



# Linear inviscid damping for shear flows near Couette in the 2D stably stratified regime

Roberta Bianchini, Michele Coti Zelati, Michele Dolce

## ► To cite this version:

Roberta Bianchini, Michele Coti Zelati, Michele Dolce. Linear inviscid damping for shear flows near Couette in the 2D stably stratified regime. 2020. hal-02619584

**HAL Id: hal-02619584**

**<https://hal.science/hal-02619584>**

Preprint submitted on 25 May 2020

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Linear inviscid damping for shear flows near Couette in the 2D stably stratified regime

Roberta Bianchini, Michele Coti Zelati, and Michele Dolce

**ABSTRACT.** We investigate the linear stability of shears near the Couette flow for a class of 2D incompressible stably stratified fluids. Our main result consists of nearly optimal decay rates for perturbations of stationary states whose velocities are monotone shear flows  $(U(y), 0)$  and have an exponential density profile. In the case of the Couette flow  $U(y) = y$ , we recover the rates predicted by Hartman in 1975, by adopting an explicit point-wise approach in frequency space. As a by-product, this implies optimal decay rates as well as Lyapunov instability in  $L^2$  for the vorticity. For the previously unexplored case of more general shear flows close to Couette, the inviscid damping results follow by a weighted energy estimate. Each outcome concerning the stably stratified regime applies to the Boussinesq equations as well. Remarkably, our results hold under the celebrated *Miles-Howard criterion* for stratified fluids.

## CONTENTS

|                            |    |
|----------------------------|----|
| 1. Introduction            | 1  |
| 2. Preliminaries           | 6  |
| 3. The Couette flow        | 10 |
| 4. Shears close to Couette | 13 |
| References                 | 27 |

## 1. Introduction

This article is concerned with the linear asymptotic stability of shear flows close to Couette for a class of two-dimensional incompressible stably stratified fluids without viscosity. Stability of *laminar equilibria*, to which the Couette flow with velocity vector  $(y, 0)$  belongs, is a widely investigated problem in the general context of hydrodynamic stability, a field where several major open questions still remain unanswered. We refer to [4, 17] for two highlighting reviews on the topic. The stability results provided here are explicitly quantified in terms of decay rates due to the phenomenon known as *inviscid damping*, caused by vorticity mixing. A brief review on the existing literature of linear and nonlinear inviscid damping is presented below.

As pointed out later on, from the rigorous mathematical viewpoint, the majority of inviscid damping results belongs to the framework of homogeneous fluids, namely incompressible flows with constant density (and zero viscosity, see [4] for a discussion about that). On the other hand, the physics literature on the stability of non-homogeneous fluids started several decades ago and provided one of the first remarkable answers in the early 60's, with a criterion for spectral stability of stratified flows due to Miles [28] and Howard [20]. The mathematical picture on the stability of shear flows for variable density fluids is therefore far from being complete and only *linear* stability results are available up to now [2, 38].

---

2000 *Mathematics Subject Classification.* 35Q35, 76F10.

*Key words and phrases.* Inviscid damping, stratified fluids, Boussinesq approximation, mixing.

A relevant class of fluids with variable density is represented by the non-homogeneous incompressible Euler equations

$$\begin{aligned}\rho(\partial_t + \mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p &= -\rho \mathbf{g}, \\ (\partial_t + \mathbf{u} \cdot \nabla) \rho &= 0, \\ \nabla \cdot \mathbf{u} &= 0,\end{aligned}\tag{1.1}$$

where  $\mathbf{u} = (u^x, u^y)$  is the velocity field,  $\rho$  is the density,  $p$  is the pressure and  $\mathbf{g} = (0, \mathbf{g})$  is (minus) gravity. Models of incompressible fluids have a large range of application, in particular in oceanography, see [31], where both incompressibility and absence of viscosity are very good approximations of reality.

The system under investigation in this paper belongs exactly to this framework. On the spatial domain  $(x, y) \in \mathbb{T} \times \mathbb{R}$  (a periodic channel), we consider

$$\begin{aligned}(\partial_t + U(y)\partial_x)(\omega - \beta\partial_y\psi) - (U''(y) - \beta U'(y))\partial_x\psi &= -R\partial_x q, \\ (\partial_t + U(y)\partial_x)q &= \partial_x\psi, \\ \Delta\psi &= \omega,\end{aligned}\tag{1.2}$$

endowed with the suitable initial data

$$\omega(0, x, y) = \omega_{in}(x, y), \quad q(0, x, y) = q_{in}(x, y).$$

The above linear system is written in terms of the unknowns  $\omega, \psi$  and  $q$ , which represent perturbations of a vorticity, a streamfunction and a (scaled) density, respectively. In particular, the velocity vector  $(v^x, v^y) = \nabla^\perp \psi = (-\partial_y \psi, \partial_x \psi)$  is recovered via the Biot-Savart law

$$v^x = -\partial_y \Delta^{-1} \omega, \quad v^y = \partial_x \Delta^{-1} \omega.\tag{1.3}$$

