The k-tuple conjecture

Yassine Lagrida

THE K-TUPLE CONJECTURE

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LAGRIDA Yassine
EMSI Marrakesh
Morocco, Marrakesh
lagyassine@gmail.com

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ABSTRACT

Let $k \in \mathbb{N}, k \geq 2$, $(h_1, h_2, \cdots, h_{k-1}) \in \mathbb{N}^{k-1}$ and consider the k-tuple $\mathcal{H}_k := (0, h_1, h_2, \cdots, h_{k-1})$ with $0 < h_1 < \cdots < h_{k-1}$.
Let $q \in \mathbb{P}$ and consider the set $\mathcal{B}_q := \{b \in \mathbb{N} \mid \gcd(b, \prod_{\substack{p \leq q \\ \text{p prime}}} p) = 1\}$.
Let $q(x)$ be the largest prime number verifiying $x \geq \Big(\prod_{\substack{p \leq q(x) \\ \text{p prime}}} p \Big)$.
Consider the functions $I_{\mathcal{H}_k}(x) := \#\{(b, b+h_1, b+h_2, \cdots, b+h_{k-1}) \in \mathcal{B}^k_{q(x)} \mid b \leq x\}$ and $\pi_{\mathcal{H}_k}(x) := \#\{(p, p+h_1, p+h_2, \cdots, p+h_{k-1}) \in \mathbb{P}^k \mid p \leq x\}$
I proved the following theorem as $x \to +\infty : I_{\mathcal{H}_k}(x) \sim \mathfrak{S}(\mathcal{H}_k)\, e^{-\gamma k}\, \dfrac{x}{\log(\log(x))^k}$.
Where γ is Euler–Mascheroni constant, and $\mathfrak{S}(\mathcal{H}_k) := \prod_{\text{p prime}} \dfrac{1 - \frac{w(\mathcal{H}_k, p)}{p}}{(1 - \frac{1}{p})^k}$ and $w(\mathcal{H}_k, p)$ is the number of distinct residues $\pmod p$ in $\mathcal{H}_k$.
Finally, i will explain why i conjecture that $I_{\mathcal{H}_k}(x) \sim \pi_{\mathcal{H}_k}(x) \big(\pi(q(x)) e^{-\gamma}\big)^k$.
If we can prove this conjecture, then we prove immediately $\pi_{\mathcal{H}_k}(x) \sim \mathfrak{S}(\mathcal{H}_k) \dfrac{x}{\log(x)^k}$.

Keywords k-tuple conjecture · prime numbers · admissible k-tuple

1 Introduction

Let $k \in \mathbb{N}, k \geq 2$.
Let $(h_1, h_2, \cdots, h_{k-1}) \in \mathbb{N}^{k-1}$.
Consider the k-tuple $\mathcal{H}_k := (0, h_1, h_2, \cdots, h_{k-1})$ with $0 < h_1 < \cdots < h_{k-1}$.

Conjecture 1 : If $\mathcal{H}_k$ is an admissible k-tuple (see the definition in the next section), then there exists infinitely many k-tuples $(p, p + h_1, \cdots, p + h_{k-1}) \in \mathbb{P}^k$.

Conjecture 2 : Consider the k-tuple $\mathcal{H}_k$, and $\pi_{\mathcal{H}_k}(x) := \#\{(p, p + h_1, \cdots, p + h_{k-1}) \in \mathbb{P}^k \,|\, p \leq x\}$, then as $x \to +\infty$:

$$\pi_{\mathcal{H}_k}(x) \sim \mathfrak{S}(\mathcal{H}_k)\frac{x}{\log(x)^k}.$$

Where $\mathfrak{S}(\mathcal{H}_k) := \prod_{\text{p prime}} \frac{1 - \frac{w(\mathcal{H}_k, p)}{p}}{(1 - \frac{1}{p})^k}$ and $w(\mathcal{H}_k, p)$ is the number of distinct residues $\pmod p$ in $\mathcal{H}_k$.

Conjecture 2 was given first time by Hardy and Littlewood in there 1923 paper **Some problems of 'Partitio numerorum'; III : On the expression of a number as a sum of primes**. [1]
The **probabilistic model for prime numbers** gives also the same predicted asymptotic formula in conjecture2.
Conjecture1 and conjecture2 are both not proven up to now, in this article we will prove a theorem and then predict the same result in conjecture2 using an elementary methods.

Let $q(x)$ be the largest prime number verifiying $x \geq \Big(\prod_{\substack{p \leq q(x) \\ \text{p prime}}} p\Big)$.
To compute prime numbers we should sieved by prime numbers less than $\sqrt{x}$.

Using **prime numbers theorem**, we have $q(x) = (1 + o(1))\log(x)$, then the target of this article is not improving sieve theory results, but to divine a **non-trivial** relation between prime numbers less than x, and numbers sieved by $2, 3, \cdots, q(x)$ and less than x.

2 Admissible k-tuples :

Let $k \in \mathbb{N}, k \geq 2$.
Let $(h_1, h_2, \cdots, h_{k-1}) \in \mathbb{N}^{k-1}$.
Consider the k-tuple $\mathcal{H}_k := (0, h_1, h_2, \cdots, h_{k-1})$ with $0 < h_1 < \cdots < h_{k-1}$.

