\delta}^2} e^{-\frac{1}{16\Lambda(t+\delta)} g(m(\delta))^2}.$$

So

$$\begin{aligned} \mathbb{E}_t |A(\delta)| &\leq c_3(1) e^{n_2} \mathbb{E}_t [e^{\frac{1}{4\Lambda(t+\delta)} \mathcal{E}_{0,t+\delta}^2} e^{-\frac{1}{16\Lambda(t+\delta)} g(m(\delta))^2}] \\ &\leq c_3(1) e^{n_2} \sqrt{\mathbb{E}_t e^{\frac{1}{2\Lambda(t+\delta)} \mathcal{E}_{0,t+\delta}^2}} \sqrt{\mathbb{E}_t e^{-\frac{1}{8\Lambda(t+\delta)} g(m(\delta))^2}}. \end{aligned}$$

Set  $n_3 := c_3(1)^2 e^{2n_2} \mathbb{E}_t [e^{\frac{1}{2\Lambda(t+\delta)} \mathcal{E}_{0,t+\delta}^2}]$  which is a finite constant when  $\varepsilon$  (and hence  $\delta$ ) is sufficiently small, and  $\xi := k^\top \Delta(t, \delta) \sim N(0, \eta^2)$  conditional on  $\mathcal{F}_t$ , where  $\eta^2 \equiv \eta(\delta)^2 =$

$\text{Var}(\xi) \leq |k|^2 \int_t^{t+\delta} |\sigma(s)|^2 ds$ . Then

$$\begin{aligned}
(\mathbb{E}_t |A(\delta)|)^2 &\leq n_3 \mathbb{E}_t [e^{-\frac{1}{8\Lambda(t+\delta)} g(m(\delta))^2}] \\
&= n_3 \mathbb{E}_t [e^{-\frac{1}{8\Lambda(t+\delta)} \ln^2 u'(v(y)-\xi)}] \\
&= \frac{n_3}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{8\Lambda(t+\delta)} \ln^2 u'(v(y)+\eta z)} e^{-z^2/2} dz \\
&= \frac{n_3}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{8\Lambda(t+\delta)} \ln^2 u'(\eta \tilde{z})} e^{-(\tilde{z}-v(y)/\eta)^2/2} d\tilde{z} \\
&= \frac{n_3}{\sqrt{2\pi}} \left[ \int_{-\infty}^{v(y)/(2\eta)} + \int_{v(y)/(2\eta)}^0 + \int_0^{+\infty} \right] e^{-\frac{1}{8\Lambda(t+\delta)} \ln^2 u'(\eta \tilde{z})} e^{-(\tilde{z}-v(y)/\eta)^2/2} d\tilde{z}.
\end{aligned}$$

We now find an integrable bound (as a function of  $y$ ) for each of the three integrals in the above.

For the first integral, take  $y_0 < 0$  small enough such that  $v(y_0) < 0$  and  $u'(v(y_0)/2) > 1$ . Then  $v(y) < 0$  and  $u'(v(y)/2) \geq u'(v(y_0)/2) > 1 \forall y \leq y_0$ . Thus

$$\int_{-\infty}^{v(y)/(2\eta)} e^{-\frac{1}{8\Lambda(t+\delta)} \ln^2 u'(\eta \tilde{z})} e^{-(\tilde{z}-v(y)/\eta)^2/2} d\tilde{z} < \sqrt{2\pi} e^{-\frac{1}{8\Lambda(t+\delta)} \ln^2 u'(v(y)/2)} \leq c_4(a_1) u'(v(y)/2)^{-a_1}, \quad y \leq y_0,$$

for any  $a_1 > 0$  and some  $c_4(a_1) < +\infty$ .<sup>7</sup>

For the second integral, we have

$$\int_{v(y)/(2\eta)}^0 e^{-\frac{1}{8\Lambda(t+\delta)} \ln^2 u'(\eta \tilde{z})} e^{-(\tilde{z}-v(y)/\eta)^2/2} d\tilde{z} \leq e^{-v(y)^2/(8\eta^2)} |v(y)/(2\eta)| \leq c_5 e^{-v(y)^2/(9\eta^2)}$$

for some constant  $c_5 > 0$ .

For the last integral, we get

$$\begin{aligned}
&\int_0^{+\infty} e^{-\frac{1}{8\Lambda(t+\delta)} \ln^2 u'(\eta \tilde{z})} e^{-(\tilde{z}-v(y)/\eta)^2/2} d\tilde{z} \\
&= \int_0^{+\infty} e^{-\frac{1}{8\Lambda(t+\delta)} \ln^2 u'(\eta \tilde{z})} e^{-\tilde{z}^2/2} e^{\tilde{z}v(y)/\eta} e^{-v(y)^2/(2\eta^2)} d\tilde{z} \\
&< e^{-v(y)^2/(2\eta^2)} \int_0^{+\infty} e^{-\tilde{z}^2/2} d\tilde{z} = \sqrt{\frac{\pi}{2}} e^{-v(y)^2/(2\eta^2)}.
\end{aligned}$$

Combining the three integrals, we conclude that, for any  $a_1 > 0, a_2 > 0$ , when  $\varepsilon \in (0, T-t)$  is small enough and  $y_1 < 0$  with  $v(y_1)$  sufficiently negative, there exists constant  $c_6 > 0$  such that for any  $\delta \in (0, \varepsilon)$ ,

$$(\mathbb{E}_t |A(\delta)|)^2 \leq c_6 [u'(v(y))^{-a_1} + e^{-a_2 v(y)^2}] := H_1(y; \gamma)^{2/\gamma} \quad \forall y \leq y_1.$$

---

<sup>7</sup>This is based on the fact (which can be easily shown) that given  $\alpha > 0$ , for any  $a > 0$  there is a constant  $c(a) > 0$  such that  $e^{-\alpha \ln^2 x} \leq c(a)x^{-a} \forall x > 1$ .

Finally, for any  $\gamma > 0$ , we can take suitable  $a_2$  and  $a_3$  (which may depend on  $\gamma$ ) such that, in view of Assumption 2.5,  $\int_{-\infty}^{\underline{y}(\gamma)} H_1(y)dy < +\infty$ , where  $\underline{y}(\gamma)$  is sufficiently negative.  $\square$

**Theorem 3.6** *There exists an integrable function  $H(\cdot)$  such that when  $\varepsilon > 0$  is sufficiently small, it holds that*

$$\left| \frac{w(t, \mathbb{E}_t[N(Y(\varepsilon, y))]) - w(t, N(Y(0, y)))}{\varepsilon} \right| \leq H(y) \quad \forall y \in \mathbb{R}.$$

*Proof.* Fix  $\varepsilon > 0$ . For any  $y \in \mathbb{R}$ , by the mean-value theorem, there exists  $\delta \in [0, \varepsilon]$  (which depends on  $t$  and  $y$ ) such that

$$\begin{aligned} \left| \frac{w(t, \mathbb{E}_t[N(Y(\varepsilon, y))]) - w(t, N(Y(0, y)))}{\varepsilon} \right| &= |w'_p(t, \mathbb{E}_t[N(Y(\delta, y))])\mathbb{E}_t[A(\delta)]| \\ &\leq w'_p(t, \mathbb{E}_t[N(Y(\delta, y))])\mathbb{E}_t[|A(\delta)|], \end{aligned}$$

where we have used the fact that, by virtue of (3.9),  $\frac{d}{d\delta}\mathbb{E}_t[N(Y(\delta, y))] = \mathbb{E}_t[A(\delta)]$ . By Assumption 3.1-(iv), we have

$$\left| \frac{w(t, \mathbb{E}_t[N(Y(\varepsilon, y))]) - w(t, N(Y(0, y)))}{\varepsilon} \right| \leq c \frac{\mathbb{E}_t[|A(\delta)|]}{(\mathbb{E}_t[N(Y(\delta, y))])^{-m}} + c \frac{\mathbb{E}_t[|A_\delta|]}{(1 - \mathbb{E}_t[N(Y(\delta, y))])^{-m}}$$

for some  $c > 0$  and  $m \in (-1, 0)$ .

Recall that  $\mathbb{E}_t[N(Y(\delta, y))] = \mathbb{P}_t(u(X(T) + k^\top \Delta(t, \delta)) > y)$ . Choose  $y_1$  and  $y_2$  such that  $\mathbb{E}_t[N(Y(\delta, y_1))] < 1/2$  and  $\mathbb{E}_t[N(Y(-\delta, y_2))] < 1/2$  for all  $\delta \in (0, \varepsilon]$ .

Using notations in Lemma 3.4 and Proposition 3.5, denote

$$\underline{y} := \underline{y}((1+m)/2) \wedge y_1, \quad \bar{y} := \bar{y}((1+m)/2) \vee y_2,$$

and

$$\begin{aligned} \hat{H}_1(y) &:= 2cH_1(y; (1+m)/2)M_1((2m)/(m-1))^{(1-m)/2}, \\ \hat{H}_2(y) &:= 2cH_2(y; (1+m)/2)M_1((2m)/(m-1))^{(1-m)/2}. \end{aligned}$$

Then:

(i) For any  $y < \underline{y}$ , we deduce

$$\begin{aligned} \left| \frac{w(t, \mathbb{E}_t[N(Y(\varepsilon, y))]) - w(t, N(Y(0, y)))}{\varepsilon} \right| &\leq 2c \frac{\mathbb{E}_t[|A(\delta)|]}{(\mathbb{E}_t[N(Y(\delta, y))])^{-m}} \\ &\leq 2c(\mathbb{E}_t[|A(\delta)|])^{(1+m)/2} \frac{(\mathbb{E}_t[|A(\delta)|])^{(1-m)/2}}{(\mathbb{E}_t[N(Y(\delta, y))])^{-m}} \\ &\leq 2cH_1(y; (1+m)/2)M_1((2m)/(m-1))^{(1-m)/2} \\ &= \hat{H}_1(y). \end{aligned}$$

(ii) Similarly, for any  $y > \bar{y}$ , we have  $\left| \frac{w(t, \mathbb{E}_t[N(Y(\varepsilon, y))] - w(t, N(Y(0, y))))}{\varepsilon} \right| \leq \hat{H}_2(y)$ .

