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CLASSIFICATION AND LIOUVILLE-TYPE THEOREMS FOR SEMILINEAR ELLIPTIC EQUATIONS IN UNBOUNDED DOMAINS.

LOUIS DUPAIGNE AND ALBERTO FARINA

1. INTRODUCTION AND MAIN RESULTS

A noncompact Riemannian manifold \mathcal{M} is said to be parabolic if every positive superharmonic function $u : \mathcal{M} \rightarrow \mathbb{R}_+$ is constant. This is equivalent to asking that there exists no positive fundamental solution of the Laplace equation or that Brownian motion is recurrent on \mathcal{M} , see e.g. [29]. \mathbb{R}^2 is parabolic, while \mathbb{R}^N is not when $N \geq 3$. Hoping that our findings could give an interesting way to study noncompact Riemannian manifolds¹ in higher dimensions, we focus in what follows on those positive superharmonic functions that are also solutions of a semilinear elliptic equation. We shall be interested more specifically in solutions having some stability property, although this assumption is not needed for some of our results. We work in the Euclidean setting and prove sharp Liouville-type theorems and classification results.

Since \mathbb{R}^N is invariant under dilations, such theorems are very much related to the corresponding regularity theory, notably some tools and results of the recent paper [6] exploiting the dilation invariance (a version of Pohozaev's classical argument [32] suited for stable solutions), the autonomous nature of the equation (a geometric Poincaré formula² discovered in [33]) and universal bounds in the C^α and H^1 norms (due to [6]). Among novelties in what follows, we are able to handle the critical dimension $N = 10$, to work without any convexity assumption on the nonlinearity and to classify solutions which are stable outside a compact set. Finally, we provide new Liouville-type theorems on half-spaces, improving those of [26], as well as on coercive epigraphs.

Our first result extends a theorem in our earlier work [15], which held for bounded stable solutions in dimension $N \leq 4$, and reads as follows.

Theorem 1. *Assume that $u \in C^2(\mathbb{R}^N)$ is bounded below and that u is a stable solution of*

$$(1) \quad -\Delta u = f(u) \quad \text{in } \mathbb{R}^N$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is locally Lipschitz and nonnegative. If $N \leq 10$, then u must be constant.

We recall that if $f \in C^1(\mathbb{R})$, a solution u of $-\Delta u = f(u)$ in an open set $\Omega \subseteq \mathbb{R}^N$ is *stable* if for every $\varphi \in C_c^1(\Omega)$, there holds

$$(2) \quad \int_{\Omega} f'(u)\varphi^2 \leq \int_{\Omega} |\nabla \varphi|^2$$

When f is locally Lipschitz, the definition³ requires more care, see [25] and [6].

Under a weaker lower bound, the previous result remains true up to dimension 9 :

Theorem 2. *Assume that $u \in C^2(\mathbb{R}^N)$ is a stable solution of (1), where $f : \mathbb{R} \rightarrow \mathbb{R}$ is locally Lipschitz and nonnegative. Assume in addition that for some $C > 0$*

$$(3) \quad u(x) \geq -C \ln(2 + |x|), \quad x \in \mathbb{R}^N.$$

If $N \leq 9$, then u must be constant.

1. at least under curvature or curvature-dimension conditions.
 2. which also holds on manifolds, see [22], and can also be reformulated in the general language of Γ -calculus, see [13]
 3. In [6], a slightly different definition than that in [25], is used. Note that when f is nondecreasing, the condition in [6] is *a priori* more restrictive. However, as observed by the authors of both papers, when restricting to test functions of the form $\varphi = |\nabla u|\psi$ or of the form $\varphi = \partial_r u \psi$, $\psi \in C_c^1(\mathbb{R}^N)$, then both definitions yield the same information. In particular, our results hold using one definition or the other.

Remark 3.

Some remarks are in order.

- Theorem 1 is sharp. Indeed, if $N \geq 11$, for $f(u) = u^p$, p sufficiently large, there exists a nontrivial positive bounded stable solution to the equation, see [19].
- Theorem 2 is also sharp. Indeed, if $N \geq 10$ and $f(u) = e^u$, there exists⁴ a radial stable solution $u \in C^2(\mathbb{R}^N)$ such that $u(x) \sim -2 \ln |x|$ as $|x| \rightarrow +\infty$.
- In the important cases where $f(u) = |u|^{p-1}u$, $p > 1$ and $f(u) = e^u$, Theorems 1 and 2 were already known to hold, see [19, 20]. In addition, in these cases, the theorems hold without assuming any bound on u , due to the scale invariance of the equations.
- In the particular case where u is radial and bounded, Theorems 1 and 2 were already known to hold, see [5, 34].
- It will be clear from the proof of Theorem 2 that the same conclusion holds true if we replace (3) by the weaker one : $u(x) \geq -C \ln^\gamma(2 + |x|)$, where $\gamma \geq 1$. Some lower bound is however needed in Theorems 1 and 2. Indeed, the function $u(x) = -|x|^2$ is a stable solution (with $f(u) = 2N$). It is bounded above, but not below.
- The assumption that u is superharmonic (i.e. $f \geq 0$) is also essential. Indeed, if $f(u) = u - u^3$, then f changes sign and for any $N \geq 1$, $u(x) = \tanh(x_1/\sqrt{2})$ is bounded and monotone (hence stable) yet non constant⁵. Similarly, if $f(u) = -2N$, then $f \leq 0$ and $u(x) = |x|^2$ is a stable solution bounded below. Still, when $f \leq 0$, one can clearly use our theorems to classify stable solutions which are bounded *above* (simply work with $-u$ in place of u).
- At last, the theorems cannot be generalized to solutions which are merely stable outside a compact set (and so to finite Morse index solutions). Indeed, if $N \geq 3$, the standard bubble is a nonconstant bounded solution of $-\Delta u = u^{\frac{N+2}{N-2}}$ in \mathbb{R}^N of Morse index 1. The same remark holds in dimension $N = 2$ for solutions stable outside a compact set satisfying the weaker bound (3), by considering the Liouville equation $-\Delta u = e^u$ in \mathbb{R}^2 (see Theorem 3 in [20]).
- Additional examples of non-trivial solutions of finite Morse index are provided by *subcritical* nonlinearities of the form $f(u) = [(u - \beta)^+]^p$, for all $p \in (1, \frac{N+2}{N-2})$, $N \geq 3$ and some $\beta > 0$, see [16].
- Note however that all these counter-examples are radial functions. We address the question of radial symmetry in Theorem 10 below.