Definition : The k-tuple $\mathcal{H}_k$ is admissible if and only if the product $p(p + h_1)\cdots(p + h_{k-1})$ haven't a fix prime divisor q for all $p \in \mathbb{P}, p \geq q$.

Example : $\mathcal{H}_k = (0, 2, 4)$ is not admissible, because p or $p + 2$ or $p + 4$ is always divisible by 3 for $p \in \mathbb{P}, p \geq 3$.
Consider this constante (in the next section we will prove it convergence) :

$$\mathfrak{S}(\mathcal{H}_k) := \prod_{\text{p prime}} \frac{1 - \frac{w(\mathcal{H}_k, p)}{p}}{(1 - \frac{1}{p})^k}.$$

Where $w(\mathcal{H}_k, p)$ is the number of distinct residues $\pmod p$ in $\mathcal{H}_k$.

Example1 : Let $\mathcal{H}_3 = (0, 2, 4) \overset{\text{mod } 3}{=} (0, 2, 1)$, then $w(\mathcal{H}_3, 3) = 3$.
Example2 : Let $\mathcal{H}_3 = (0, 2, 6) \overset{\text{mod } 3}{=} (0, 2, 0)$, then $w(\mathcal{H}_3, 3) = 2$.

It's clear that $\forall p \in \mathbb{P}$, $1 \leq w(\mathcal{H}_k, p) \leq p$ and if $p > h_{k-1}$ then $w(\mathcal{H}_k, p) = k$.

Theorem 2.1. *$\mathcal{H}_k$ is not admissible* $\iff \exists q \in \mathbb{P} : w(\mathcal{H}_k, q) = q$
Consequence : $\mathcal{H}_k$ is not admissible $\iff \mathfrak{S}(\mathcal{H}_k) = 0$.

Proof :
$\Longrightarrow$) Suppose that $\mathcal{H}_k$ is not admissible
Then $(\exists q \in \mathbb{P})(\forall p \in \mathbb{P}, p \geq q) \ : \ q | p(p+h_1)\cdots(p+h_{k-1})$
We have q is a prime number, then for $p \in \mathbb{P}$ we have $q = p$ or q devide $p + h_i$ where $i \in \{1, 2, \cdots, k-1\}$
According to **Dirichlet's theorem on arithmetic progressions** and for $p \neq q, p \pmod q$ can take values $1, 2, \cdots, q-1$ infinitly many times whene $p \in \mathbb{P}$
Taking r equals each time $1, 2, \cdots, q-1$ and $p = r \pmod q$, we have $p + h_i = 0 \pmod q$ for $i \in \{1, 2, \cdots, k-1\}$
Then $h_i \pmod q$ can take all values $1, 2, \cdots, q-1$
And adding the case $q = p$ we have $w(\mathcal{H}_k, q) = q$
$\Longleftarrow$) suppose that $\exists q \in \mathbb{P} \ : \ w(\mathcal{H}_k, q) = q$
Let $p \in \mathbb{P}$ and $p = r \pmod q$
If $r = 0$ then $p = q$ then q devide p
If $r \neq 0$ we have $1 \leq q - r \leq q - 1$
$w(\mathcal{H}_k, q) = q \implies \exists i \in \{1, 2, \cdots, k-1\}, \ h_i = q - r \pmod q$
Then q devide $p + h_i$
Then $\mathcal{H}_k$ is not admissible.

Theorem 2.2. *If $\exists q \in \mathbb{P} \ : \ w(\mathcal{H}_k, q) = q$ then $q \leq k$.*

Proof : The proof is simple, because we have k elements in the k-tuple $\mathcal{H}_k$.

Theorem 2.3.

$$\mathcal{H}_k \textit{ is admissible } \iff \forall p \in \mathbb{P}, p \leq k \ : \ w(\mathcal{H}_k, p) \neq p \tag{1}$$

Example1 : $\mathcal{H}_k = (0, 4, 6)$ is admissible :
$(0, 4, 6) \overset{\text{mod } 2}{=} (0, 0, 0) \implies w(\mathcal{H}_3, 2) = 1$
$(0, 4, 6) \overset{\text{mod } 3}{=} (0, 1, 0) \implies w(\mathcal{H}_3, 3) = 2$

Example2 : $\mathcal{H}_5 = (0, 4, 12, 16, 18)$ is not admissible :
$(0, 4, 12, 16, 18) \overset{\text{mod } 2}{=} (0, 0, 0, 0, 0) \implies w(\mathcal{H}_5, 2) = 1$
$(0, 4, 12, 16, 18) \overset{\text{mod } 3}{=} (0, 1, 0, 1, 0) \implies w(\mathcal{H}_5, 3) = 2$
$(0, 4, 12, 16, 18) \overset{\text{mod } 5}{=} (0, 4, 2, 1, 3) \implies w(\mathcal{H}_5, 5) = 5$

3 $\Upsilon_{b,q}(k)$ function :

Let $b, q \in \mathbb{P}, k \in \mathbb{N}$ and $k < b$.
Consider those functions :

$$\Upsilon_{b,q}(k) := \prod_{\substack{b \leq p \leq q \\ \text{p prime}}} \left(1 - \frac{k}{p}\right).$$

$$F_b(k) := \prod_{\substack{b \leq p \\ \text{p prime}}} \left(1 - \frac{k^2}{p^2}\right), \quad C_b(k) := \prod_{\substack{b \leq p \\ \text{p prime}}} \left(1 - \left(\frac{k-1}{p-1}\right)^2\right).$$