(iii) Finally,  $\left| \frac{w(t, \mathbb{E}_t[N(Y(\varepsilon, y))] - w(t, N(Y(0, y))))}{\varepsilon} \right| \leq 2cM_1(-m)$  when  $y \in [\underline{y}, \bar{y}]$ .

Then  $H(\cdot)$ , where  $H(y) = \hat{H}_1(y)\mathbf{1}_{y < \underline{y}} + 2cM_1(-m) + \hat{H}_2(y)\mathbf{1}_{y \geq \bar{y}}$ ,  $y \in \mathbb{R}$ , is the desired function.  $\square$

## 4 Sufficient Conditions

Theorem 3.1 holds under three major assumptions: the existence of a positive constant  $\kappa$  such that the budget constraint (3.6) holds, the existence of a positive solution to the ODE (3.3), and the validity of the inequality (3.5). It is hard to verify these assumptions (and hence hard to tell if they are reasonable) because they are not imposed directly on the model primitives. This section explores equivalent or at least sufficient conditions, that are shown to be reasonable economically, under which these three assumptions are satisfied respectively.

### 4.1 Budget constraint (3.6)

The equation (3.6) is *not* a “standard” budget constraint appearing in the classical Merton problem because  $\rho(T)$  and  $\bar{\rho}(T)$  are in general different. However, we have the following result.

**Theorem 4.1** *The following two statements are equivalent:*

(i) *There exists  $\kappa > 0$  such that (3.6) holds.*

(ii) *There exists  $\bar{\kappa} > 0$  such that*

$$\mathbb{E}[\bar{\rho}(T)I(\bar{\kappa}\bar{\rho}(T))] = x_0. \quad (4.1)$$

*Proof.* Let  $\mathbb{Q}$  and  $\bar{\mathbb{Q}}$  be respectively the equivalent martingale measures corresponding to  $\rho(T)$  and  $\bar{\rho}(T)$ ; namely,

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \rho(T), \quad \frac{d\bar{\mathbb{Q}}}{d\mathbb{P}} = \bar{\rho}(T).$$

By Girsanov’s theorem,  $\tilde{W}(\cdot)$  and  $\bar{W}(\cdot)$  are respectively Brownian motions under  $\mathbb{Q}$  and  $\bar{\mathbb{Q}}$ , where

$$\tilde{W}(t) := W(t) + \int_0^t \theta(s)ds, \quad \bar{W}(t) := W(t) + \int_0^t \lambda(s)\theta(s)ds, \quad t \in [0, T].$$

Now, we have

$$\begin{aligned}
\mathbb{E}[\rho(T)I(\kappa\bar{\rho}(T))] &= \mathbb{E}^{\mathbb{Q}} \left[ I(\kappa e^{-\frac{1}{2} \int_0^T |\lambda(s)\theta(s)|^2 ds - \int_0^T \lambda(s)\theta(s)^\top dW(s)}) \right] \\
&= \mathbb{E}^{\mathbb{Q}} \left[ I(\kappa e^{-\frac{1}{2} \int_0^T |\lambda(s)\theta(s)|^2 ds + \int_0^T \lambda(s)|\theta(s)|^2 ds - \int_0^T \lambda(s)\theta(s)^\top d\tilde{W}(s)}) \right] \\
&= \mathbb{E} \left[ I(\kappa e^{-\frac{1}{2} \int_0^T |\lambda(s)\theta(s)|^2 ds + \int_0^T \lambda(s)|\theta(s)|^2 ds - \int_0^T \lambda(s)\theta(s)^\top dW(s)}) \right] \\
&= \mathbb{E}^{\bar{\mathbb{Q}}} \left[ I(\kappa e^{-\frac{1}{2} \int_0^T |\lambda(s)\theta(s)|^2 ds + \int_0^T \lambda(s)|\theta(s)|^2 ds - \int_0^T \lambda(s)\theta(s)^\top d\bar{W}(s)}) \right] \\
&= \mathbb{E} \left[ \bar{\rho}(T) I(\kappa e^{-\frac{1}{2} \int_0^T |\lambda(s)\theta(s)|^2 ds + \int_0^T \lambda(s)|\theta(s)|^2 ds - \int_0^T \lambda(s)\theta(s)^\top d\bar{W}(s)}) \right] \\
&= \mathbb{E} \left[ \bar{\rho}(T) I \left( \kappa e^{\int_0^T \lambda(s)(1-\lambda(s))|\theta(s)|^2 ds} \bar{\rho}(T) \right) \right].
\end{aligned}$$

This establishes the desired equivalence with  $\bar{\kappa} = \kappa e^{\int_0^T \lambda(s)(1-\lambda(s))|\theta(s)|^2 ds}$ .  $\square$

So the assumption in Theorem 3.1 regarding the existence of a positive constant  $\kappa$  satisfying (3.6) boils down to the familiar condition (4.1). The latter condition is standard in the classical Merton problem in which the pricing kernel is  $\bar{\rho}(T)$  or, equivalently, the market price of risk process is  $\lambda(\cdot)\theta(\cdot)$ . Note that because the probability weighting function  $w$  has been embedded into  $\lambda(\cdot)$ , the existence of a positive  $\bar{\kappa}$  satisfying (4.1) becomes a condition on the utility function  $u$  only, which is satisfied by, say, the exponential utility.<sup>8</sup>

More important, with Theorem 4.1, Theorem 3.1 shows that the investment behavior of the sophisticated RDU agent is *indistinguishable* from an EUT maximizer in a market where the market price of risk is revised from  $\theta(\cdot)$  to  $\lambda(\cdot)\theta(\cdot)$ . This finding may have important economic implications especially in the study of market equilibria.

## 4.2 Existence of positive solutions to (3.3)

The main result of this paper depends crucially on the existence of a positive solution to the ODE (3.3). Note that this equation is highly nonlinear, and singular at  $t = T$  in that the denominator of the right hand side of the equation is 0 at  $T$ . In this subsection we provide conditions under which (3.3) admits positive solutions, by applying a general existence result for a class of ODEs with singular initial/terminal values (see Appendix A).

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<sup>8</sup>Indeed, if this condition fails, than it is usually an indication that the original problem is not well-posed and/or an optimal solution is not attainable; see Jin et al (2008) for a detailed analysis on this constraint.



Setting  $y(t) = \Lambda(T - t)$ , (3.3) is equivalent to

$$\begin{cases} y'(t) = |\theta(T - t)|^2 \left( \frac{h(T-t, \sqrt{y(t)})}{h'_x(T-t, \sqrt{y(t)})} \right)^2 y(t), & t \in (0, T], \\ y(0) = 0. \end{cases} \quad (4.2)$$

In the rest of this subsection we study equation (4.2) instead of (3.3). The key idea is to first establish the local existence in the right neighborhood of  $t = 0$ , and then extend it globally to the whole time interval  $[0, T]$ .

We introduce the following assumption on the function  $h(\cdot, \cdot)$  (which depends directly on the probability weighting function) and on the market represented by  $\theta(\cdot)$ :

**Assumption 4.2** (i)  $h'_x(t, 0) \geq 0$  and  $h'''_x(t, 0) \geq 0 \forall t \in [0, T]$ .

(ii)  $\limsup_{t \uparrow T} \frac{h'_x(t, 0)}{\sqrt{|\theta(t)|^2(T-t)}} < 1$  and  $\liminf_{t \uparrow T} |\theta(t)|^2 > 0$ .

(iii)  $\sup_{t \in [0, T]} h(t, 1) < +\infty$  and  $\limsup_{t \uparrow T} h''_x(t, 1) < +\infty$ .

(iv)  $\inf_{t \in [0, T]} h''_x(t, 0) > 0$ .

Assumption 4.2-(iii) and -(iv) are mild. Assumption 4.2-(i) can be relaxed with a more subtle analysis than the one to be given below; although we will not pursue in that direction. The first part of Assumption 4.2-(ii) is the most important of all, which regulates how the probability weighting function  $w(t, \cdot)$  should behave, given the market, when  $t$  is sufficiently close to the terminal time  $T$ .<sup>9</sup> Mathematically, this terminal behavior of weighting functions is translated into the singularity of the ODE (3.3) at  $T$ , which is why Assumption 4.2 is needed for a proof of the existence of (3.3). Luckily, we will show in Appendix B that all the parts of Assumption 4.2 are satisfied by a family of time-varying Tversky–Kahnmen’s weighting functions.

We first strengthen Lemma 2.7 under Assumption 4.2-(i).