As important corollaries, we obtain the following Liouville-type results on half-spaces and on coercive epigraphs.

Theorem 4. *Let $u \in C^2(\overline{\mathbb{R}_+^N})$ be a bounded solution of*

$$(4) \quad \begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}_+^N, \\ u > 0 & \text{in } \mathbb{R}_+^N, \\ u = 0 & \text{on } \partial\mathbb{R}_+^N. \end{cases}$$

Assume $f \in C^1(\mathbb{R})$ and

(1) either $f(t) \geq 0$ for $t \geq 0$,

(2) or there exists $z > 0$ such that $f(t) \geq 0$ for $t \in [0, z]$ and $f(t) \leq 0$ for $t \geq z$.

If $2 \leq N \leq 11$, then u must be one-dimensional and monotone (i.e., $u = u(x_N)$ and $\frac{\partial u}{\partial x_N} > 0$ in \mathbb{R}_+^N).

Theorem 4 recovers and improves upon a result of [26], which held for $N \leq 5$ (see also [2, 3, 8, 11, 12, 18, 19, 21, 23–25] for some other results concerning problem (4).)

For the Neumann boundary condition, the following result holds.

4. To prove this, using Emden's transformation, the equation is equivalent to an autonomous ode having a unique stationary point, which corresponds to the singular solution $u_s(x) = \ln \frac{2(N-2)}{|x|^2}$ and is attractive, see e.g. pp. 36-37 in Chapter 2 in [14]. In addition, u_s is stable for $N \geq 10$, thanks to Hardy's inequality. Since u_s is singular at the origin, it is clear that $u \leq u_s$ in some ball B_R . In fact, the inequality holds throughout \mathbb{R}^N and so u is also stable. Otherwise, there would exist R' such that $u_s - u = 0$ on $\partial B_{R'}$. Using $\varphi = u_s - u \in H_0^1(B_{R'})$ as a test function in (2) would then contradict the stability of u_s , using (the proof of) Proposition 3.2.1 in [14].

5. One could ask whether other kinds of solutions exist. This is a delicate question deeply related to the celebrated De Giorgi conjecture, see e.g. [7, 14, 27] for more on this subject.

Theorem 5. Let $u \in C^2(\overline{\mathbb{R}_+^N})$ be a stable solution of

$$(5) \quad \begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}_+^N, \\ \partial_n u = 0 & \text{on } \partial\mathbb{R}_+^N. \end{cases}$$

Assume in addition that $f : \mathbb{R} \rightarrow \mathbb{R}$ is locally Lipschitz and nonnegative and u is bounded below. If $N \leq 10$, then u must be constant.

Here, a solution of (5) is said to be stable if (2) holds with $\Omega = \mathbb{R}_+^N$ for all $\varphi \in C_c^1(\overline{\mathbb{R}_+^N})$.

At last, for coercive epigraphs, the following result holds true.

Theorem 6. Let $\Omega \subset \mathbb{R}^N$ denote a locally Lipschitz coercive epigraph and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a bounded solution of

$$(6) \quad \begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Assume that $f \in C^1(\mathbb{R})$, $f(t) > 0$ for $t > 0$ and $2 \leq N \leq 11$. Then, $f(0) = 0$ and $u = 0$.

Before stating our results on solutions which are merely stable outside a compact set, it will be useful to discuss the proof of Theorem 2, which relies on the following *a priori* estimate, recently established in [6].

Theorem A ([6]). Let B_1 be the unit ball of \mathbb{R}^N , $N \geq 1$. Assume that $u \in C^2(B_1)$ is a stable solution of

$$-\Delta u = f(u) \quad \text{in } B_1,$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is locally Lipschitz and nonnegative. If $N \leq 9$, then

$$(7) \quad \|u\|_{C^\alpha(\overline{B_{1/2}})} \leq C \|u\|_{L^1(B_1)},$$

where $\alpha \in (0, 1)$, $C > 0$ are dimensional constants.

Since the proof of Theorem 2 is very short, we provide it without further ado.

Proof of Theorem 2.

Given $R > 2$, apply (7) to $u_R(x) = u(Rx)$, leading to

$$|u(x) - u(y)| \leq CR^{-\alpha} |x - y|^\alpha \int_{B_R} |u| \quad \text{for } x, y \in B_{R/2}.$$

Since u satisfies the lower bound (3), observe that

$$|u| = |u + C \ln(2 + |x|) - C \ln(2 + |x|)| \leq |u + C \ln(2 + |x|)| + |C \ln(2 + |x|)| = u + 2C \ln(2 + |x|).$$

So, recalling that u is superharmonic,

$$0 \leq \int_{B_R} |u| \leq \int_{B_R} u + 2C \ln(2 + R) \leq u(0) + 2C \ln(2 + R)$$

and so

$$|u(x) - u(y)| \leq CR^{-\alpha} |x - y|^\alpha (u(0) + 2C \ln(2 + R)).$$

Let $R \rightarrow +\infty$ to conclude that $u(x) = u(y)$ for all $x, y \in \mathbb{R}^N$. \square

Remark 7. Since Theorem 2 fails for finite Morse index solutions, it follows from the proof above that the *a priori* estimate in Theorem A cannot hold either for such solutions.

As already observed in Remark 3, Theorems 1 and 2 fail for solutions with positive and finite Morse index. Nevertheless, we can prove radial symmetry and sharp asymptotic behavior at infinity of such solutions. More precisely, we have the following two results.

Theorem 8. Let $u \in C^2(\mathbb{R}^N)$ be a solution of

$$(8) \quad -\Delta u = f(u) \quad \text{in } \mathbb{R}^N,$$

which is stable outside the ball B_1 and bounded below. Assume that $f \geq 0$ is locally Lipschitz continuous and $1 \leq N \leq 10$. Then,

i) if $N = 1, 2$, then u is constant.

ii) if $3 \leq N \leq 9$, there exists a constant $C > 0$ depending on u and N only such that

$$(9) \quad |u(x) - \inf_{\mathbb{R}^N} u| \leq C|x|^{-\frac{N}{2} - \sqrt{N-1} + 2}, \quad \text{for all } x \in \mathbb{R}^N,$$

$|\nabla u| \in L^2(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N \setminus B_R} |\nabla u|^2 \leq CR^{-2(\sqrt{N-1}-1)}, \quad \text{for all } R > 2.$$

iii) if $N = 10$, for any $\varepsilon > 0$ (small enough) there exists a constant $C_\varepsilon > 0$ depending on u , N and ε only such that

$$(10) \quad |u(x) - \inf_{\mathbb{R}^N} u| \leq C_\varepsilon |x|^{-\frac{N}{2} - \sqrt{N-1} + 2 + \varepsilon}, \quad \text{for all } x \in \mathbb{R}^N,$$

$|\nabla u| \in L^2(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N \setminus B_R} |\nabla u|^2 \leq C_\varepsilon R^{-2(\sqrt{N-1}-1) + \varepsilon}, \quad \text{for all } R > 2.$$

Furthermore, $f(\inf_{\mathbb{R}^N} u) = 0$ and $(u - \inf_{\mathbb{R}^N} u) \in L^{2^*}(\mathbb{R}^N)$.