In this section we will find the asymptotic formula of $\Upsilon_{b,q}(k)$ as $q \to +\infty$.
We have :

$$\Upsilon_{b,q}(k) = \prod_{\substack{b \leq p \leq q \\ \text{p prime}}} \left(1 - \left(\frac{k-1}{p-1}\right)^2\right) \prod_{\substack{b \leq p \leq q \\ \text{p prime}}} \left(1 - \frac{1}{p}\right)^2 \prod_{\substack{b \leq p \leq q \\ \text{p prime}}} \left(1 + \frac{k-2}{p}\right)^{-1} \tag{2}$$

Consider this constant $A_b := \prod_{\substack{p < b \\ \text{p prime}}} \left(1 - \frac{1}{p}\right)$.
Consider **Mertens' third theorem** :

$$\prod_{\substack{p \leq q \\ \text{p prime}}} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log(q)} \tag{3}$$

Where γ is Euler–Mascheroni constant : $\gamma = \lim_{n\to+\infty} \sum_{k=1}^{n} \frac{1}{k} - \log(n)$.

Using (2) and (3) we have as $q \to +\infty$:

$$\Upsilon_{b,q}(k) \sim \begin{cases} 1 & \text{if } k = 0 \\ \dfrac{e^{-\gamma}}{A_b \log(q)} & \text{if } k = 1 \\ \dfrac{C_b(k)\, e^{-2\gamma}}{A_b^2\, F_b(k-2)\, \log^2(q)} \Upsilon_{b,q}(k-2) & \text{if } k \geq 2 \end{cases}$$

Then for $k \geq 2$ we have :

$$\Upsilon_{b,q}(k) \sim \begin{cases} \dfrac{C_b(2)C_b(4)\cdots C_b(k)\, e^{-k\gamma}}{F_b(0)F_b(2)\cdots F_b(k-2)A_b^k} \dfrac{1}{\log^k(q)} & \text{if k is even} \\ \dfrac{C_b(3)C_b(5)\cdots C_b(k)\, e^{-k\gamma}}{F_b(1)F_b(3)\cdots F_b(k-2)A_b^k} \dfrac{1}{\log^k(q)} & \text{if k is odd} \end{cases}$$

We should simplify this formula, we have :

$$F_b(k) = \prod_{\substack{b\leq p \\ \text{p prime}}} \frac{(p-k)(p+k)}{p^2}\,, \quad C_b(k) = \prod_{\substack{b\leq p \\ \text{p prime}}} \frac{(p-k)(p+k-2)}{(p-1)^2}.$$

For $k \geq 2$ we have :

$$\prod_{\substack{b\leq p \\ \text{p prime}}} \frac{p^{k-1}(p-k)}{(p-1)^k} = \begin{cases} \dfrac{C_b(2)C_b(4)\cdots C_b(k)}{F_b(0)F_b(2)\cdots F_b(k-2)} & \text{if k is even} \\ \dfrac{C_b(3)C_b(5)\cdots C_b(k)}{F_b(1)F_b(3)\cdots F_b(k-2)} & \text{if k is odd} \end{cases} \tag{4}$$

And we have :

$$\prod_{\substack{b\leq p \\ \text{p prime}}} \frac{p^{k-1}(p-k)}{(p-1)^k} = \prod_{\substack{b\leq p \\ \text{p prime}}} \frac{1-\frac{k}{p}}{(1-\frac{1}{p})^k}$$

Then, formula (4) show that for $k < b$ the product $\prod_{\substack{b\leq p \\ \text{p prime}}} \frac{1-\frac{k}{p}}{(1-\frac{1}{p})^k}$ is convergente and not null.

Theorem 3.1. *For a fixed $k \in \mathbb{N}, b \in \mathbb{P}$ and $q \to +\infty$ we have*

$$\Upsilon_{b,q}(k) \sim \left(\prod_{\substack{b\leq p \\ p \text{ prime}}} \frac{1-\frac{k}{p}}{(1-\frac{1}{p})^k} \right) \frac{e^{-k\gamma}}{A_b^k} \frac{1}{\log(q)^k}$$

4 Chinese remainder theorem :

Let $q \in \mathbb{P}, q \geq 5$.
Consider the set :

$$\mathcal{B}_q := \{b \in \mathbb{N} \mid \gcd(b, \prod_{\substack{p\leq q \\ \text{p prime}}} p) = 1\}$$

$\mathcal{B}_q$ is the set of numbers coprime to $2, 3, \cdots, q$.
Let $x \in \mathbb{R}_+$ and Consider the function :

$$I_{\mathcal{H}_k}(x, q) := \#\{(b, b+h_1, b+h_2, \cdots, b+h_{k-1}) \in \mathcal{B}_q^k \mid b \leq x\}.$$

$I_{\mathcal{H}_k}(x,q)$ is the number of k-tuples coprime to $2,3,\cdots,q$ and less than x.
In this section we will compute $I_{\mathcal{H}_k}(x,q)$ for $x=\Big(\prod\limits_{\substack{p\leq q\\ \text{p prime}}} p\Big)$.