**Lemma 4.3** *Under Assumption 4.2-(i) in addition to the same assumption of Lemma 2.7, for any  $t \in [0, T]$ ,  $h(t, \cdot)$  has the following properties:*

(i)  $h(t, x)$  and  $h''(t, x)$  are both increasing in  $x \geq 0$ , and  $h'_x(t, x)$  is convex in  $x \geq 0$ .

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<sup>9</sup>We believe that this is a distinctive feature of the continuous-time setting. In the discrete-time case, there is no infinitesimal issue of the weighting functions close to the terminal time.

(ii)  $xh_x''(t, 0) \leq h_x'(t, x) \leq h_x'(t, 0) + xh_x''(t, x) \forall x \geq 0$ .

*Proof.* (i) It follows from Lemma 2.7-(iii) that  $h_x'(t, x)$  and  $h_x^{(3)}(t, x)$  are both increasing in  $x \geq 0$ . Assumption 4.2-(i) then leads to the desired results readily.

(ii) Applying the Taylor expansion, we have for any  $x > 0$ , there exists  $\zeta \in [0, x]$  such that

$$h_x'(t, x) = h_x'(t, 0) + xh_x''(t, 0) + \frac{1}{2}x^2h_x^{(3)}(t, \zeta) \geq xh_x''(t, x).$$

On the other hand, the convexity of  $h_x'(t, x)$  in  $x \geq 0$  implies that

$$h_x'(t, x) \leq h_x'(t, 0) + xh_x''(t, x).$$

The proof is complete. □

**Lemma 4.4** *Under Assumption 4.2, there exist  $k_1 > 0$  greater than any given number and  $\delta_1 > 0$  such that*

$$|\theta(T-t)|^2 \left( \frac{\sqrt{t}h(T-t, \sqrt{k_1t})}{h_x'(T-t, \sqrt{k_1t})} \right)^2 < 1 \quad \forall t \in (0, \delta_1]. \quad (4.3)$$

Moreover, there exist  $k_2 > 0$  less than any given positive number and  $\delta_2 > 0$  such that

$$|\theta(T-t)|^2 \left( \frac{\sqrt{t}h(T-t, \sqrt{k_2t})}{h_x'(T-t, \sqrt{k_2t})} \right)^2 > 1 \quad \forall t \in (0, \delta_2]. \quad (4.4)$$

*Proof.* By Lemma 4.3-(i),  $h(T-t, x)$  is increasing when  $x \geq 0$ ; hence

$$\limsup_{t \downarrow 0} h(T-t, \sqrt{kt}) \leq \limsup_{t \downarrow 0} h(T-t, 1) < +\infty, \quad (4.5)$$

where the finiteness is due to Assumption 4.2-(iii). On the other hand, it follows from Lemma 4.3-(ii) that, for any  $k > 0$ ,

$$\frac{h_x'(T-t, \sqrt{kt})}{\sqrt{t}} \geq \sqrt{k}h_x''(T-t, 0) \geq \sqrt{k} \inf_{s \in [0, T]} h_x''(s, 0) > 0, \quad (4.6)$$

where the last inequality is due to Assumption 4.2-(iv). Combining (4.5) and (4.6) and noting the boundedness of  $\theta(\cdot)$ , we conclude that there is  $k = k_1$  greater than any given number such that (4.3) is satisfied.

Next, by Lemma 4.3-(ii), we deduce that for sufficiently small  $k > 0$ ,

$$\begin{aligned} \limsup_{t \downarrow 0} \frac{h_x'(T-t, \sqrt{kt})}{\sqrt{|\theta(T-t)|^2 t}} &\leq \limsup_{t \downarrow 0} \left\{ \frac{h_x'(T-t, 0)}{\sqrt{|\theta(T-t)|^2 t}} + \sqrt{k} \frac{h_x''(T-t, \sqrt{kt})}{\sqrt{|\theta(T-t)|^2}} \right\} \\ &\leq \limsup_{t \downarrow 0} \frac{h_x'(T-t, 0)}{\sqrt{|\theta(T-t)|^2 t}} + \sqrt{k} \limsup_{t \downarrow 0} \frac{h_x''(T-t, 1)}{\sqrt{|\theta(T-t)|^2}} \\ &< 1, \end{aligned}$$

where the last inequality follows from Assumption 4.2-(ii) and -(iii). Since  $h(T-t, \sqrt{kt}) \geq h(T-t, 0) = 1$ , we conclude that there is  $k = k_2$  less than any given positive number so that (4.4) is satisfied.  $\square$

The following establishes the local existence of the ODE (4.2).

**Proposition 4.5** *Under Assumption 4.2, equation (4.2) admits a solution  $y(\cdot) \in C[0, \delta] \cap C^1(0, \delta]$  on a time interval  $[0, \delta]$  for some  $\delta > 0$ , with  $y(t) > 0 \forall t \in (0, \delta]$ .*

*Proof.* Fix  $k_1 > 0$  and  $0 < k_2 < k_1$  as in Lemma 4.4, and take  $\delta = \delta_1 \wedge \delta_2$ . Then it is easy to check that  $\beta(t) = k_1 t$  and  $\alpha(t) = k_2 t$  satisfy all the requirements in Theorem A.2 on  $(0, \delta]$ ; hence the result.  $\square$

Next we extend the local solution  $y(\cdot)$  obtained in Proposition 4.5 to the whole time interval  $[0, T]$ . Denote  $y(\delta) = x_1 > 0$ . Without loss of generality, we assume  $x_1 < 1$  (otherwise, we can make  $\delta$  closer to 0 to reduce  $x_1$ ).

**Proposition 4.6** *Under Assumption 4.2-(i) and (iv), for any  $\delta > 0$  and  $0 < x_1 < 1$ , the equation*

$$\begin{cases} y'(t) = |\theta(T-t)|^2 \left( \frac{h(T-t, \sqrt{y(t)})}{h'_x(T-t, \sqrt{y(t)})} \right)^2 y(t), & t \in (\delta, T], \\ y(\delta) = x_1 \end{cases} \quad (4.7)$$

*has a solution  $y(\cdot) \in C^1[\delta, T]$  with  $y(t) > 0 \forall t \in [\delta, T]$ .*

*Proof.* Since  $\ln h(x, t)$  is convex in  $x \geq 0$  (by Lemma 2.7-(iv)), we have

$$\frac{h'_x(t, x)}{h(t, x)} \geq \frac{h'_x(t, \sqrt{x_1})}{h(t, \sqrt{x_1})} \geq \sqrt{x_1} \frac{h''_x(t, 0)}{h(t, \sqrt{x_1})} \geq \sqrt{x_1} \frac{h''_x(t, 0)}{h(t, 1)} \geq c_1 > 0 \quad \forall t \in [0, T], \quad x \geq \sqrt{x_1},$$

where the second inequality is due to Lemma 4.3-(ii) and  $c_1$  (which may depend on  $x_1$ ) is a constant arising from Assumption 4.2-(iii) and -(iv). Hence for any  $y \geq x_1$ , we have

$$|\theta(T-t)|^2 \left( \frac{h(T-t, \sqrt{y})}{h'_x(T-t, \sqrt{y})} \right)^2 y \leq c_2 y,$$

where  $c_2 > 0$  is a constant depending on  $c_1$  and the bound of  $|\theta(\cdot)|^2$ .

Denote  $c_3 = x_1 e^{c_2 T} < +\infty$ , and a truncation function  $r(y) = (y \vee x_1) \wedge c_3$  for  $y \geq 0$ .

Consider the ODE

$$\begin{cases} y'(t) = f(t, y(t)), & t \in (\delta, T], \\ y(\delta) = x_1 \end{cases} \quad (4.8)$$

where

$$f(t, y) := |\theta(T - t)|^2 \left( \frac{h(T - t, \sqrt{r(y)})}{h'_x(T - t, \sqrt{r(y)})} \right)^2 r(y).$$

It is easy to show that  $f(\cdot, \cdot)$  satisfies the conditions in Theorem A.3 on  $(\delta, T]$ ; hence (4.8) admits a solution  $y(\cdot)$ . Moreover, since  $y'(t) \geq 0$  we have  $y(t) \geq x_1 \forall t \in [\delta, T]$ .

Take  $t_1 := \inf\{t \in (\delta, T] : y(t) \geq c_3\} \wedge T$ . If  $t_1 < T$ , then  $y(t_1) = c_3$  and  $y(t) < c_3 \forall t < t_1$ . Consequently  $y'(t) = |\theta(T - t)|^2 \left( \frac{h(T - t, \sqrt{y(t)})}{h'_x(T - t, \sqrt{y(t)})} \right)^2 y(t) \leq c_2 y(t)$ ,  $\forall t \in (\delta, t_1)$ . Grownwall's inequality then yields  $y(t_1) \leq x_1 e^{c_2 t_1} < c_3$ , which is a contradiction. Hence  $t_1 = T$  and  $y(t) \leq c_3$  on  $[\delta, T]$ . In this case equations (4.8) and (4.7) coincide.  $\square$

Combing Propositions 4.5 and 4.6, we arrive at

**Theorem 4.7** *Under Assumption 4.2, the equation (3.3) admits a solution  $\Lambda(\cdot) \in C[0, T] \cap C^1[0, T]$  satisfying  $\Lambda(t) > 0 \forall t \in [0, T]$ .*

### 4.3 Inequality (3.5)

We now provide conditions on the model primitives under which the inequality (3.5) holds. We assume that the ODE (3.3) admits a positive solution  $\Lambda(\cdot) \in C[0, T] \cap C^1[0, T]$ , and introduce the following additional assumption.