Remark 9.

- Clearly, Theorem 8 remains true for solutions stable outside a compact set.
- Theorem 8 complements some results established in [15, 21]. It was already known to hold for any $N \geq 3$ in the particular case where u is bounded and radial, see [35].
- Theorem 8 fails if $N \geq 12$ for nonradial solutions. Indeed, if $n \geq 11$, there exists a nontrivial bounded stable positive radial solution v for the nonlinearity $f(v) = v^p$, p large, see [19]. Then, the function $u(x', x_N) = v(x')$ for $(x', x_N) \in \mathbb{R}^n \times \mathbb{R}$ is a bounded stable positive solution (of the same problem) in dimension $N = n + 1 \geq 12$. In addition, $u \not\rightarrow 0$ as $|x| \rightarrow \infty$ and $\nabla u \notin L^2(\mathbb{R}^N)$.
- Whether Theorem 8 holds in dimension $N = 11$ is an open problem.

Theorem 10. *Let $u \in C^2(\mathbb{R}^N)$ be a solution of (1) which is stable outside a compact set and bounded below. Assume that $f \in C_{loc}^{3,1}(\mathbb{R})$, $f \geq 0$ and $N \leq 10$. Then, u is radially symmetric about some point and radially decreasing (and radially strictly decreasing if u is not a constant).*

Remark 11.

- The assumption $f \in C_{loc}^{3,1}(\mathbb{R})$ is most likely technical. It is needed only at every zero of f . As follows from its proof, the theorem remains true if $f \in C_{loc}^{1,1}(\mathbb{R})$ and $5 \leq N \leq 10$ and if $f \in C_{loc}^{2,1}(\mathbb{R})$ and $N = 4$.
- Whether the result remains true in dimension $N = 11$ is an open problem, even for the nonlinearity $f(u) = u^p$, p large. However, the positive cylindrical solution mentioned in Remark 9 shows that Theorem 10 fails in dimension $N \geq 12$. Working similarly with the exponential nonlinearity, Theorem 10 also fails in dimension $N \geq 11$ under the weaker lower-bound (3).
- Thanks to the result of [35], (10) holds true for $\varepsilon = 0$, that is, optimal asymptotic bounds also hold in dimension $N = 10$.

2. STABLE SOLUTIONS

Let us prove Theorem 1.

Proof of Theorem 1.

We start by noting that, if u is a stable solution of (1) in \mathbb{R}^n , $n \geq 1$, then the function $v_k := v_k(x_1, \dots, x_n, \dots, x_{n+k}) := u(x_1, \dots, x_n)$ is a stable solution of (1) in \mathbb{R}^{n+k} for any $k \geq 1$. Therefore, it is enough to prove the claim for solutions u in dimension $N = 10$.

Since u is superharmonic and bounded below, its spherical average $\int_{\partial B_R} u$ decreases to a limit $\ell \in \mathbb{R}$, as $R \rightarrow +\infty$. Replacing u by $u - \ell$, we may assume from here on that

$$(11) \quad \int_{\partial B_R} u \searrow 0, \quad \text{as } R \rightarrow +\infty$$

(note that $u - \ell$ is a bounded below stable solution of (1) with f replaced by the non negative nonlinearity $f(\cdot + \ell)$.)

We begin by proving the following lemma (which holds in any dimension) :

Lemma 1.

$$(12) \quad \int_{B_R} |x|^{2-N} |\nabla u|^2 = o(\ln R), \quad \text{as } R \rightarrow +\infty.$$

Proof. According to Proposition 2.5 in [6], the following H^1 -bound holds true

$$\|\nabla u\|_{L^2(B_1)} \leq C \|u\|_{L^1(B_2)}$$

Applying the above to $u_R(x) = u(Rx)$ (as we may) yields

$$(13) \quad \int_{B_R} |\nabla u|^2 \leq CR^{N-2} \left(\int_{B_{2R}} u \right)^2.$$

Here, we have used that $\int_{\partial B_r} u \geq 0$, thanks to (11). Using polar coordinates and integration by parts, we find

$$\int_{B_R \setminus B_1} |x|^{2-N} |\nabla u|^2 = \int_1^R r^{2-N} \int_{\partial B_r} |\nabla u|^2 = \left[r^{2-N} \int_{B_r} |\nabla u|^2 \right]_{r=1}^{r=R} - (2-N) \int_1^R r^{1-N} \int_{B_r} |\nabla u|^2$$

Let us inspect each term in the right-hand side. Since u is superharmonic, $\int_{B_r} u \leq u(0)$. By (13), we deduce that the first term is bounded. For the second term, either $\int_{B_r} u$ is bounded and so

$$\lim_{r \rightarrow +\infty} r^{-N} \int_{B_r} u = 0$$

or using L'Hôpital's rule and (11) we have, once again,

$$\lim_{r \rightarrow +\infty} r^{-N} \int_{B_r} u = \lim_{r \rightarrow +\infty} N^{-1} r^{1-N} \int_{\partial B_r} u = 0$$

Hence, by (13) and the above

$$r^{1-N} \int_{B_r} |\nabla u|^2 \leq C \frac{1}{r} \left(\int_{B_{2r}} u \right)^2 = o(1/r), \quad \text{as } r \rightarrow +\infty.$$

Hence,

$$\int_1^R r^{1-N} \int_{B_r} |\nabla u|^2 = o \left(\int_1^R \frac{1}{r} \right) = o(\ln R),$$

and the lemma follows since $\int_{B_1} |x|^{2-N} |\nabla u|^2$ is bounded. \square