Theorem 4.1.

$$I_{\mathcal{H}_k}(\prod_{\substack{p\leq q\\ p\ prime}} p,q)=\prod_{\substack{p\leq q\\ p\ prime}}(p-w(\mathcal{H}_k,p))$$

Where $w(\mathcal{H}_k,p)$ is the number of distinct residues $\pmod p$ in $\mathcal{H}_k$.

proof : Let p_i be a prime number verify $p_i\leq q$
To compute k-tuples coprime to $\Big(\prod\limits_{\substack{p\leq q\\ \text{p prime}}} p\Big)$ and less than $\Big(\prod\limits_{\substack{p\leq q\\ \text{p prime}}} p\Big)$ we should compute the number of k-tuples verifying :

$$\begin{aligned} b &= r_{0,i} \pmod{p_i}\\ b+h_1 &= r_{1,i} \pmod{p_i}\\ &\cdots\\ b+h_{k-1} &= r_{k-1,i} \pmod{p_i}\end{aligned} \tag{5}$$

With $b\leq \prod\limits_{\substack{p\leq q\\ \text{p prime}}} p$ and $\forall j\in\{0,1,2,\cdots,k-1\}\ :\ r_{j,i}\neq 0$ for every $p_i\leq q$
If $h_j=h_l \pmod{p_i}$ for some $j\neq l$ then they represent the same equations in (5)
Then we have $p_i-w(\mathcal{H}_k,p_i)$ possibilities that $r_{j,i}\neq 0$ for $j\in\{0,1,\cdots,k-1\}$ Then using **Chinese remainder theorem** and **fundamental counting principle** we have :

$$I_{\mathcal{H}_k}(\prod_{\substack{p\leq q\\ \text{p prime}}} p,q)=\prod_{\substack{p\leq q\\ \text{p prime}}}(p-w(\mathcal{H}_k,p))$$

5 Asymptotic formula :

Let $q\in\mathbb{P}, q\geq 5$, and $\mathcal{B}_q:=\{b\in\mathbb{N}\,|\,\gcd(b,\prod\limits_{\substack{p\leq q\\ \text{p prime}}} p)=1\}$.

Let $x\in\mathbb{R}_+$ and consider the function $I(x,q):=\#\{b\in\mathcal{B}_q\,|\,b\leq x\}$.
$I(x,q)$ is the number of numbers coprime to $2,3,\cdots,q$ and less than x.

Theorem 5.1. *Let $x\geq \prod\limits_{\substack{p\leq q\\ p\ prime}} p$, then as $q\to+\infty$:*

$$\frac{I(x,q)}{x}\sim\prod_{\substack{p\leq q\\ p\ prime}}\left(1-\frac{1}{p}\right) \tag{6}$$

Sketch of the proof : Let $x\geq \prod\limits_{\substack{p\leq q\\ \text{p prime}}} p$, and let q_0 be the previous prime of q, then we prove :

1) $q_0\to+\infty$, $\dfrac{I(x,q_0)}{x}\sim\prod\limits_{\substack{p\leq q_0\\ \text{p prime}}}\left(1-\dfrac{1}{p}\right)$

2) For $F(x)=I(x,q_0)-I(x,q)$ we have $\lim\limits_{q_0\to+\infty}\prod\limits_{\substack{p\leq q_0\\ \text{p prime}}}\left(1-\dfrac{1}{p}\right)^{-1}\dfrac{F(x)}{x}=0$

3) conclusion : $q\to+\infty$, $\dfrac{I(x,q)}{x}\sim\prod\limits_{\substack{p\leq q\\ \text{p prime}}}\left(1-\dfrac{1}{p}\right)$.

proof : 1) Let $n \in \mathbb{N}^*$
We have $I(n \prod\limits_{\substack{p \leq q_0 \\ \text{p prime}}} p, q_0) = n \prod\limits_{\substack{p \leq q_0 \\ \text{p prime}}} (p-1)$
The function $x \to I(x, q_0)$ is increasing
Then for $n \cdot (\prod\limits_{\substack{p \leq q_0 \\ \text{p prime}}} p) \leq x \leq (n+1) \cdot (\prod\limits_{\substack{p \leq q_0 \\ \text{p prime}}} p)$, we have :

$$n \prod_{\substack{p \leq q_0 \\ \text{p prime}}} (p-1) \leq I(x, q_0) \leq (n+1) \prod_{\substack{p \leq q_0 \\ \text{p prime}}} (p-1)$$

We have :

$$n \cdot (\prod_{\substack{p \leq q_0 \\ \text{p prime}}} p) \leq x < (n+1) \cdot (\prod_{\substack{p \leq q_0 \\ \text{p prime}}} p) \iff \frac{1}{(n+1) \cdot \prod\limits_{\substack{p \leq q_0 \\ \text{p prime}}} p} < \frac{1}{x} \leq \frac{1}{n \cdot \prod\limits_{\substack{p \leq q_0 \\ \text{p prime}}} p}$$

Then :

$$\frac{n}{n+1} \leq \prod_{\substack{p \leq q_0 \\ \text{p prime}}} \left(1 - \frac{1}{p}\right)^{-1} \frac{I(x, q_0)}{x} \leq \frac{n+1}{n}$$