**Assumption 4.8** *For a.e.  $t \in [0, T)$ ,  $w(t, \cdot)$  is either convex or inverse S-shaped.*

A convex weighting function captures risk aversion in terms of exaggerating the small probability of very “bad” events while downplaying very “good” events; see Yaari (1987). On the other hand, as discussed in Introduction, an inverse S-shaped probability weighting function is more interesting as it reflects the tendency of inflating the small probabilities of *both* tails which are consistent with the conclusions of many experimental and empirical works.

**Lemma 4.9** *Under Assumptions 4.2 and 4.8, we have*

$$\lim_{|y| \rightarrow +\infty} \frac{w'_p(t, N(y))N'(y)}{I'(e^{\sqrt{\Lambda(t)}y - c})} = 0 \quad \forall c \in \mathbb{R}, \forall t \in [0, T]. \quad (4.9)$$

*Proof.* It follows from (3.16) that

$$0 \leq \frac{w'_p(t, N(y))}{-I'(e^{\sqrt{\Lambda(t)}y - c})} \leq K w'_p(t, N(y)) (e^{\alpha \sqrt{\Lambda(t)}y - \alpha c} + 1).$$

However

$$\int_{-\infty}^{+\infty} w'_p(t, N(y))(e^{\alpha\sqrt{\Lambda(t)}y-\alpha c} + 1)N'(y)dy = e^{-\alpha c}h(t, \alpha\sqrt{\Lambda(t)}) + 1 < +\infty,$$

implying

$$\lim_{|y|\rightarrow+\infty} w'_p(t, N(y))(e^{\alpha\sqrt{\Lambda(t)}y-\alpha c} + 1)N'(y) = 0.$$

This completes the proof.  $\square$

We now analyze the integral in (3.5) by decomposing it into  $M_0 + M$ , where

$$M_0 = \int_{-\infty}^{+\infty} w'_p \left( t, N \left( \frac{c-g(x)}{\sqrt{\Lambda(t)}} \right) \right) N' \left( \frac{c-g(x)}{\sqrt{\Lambda(t)}} \right) \frac{c-g(x)}{\Lambda(t)} g'(x)^2 du(x),$$

and

$$\begin{aligned} M &= \int_{-\infty}^{+\infty} w'_p \left( t, N \left( \frac{c-g(x)}{\sqrt{\Lambda(t)}} \right) \right) N' \left( \frac{c-g(x)}{\sqrt{\Lambda(t)}} \right) g''(x) du(x) \\ &= \int_{-\infty}^{+\infty} w'_p \left( t, N \left( \frac{c-g(x)}{\sqrt{\Lambda(t)}} \right) \right) N' \left( \frac{c-g(x)}{\sqrt{\Lambda(t)}} \right) u'(x) dg'(x). \end{aligned}$$

Applying integration by parts and noting that  $N''(x) = -xN'(x)$ , we can further decompose  $M$  into  $-(M_1 + M_2 + M_3)$  where

$$\begin{aligned} M_1 &= \int_{-\infty}^{+\infty} g'(x) w''_p \left( t, N \left( \frac{c-g(x)}{\sqrt{\Lambda(t)}} \right) \right) \left( N' \left( \frac{c-g(x)}{\sqrt{\Lambda(t)}} \right) \right)^2 u'(x) \frac{-g'(x)}{\sqrt{\Lambda(t)}} dx \\ M_2 &= \int_{-\infty}^{+\infty} g'(x) w'_p \left( t, N \left( \frac{c-g(x)}{\sqrt{\Lambda(t)}} \right) \right) N'' \left( \frac{c-g(x)}{\sqrt{\Lambda(t)}} \right) u'(x) \frac{-g'(x)}{\sqrt{\Lambda(t)}} dx \\ &= - \int_{-\infty}^{+\infty} g'(x) w'_p \left( t, N \left( \frac{c-g(x)}{\sqrt{\Lambda(t)}} \right) \right) N' \left( \frac{c-g(x)}{\sqrt{\Lambda(t)}} \right) u'(x) \frac{c-g(x)}{\sqrt{\Lambda(t)}} \frac{-g'(x)}{\sqrt{\Lambda(t)}} dx \\ &= \int_{-\infty}^{+\infty} g'(x)^2 w'_p \left( t, N \left( \frac{c-g(x)}{\sqrt{\Lambda(t)}} \right) \right) N' \left( \frac{c-g(x)}{\sqrt{\Lambda(t)}} \right) u'(x) \frac{c-g(x)}{\sqrt{\Lambda(t)}} dx \\ &\equiv M_0, \\ M_3 &= \int_{-\infty}^{+\infty} g'(x) w'_p \left( t, N \left( \frac{c-g(x)}{\sqrt{\Lambda(t)}} \right) \right) N' \left( \frac{c-g(x)}{\sqrt{\Lambda(t)}} \right) u''(x) dx, \end{aligned}$$

assuming that for any  $c \in \mathbb{R}$ ,

$$w'_p \left( t, N \left( \frac{c-g(x)}{\sqrt{\Lambda(t)}} \right) \right) N' \left( \frac{c-g(x)}{\sqrt{\Lambda(t)}} \right) u'(x) g'(x) = 0 \text{ when } |x| \rightarrow +\infty. \quad (4.10)$$

Since  $g'(x) = -\frac{u''(x)}{u'(x)}$ , (4.10) can be written as

$$w'_p \left( t, N \left( \frac{c - g(x)}{\sqrt{\Lambda(t)}} \right) \right) N' \left( \frac{c - g(x)}{\sqrt{\Lambda(t)}} \right) u''(x) = 0 \text{ when } |x| \rightarrow \infty,$$

which is equivalent to (4.9).

Now, denoting  $y = \frac{c - g(x)}{\sqrt{\Lambda(t)}}$ , we have

$$\begin{aligned} M_0 + M &= -(M_1 + M_3) \\ &= - \int_{-\infty}^{+\infty} g'(x) N'(y) \left( w''_p(t, N(y)) N'(y) u'(x) \frac{dy}{dx} + w'_p(t, N(y)) \right) u''(x) dx \\ &= - \int_{-\infty}^{+\infty} g'(x) N'(y) d [w'_p(t, N(y)) u'(x)] \\ &= - \int_{+\infty}^{-\infty} g'(f(c - \sqrt{\Lambda(t)y})) N'(y) d [w'_p(t, N(y)) u'(f(c - \sqrt{\Lambda(t)y}))] \\ &= \int_{-\infty}^{+\infty} \left[ \frac{N'(y)}{f'(c - \sqrt{\Lambda(t)y})} \right] d [w'_p(t, N(y)) u'(f(c - \sqrt{\Lambda(t)y}))]. \end{aligned}$$

Hence, inequality (3.5) is equivalent to

$$\int_{-\infty}^{+\infty} \frac{N'(y)}{f'(c - \sqrt{\Lambda(t)y})} d [w'_p(t, N(y)) u'(f(c - \sqrt{\Lambda(t)y}))] \geq 0 \quad (4.11)$$

for any  $c \in \mathbb{R}$ , a.e.  $t \in [0, T]$ .

An obvious sufficient condition for (4.11) is that

$$w'_p(t, N(y)) u'(f(c - \sqrt{\Lambda(t)y})) \equiv w'_p(t, N(y)) e^{\sqrt{\Lambda(t)y} - c}$$

is increasing in  $y$  for any  $c \in \mathbb{R}$ , which holds automatically if  $w(t, \cdot)$  is convex.

**Theorem 4.10** *Under Assumptions 4.2 and 4.8, the inequality (3.5) holds.*

*Proof.* We have shown above that (3.5) holds at  $t$  when  $w(t, \cdot)$  is convex. Let us now focus on the case when  $w(t, \cdot)$  is inverse S-shaped.

First of all, it follows from Theorem 4.7 that the ODE (3.3) admits a positive solution  $\Lambda(\cdot) \in C[0, T] \cap C^1[0, T]$ .

Fix  $t \in [0, T]$ . Noting  $f'(x) = -\tilde{\kappa} I'(\tilde{\kappa} e^{-x}) e^{-x}$  where  $\tilde{\kappa} = \kappa e^{-\frac{1}{2}\Lambda(0)} > 0$ , we can rewrite

condition (4.11) as

$$\begin{aligned}
0 &\leq \int_{-\infty}^{+\infty} \frac{N'(y)}{f'(c - \sqrt{\Lambda(t)y})} d \left[ w'_p(t, N(y)) u'(f(c - \sqrt{\Lambda(t)y})) \right] \\
&= \int_{-\infty}^{+\infty} \frac{N'(y)}{-\tilde{\kappa} I'(\tilde{\kappa} e^{\sqrt{\Lambda(t)y-c}}) e^{\sqrt{\Lambda(t)y-c}}} d \left[ w'_p(t, N(y)) e^{\sqrt{\Lambda(t)y-c}} \right] \\
&= \int_{-\infty}^{+\infty} \frac{N'(y)}{-\tilde{\kappa} I'(\tilde{\kappa} e^{\sqrt{\Lambda(t)y-c}})} \left[ w'_p(t, N(y)) \sqrt{\Lambda(t)} + \frac{\partial w'_p(t, N(y))}{\partial y} \right] dy \\
&= \sqrt{\Lambda(t)} M_4 + M_5,
\end{aligned}$$

where  $M_4 = \int_{-\infty}^{+\infty} \frac{w'_p(t, N(y)) N'(y)}{-\tilde{\kappa} I'(\tilde{\kappa} e^{\sqrt{\Lambda(t)y-c}})} dy > 0$ , and

$$\begin{aligned}
M_5 &= \int_{-\infty}^{+\infty} \frac{\frac{\partial w'_p(t, N(y))}{\partial y}}{-\tilde{\kappa} I'(\tilde{\kappa} e^{\sqrt{\Lambda(t)y-c}})} N'(y) dy \\
&= \mathbb{E}[F(\xi)G(\xi)]
\end{aligned}$$

with  $F(y) = \frac{\partial w'_p(t, N(y))}{\partial y}$  and  $G(y) = \frac{1}{-\tilde{\kappa} I'(\tilde{\kappa} e^{\sqrt{\Lambda(t)y-c}})} > 0$ .