Next, we use inequality⁶ (2.2) in [6], which, up to rescaling, reads as follows : for all $\zeta \in C_c^{0,1}(\mathbb{R}^N)$, if $2 \leq N \leq 10$, there holds

$$0 \leq \int -2|x|^{2-N} |\nabla u|^2 \zeta(x \cdot \nabla \zeta) + \int 4|x|^{2-N} (x \cdot \nabla u) \zeta \nabla u \cdot \nabla \zeta + \int (2-N)|x|^{-N} |x \cdot \nabla u|^2 \zeta(x \cdot \nabla \zeta) + \int |x|^{2-N} |x \cdot \nabla u|^2 |\nabla \zeta|^2$$

In particular, if ζ is radial and $r = |x|$, the above inequality reduces to

$$2 \int r^{2-N} |\nabla u|^2 \zeta r \zeta' \leq \int r^{2-N} \left(\frac{\partial u}{\partial r} \right)^2 r \zeta' \{ (6-N)\zeta + r \zeta' \}$$

⁶ This inequality is derived by computing the second variation of energy along dilations, similarly yet differently from what can be done in Pohozaev's identity or certain monotonicity formulae.

Choose ζ as follows. Given $R > R_1 > 2$, $r \in \mathbb{R}_+$,

$$\zeta(r) = \begin{cases} r^4 & \text{in } [0, R_1), \\ R_1^4 & \text{in } [R_1, R), \\ R_1^4 \frac{\ln \frac{r}{R^2}}{\ln \frac{1}{R}} & \text{in } [R, R^2), \\ 0 & \text{otherwise} \end{cases}$$

Since $N = 10$, we have that

$$r\zeta' \{(6 - N)\zeta + r\zeta'\} = 0 \quad \text{in } [0, R],$$

which leads us to an inequality of the form

$$8 \int_{B_{R_1}} |\nabla u|^2 \leq CR_1^8 \int_{B_{R^2} \setminus B_R} |x|^{2-N} |\nabla u|^2 ((\ln R)^{-1} + (\ln R)^{-2})$$

Applying Lemma 1 and letting $R \rightarrow +\infty$, we arrive at

$$\int_{B_{R_1}} |\nabla u|^2 = 0.$$

Letting $R_1 \rightarrow +\infty$, we deduce that u is constant. \square

Proof of Theorem 4. Since $f(0) \geq 0$ and u is bounded, we have that $\frac{\partial u}{\partial x_N} > 0$ on \mathbb{R}_+^N (see [3], [11]). Therefore $\frac{\partial u}{\partial x_N}$ is a positive solution of the linearized equation $-\Delta w - f'(u)w = 0$ on \mathbb{R}_+^N and so u is a stable solution of $-\Delta w = f(w)$ on \mathbb{R}_+^N (see for instance [28, 31]).

The boundedness of u , standard elliptic estimates and the monotonicity of u with respect to the variable x_N , imply that the function

$$v(x_1, \dots, x_{N-1}) := \lim_{x_N \rightarrow +\infty} u(x)$$

satisfies

$$(14) \quad \begin{cases} v \in C^2(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1}), \\ -\Delta v = f(v) & \text{in } \mathbb{R}^{N-1}, \\ v \geq 0 & \text{in } \mathbb{R}^{N-1}. \end{cases}$$

In addition v is a stable solution of the above problem (see for instance [1, 3]). Here we have used the continuity of f' .

If f satisfies 1. we can apply Theorem 1 to (14) to infer that $v \equiv c = \text{const}$. Here we have used that $N - 1 \leq 9$. The equation then yields $f(c) = 0$. To conclude we observe that $\frac{\partial u}{\partial x_N} > 0$ on \mathbb{R}_+^N implies $\sup_{\mathbb{R}_+^N} u = \sup_{\mathbb{R}^N} v (= c)$ and so $f(\sup_{\mathbb{R}_+^N} u) = 0$. The one-dimensional symmetry of u is then a consequence of Theorem 1 in [2].

When 2. is in force, we observe that $u(x) \leq z$ for any $x \in \mathbb{R}_+^N$, thanks to Lemma 2.4 in [26]. The latter and the definition of v imply that $v(x) \leq z$ for any $x \in \mathbb{R}_+^N$ and $\sup_{\mathbb{R}_+^N} u = \sup_{\mathbb{R}^N} v \leq z$. As before, to conclude, it is enough to prove that $f(\sup_{\mathbb{R}_+^N} v) = 0$. If $\sup_{\mathbb{R}_+^N} v = z$ we are done since $f(\sup_{\mathbb{R}_+^N} v) = f(z) = 0$. If $\sup_{\mathbb{R}_+^N} v < z$, then v is a stable solution of

$$(15) \quad \begin{cases} v \in C^2(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1}), \\ -\Delta v = g(v) & \text{in } \mathbb{R}^{N-1}, \end{cases}$$

where g is any nonnegative function of class $C^1(\mathbb{R}, \mathbb{R})$ which coincide with f on $(-\infty, \beta)$, for a fixed $\beta \in (\sup_{\mathbb{R}_+^N} v, z)$. Therefore an application of Theorem 1 to (15) yields $v \equiv c = \text{const}$. and so $f(c) = 0$. Hence $f(\sup_{\mathbb{R}_+^N} u) = f(\sup_{\mathbb{R}^N} v) = f(c) = 0$, which concludes the proof. \square

Proof of Theorem 5. By proceeding as in the first part of the proof of Theorem 2 in [12] we get that the even extension of u to \mathbb{R}^N is a bounded below stable solution of (1) in \mathbb{R}^N , $N \leq 10$. This function must be constant by Theorem 1. The latter implies the desired conclusion. \square

Proof of Theorem 6. Assume by contradiction that $u > 0$ in Ω . According to [4] (see also [17] for a prior result concerning smooth epigraphs) u is monotone. Therefore u is a stable solution and working as in the proof of Theorem 4, we deduce that

$$v(x_1, \dots, x_{N-1}) := \lim_{x_N \rightarrow +\infty} u(x)$$

is a positive stable solution of $-\Delta u = f(u)$ in \mathbb{R}^{N-1} , with $N-1 \leq 10$, and so it must be constant by Theorem 1. But this is in contradiction with $f(t) > 0$ for $t > 0$. Hence u must vanish somewhere in Ω and so it must be identically zero by the strong maximum principle. The latter then implies that $f(0) = 0$. \square

3. SOLUTIONS WHICH ARE STABLE OUTSIDE A COMPACT SET

Proof of Theorem 8. If $N \leq 2$ the solution u must be constant since it is bounded and superharmonic. We may therefore suppose that $N \geq 3$.