We have $x \geq \left(\prod\limits_{\substack{p \leq q \\ \text{p prime}}} p \right)$ then $n \geq q$
Then $q_0 \to +\infty \implies q \to +\infty \implies n \to +\infty$
Then $\lim\limits_{q_0 \to +\infty} \frac{n}{n+1} = \lim\limits_{q_0 \to +\infty} \frac{n+1}{n} = 1$
Conclusion :

$$q_0 \to \infty, \ \frac{I(x, q_0)}{x} \sim \prod_{\substack{p \leq q_0 \\ \text{p prime}}} \left(1 - \frac{1}{p}\right)$$

2) Let $n \in \mathbb{N}^*$, we have :

$$\begin{aligned}
F(n \cdot \prod_{\substack{p \leq q \\ \text{p prime}}} p) &= I(n \cdot \prod_{\substack{p \leq q \\ \text{p prime}}} p, \ q_0) - I(n \cdot \prod_{\substack{p \leq q \\ \text{p prime}}} p, \ q) \\
&= I(nq \cdot \prod_{\substack{p \leq q_0 \\ \text{p prime}}} p, \ q_0) - I(n \cdot \prod_{\substack{p \leq q \\ \text{p prime}}} p, \ q) \\
&= nq \prod_{\substack{p \leq q_0 \\ \text{p prime}}} (p-1) - n \prod_{\substack{p \leq q \\ \text{p prime}}} (p-1) \\
&= n \prod_{\substack{p \leq q_0 \\ \text{p prime}}} (p-1) \cdot (q - (q-1)) \\
&= n \prod_{\substack{p \leq q_0 \\ \text{p prime}}} (p-1)
\end{aligned}$$

We have $x \to F(x)$ an increasing function (F(x) is the number of numbers less than x and has q as little divisor)
For $n \cdot (\prod\limits_{\substack{p \leq q \\ \text{p prime}}} p) \leq x < (n+1) \cdot (\prod\limits_{\substack{p \leq q \\ \text{p prime}}} p)$ we have :

$$n \prod_{\substack{p \leq q_0 \\ \text{p prime}}} (p-1) \leq F(x) \leq (n+1) \prod_{\substack{p \leq q_0 \\ \text{p prime}}} (p-1)$$

We have :

$$n \cdot (\prod_{\substack{p \leq q \\ \text{p prime}}} p) \leq x < (n+1) \cdot (\prod_{\substack{p \leq q \\ \text{p prime}}} p) \iff \frac{1}{(n+1) \cdot \prod\limits_{\substack{p \leq q \\ \text{p prime}}} p} < \frac{1}{x} \leq \frac{1}{n \cdot \prod\limits_{\substack{p \leq q \\ \text{p prime}}} p}$$

Then :

$$\frac{n}{n+1}\frac{1}{q} \leq \prod_{\substack{p\leq q_0\\ \text{p prime}}}\left(1-\frac{1}{p}\right)^{-1}\frac{F(x)}{x} \leq \frac{n+1}{n}\frac{1}{q}$$

We have $x \geq \left(\prod_{\substack{p\leq q\\ \text{p prime}}} p\right)$, then :

$$\lim_{q_0\to+\infty}\prod_{\substack{p\leq q_0\\ \text{p prime}}}\left(1-\frac{1}{p}\right)^{-1}\frac{F(x)}{x} = 0$$

3) We have $x \geq \left(\prod_{\substack{p\leq q\\ \text{p prime}}} p\right)$

We proved that $\lim_{q_0\to+\infty}\prod_{\substack{p\leq q_0\\ \text{p prime}}}\left(1-\frac{1}{p}\right)^{-1}\frac{I(x,q_0)-I(x,q)}{x} = 0$ and $\lim_{q_0\to+\infty}\prod_{\substack{p\leq q_0\\ \text{p prime}}}\left(1-\frac{1}{p}\right)^{-1}\frac{I(x,q_0)}{x} = 1$

Conclusion :

$$q\to+\infty,\ \frac{I(x,q)}{x} \sim \prod_{\substack{p\leq q\\ \text{p prime}}}\left(1-\frac{1}{p}\right)$$

.

Theorem 5.2. *Let $x \geq \left(\prod_{\substack{p\leq q\\ p\ prime}} p\right)$, then as $q\to+\infty$:*

$$I(x,q) \sim e^{-\gamma}\frac{x}{\log(q)}$$

Proof : Let $x \geq \left(\prod_{\substack{p\leq q\\ \text{p prime}}} p\right)$, from theorem 5.1 we have as $q\to+\infty$:

$$\frac{I(x,q)}{x} \sim \prod_{\substack{p\leq q\\ \text{p prime}}}\left(1-\frac{1}{p}\right)$$

Consider **Mertens' third theorem** :

$$\prod_{\substack{p\leq q\\ \text{p prime}}}\left(1-\frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log(q)}$$

Then we have the resulte.

Let $k\in\mathbb{N}, k\geq 2$.
Consider the k-tuple $\mathcal{H}_k = (0,h_1,h_2,\cdots,h_{k-1})$.
Consider the function :

$$I_{\mathcal{H}_k}(x,q) := \#\{(b,b+h_1,\cdots,b+h_{k-1}) \in \mathcal{B}_q^k \mid b\leq x\}$$

$I_{\mathcal{H}_k}(x,q)$ is the number of k-tuples less than x and coprime to $2,3,\cdots,q$.