Since  $w(t, \cdot)$  is inverse S-shaped, there exists  $q \in (0, 1)$  such that  $w''_p(t, p) \leq 0$  on  $(0, q)$ ,  $w''_p(t, p) \geq 0$  on  $(q, 1)$ , and  $w''_p(t, q) = 0$ . Furthermore, for any  $b > 0$ ,

$$\begin{aligned}
&\int_0^b \frac{\partial w'_p(t, N(y))}{\partial y} N'(y) dy \\
&= \int_0^b N'(y) d[w'_p(t, N(y))] \\
&= (w'_p(t, N(y)) N'(y)) \Big|_{y=0}^{y=b} - \int_a^b w'_p(t, N(y)) dN'(y) \\
&= w'_p(t, N(b)) N'(b) - \frac{1}{\sqrt{2\pi}} w'_p(t, \frac{1}{2}) + \int_0^b w'_p(t, N(y)) y dN(y).
\end{aligned}$$

Since  $w''_p(t, p) \geq 0$  for sufficiently large  $p$ ,  $\int_0^b \frac{\partial w'_p(t, N(y))}{\partial y} N'(y) dy$  is increasing in  $b$  for sufficiently large  $b$ . Hence  $\lim_{b \uparrow +\infty} \int_0^b \frac{\partial w'_p(t, N(y))}{\partial y} N'(y) dy$  exists. However,  $+\infty > h'_x(t, 0) \equiv \mathbb{E}[w'_p(t, N(\xi))\xi]$ , we conclude that  $\lim_{b \uparrow +\infty} \int_0^b w'_p(t, N(y)) y dN(y)$  exists and is finite. This implies that  $a := \lim_{b \uparrow +\infty} w'_p(t, N(b)) N'(b)$  exists. It then follows from the fact that  $+\infty > h(t, 0) \equiv \mathbb{E}[w'_p(t, N(\xi))]$  that  $a = 0$ , i.e.,

$$\int_0^{+\infty} \frac{\partial w'_p(t, N(y))}{\partial y} N'(y) dy = -\frac{1}{\sqrt{2\pi}} w'_p \left( t, \frac{1}{2} \right) + \int_0^{+\infty} w'_p(t, N(y)) y dN(y).$$

Similarly, we can show that

$$\int_{-\infty}^0 \frac{\partial w'_p(t, N(y))}{\partial y} N'(y) dy = \frac{1}{\sqrt{2\pi}} w'_p \left( t, \frac{1}{2} \right) + \int_{-\infty}^0 w'_p(t, N(y)) y dN(y).$$

As a result,

$$\mathbb{E}[F(\xi)] = \int_{-\infty}^{+\infty} \frac{\partial w'_p(t, N(y))}{\partial y} N'(y) dy = \int_{-\infty}^{+\infty} w'_p(t, N(y)) y dN(y) = h'_x(t, 0) \geq 0.$$

Denote  $c = N^{-1}(q)$ . Then  $F(\xi)$  is negative on  $(-\infty, c)$ , positive on  $(c, +\infty)$ , and  $F(c) = 0$ . Since  $u''(x)$  is increasing in  $x$ , so is  $I'(x) = 1/u''(I(x))$ ; hence  $G(\cdot)$  is increasing. Then we have  $F(\xi)(G(\xi) - G(c)) \geq 0$ . Consequently,

$$0 \leq \mathbb{E}[F(\xi)(G(\xi) - G(c))] = \mathbb{E}[F(\xi)G(\xi)] - G(c)\mathbb{E}[F(\xi)] = \mathbb{E}[F(\xi)G(\xi)] - G(c)h'_x(t, 0),$$

which implies

$$\mathbb{E}[F(\xi)G(\xi)] \geq G(c)h'_x(t, 0) \geq 0.$$

The proof is complete. □

## 5 An Example

In this section, we give a concrete example to demonstrate our results.

Define

$$F(x) := N(x/\sqrt{2}), \text{ and } w(t, p) := F(N^{-1}(p)),$$

where  $N$  is the probability distribution function of a standard normal. So in this example our weighting function is time-invariant, and we will drop  $t$  and write  $w(p) = w(t, p)$  throughout this section.

It is clear that  $w(0) = 0, w(1) = 1, w'(p) = \frac{F'(N^{-1}(p))}{N'(N^{-1}(p))} > 0$ . So  $w$  is a probability weighting function satisfying Assumption 2.3-(i). Moreover, we have

$$\begin{aligned} w(N(x)) &= F(x) = N(x/\sqrt{2}), \\ w'(N(x)) &= \frac{F'(x)}{N'(x)} = \frac{\frac{1}{\sqrt{2}}N'(x/\sqrt{2})}{N'(x)} = \frac{1}{\sqrt{2}}e^{x^2/4}, \\ h(t, x) &\equiv \mathbb{E}[w'(N(\xi))e^{x\xi}] = \frac{1}{\sqrt{2}} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{xz} e^{-z^2/4} dz = e^{x^2}. \end{aligned}$$



Now, take any  $m \in (-1, -1/2)$ . Then

$$\begin{aligned}
\lim_{p \downarrow 0} \frac{w'(p)}{p^m} &= \lim_{x \rightarrow -\infty} \frac{w'(N(x))}{N(x)^m} \\
&= \lim_{x \rightarrow -\infty} \frac{1}{\sqrt{2}} \frac{e^{x^2/4}}{N(x)^m} \\
&= \frac{1}{\sqrt{2}} \left( \lim_{x \rightarrow -\infty} \frac{e^{x^2/(4m)}}{N(x)} \right)^m \\
&= \frac{1}{\sqrt{2}} \left( \lim_{x \rightarrow -\infty} \frac{x e^{x^2/(4m)}/2m}{N'(x)} \right)^m \\
&= \frac{1}{\sqrt{2}} \left( \frac{1}{2m} \lim_{x \rightarrow -\infty} x e^{x^2(\frac{1}{4m} + \frac{1}{2})} \right)^m \\
&= 0.
\end{aligned}$$

Also, we have  $\lim_{p \uparrow 0} \frac{w'(p)}{(1-p)^m} = \lim_{q \downarrow 0} \frac{w'(1-q)}{q^m} = \lim_{q \downarrow 0} \frac{w'(q)}{q^m} = 0$ . So Assumption 2.4-(iv) holds.

Since  $h(t, x) = e^{x^2}$ , it is straightforward to verify that Assumption 4.2 is satisfied. Moreover,  $w'(p) = \frac{1}{2} e^{(N^{-1}(p))^2/4}$ ; so  $w$  is concave on  $[0, 1/2]$  and convex on  $[1/2, 1]$ . It is therefore inverse S-shaped, satisfying Assumption 4.8. We have now checked the validity of all the assumptions on  $w$  in this paper.

With the explicit form of  $h$ , the ODE (3.3) becomes

$$\Lambda'(t) = -\frac{1}{4} |\theta(t)|^2, \quad t \in [0, T), \quad \Lambda(T) = 0,$$

whose solution is  $\Lambda(t) = \frac{1}{4} \int_t^T |\theta(s)|^2 ds$ . Thus,

$$\lambda(t) = \sqrt{-\Lambda'(t)/|\theta(t)|^2} = 1/2.$$

As discussed at the end of Section 4.1,  $\lambda(t) = 1/2$  means that the RDU agent in this example acts like an EUT agent with the market price of risk is revised to half of the original one. In other words, because of the probability weighting and the consistent planning (recall that consistent planning amounts to a heavily constrained optimization), the agent will lose half of the risk premium or Sharpe ratio.

To complete this example, we present the equilibrium portfolio  $\pi^*$  when the exponential utility function is  $u(x) = 1 - e^{-\alpha x}$ ,  $x \in \mathbb{R}$ , for some  $\alpha > 0$ , which clearly satisfies all the assumptions on the utility function in this paper.

In this case,  $u'(x) = \alpha e^{-\alpha x}$  and hence  $I(x) = \frac{\ln \alpha - \ln x}{\alpha}$ ,  $x > 0$ . It is easy to check that there exists  $\kappa > 0$  satisfying the budget constraint (3.6). Hence the desired terminal wealth is

$$X^*(T) = I(\kappa \bar{\rho}(T)) = c + \frac{1}{\alpha} \int_0^T \lambda(t) \theta(t)^\top d\tilde{W}(t),$$

where  $\tilde{W}(t) = W(t) + \int_0^t \theta(s) ds$  is a standard Brownian motion under the risk-neutral measure,  $\mathbb{Q}$ , of the original market, and  $c$  is a constant dependent of the market parameters.