Moreover,  $R > 0$  and  $\beta \geq 0$  are *a priori* two constant and independent parameters. As explained in the subsequent Sections 1.3-1.4, the potential interdependence of the constants  $R$  and  $\beta$  establishes the distinction between two different physically relevant models: the case of shear flows in the Boussinesq approximation (for  $\beta = 0$  and  $R > 1/4$ ) and the case of an exponentially stratified fluid (for  $\beta > 0$  and  $R = \beta \mathbf{g} > 1/4$ ).

**1.1. Linear inviscid damping and main results.** Since the  $x$ -average is a conserved quantity for the dynamics of (1.2), it is convenient to introduce the notation

$$\langle f \rangle_x = \frac{1}{2\pi} \int_{\mathbb{T}} f(x, y) dx$$

to denote the  $x$ -average of a function  $f : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$ . We prove a quantitative asymptotic stability result when the background shear is close to the Couette flow  $U(y) = y$ . The main result of this paper reads as follows.

**THEOREM 1.1.** *Let  $R > 1/4$  and  $\beta \geq 0$  be arbitrarily fixed. There exists a small constant  $\varepsilon_0 = \varepsilon_0(\beta, R) \in (0, 1)$  with the following property. If  $\varepsilon \in (0, \varepsilon_0]$  and*

$$\|U' - 1\|_{H^6} + \|U''\|_{H^5} \leq \varepsilon,\tag{1.4}$$

*then for every  $t \geq 0$  we have*

$$\|q(t) - \langle q \rangle_x\|_{L^2} + \|v^x(t) - \langle v^x \rangle_x\|_{L^2} \lesssim \frac{1}{\langle t \rangle^{\frac{1}{2} - \delta_\varepsilon}} (\|\omega_{in} - \langle \omega_{in} \rangle_x\|_{L^2} + \|q_{in} - \langle q_{in} \rangle_x\|_{H^1}),\tag{1.5}$$

$$\|v^y(t)\| \lesssim \frac{1}{\langle t \rangle^{\frac{3}{2} - \delta_\varepsilon}} \left( \|\omega_{in} - \langle \omega_{in} \rangle_x\|_{H^{\frac{1}{4}}} + \|q_{in} - \langle q_{in} \rangle_x\|_{H^{\frac{5}{4}}} \right),\tag{1.6}$$

where  $\delta_\varepsilon = 2\sqrt{\varepsilon}$ .

Assumption (1.4) is a quantification of how close the background shear is to the linear profile given by the Couette flow  $U(y) = y$ , which corresponds to  $\varepsilon = 0$ . Theorem 1.1 is a consequence of more general stability estimates in  $H^s$ , for any  $s \geq 0$  (see Theorems 3.1 and 4.1 below). It is worth mentioning here that the constants hidden in the symbol  $\lesssim$  in (1.5)-(1.6) blow up exponentially fast as  $R \rightarrow 1/4$ . Moreover, the choice of  $\delta_\varepsilon$  is not dictated by any scaling reasons: it can be actually taken proportional to  $\varepsilon$  itself, at the cost of a few more technicalities (see Remark 4.9).

The decay estimates (1.5)-(1.6) are an example of *inviscid damping*. This terminology, introduced in analogy with Landau damping [29], refers to a stability mechanism due to vorticity mixing. Studied since Kelvin [25] and Orr [30], the physics literature is quite vast (see [15, 16, 32, 37] and references therein). Of interest to the case treated here, we mention that the mixing phenomenon behind inviscid damping also appears in stratified shear flows [8, 26].

In the case of the 2D Euler equations with constant density, the *nonlinear* asymptotic stability of the Couette flow was obtained in [5], and much more recently the cases of monotone shears [22, 23, 27] and the point-vortex [21] were treated. In the linearized setting, more general flows can be analyzed [3, 11, 18, 24, 33–36, 39–42].

However, in the density-dependent case, vorticity mixing is not the only relevant effect. This is clarified in the special case of the Couette flow  $U(y) = y$ , where a proof which is *point-wise* in frequency space can be obtained, and gives the following instability result for the vorticity.