Theorem 5.3. *Let $x \geq \prod_{\substack{p\leq q\\ p\ prime}} p$, then as $q\to+\infty$:*

$$\frac{I_{\mathcal{H}_k}(x,q)}{x} \sim \prod_{\substack{p\leq q\\ p\ prime}}\left(1-\frac{w(\mathcal{H}_k,p)}{p}\right)$$

Proof : We proceed like the proof of theorem 5.1 and we prove : Let $x \geq \Big(\prod\limits_{\substack{p \leq q \\ \text{p prime}}} p \Big)$

Let q_0 be the previous prime of q, then we prove :
1) $q_0 \to +\infty$, $\dfrac{I_{\mathcal{H}_k}(x, q_0)}{x} \sim \prod\limits_{\substack{p \leq q_0 \\ \text{p prime}}} \left(1 - \dfrac{w(\mathcal{H}_k, p)}{p}\right)$

2) For $F(x) = I_{\mathcal{H}_k}(x, q_0) - I_{\mathcal{H}_k}(x, q)$ we have $\lim\limits_{q_0 \to +\infty} \prod\limits_{\substack{p \leq q_0 \\ \text{p prime}}} \left(1 - \dfrac{w(\mathcal{H}_k, p)}{p}\right)^{-1} \dfrac{F(x)}{x} = 0$

3) Conclusion : $\dfrac{I_{\mathcal{H}_k}(x, q)}{x} \sim \prod\limits_{\substack{p \leq q \\ \text{p prime}}} \left(1 - \dfrac{w(\mathcal{H}_k, p)}{p}\right)$.

Let $\mathfrak{S}(\mathcal{H}_k) := \prod\limits_{\text{p prime}} \dfrac{1 - \frac{w(\mathcal{H}_k, p)}{p}}{(1 - \frac{1}{p})^k}$

Theorem 5.4. *Let $x \geq \Big(\prod\limits_{\substack{p \leq q \\ p \text{ prime}}} p \Big)$, then as $q \to +\infty$:*

$$I_{\mathcal{H}_k}(x, q) \sim e^{-\gamma k}\, \mathfrak{S}(\mathcal{H}_k) \frac{x}{\log(q)^k}$$

Proof : Let $x \geq \Big(\prod\limits_{\substack{p \leq q \\ \text{p prime}}} p \Big)$, in theorem 5.3 we proved as $q \to +\infty$:

$$\frac{I_{\mathcal{H}_k}(x, q)}{x} \sim \prod_{\substack{p \leq q \\ \text{p prime}}} \left(1 - \frac{w(\mathcal{H}_k, p)}{p}\right)$$

If $p \in \mathbb{P}$ and $p > h_{k-1}$ we have $w(\mathcal{H}_k, p) = k$
Let $b \in \mathbb{P}$ be the smallest prime number verifying $b > h_{k-1}$, then using $\Upsilon_{b,q}(k)$ function in section 3 :

$$\begin{aligned} \prod_{\substack{p \leq q \\ \text{p prime}}} \left(1 - \frac{w(\mathcal{H}_k, p)}{p}\right) &= \prod_{\substack{p < b \\ \text{p prime}}} \left(1 - \frac{w(\mathcal{H}_k, p)}{p}\right) \prod_{\substack{b \leq p \leq q \\ \text{p prime}}} \left(1 - \frac{k}{p}\right) \\ &= \prod_{\substack{p < b \\ \text{p prime}}} \left(1 - \frac{w(\mathcal{H}_k, p)}{p}\right) \Upsilon_{b,q}(k) \end{aligned}$$

In theorem 3.1 we proved that as $q \to +\infty$:

$$\Upsilon_{b,q}(k) \sim \left(\prod_{\substack{b \leq p \\ \text{p prime}}} \frac{1 - \frac{k}{p}}{(1 - \frac{1}{p})^k} \right) \frac{e^{-k\gamma}}{A_b^k} \frac{1}{\log(q)^k}$$

With $A_b = \prod\limits_{\substack{p < b \\ \text{p prime}}} \left(1 - \dfrac{1}{p}\right)$

Then we have the result as $q \to +\infty$:

$$\prod_{\substack{p \leq q \\ \text{p prime}}} \left(1 - \frac{w(\mathcal{H}_k, p)}{p}\right) \sim \left(\prod_{\text{p prime}} \frac{1 - \frac{w(\mathcal{H}_k, p)}{p}}{(1 - \frac{1}{p})^k} \right) \frac{e^{-k\gamma}}{\log(q)^k}$$

Finally as $q \to +\infty$:

$$I_{\mathcal{H}_k}(x,q) \sim \mathfrak{S}(\mathcal{H}_k)\, e^{-\gamma k}\, \frac{x}{\log(q)^k}$$

.

Let $q(x)$ be the largest prime number verifiying $x \geq \Big(\prod_{\substack{p \leq q(x) \\ \text{p prime}}} p \Big)$.