By the pricing theory, the replicating wealth process  $X^*(\cdot)$  of  $X^*(T)$  is a  $\mathbb{Q}$ -martingale (recall that the risk-free rate has been assumed to be 0); hence

$$X^*(t) = c + \frac{1}{\alpha} \int_0^t \lambda(s) \theta(s)^\top d\tilde{W}(s), \quad t \in [0, T].$$

Matching the above with the wealth equation  $dX^*(t) = \pi^*(t)^\top \sigma(t) d\tilde{W}(t)$ , we obtain the equilibrium portfolio

$$\pi^*(t) = \frac{1}{\alpha} \lambda(t) (\sigma(t)^\top)^{-1} \theta(t).$$

Recall that the optimal portfolio of an EUT agent with the same exponential utility is  $\pi^{EUT}(t) = \frac{1}{\alpha} (\sigma(t)^\top)^{-1} \theta(t)$ . Hence the risky investment of the RDU agent at  $t$  is that of the EUT agent multiplied by  $\lambda(t)$ . Note that this result does not depend on the specific form of the weighting function so long as  $\lambda(\cdot)$  exists. In the special case when  $w(t, p) = F(N^{-1}(p))$ ,  $\lambda(t) \equiv 1/2$ ; so the risk exposure is reduced by half.

## 6 Reduction in Risk Premium

We have seen that the sophisticated RDU agent behaves as if the risk premium is factored by  $\lambda(\cdot)$ . In the example with the specific probability weighting function presented in Section 5,  $\lambda(t) = 1/2$ ; so the risk premium is reduced and the agent acts more cautiously than her EUT counterpart. In this section, we answer the general question of when there is a reduction in risk premium or, equivalently, when  $\lambda(t) < 1$ .

We assume  $\Lambda(\cdot)$  exists. It follows from (3.3) that

$$\lambda(t) = \left| \frac{\sqrt{\Lambda(t)} h(t, \sqrt{\Lambda(t)})}{h_x(t, \sqrt{\Lambda(t)})} \right| > 0, \quad t \in [0, T].$$

For any  $x \in \mathbb{R}^+$ , define a probability measure  $\mathbb{Q}^x$  by  $\frac{d\mathbb{Q}^x}{d\mathbb{P}} = e^{-\frac{x^2}{2} + x\xi}$ , under which  $\xi - x \sim N(0, 1)$ . So

$$\begin{aligned} h(t, x) &= e^{\frac{x^2}{2}} \mathbb{E}[w'_p(t, N(\xi))e^{-\frac{x^2}{2} + x\xi}] \\ &= e^{\frac{x^2}{2}} \mathbb{E}^{\mathbb{Q}^x}[w'_p(t, N(\xi - x + x))] \\ &= e^{\frac{x^2}{2}} \mathbb{E}[w'_p(t, N(\xi + x))] \\ &= e^{\frac{x^2}{2}} H(t, x), \end{aligned}$$

where  $H(t, x) := \mathbb{E}[w'_p(t, N(\xi + x))] \geq 0$ . The following characterizes the condition  $\lambda(t) < 1$ .

**Theorem 6.1** *Assume that  $\Lambda(\cdot)$  exists. Then, for any  $t \in [0, T)$ ,  $\lambda(t) < 1$  if and only if  $H'_x(t, \sqrt{\Lambda(t)}) > 0$ .*

*Proof.* We have

$$\frac{xh(t, x)}{h'_x(t, x)} = \frac{xe^{\frac{x^2}{2}} H(t, x)}{e^{\frac{x^2}{2}} H'_x(t, x) + xe^{\frac{x^2}{2}} H(t, x)} = \frac{xH(t, x)}{H'_x(t, x) + xH(t, x)}.$$

Noting  $\lambda(t) > 0 \forall t \in [0, T)$ , we conclude that, for any  $t \in [0, T)$ ,  $\lambda(t) = \frac{xh(t, x)}{h'_x(t, x)}|_{x=\sqrt{\Lambda(t)}} < 1$  if and only if  $H'_x(t, x)|_{x=\sqrt{\Lambda(t)}} \equiv H'_x(t, \sqrt{\Lambda(t)}) > 0$ .  $\square$

**Corollary 6.2** *For any  $t \in [0, T)$ ,  $\lambda(t) < 1$  if  $H(t, x)$  is strictly increasing in  $x \in \mathbb{R}^+$ . In particular, if  $w(t, p)$  is strictly convex in  $p$ , then  $\lambda(t) < 1$ .*

*Proof.* This is obvious.  $\square$

In the example in Section 5,  $H(t, x) = e^{-x^2/2}h(t, x) = e^{x^2/2}$ , which is strictly increasing in  $x \in \mathbb{R}^+$ ; hence Corollary 6.2 applies. On the other hand, as explained earlier, a strict convexity of  $w(t, p)$  in  $p$  underlines strict overweighting of the left tail (i.e. the bad events) and strict underweighting of the right tail (i.e. the good events); so it enhances the level of risk aversion leading to a smaller risk premium and less risky exposure.

Even if  $w$  is more general including being inverse S-shaped, it is still possible that  $\lambda(t) < 1$ , as demonstrated by the example in Section 5. In this case, the equivalent condition  $H'_x(t, \sqrt{\Lambda(t)}) > 0$  can be used to uncover the balance between the risk-averse component and the risk-seeking component in probability weighting needed to render an overall reduction in risk premium. We leave a detailed study along this direction to interested readers.

## 7 Concluding Remarks

A continuous-time RDU portfolio selection problem is inherently time inconsistent. A sophisticated agent, realizing that in the future she herself might disagree to her current planning, resorts to consistent investment by implementing intra-personal equilibrium strategies from which she will have no incentive to deviate at any point in time. We have solved the open problem of deducing such equilibrium strategies, by developing an approach that to our best knowledge is new to the literature. The main technical thrust of our approach is to express the first-order derivative of the small deviation from an equilibrium as a quadratic function of the deviating amount. The definition of the equilibrium requires the quadratic function to have a constant sign whatever the amount might be. This leads to an equality (which in turn leads to an ODE) and an inequality, the two constituting the main sufficient conditions for deriving explicitly the final wealth profile and, hence, the resulting strategy courtesy of the market completeness.

With an intra-personal equilibrium strategy, the agent, at any given time, in effect solves a *constrained* dynamically optimal RDU model in which the constraint is to honor *all* her future strategies. One may think this would lead to an extremely complicated terminal wealth profile. Our result, however, shows that the terminal wealth is surprisingly simple – it resembles that of an optimal Merton portfolio, except that the investment opportunity set needs to be modified properly. In other words, the RDU agent behaves as if she was an EUT agent, only that she is in a fictions market where she blends her probability weighting function into the market price of risk. This observation may in turn shed lights on finding intertemporal market equilibria for markets where all the agents are EUT and/or RDU consistent planners.

We derived our equilibrium strategy based on an Ansatz; as such, our results do not rule out the possibility of having other equilibria beyond our Ansatz. In general, uniqueness of intra-personal equilibrium for time-inconsistent problems remains a very challenging research question.<sup>10</sup> It is particularly the case for the RDU model, or so we believe.

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<sup>10</sup>To our best knowledge, Hu et al (2017) is the only paper that addresses the uniqueness in continuous time.

# Appendices

## A Existence of Solutions to a Class of ODEs

In this appendix we present some general existing results on a class of ODEs, taken from Agarwal and O'Regan (2004). Consider the following ODE

$$\begin{cases} y'(t) = f(t, y(t)), & t \in (0, T], \\ y(0) = 0 \in \mathbb{R}, \end{cases} \quad (\text{A.1})$$

where  $f$  may not be defined at  $t = 0$ .

Denote by  $AC[0, s]$  the set of absolutely continuous functions on  $[0, s]$  where  $s > 0$ . The following assumption is introduced in Agarwal and O'Regan (2004).

**Assumption A.1** *There exists  $t_0 \in (0, T]$  such that*

1. *There is  $\beta \in AC[0, t_0] \cap C^1(0, t_0]$  with  $\beta(0) \geq 0$  such that*

$$\beta'(t) \geq f(t, \beta(t)), \quad t \in (0, t_0].$$

2. *There is  $\alpha \in AC[0, t_0] \cap C^1(0, t_0]$  with  $\alpha(t) \leq \beta(t) \forall t \in [0, t_0]$  and  $\alpha(0) \leq 0$  such that*

$$\alpha'(t) \leq f(t, \alpha(t)), \quad t \in (0, t_0].$$

3. *The function*

$$f^*(t, y) = \begin{cases} f(t, \beta(t)) + g(\beta(t) - y) & y \geq \beta(t), \\ f(t, y) & \alpha(t) < y < \beta(t), \\ f(t, \alpha(t)) + g(\alpha(t) - y) & y \leq \alpha(t), \end{cases}$$

*is, in the region  $(0, t_0] \times \mathbb{R}$ , continuous in  $y$  for any  $t$  and measurable in  $t$  for any  $y$ , where  $g(x) = x\mathbf{1}_{|x| \leq 1} + \text{sign}(x)\mathbf{1}_{|x| > 1}$  is the radial retraction.<sup>11</sup>*

The following two propositions, both drawn from Agarwal and O'Regan (2004), concern the local and global existence of the ODE (A.1) respectively.