**COROLLARY 1.2.** *Let  $U(y) = y$  in (1.2), and assume that  $R > 1/4$  and  $\beta \geq 0$ . If  $\omega_{in} \in L^2$  and  $q_{in} \in H^{\frac{1}{2}}$ , then*

$$\|\omega(t)\|_{L^2} \gtrsim \langle t \rangle^{\frac{1}{2}} \left( \|\omega_{in}\|_{L^2} + \|q_{in}\|_{H^{\frac{1}{2}}} \right), \quad (1.7)$$

for every  $t \geq 0$ .

Corollary 1.2 highlights the presence of a linear Lyapunov-type instability in  $L^2$  for the vorticity. A similar type of instability was also recently observed in the compressible case [2] and in [43] in the simplified case of a viscous fluid under the Boussinesq approximation. We point out that this instability result for the vorticity is the reason why the inviscid damping rates of Theorem 1.1 are slower than the ones of the constant density case. Furthermore, unlike the case of homogeneous fluids, the fact that the bound (1.7) holds pointwise for each fixed frequency (see (3.4)) shows in particular a *transient decay* for times below the critical ones. For homogeneous fluids linearized around Couette, a completely different (linear) model was introduced in [13], where the authors employ *ad hoc* initial data, which are built in such a way that the vorticity is unstable in any  $H^s$ ,  $s > 2$  but the velocity still enjoys inviscid damping. This is related to the so-called mechanism of *echoes*.

**1.2. Proof strategy and comparison with previous works.** To the best of our knowledge, a linear asymptotic stability analysis of shear flows in the non-constant density setting appears in [38] and [2].

In [38], the authors treat the case of the Couette flow  $U(y) = y$  and an exponential density profile as the one in (1.11), for an arbitrary  $R > 0$ . In the case  $R > 1/4$ , an improvement of the statement of [38] is recovered by Theorem 1.1 by simply setting  $\varepsilon = 0$ . In particular no (exponential) weights are needed in the norms. However, it is shown in [38] that for  $R \leq 1/4$  the decay rates undergo (at least) a logarithmic correction. The methods of proof in our work are completely different from those employed in [38]. There, the analysis is based on the use of hypergeometric functions (an approach that dates back to Hartman [19]), which were needed to solve the system (1.2) in frequency space. In fact, by performing the change of variables to follow the background shear, it is possible to rewrite (1.2) as a second order linear ODE in terms of  $\psi$  that can be solved with the use of hypergeometric functions.

In our paper, we instead take inspiration from the recently treated compressible case [2]. Namely, we consider (1.2) as a  $2 \times 2$  non-autonomous dynamical system at each fixed frequency. Then, by exploiting a proper symmetrization, we present a proof of Theorem 1.1 in the Couette case (i.e. for  $\varepsilon = 0$ ), which will be propaedeutic to the study of shears close to Couette.

Finally, our method of proof relies on the use of an energy functional endowed with certain suitable weights, among which one is inspired by the Alinhac ghost weight energy method [1] and was already used by Zillinger (see [40] and references therein) in the constant density case, and in [2] in the compressible case. Notice that the analysis of this model requires a delicate tracking of the constants depending on  $\beta$ , which will be involved in the definition of one of the weights of our energy functional. In fact, this is the idea which allows us to handle the general case of exponentially stratified fluids near Couette. We would also like to point out that the shears-near-Couette case could be seen as a step forward aiming at investigating *nonlinear* stability of the Couette flow for stratified fluids, as a good understanding of the dynamics of the background shear flow was a key point for the nonlinear stability in the constant density case, see [5].

**1.3. Stratified flows in the Boussinesq approximation.** The mathematical description of the dynamics of the ocean can be modeled by system (1.1), under some additional hypotheses. Specifically, in most physical non-homogeneous fluids, the variations of the density profile are negligible compared to its (constant) average. It is then customary to assume that at equilibrium the stratification (or density variation) is given by a *stable* profile  $\bar{\rho}(x, y) = \bar{\rho}(y)$  so that  $\partial_y \bar{\rho}(y) < 0$ . Among all the possible stratification's equilibria, one often assumes in addition that the stratification is locally affine so that  $\partial_y \bar{\rho}(y)$  is constant, see [7, 14] and references therein for an interesting discussion on the benefits and limitations of this approximation. Thus we consider a *linear background stratification*  $\bar{\rho}(y) = 1 - \gamma y$ , with  $0 < \gamma \ll 1$ . Notice that this profile is stable and locally affine and, at least for small  $\gamma$ 's, one can locally rely on the asymptotic  $\bar{\rho}(y) = 1 - \gamma y \approx e^{-\gamma y}$ . This way, we linearize system (1.1) around the equilibrium