Theorem 5.5. *As $x \to +\infty$, we have :*

$$q(x) \sim \log(x)$$

Proof : We have $q(x)$ is the largest prime number verifiying $x \geq \Big(\prod_{\substack{p \leq q(x) \\ \text{p prime}}} p \Big)$, then :

$$\left(\prod_{\substack{p \leq q(x) \\ \text{p prime}}} p \right) \leq x < \left(\prod_{\substack{p \leq q_1(x) \\ \text{p prime}}} p \right)$$

Where $q_1(x)$ is the next prime to $q(x)$, then :

$$\frac{\log\left(\prod_{\substack{p \leq q(x) \\ \text{p prime}}} p \right)}{q(x)} \leq \frac{\log(x)}{q(x)} < \frac{\log\left(\prod_{\substack{p \leq q_1(x) \\ \text{p prime}}} p \right)}{q(x)}$$

Using **Prime numbers theorem** we have as $x \to +\infty$:

$$\log\left(\prod_{\substack{p \leq q(x) \\ \text{p prime}}} p \right) \sim q(x) \ , \ q(x) \sim q_1(x)$$

Then we have the result :

$$q(x) \sim \log(x)$$

.

Theorem 5.6. *Let $I(x) = I(x, q(x))$, we have $x \geq \left(\prod_{\substack{p \leq q(x) \\ p \text{ prime}}} p \right)$ then using theorem 5.2 as $x \to +\infty$:*

$$I(x) \sim e^{-\gamma} \frac{x}{\log(\log(x))}$$

.

Theorem 5.7. *Let $I_{\mathcal{H}_k}(x) = I_{\mathcal{H}_k}(x, q(x))$, we have $x \geq \left(\prod_{\substack{p \leq q(x) \\ p \text{ prime}}} p \right)$ then using theorem 5.4 as $x \to +\infty$:*

$$I_{\mathcal{H}_k}(x) \sim \mathfrak{S}(\mathcal{H}_k)\, e^{-\gamma k}\, \frac{x}{\log(\log(x))^k}$$

.

Notice : we can prove theorems 5.6 and 5.7 without using **prime numbers theorem**, we can use **Chebyshev theorems** :

$$\log\left(\prod_{\substack{p \leq q(x) \\ \text{p prime}}} p \right) \asymp q(x)$$

Then we have :

$$\log\log\left(\prod_{\substack{p\leq q(x)\\ \text{p prime}}} p\right)\sim \log q(x)$$

We have :

$$\pi(x)\asymp \frac{x}{\log(x)}$$

Then :

$$\log\pi(x)\sim\log(x)\implies \log p_n\sim\log n$$

We can deduce that (independetely to **prime numbers theorem**) :

$$\log q(x)\sim\log\log(x)$$

6 The link with the k-tuple conjecture :

We proved in theorem 5.6 as $x\to+\infty$:

$$I(x)\sim e^{-\gamma}\frac{x}{\log(\log(x))} \tag{7}$$

Where $I(x)$ is the number of numbers less than x and coprime to $2,3,\cdots,q(x)=(1+o(1))\log(x)$, with $q(x)$ is the largest prime number verifying $x\geq\Big(\prod_{\substack{p\leq q(x)\\ \text{p prime}}} p\Big)$.

We can see that :

$$\frac{x}{\log(\log(x))}=\frac{x}{\log(x)}\frac{\log(x)}{\log(\log(x))}$$

Then using **prime numbers theorem** we can rewrite formula (7) as :

$$I(x)\sim\pi(x)\big(\pi(q(x))e^{-\gamma}\big) \tag{8}$$

The number of numbers less than x and coprime to $2,3,\cdots,q(x)$ is giving asymptoticaly by the number of prime numbers less than x multiplied by the number of prime numbers less than $q(x)$ multiplied by the constant $e^{-\gamma}$.

Then, formula (8) can represent a **non-trivial** relation between prime numbers less than x and numbers less than x and sieved by $2,3,\cdots,q(x)$.

It is legitimate to think that, if we denote $\mathbb{P}(x)$ the set of prime number less than x ($\pi(x)=\#\mathbb{P}(x)$) and $I_1(x)$ the set of numbers coprime to $2,3,\cdots,q(x)$ and less than x ($I(x)=\#I_1(x)$) we have :

$$(\mathbb{P}(x)\smallsetminus\mathbb{P}(q(x)))\subset I_1(x)$$

The **trivial** relation between $\mathbb{P}(x)$ and $I_1(x)$ is that every prime number p verify $q(x)<p\leq x$ is coprime to $2,3,\cdots,q(x)$ and then $p\in I_1(x)$.

Let $\pi_{\mathcal{H}_k}(x)=\#\{(p,p+h_1,\cdots,p+h_{k-1})\in\mathbb{P}^k\,|\,p\leq x\}$.

Conjecture : As $x\to+\infty$:

$$I_{\mathcal{H}_k}(x)\sim\pi_{\mathcal{H}_k}(x)\big(\pi(q(x))e^{-\gamma}\big)^k. \tag{9}$$

This conjecture and **prime numbers theorem** give immediately as $x\to+\infty$:

$$\pi_{\mathcal{H}_k}(x)\sim\mathfrak{S}(\mathcal{H}_k)\,\frac{x}{\log(x)^k}. \tag{10}$$

The same conjecture founded by the **probabilistic model for prime numbers** and **Harddy-Littlewood circle method**.
To prove formula (9) (and then proving the k-tuple conjecture) we should to prove formula (8) without using **prime numbers theorem**.

We proved formula (7) without using **prime numbers theorem**, then if we arrive to prove also (8) without using **prime numbers theorem** we will do the same work to prove (9).