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<sup>11</sup>In [1],  $f^*$  is assumed to be (jointly) continuous; but from the proofs therein, we can easily weaken it to this current version.

**Proposition A.2 (Agarwal and O’Regan 2004, Theorem 3.1)** *Under Assumption A.1, the ODE (A.1) admits a solution  $y \in AC[0, t_0]$  satisfying  $\alpha(t) \leq y(t) \leq \beta(t)$  for  $t \in [0, t_0]$ .*

**Proposition A.3 (Agarwal and O’Regan 2004, Theorem 1.4)** *Given  $t_0 \in (0, T]$ , assume that  $f(t, y)$  is continuous in  $y$  for any  $t \in [t_0, T]$ , measurable in  $t$  for any  $y \in \mathbb{R}$ , and there exists  $g(\cdot) \in L^1[t_0, T]$  such that  $|f(t, y)| \leq g(t) \forall (t, y) \in [t_0, T] \times \mathbb{R}$ . Then the equation*

$$\begin{cases} y'(t) = f(t, y(t)), & t \in (t_0, T], \\ y(t_0) = a \in \mathbb{R}, \end{cases} \quad (\text{A.2})$$

*admits a solution  $y \in AC[t_0, T]$ .*

## B Tversky–Kahneman’s Probability Weighting Functions

In this appendix we verify that a class of time-varying Tversky–Kahneman (TK) probability weighting functions satisfy all the technical assumptions required in the paper.

First of all, the original TK weighting function, introduced in Tversky and Kahneman (1992), is

$$w_{TK}(p; \delta) := \frac{p^\delta}{[p^\delta + (1-p)^\delta]^{1/\delta}}, \quad p \in [0, 1], \quad (\text{B.1})$$

where  $\delta \in (0, 1]$  is a parameter. This is an inverse S-shaped function, with  $w'_{TK}(p; \delta) > 1$  when  $p$  is close to both 0 and 1. Moreover, a smaller  $\delta$  implies a stronger degree of probability weighting. When  $\delta = 1$ , there is no probability weighting.

We now vary the parameter  $\delta$  over time to generate a family of time-dependent TK functions. Given a measurable function  $\delta : [0, T] \mapsto (0, 1]$  with  $0 < \delta(t) < 1$  for  $t \in [0, T)$  and  $\delta(T) = 1$ , define

$$\tilde{w}(t, p) := w_{TK}(p; \delta(t)), \quad (t, p) \in [0, T] \times [0, 1]. \quad (\text{B.2})$$

The purpose of this appendix is to show that  $\tilde{w}$  satisfies all the assumptions in the paper, under proper conditions on  $\delta(\cdot)$ .

Clearly,  $\tilde{w}$  satisfies Assumption 2.3-(i).

**Proposition B.1**  *$\tilde{w}$  satisfies Assumption 2.4-(iv).*

*Proof.* As  $t \in [0, T)$  is fixed in Assumption 2.4-(iv), from the construction of  $\tilde{w}$  it suffices to prove the conclusion for  $w_{TK}(\cdot; \delta)$  with  $\delta \in (0, 1)$  fixed.

Denote a function

$$\alpha(p; \delta) := [p^\delta + (1-p)^\delta]^{-\frac{1}{\delta}}, \quad p \in [0, 1], \quad (\text{B.3})$$

with parameter  $\delta \in (0, 1)$ . It is easy to see that  $1 \leq p^\delta + (1-p)^\delta < 2$ ; hence

$$2^{-\frac{1}{\delta}} < \alpha(p; \delta) \leq 1. \quad (\text{B.4})$$

Moreover,  $\alpha(p; \delta) = \alpha(1-p; \delta)$ ,  $w_{TK}(p; \delta) = p^\delta \alpha(p; \delta)$ . Hence

$$\begin{aligned} \alpha'(p; \delta) &= -\frac{1}{\delta} [p^\delta + (1-p)^\delta]^{-\frac{1}{\delta}-1} \delta [p^{\delta-1} - (1-p)^{\delta-1}] \\ &= -\alpha(p; \delta)^{1+\delta} [p^{\delta-1} - (1-p)^{\delta-1}], \\ w'_{KT}(p; \delta) &= \delta p^{\delta-1} \alpha(p; \delta) + p^\delta \alpha'(p; \delta) \\ &= \delta \alpha(p; \delta) p^{\delta-1} + \alpha(p; \delta)^{1+\delta} p^\delta (1-p)^{\delta-1} - \alpha(p; \delta)^{1+\delta} p^{2\delta-1} \\ &< p^{\delta-1} + (1-p)^{\delta-1}, \end{aligned}$$

where the last inequality is by (B.4). This completes the proof.  $\square$

Before we move to the next assumption, for any probability weighting function  $w : [0, 1] \mapsto [0, 1]$ , define (with a slight abuse of notation)  $h(x; w) := \mathbb{E}[w'(N(\xi))e^{x\xi}]$ ,  $x \in \mathbb{R}$ . It then follows from Lemma 2.7-(ii) that, for any *odd* number  $n \geq 1$ ,

$$h^{(n)}(0; w) = \mathbb{E}[w'(N(\xi); \delta)\xi^n] \quad (\text{B.5})$$

$$\begin{aligned} &= \int_{-\infty}^0 w'(N(y))y^n N'(y)dy + \int_0^{+\infty} w'(N(y))y^n N'(y)dy \\ &= \int_{+\infty}^0 w'(N(-y))(-y)^n N'(-y)d(-y) + \int_0^{+\infty} w'(N(y))y^n N'(y)dy \\ &= -\int_0^{+\infty} w'(1-N(y))y^n N'(y)dy + \int_0^{+\infty} w'(N(y))y^n N'(y)dy \\ &= \int_0^{+\infty} [w'(N(y)) - w'(1-N(y))]y^n N'(y)dy, \end{aligned} \quad (\text{B.6})$$

and

$$\begin{aligned} h''(0; w) &= \int_{-\infty}^0 w'(N(y))y^2 N'(y)dy + \int_0^{+\infty} w'(N(y))y^2 N'(y)dy \\ &= \int_{+\infty}^0 w'(N(-y))(-y)^2 N'(-y)d(-y) + \int_0^{+\infty} w'(N(y))y^2 N'(y)dy \\ &= \int_0^{+\infty} w'(1-N(y))y^2 N'(y)dy + \int_0^{+\infty} w'(N(y))y^2 N'(y)dy \\ &= \int_0^{+\infty} [w'(N(y)) + w'(1-N(y))]y^2 N'(y)dy. \end{aligned} \quad (\text{B.7})$$

**Proposition B.2**  $\tilde{w}$  satisfies Assumption 4.2-(i).

*Proof.* With a slight abuse of notation, define  $h(x; \delta) := \mathbb{E}[w'_{KT}(N(\xi); \delta)e^{x\xi}]$ ,  $x \in \mathbb{R}$ , where  $\delta \in (0, 1]$ . Again, as  $t \in [0, T]$  is fixed in Assumption 4.2-(i), we can drop  $t$  and need only to show that  $h'(0; \delta) \geq 0$  and  $h'''(0; \delta) \geq 0$ .

Noting that  $w'_{TK}(1-p; \delta) = \delta(1-p)^{\delta-1}\alpha(p; \delta) - (1-p)^\delta\alpha'(p; \delta)$ , we have

$$\begin{aligned}
w'_{TK}(p; \delta) - w'_{TK}(1-p; \delta) &= \delta p^{\delta-1}\alpha(p; \delta) + p^\delta\alpha'(p; \delta) - [\delta(1-p)^{\delta-1}\alpha(p; \delta) - (1-p)^\delta\alpha'(p; \delta)] \\
&= \delta\alpha(p; \delta)[p^{\delta-1} - (1-p)^{\delta-1}] + \alpha'(p; \delta)[p^\delta + (1-p)^\delta] \\
&= \delta\alpha(p; \delta)[p^{\delta-1} - (1-p)^{\delta-1}] + \alpha'(p; \delta)\alpha(p; \delta)^{-\delta} \\
&= \delta\alpha(p; \delta)[p^{\delta-1} - (1-p)^{\delta-1}] + \alpha(p; \delta)[(1-p)^{\delta-1} - p^{\delta-1}] \\
&= (\delta-1)\alpha(p; \delta)[p^{\delta-1} - (1-p)^{\delta-1}] > 0 \quad \forall p \in (1/2, 1].
\end{aligned} \tag{B.8}$$

Now Assumption 4.2-(i) holds by virtue of (B.6) where we take  $w = w_{PK}(\cdot; \delta)$ .  $\square$

**Proposition B.3** If  $\inf_{t \in [0, T]} \delta(t) > 0$ , then  $\tilde{w}$  satisfies Assumption 4.2-(iv).

*Proof.* We have

$$\begin{aligned}
w'_{TK}(p; \delta) + w'_{TK}(1-p; \delta) &= \delta p^{\delta-1}\alpha(p; \delta) + p^\delta\alpha'(p; \delta) + [\delta(1-p)^{\delta-1}\alpha(p; \delta) - (1-p)^\delta\alpha'(p; \delta)] \\
&= \delta\alpha(p; \delta)[p^{\delta-1} + (1-p)^{\delta-1}] + \alpha'(p; \delta)[p^\delta - (1-p)^\delta] \\
&> \delta\alpha(p; \delta)[p^{\delta-1} + (1-p)^{\delta-1}] \\
&> 2^{-\frac{1}{\delta}}\delta \quad \forall p \in (1/2, 1].
\end{aligned}$$

It hence follows from (B.7) that

$$h''_x(t, 0) > \int_0^{+\infty} 2^{-\frac{1}{\delta(t)}} \delta(t) y^2 dN(y) = 2^{-\frac{1}{\delta(t)}} \delta(t) / 2 \geq \inf_{t \in [0, T]} [2^{-\frac{1}{\delta(t)}} \delta(t)] / 2 > 0,$$

owing to the condition that  $\inf_{t \in [0, T]} \delta(t) > 0$ .  $\square$

For Assumption 4.2-(iii), we need the following lemma.