We begin by adapting Lemma 1 to solutions which are stable outside a compact set.

Lemma 2. *Assume that $N \geq 3$ and that u is a solution which is stable outside a compact set. Then,*

$$(16) \quad \int_{B_R} |x|^{2-N} |\nabla u|^2 = o(\ln R), \quad \text{as } R \rightarrow +\infty.$$

Proof. Assume without loss of generality that (11) holds and that $R = 2^n$ for some $n \in \mathbb{N}^*$. Then,

$$\int_{B_R \setminus B_2} |\nabla u|^2 = \int_{B_{2^n} \setminus B_2} |\nabla u|^2 = \sum_{k=2}^n \int_{B_{2^k} \setminus B_{2^{k-1}}} |\nabla u|^2.$$

Given $k \in \{2, \dots, n\}$, let $v(x) = u(2^{k-1}x)$, $x \in \mathbb{R}^N \setminus B_{2^{-(k-1)}}$. Then, v is stable outside the ball of radius $B_{2^{-(k-1)}}$ and

$$\int_{B_{2^k} \setminus B_{2^{k-1}}} |\nabla u|^2 = 2^{(k-1)(N-2)} \int_{B_2 \setminus B_1} |\nabla v|^2$$

The annulus $B_2 \setminus B_1$ can be covered by finitely many balls of radius $\frac{1}{2}$. On each of these balls, v is stable. Applying Proposition 2.5 in [6], we deduce that

$$\int_{B_2 \setminus B_1} |\nabla v|^2 \leq C \int_{B_4 \setminus B_{1/2}} |v| = C \int_{B_{2^{k+1}} \setminus B_{2^{k-2}}} u = o(1) \quad \text{as } k \rightarrow +\infty,$$

where we used (11) for the last equality. So,

$$(17) \quad \int_{B_R \setminus B_2} |\nabla u|^2 \leq C \sum_{k=2}^n o(2^{(k-1)(N-2)}) = o(R^{N-2}) \quad \text{as } R = 2^n \rightarrow +\infty.$$

Integrate by parts $\int_{B_R \setminus B_1} |x|^{2-N} |\nabla u|^2$ exactly as in the proof of Lemma 1 and (16) follows. \square

Next, we prove that $\nabla u \in L^2(\mathbb{R}^N)$. Assume that $3 \leq N \leq 9$. According to Lemma 2.1, inequality (2.2) in [6], for all $\zeta \in C_c^{0,1}(\mathbb{R}^N)$, ζ radial with support outside the ball of radius 1, there holds

$$(18) \quad \begin{aligned} \frac{(N-2)(10-N)}{4} \int_{\mathbb{R}^N} r^{2-N} \left(\frac{\partial u}{\partial r} \right)^2 \zeta^2 &\leq \\ &\leq -2 \int_{\mathbb{R}^N} r^{2-N} |\nabla u|^2 \zeta r \zeta' + \int_{\mathbb{R}^N} r^{2-N} \left(\frac{\partial u}{\partial r} \right)^2 r \zeta' \{(6-N)\zeta + r \zeta'\} \end{aligned}$$

By analogy with a strategy found in [35], we choose ζ as follows. Fix $R_2 > R > 2$ and

$$(19) \quad \alpha = \frac{N}{2} + \sqrt{N-1} - 2.$$

Given $r \in \mathbb{R}_+$, let

$$\zeta(r) = \begin{cases} 2^\alpha(r-1) & \text{in } [1, 2), \\ r^\alpha & \text{in } [2, R) \\ R^\alpha & \text{in } [R, R_2) \\ R^\alpha \frac{\ln \frac{r}{R_2}}{\ln \frac{1}{R_2}} & \text{in } [R_2, R_2^2), \\ 0 & \text{otherwise} \end{cases}$$

All integral terms in (18) in the region $[1 \leq |x| \leq 2]$ are controlled by a constant C depending on u and N only. For the region $[2 \leq |x| \leq R]$, our choice of α leads to the cancellation of all terms involving the radial derivative of u , so that all remains is the *negative* term (in the right-hand side)

$$-2\alpha \int_{B_R \setminus B_2} r^{2-N+2\alpha} |\nabla_T u|^2,$$

where we denoted the tangential part of the gradient by $\nabla_T u = \nabla u - \frac{\partial u}{\partial r} e_r$, $e_r = x/|x|$. In the region $[R, R_2)$, all terms disappear except the left-hand side :

$$R^{2\alpha} \frac{(N-2)(10-N)}{4} \int_{B_{R_2} \setminus B_R} r^{2-N} \left(\frac{\partial u}{\partial r} \right)^2$$

Finally, in the region $[R_2 \leq |x| \leq R_2^2]$, all terms can be bounded above by a constant $C = C(R, N)$ times

$$\left(\int_{B_{R_2^2} \setminus B_{R_2}} |x|^{2-N} |\nabla u|^2 \right) ((\ln R_2)^{-1} + (\ln R_2)^{-2}).$$

This quantity converges to 0 as $R_2 \rightarrow +\infty$, thanks to Lemma 2. So, in the limit $R_2 \rightarrow +\infty$, inequality (18) reduces to

$$\frac{(N-2)(10-N)}{4} R^{2\alpha} \int_{\mathbb{R}^N \setminus B_R} |x|^{2-N} \left(\frac{\partial u}{\partial r} \right)^2 + 2\alpha \int_{B_R \setminus B_2} |x|^{2-N+2\alpha} |\nabla_T u|^2 \leq C.$$

The above inequality being true for all $R > 2$, we readily deduce that for $3 \leq N \leq 9$,

$$(20) \quad \int_{B_{2R} \setminus B_R} |\nabla u|^2 \leq CR^{-(2-N+2\alpha)}$$

When $N = 10$, we only have

$$\int_{B_{2R} \setminus B_R} |\nabla_T u|^2 \leq CR^{-(2-N+2\alpha)}$$

However, replacing α by $\alpha_\epsilon = \alpha - \epsilon$, $\epsilon > 0$ small, in the definition of ζ leads to an inequality of the form

$$\int_{B_{2R} \setminus B_R} |\nabla u|^2 \leq C_\epsilon R^{-(2-N+2\alpha_\epsilon)},$$

Note that $2 - N + 2\alpha = 2(\sqrt{N-1} - 1) > 0$ for $N \geq 3$. So, applying this inequality with $2^k R$, $k \in \mathbb{N}$, in place of R and summing over k implies that $\nabla u \in L^2(\mathbb{R}^N)$ and

$$(21) \quad \int_{\mathbb{R}^N \setminus B_R} |\nabla u|^2 \leq CR^{-2(\sqrt{N-1}-1)},$$

if $3 \leq N \leq 9$ (and $\nabla u \in L^2(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N \setminus B_R} |\nabla u|^2 \leq C_\epsilon R^{-2(\sqrt{N-1}-1)+\epsilon}$ if $N = 10$).