Other Examples :
For n even number, let :

$$G(n) := \#\{(p, n-p) \in \mathbb{P}^2 \,|\, p \leqslant n\}.$$

Let :

$$L(x) := \#\{p = n^2 + 1 \in \mathbb{P} \mid p \leqslant x\}.$$

Let :

$$G_b(n) := \#\Big\{(b, n-b) \in \mathcal{B}^2_{q(n)} \,|\, b \leqslant \prod_{\substack{p \leq q(n) \\ \text{p prime}}} p\Big\}$$

$G_b(n)$ is the number of 2-tuples $(b, n-b)$ coprime to $2, 3, \cdots, q(n)$ and b less than $\prod_{\substack{p \leq q(n) \\ \text{p prime}}} p$.

Let :

$$L_b(x) := \#\Big\{b \in \mathcal{B}_{q(x)} \,|\, b = n^2 + 1,\, b \leqslant x\Big\}.$$

$L_b(x)$ is the number of numbers coprime to $2, 3, \cdots, q(x)$ and written as $n^2 + 1$ and less than x.

Using **Chinese remainder theorem** we prove that :

$$\begin{aligned} G_b(n) &= \prod_{\substack{3 \leq p \leq q(n) \\ \text{p prime},\, p \mid n}} (p-1) \prod_{\substack{3 \leq p \leq q(n) \\ \text{p prime},\, p \nmid n}} (p-2) \\ &= \Big(\prod_{\substack{p \mid n \\ \text{p prime} \\ 3 \leq p \leq q(n)}} \frac{p-1}{p-2}\Big)\Big(\prod_{\substack{3 \leq p \leq q(n) \\ \text{p prime}}} (p-2)\Big). \end{aligned}$$

Using **Chinese remainder theorem** we prove that :

$$L_b\Big(\Big(\prod_{\substack{p \leq q(x) \\ \text{p prime}}} p\Big)^2 + 1\Big) = \Big(\prod_{\substack{p \leqslant q(x) \\ p \equiv 3[4] \\ p \text{ prime}}} p\Big) \prod_{\substack{p \leqslant q(x) \\ p \equiv 1[4] \\ p \text{ prime}}} (p-2)$$

Then we prove as $n \to +\infty$:

$$\frac{G_b(n)}{n} \sim \Big(\prod_{\substack{p \mid n \\ \text{p prime} \\ 3 \leq p \leq q(n)}} \frac{p-1}{p-2}\Big) \prod_{\substack{3 \leq p \leq q(n) \\ \text{p prime}}} \Big(1 - \frac{2}{p}\Big)$$

Then we prove as $x \to +\infty$:

$$\begin{aligned} \frac{L_b(x^2)}{x} &\sim \Big(\prod_{\substack{p \leq q(x) \\ \text{p prime}}} p\Big)^{-1} \Big(\prod_{\substack{p \leqslant q(x) \\ p \equiv 3[4] \\ p \text{ prime}}} p\Big) \prod_{\substack{p \leqslant q(x) \\ p \equiv 1[4] \\ p \text{ prime}}} (p-2) \\ &\sim \prod_{\substack{p \leq q(x) \\ \text{p prime}}} \Big(1 - \frac{1}{p}\Big) \Big(\prod_{\substack{p \leqslant q(x) \\ p \equiv 3[4] \\ p \text{ prime}}} \frac{p}{p-1}\Big)\Big(\prod_{\substack{p \leqslant q(x) \\ p \equiv 1[4] \\ p \text{ prime}}} \frac{p-2}{p-1}\Big) \\ &\sim \prod_{\substack{p \leq q(x) \\ \text{p prime}}} \Big(1 - \frac{1}{p}\Big) \prod_{\substack{p \leq q(x) \\ \text{p prime}}} \Big(1 - \frac{(-1)^{(p-1)/2}}{p-1}\Big) \end{aligned}$$

Then we have the result :

$$G_b(n) \sim 2C_2\Big(\prod_{\substack{p|n \\ \text{p prime}}} \frac{p-1}{p-2}\Big)e^{-2\gamma}\frac{n}{\log(\log(n))^2}.$$

Where $C_2 = \prod_{\substack{3\leq p \\ \text{p prime}}} \Big(1 - \frac{1}{(p-1)^2}\Big)$.

We have the result :

$$L_b(x) \sim \prod_{\substack{3\leq p \\ \text{p prime}}} \left(1 - \frac{(-1)^{(p-1)/2}}{p-1}\right) e^{-\gamma}\,\frac{\sqrt{x}}{\log(\log(x))}.$$

Using the conjecture about the non-trivial relation between prime numbers less than x and numbers coprime to 2,3,...,q(x) and less than x :

$$G_b(n) \sim G(n)\big(\pi(q(n))e^{-\gamma}\big)^2$$

And :

$$L_b(x) \sim L(x)\big(\pi(q(x))e^{-\gamma}\big)$$

Then we have the results :

$$G(n) \sim 2C_2\Big(\prod_{\substack{p|n \\ \text{p prime}}} \frac{p-1}{p-2}\Big)\frac{n}{\log(n)^2}.$$

And :

$$L(x) \sim \prod_{\substack{3\leq p \\ \text{p prime}}} \left(1 - \frac{(-1)^{(p-1)/2}}{p-1}\right) \frac{\sqrt{x}}{\log(x)}.$$

Références

[1] Gordon H. Hardy and John Edensor Littlewood, Some problems of 'Partitio numerorum' ; III : On the expression of a number as a sum of primes, 61. Acta Math, place (1923)