**Lemma B.4** For any  $\epsilon \in (0, 1)$ , we have  $\mathbb{E}[N(\xi)^{-\epsilon} e^{x\xi}] < +\infty \quad \forall x \in \mathbb{R}$ .

*Proof.* The statement is symmetric for  $x \leq 0$  and  $x \geq 0$ ; hence it suffices to prove for the case when  $x < 0$  (the case of  $x = 0$  is trivial).



Since  $\mathbb{E}[N(\xi)^{-\epsilon} e^{x\xi} \mathbf{1}_{\xi \geq -1}] \leq e^x \mathbb{E}[N(\xi)^{-\epsilon}] = \frac{e^x}{1-\epsilon}$ , we only need to focus on  $\mathbb{E}[N(\xi)^{-\epsilon} e^{x\xi} \mathbf{1}_{\xi < -1}]$ . By the fact that for any  $\gamma > 0$ ,  $\lim_{y \rightarrow -\infty} N(y)e^{\gamma y} = \lim_{y \rightarrow -\infty} \frac{N'(y)}{-\gamma e^{-\gamma y}} = 0$ , we deduce that there exists a constant  $K > 0$  such that  $N(y)^{(1-\epsilon)/2} e^{xy} \leq K \forall y < -1$ . Thus

$$\mathbb{E}[N(\xi)^{-\epsilon} e^{x\xi} \mathbf{1}_{\xi < -1}] \leq K \mathbb{E}[N(\xi)^{-(1+\epsilon)/2}] < +\infty.$$

The proof is complete.  $\square$

**Proposition B.5** *If  $\inf_{t \in [0, T]} \delta(t) > 0$ , then  $\tilde{w}$  satisfies Assumption 4.2-(iii).*

*Proof.* Take  $\epsilon := 1 \wedge \inf_{t \in [0, T]} \delta(t) > 0$ . For any  $t \in [0, T]$ , it follows from the bound of  $w'_{TK}(p; \delta)$  (see the proof of Proposition B.1) that

$$\begin{aligned} h(t, 2) &= \mathbb{E}[w'_{TK}(N(\xi); \delta(t)) e^{2\xi}] \\ &\leq \mathbb{E}[N(\xi)^{\delta(t)-1} e^{2\xi}] + \mathbb{E}[N(-\xi)^{\delta(t)-1} e^{2\xi}] \\ &= \mathbb{E}[N(\xi)^{\delta(t)-1} e^{2\xi}] + \mathbb{E}[N(\xi)^{\delta(t)-1} e^{-2\xi}] \\ &\leq \mathbb{E}[N(\xi)^{\epsilon-1} e^{2\xi}] + \mathbb{E}[N(\xi)^{\epsilon-1} e^{-2\xi}]. \end{aligned}$$

According to Lemma B.4, we know  $h(t, 2) \leq K$  for some constant  $K$  independent of  $t$ . However,  $h(t, x)$  is increasing in  $x$ ; so  $\sup_{t \in [0, T]} h(t, 1) \leq \sup_{t \in [0, T]} h(t, 2) < K$ .

Next, by the fact that  $\xi^2 < e^\xi + e^{-\xi}$  we have  $h''_x(t, 1) \leq h(t, 2) + h(t, 0) < K + 1$ .  $\square$

Finally, we check Assumption 4.2-(ii). We first need two lemmas.

**Lemma B.6** *There exists  $\bar{\delta} \in (0, 1)$  such that*

$$\sup_{\delta \in [\bar{\delta}, 1]} \int_0^{+\infty} \ln[N(y)^\delta + N(-y)^\delta] dy < +\infty.$$

*Proof.* Fix  $\delta \in (0, 1]$ . Observe that

$$\begin{aligned} \lim_{y \rightarrow +\infty} \frac{\ln[N(y)^\delta + N(-y)^\delta]}{y^{-2}} &= \lim_{y \rightarrow +\infty} \frac{\delta N'(y)}{y^{-3}} \frac{1}{-2} \frac{N(y)^{\delta-1} - N(-y)^{\delta-1}}{N(y)^\delta + N(-y)^\delta} \\ &\leq \lim_{y \rightarrow +\infty} \frac{\delta N'(y)}{y^{-3}} \frac{1}{2} \frac{N(-y)^{\delta-1}}{N(y)^\delta + N(-y)^\delta} \\ &\leq \lim_{y \rightarrow +\infty} \frac{N'(y)}{N(-y)^{1-\delta} y^{-3}} \frac{\delta}{2} \\ &= \lim_{y \rightarrow +\infty} \frac{\delta N'(y)^{1-\delta/2} N'(y)^{\delta/2}}{2 N(-y)^{1-\delta} y^{-3}} \\ &= 0. \end{aligned}$$

Hence, there exists a constant  $K > 0$  (which may depend on  $\delta$ ), such that

$$\ln[N(y)^\delta + N(-y)^\delta] < Ky^{-2} \quad \forall y > 1.$$

As a result,

$$\int_0^{+\infty} \ln[N(y)^\delta + N(-y)^\delta] dy \leq \int_0^1 \ln[N(y)^\delta + N(-y)^\delta] dy + K \int_1^T y^{-2} dy < +\infty.$$

Note that the integrand on the left hand side is decreasing in  $\delta$ ; hence the above finiteness is uniform when  $\delta$  is sufficiently close to 1.  $\square$

Recall we have defined  $h(x; \delta) = \mathbb{E}[w'_{TK}(N(\xi); \delta)e^{x\xi}]$ ,  $x \in \mathbb{R}$ ,  $\delta \in (0, 1]$ .

**Lemma B.7** *We have*

$$h'(0; \delta) = \frac{1-\delta}{\delta} \int_0^{+\infty} \ln[N(y)^\delta + N(-y)^\delta] dy + o\left(\frac{1-\delta}{\delta}\right)$$

when  $\delta$  is sufficiently close to 1.

*Proof.* By (B.6) and (B.8), we have

$$\begin{aligned} h'(0; \delta) &= \int_0^{+\infty} (\delta-1)\alpha(N(y); \delta)[N(y)^{\delta-1} - N(-y)^{\delta-1}]y dN(y) \\ &= \frac{\delta-1}{\delta} \int_0^{+\infty} \alpha(N(y); \delta)y d[N(y)^\delta + N(-y)^\delta] \\ &= \frac{\delta-1}{\delta} \frac{\delta}{\delta-1} \int_0^{+\infty} y d[N(y)^\delta + N(-y)^\delta]^{1-1/\delta} \\ &= \int_0^{+\infty} \{1 - [N(y)^\delta + N(-y)^\delta]^{1-1/\delta}\} dy, \end{aligned}$$

where we have used the fact that  $\lim_{y \rightarrow +\infty} \{1 - [N(y)^\delta + N(-y)^\delta]^{1-1/\delta}\}y = 0$ .

Applying the general Taylor expansion  $x^\epsilon = 1 + \epsilon \ln x + o(|\epsilon|)(\ln x)^2$  for  $x > 0$  and sufficiently small  $|\epsilon|$ , we deduce

$$\int_0^{+\infty} \{1 - [N(y)^\delta + N(-y)^\delta]^{1-1/\delta}\} dy = \frac{1-\delta}{\delta} \int_0^{+\infty} \ln[N(y)^\delta + N(-y)^\delta] dy + o\left(\frac{1-\delta}{\delta}\right),$$

where we have used the finiteness  $\int_0^{+\infty} \{\ln[N(y)^\delta + N(-y)^\delta]\}^2 dy < +\infty$ , which follows from the inequalities  $0 < \ln[N(y)^\delta + N(-y)^\delta] < \ln 2$  and Lemma B.6.  $\square$

**Proposition B.8** *If*

$$\limsup_{t \uparrow T} \frac{1 - \delta(t)}{\delta(t)\sqrt{T-t}} = 0 \text{ and } \liminf_{t \uparrow T} |\theta(t)|^2 > 0,$$

then  $\tilde{w}$  satisfies Assumption 4.2-(ii).

*Proof.* This follows immediately from Lemmas B.6 and B.7. □

The second condition in the above is satisfied if we assume that  $\theta(\cdot)$  is left continuous at  $T$  and  $\theta(T) \neq 0$ , which means that we do not have a trivial market at  $T$ . The first condition means that as  $t$  approaches the terminal time  $T$ ,  $\delta(t)$  should approach  $\delta(T) \equiv 1$  faster than  $\sqrt{T - t}$ .

To summarize, the family of time-varying TK weighting functions satisfy all the assumptions of the paper if the measurable function  $\delta(\cdot)$  satisfies

$$\inf_{t \in [0, T]} \delta(t) > 0, \quad \limsup_{t \uparrow T} \frac{1 - \delta(t)}{\delta(t)\sqrt{T - t}} = 0.$$

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