Next, fix a point $x \in \mathbb{R}^N \setminus \overline{B_2}$ and let $R = |x|/2$. Take another point $y \in \partial B(x, r)$, $r \leq R$, and apply the fundamental theorem of calculus :

$$|u(x) - u(y)| \leq \int_0^1 \left| \frac{d}{dt} u(x + t(y-x)) \right| dt \leq r \int_0^1 |\nabla u(x + t(y-x))| dt$$

Integrating over $\partial B(x, r)$, we deduce that

$$\int_{\partial B(x, r)} |u(x) - u(y)| d\sigma(y) \leq \int_{B(x, r)} |\nabla u| \leq \int_{B(x, R)} |\nabla u|$$

Integrating once more in $r \in (0, R)$, using Cauchy-Schwarz and applying at last (21), for $3 \leq N \leq 9$ we get

$$(22) \quad \begin{aligned} \int_{B(x,R)} |u(x) - u(y)| \, dy &\leq CR^{1-N} \int_{B(x,R)} |\nabla u| \leq CR^{1-N/2} \left(\int_{B(x,R)} |\nabla u|^2 \right)^{\frac{1}{2}} \\ &\leq CR^{1-N/2} \left(\int_{B(0,3R) \setminus B(0,R)} |\nabla u|^2 \right)^{\frac{1}{2}} \leq CR^{-\frac{N}{2} - \sqrt{N-1} + 2}, \end{aligned}$$

(for $N = 10$ the latter is replaced by $C_\varepsilon R^{-2(\sqrt{N-1}-1)+\varepsilon}$).

We may draw two conclusions from the above inequality. Firstly,

$$\lim_{|x| \rightarrow +\infty} u(x) = 0.$$

Recalling that u is superharmonic, we have indeed

$$\begin{aligned} u(x) &\leq \int_{B(x,R)} |u(x) - u(y)| \, dy + \int_{B(x,R)} u \leq CR^{-\frac{N}{2} - \sqrt{N-1} + 2} + C \int_{B(0,3R)} u(y) \, dy \\ &\leq CR^{-\frac{N}{2} - \sqrt{N-1} + 2} + Cu(0) \leq C. \end{aligned}$$

So, u is bounded. By elliptic regularity, so is ∇u . We also know that $\nabla u \in L^2(\mathbb{R}^N)$, whence $u - \xi \in L^{2^*}(\mathbb{R}^N)$, for some $\xi \in \mathbb{R}$, thanks to Theorem 1.78 of [30]. The proof of this result also implies that $\xi = \lim_{j \rightarrow \infty} \int_{B(0,2^j)} u$ and so $\xi = 0$ by (11) (as already seen in the proof of Lemma 1.) Therefore, $u \in L^{2^*}(\mathbb{R}^N)$ and thus u^{2^*} is integrable and globally Lipschitz on \mathbb{R}^N . This clearly implies that $\lim_{|x| \rightarrow +\infty} u(x) = 0$.

Secondly, similarly to (22), we have for $3 \leq N \leq 9$

$$\begin{aligned} \int_{B(2x,R)} |u(2x) - u(y)| \, dy &\leq CR^{-\frac{N}{2} - \sqrt{N-1} + 2} \quad \text{and} \\ \int_{B(\frac{3}{2}x, 2R)} \left| u\left(\frac{3}{2}x\right) - u(y) \right| \, dy &\leq CR^{-\frac{N}{2} - \sqrt{N-1} + 2}. \end{aligned}$$

Using the notation $u_{z,r} = \int_{B(z,r)} u$ for the average of u on a given ball $B(z,r)$, it follows that

$$\begin{aligned} |u(x) - u(2x)| &\leq |u(x) - u_{x,R}| + \left| u_{x,R} - u\left(\frac{3}{2}x\right) \right| + \left| u\left(\frac{3}{2}x\right) - u_{2x,R} \right| + |u_{2x,R} - u(2x)| \\ &\leq CR^{-\frac{N}{2} - \sqrt{N-1} + 2} + \int_{B(x,R)} \left| u\left(\frac{3}{2}x\right) - u(y) \right| \, dy + \int_{B(2x,R)} \left| u\left(\frac{3}{2}x\right) - u(y) \right| \, dy \\ &\leq CR^{-\frac{N}{2} - \sqrt{N-1} + 2} + C \int_{B(\frac{3}{2}x, 2R)} \left| u\left(\frac{3}{2}x\right) - u(y) \right| \, dy \\ &\leq C|x|^{-\frac{N}{2} - \sqrt{N-1} + 2} \end{aligned}$$

Applying the above inequality to $2^k x$, $k \in \mathbb{N}$, in place of x and summing over k , we deduce that the sequence (u_k) defined by $u_k(x) = u(2^k x)$, converges in $C_{loc}^0(\mathbb{R}^N \setminus \overline{B}_2)$ to a limit v as $k \rightarrow +\infty$ and

$$|u(x) - v(x)| \leq C|x|^{-\frac{N}{2} - \sqrt{N-1} + 2} \quad \text{if } 3 \leq N \leq 9,$$

(resp. $|u(x) - v(x)| \leq C_\varepsilon|x|^{-\frac{N}{2} - \sqrt{N-1} + 2 + \varepsilon}$ if $N = 10$). Since $\lim_{|x| \rightarrow +\infty} u(x) = 0$ we necessarily have that $v \equiv 0$ and so

$$|u(x)| \leq C|x|^{-\frac{N}{2} - \sqrt{N-1} + 2} \quad \text{if } 3 \leq N \leq 9,$$

(resp. $|u(x)| \leq C_\varepsilon|x|^{-\frac{N}{2} - \sqrt{N-1} + 2 + \varepsilon}$ if $N = 10$).

In order to establish (9) and (10), it remains to prove that $\inf_{\mathbb{R}^N} u = 0$. We consider a sequence (x_n) such that $u(x_n) \rightarrow \inf_{\mathbb{R}^N} u$. Then, either $|x_n| \rightarrow +\infty$ or (x_n) possesses a bounded subsequence (still called (x_n)) such that $x_n \rightarrow \bar{x}$. In the first case we clearly have $\inf_{\mathbb{R}^N} u = 0$, while in the second case we get that $u(\bar{x}) = \inf_{\mathbb{R}^N} u$ and so the nonnegative subharmonic function $v := u - u(\bar{x})$ vanishes at \bar{x} . Therefore v must be zero by the strong maximum principle which, in turn, yields that $u \equiv \inf_{\mathbb{R}^N} u = \lim_{|x| \rightarrow +\infty} u(x) = 0$. \square

Proof of Theorem 10.

If $N \leq 2$, u is constant by Theorem 8 and so we are done. When $N \geq 3$ we can assume that u is not constant (otherwise the claim is trivially true). As before, up to replacing u by $u - \inf_{\mathbb{R}^N} u$, we may and do suppose that $\inf_{\mathbb{R}^N} u = 0$ and so $u > 0$ in \mathbb{R}^N by the strong maximum principle. Also, thanks to the asymptotics of Theorem 8 we have

$$(23) \quad \begin{cases} u \in L^{\frac{N}{2}}(\mathbb{R}^N) & \text{if } N \geq 5, \\ u^2 \in L^{\frac{N}{2}}(\mathbb{R}^N) & \text{if } N = 4, \\ u^3 \in L^{\frac{N}{2}}(\mathbb{R}^N) & \text{if } N = 3. \end{cases}$$

Moreover, since $f \geq 0$ by assumption and $f(0) = 0$ by Theorem 8, we must have $f'(0) = 0$ and $f''(0) \geq 0$. We also observe that in dimension $N = 3, 4$ we must have $f''(0) = f'''(0) = 0$. Indeed, if $f''(0) > 0$, then $\liminf_{t \rightarrow 0^+} \frac{f(t)}{t^{\frac{N}{N-2}}} \in (0, +\infty]$ and Theorem 3.5 of [9] implies that u must be identically zero, in contradiction with $u > 0$. Thus, $f''(0) = 0$ and so $f'''(0) = 0$ (again by $f \geq 0$). In particular we have

$$(24) \quad \forall t \in [0, \max_{\mathbb{R}^N} u] \quad |f'(t)| \leq \begin{cases} Ct & \text{if } N \geq 5, \\ Ct^2 & \text{if } N = 4, \\ Ct^3 & \text{if } N = 3, \end{cases}$$

where $C > 0$ is a constant depending only on f .

We are now ready to apply the moving planes method. Given $\lambda \in \mathbb{R}$, set $\Sigma_\lambda = \{(x_1, x') \in \mathbb{R}^N : x_1 < \lambda\}$ and for $x \in \Sigma_\lambda$, let as usual $x^\lambda = (2\lambda - x_1, x')$, $u_\lambda(x) = u(x^\lambda)$ and $w_\lambda = u_\lambda - u$. Since $u > 0$ and $\lim_{|x| \rightarrow +\infty} u(x) = 0$ we may and do suppose that (up to translation) $u(0) = \max_{\mathbb{R}^N} u > 0$ and then we prove that $u \equiv u_0$ in Σ_0 . Since the coordinate axis x_1 can be chosen arbitrarily, we will then conclude that u is radially symmetric about the origin. That u is radially strictly decreasing is also a standard consequence of the moving planes procedure and the strong maximum principle. This will provide the desired result.

We claim that $w_\lambda \geq 0$ in Σ_λ if $\lambda \leq -K$ for some $K > 0$. Indeed we have that

$$(25) \quad -\Delta w_\lambda = f(u_\lambda) - f(u) = a_\lambda w_\lambda \quad \text{in } \Sigma_\lambda,$$

where

$$(26) \quad a_\lambda := \begin{cases} \frac{f(u_\lambda) - f(u)}{u_\lambda - u} & \text{if } u - u_\lambda > 0, \\ 0 & \text{if } u - u_\lambda \leq 0, \end{cases}$$

belongs to $L^{N/2}(\Sigma_\lambda)$. To see this we observe that, by the mean value theorem, for every $x \in \Sigma_\lambda$ we have $|a_\lambda(x)| = |f'(\xi(x, \lambda))| \mathbf{1}_{\{u - u_\lambda > 0\}}(x)$, where $\xi(x, \lambda) \in (u_\lambda(x), u(x))$. Combining the latter with (24) we get

$$(27) \quad |a_\lambda| \leq \begin{cases} Cu & \text{if } N \geq 5, \\ Cu^2 & \text{if } N = 4, \\ Cu^3 & \text{if } N = 3, \end{cases} \quad \text{on } \Sigma_\lambda$$

where $C > 0$ is a constant depending only on f . The latter and (23) imply $a_\lambda \in L^{N/2}(\Sigma_\lambda)$ for any λ . Therefore, given any $\varepsilon > 0$ we can find $K > 0$ such that $\|a_\lambda\|_{L^{N/2}(\Sigma_\lambda)} < \varepsilon$, for any $\lambda \leq -K$.

Let $\chi_R(x) = \chi(x/R)$ be a standard cut-off function, with $R > 1$. Multiplying (25) by $w_\lambda^- \chi_R^2$, integrating by parts and making use of Sobolev's inequality we find

$$\left(\int_{\Sigma_\lambda} |w_\lambda^- \chi_R|^{2^*} \right)^{2/2^*} \leq C \int_{\Sigma_\lambda} |\nabla [w_\lambda^- \chi_R]|^2 \leq C \left\{ \int_{\Sigma_\lambda} (w_\lambda^-)^2 |\nabla \chi_R|^2 + \|a_\lambda\|_{L^{N/2}(\Sigma_\lambda)} \left(\int_{\Sigma_\lambda} |w_\lambda^- \chi_R|^{2^*} \right)^{2/2^*} \right\}$$

Letting $R \rightarrow +\infty$ and then choosing $\|a_\lambda\|_{L^{N/2}(\Sigma_\lambda \cap \{u > u_\lambda\})} < \varepsilon$ small enough, we deduce that $(w_\lambda)^- \equiv 0$ as claimed. Here we also used that $(w_\lambda^-)^2 \in L^{\frac{N}{N-2}}(\mathbb{R}^N)$ to show that $\int_{\Sigma_\lambda} (w_\lambda^-)^2 |\nabla \chi_R|^2 \rightarrow 0$ as $R \rightarrow \infty$ (recall that $u \in L^{2^*}(\mathbb{R}^N)$ by Theorem 8).

Let us now finish the moving planes method by setting

$$\lambda_0 := \sup\{\lambda < 0 : w_t \geq 0 \text{ in } \Sigma_t \quad \forall t \leq \lambda\}.$$

To conclude the proof it is enough to prove that $\lambda_0 = 0$. We argue by contradiction and suppose that $\lambda_0 < 0$. By continuity, $w_{\lambda_0} \geq 0$ in Σ_{λ_0} . By the strong maximum principle we deduce that either $w_{\lambda_0} > 0$ in Σ_{λ_0} or $w_{\lambda_0} \equiv 0$ in Σ_{λ_0} . The latter is not possible if $\lambda_0 < 0$ since in this case we would have $u(2\lambda_0, 0') = u(0) = \max_{\mathbb{R}^N} u > 0$ and so $w_\lambda \equiv 0$ in Σ_λ for any $\lambda \in [2\lambda_0, \lambda_0]$ by the strong maximum principle and the Hopf lemma. By repeating (infinitely many times) the previous argument we would find $u(s, 0') = u(0) = \max_{\mathbb{R}^N} u > 0$ for any $s < \lambda_0$, contradicting thus the assumption $\lim_{|x| \rightarrow +\infty} u(x) = 0$. Therefore $w_{\lambda_0} > 0$ in Σ_{λ_0} .

Now we achieve a contradiction by proving the existence of $\tau_0 > 0$ such that for any $0 < \tau < \tau_0$ we have $w_{\lambda_0+\tau} \geq 0$ in $\Sigma_{\lambda_0+\tau}$.

To this end we are going to show that, for every $\delta > 0$ there are $\tau_0 > 0$ and a compact set $K \subset \Sigma_{\lambda_0}$ (both depending on δ and λ_0) such that

$$(28) \quad \|a_\lambda\|_{L^{N/2}(\Sigma_\lambda \setminus K)} < \delta, \quad w_\lambda > 0 \text{ in } K, \quad \forall \lambda \in [\lambda_0, \lambda_0 + \tau_0].$$

We prove this for $N \geq 5$ (the case $N = 3, 4$ is obtained in the same way by using (23) and (27) for $N = 3, 4$). First pick a compact set $K \subset \Sigma_{\lambda_0}$ such that $\int_{\Sigma_{\lambda_0 \setminus K}} (Cu)^{\frac{N}{2}} < (\frac{\delta}{2})^{\frac{N}{2}}$ and then fix a $\tau_1 = \tau_1(\delta, \lambda_0) > 0$ such that $\int_{\Sigma_{\lambda \setminus \lambda_0}} (Cu)^{\frac{N}{2}} < (\frac{\delta}{2})^{\frac{N}{2}}$ for every $\lambda \in [\lambda_0, \lambda_0 + \tau_1]$, where C is the constant appearing in (27) (this choice is clearly possible in view of (23)). Combining these information with (27) we obtain that $\|a_\lambda\|_{L^{N/2}(\Sigma_\lambda \setminus K)} < \delta$ for any $\lambda \in [\lambda_0, \lambda_0 + \tau_1]$. Then, the uniform continuity of the function $u(2\lambda - x_1, x') - u(x)$ on the compact set $K \times [\lambda_0, \lambda_0 + \tau_1]$ and the fact that $w_{\lambda_0} > 0$ in Σ_{λ_0} ensure that $u_{\lambda_0+\tau} - u > 0$ in K for any $0 \leq \tau < \tau_2$, for some $\tau_2 = \tau_2(\delta, \lambda_0) < \tau_1$. Hence, (28) holds true with $\tau_0 \in (0, \tau_2)$.

As before, we multiply the equation (25) by $w_\lambda^- \chi_R^2$, we integrate by parts and use Sobolev's inequality to get

$$\left(\int_{\Sigma_\lambda} |w_\lambda^- \chi_R|^{2^*} \right)^{2/2^*} \leq C_S^2 \int_{\Sigma_\lambda} |\nabla[w_\lambda^- \chi_R]|^2 \leq C_S^2 \left\{ \int_{\Sigma_\lambda} (w_\lambda^-)^2 |\nabla \chi_R|^2 + \|a_\lambda\|_{L^{N/2}(\Sigma_\lambda)} \left(\int_{\Sigma_\lambda} |w_\lambda^- \chi_R|^{2^*} \right)^{2/2^*} \right\}$$

which, in view of (28) with $\delta = \frac{1}{2C_S^2}$, gives for every $\lambda \in [\lambda_0, \lambda_0 + \tau_0]$,

$$\begin{aligned} \left(\int_{\Sigma_\lambda \setminus K} |w_\lambda^- \chi_R|^{2^*} \right)^{2/2^*} &\leq C_S^2 \left\{ \int_{\Sigma_\lambda \setminus K} (w_\lambda^-)^2 |\nabla \chi_R|^2 + \|a_\lambda\|_{L^{N/2}(\Sigma_\lambda)} \left(\int_{\Sigma_\lambda \setminus K} |w_\lambda^- \chi_R|^{2^*} \right)^{2/2^*} \right\} \\ &\leq C_S^2 \int_{\Sigma_\lambda \setminus K} (w_\lambda^-)^2 |\nabla \chi_R|^2 + \frac{1}{2} \left(\int_{\Sigma_\lambda \setminus K} |w_\lambda^- \chi_R|^{2^*} \right)^{2/2^*}. \end{aligned}$$

Then

$$\left(\int_{\Sigma_\lambda \setminus K} |w_\lambda^- \chi_R|^{2^*} \right)^{2/2^*} \leq 2C_S^2 \int_{\Sigma_\lambda \setminus K} (w_\lambda^-)^2 |\nabla \chi_R|^2 \rightarrow 0 \quad \text{when } R \rightarrow \infty,$$

since $u \in L^{2^*}(\mathbb{R}^N)$. Therefore, $\int_{\Sigma_\lambda \setminus K} |w_\lambda^-|^{2^*} = 0$ and so $w_\lambda \geq 0$ in Σ_λ , for every $\lambda \in [\lambda_0, \lambda_0 + \tau_0]$. This contradicts the definition of λ_0 and so $\lambda_0 = 0$ which, in turn, yields $u \leq u_0$ in Σ_0 . Now we can apply the same procedure to the function $v(x) := u(-x_1, x') (= u_0(x))$ to find that $v \leq v_0$ in Σ_0 , i.e., $u_0 \leq u$ in Σ_0 . This proves that $u_0 \equiv u$ on Σ_0 and concludes the proof. □

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