$$\bar{\rho} = \bar{\rho}(y) = 1 - \gamma y, \quad \partial_y \bar{\rho} = -\mathbf{g} \bar{\rho},$$

with zero velocity, and we consider small disturbances of this equilibrium, which create small fluctuations of the density  $\kappa \tilde{\rho}$  and a (zero divergence) velocity field  $\kappa \tilde{\mathbf{u}}$  as well, with  $\kappa \ll 1$ . Then we plug in system (1.1) the following expansions

$$\mathbf{u} = \kappa \tilde{\mathbf{u}}, \quad \rho(t, x, y) = 1 - \gamma y + \kappa \tilde{\rho}, \quad p = -\mathbf{g}(1 - \gamma y) - \kappa \tilde{p},$$

so obtaining the *Boussinesq equations*

$$\begin{aligned} (\partial_t + \kappa \tilde{\mathbf{u}} \cdot \nabla) \tilde{\mathbf{u}} + \nabla \tilde{p} &= -\frac{\tilde{\rho}}{\bar{\rho}} \mathbf{g}, \\ (\partial_t + \kappa \tilde{\mathbf{u}} \cdot \nabla) \tilde{\rho} - \kappa \gamma v^y &= 0, \\ \nabla \cdot \tilde{\mathbf{u}} &= 0. \end{aligned}$$

The above 2D system is widely investigated and besides referring to [31] for a more physically detailed derivation, we just remark that it describes perturbations of a steady solution where gravity compensates for the vertical derivative of the pressure ( $\partial_y \bar{p} = -\mathbf{g} \bar{\rho}$ ), which translates in the fact that, in this Boussinesq regime, the restoring force of equilibrium's fluctuations is exactly gravity (*Archimedes' principle*).

Rather than a well-posedness and stability theory of small (and finite energy) perturbations of stratified fluids, see for instance [7, 9, 10, 12], here we are interested in the stability of shear flows. Thus we apply a further linearization near

$$\bar{\mathbf{u}} = (U(y), 0),$$

and hitting with  $\nabla^\perp \cdot = (-\partial_y, \partial_x) \cdot$  the momentum equation yields

$$\begin{aligned} (\partial_t + \kappa U(y) \partial_x) \omega - \kappa U''(y) \partial_x \psi &= -\mathbf{g} \frac{\partial_x \tilde{\rho}}{\bar{\rho}}, \\ (\partial_t + \kappa U(y) \partial_x) \tilde{\rho} &= \kappa \gamma \partial_x \psi, \\ \Delta \psi &= \omega. \end{aligned} \tag{1.8}$$

The last step is to scale time as  $\tau = \kappa t$  and define the scaled density

$$q = \frac{\tilde{\rho}}{\gamma},$$

so that, neglecting the small departures from 1 in  $\bar{\rho}(y) = 1 - \gamma y$  in the denominator of the first equation in (1.8), we end up with

$$\begin{aligned} (\partial_t + U(y)\partial_x)\omega - U''(y)\partial_x\psi &= -R\partial_x q, \\ (\partial_t + U(y)\partial_x)q &= \partial_x\psi, \\ \Delta\psi &= \omega, \end{aligned} \tag{1.9}$$

and

$$R = \frac{\gamma g}{\kappa}.$$

Notice that (1.9) is exactly (1.2) with  $\beta = 0$ , so that Theorem 1.1 gives asymptotic stability for system (1.9) in the regime  $R > 1/4$ .

**1.4. The stably stratified regime.** A model for the stably stratified regime can be heuristically derived from the Euler equations (1.1) as follows. We consider an *exponential stratification*, where the *stable* background density is  $\bar{\rho}(y) = e^{-\beta y}$  for some  $\beta > 0$ , and we linearize system (1.1) around this steady profile. Again, we are interested in studying stability of shear flows. Putting both the approximations together, we thus linearize (1.1) near

$$\bar{\mathbf{u}} = (U(y), 0), \quad \bar{\rho} = \bar{\rho}(y) = e^{-\beta y}, \quad \partial_y \bar{\rho} = -g\bar{\rho}.$$

Hitting the equation for the velocity with  $\nabla^\perp \cdot$ , we obtain

$$\begin{aligned} (\partial_t + U(y)\partial_x)\omega - U''(y)\partial_x\psi + \beta(\partial_t v^x + U(y)\partial_x v^x + U'(y)\partial_x\psi) &= -g\frac{\partial_x \rho}{\bar{\rho}}, \\ (\partial_t + U(y)\partial_x)\frac{\rho}{\bar{\rho}} &= \beta\partial_x\psi, \\ \Delta\psi &= \omega, \end{aligned} \tag{1.10}$$

where the velocity perturbation is recovered again from the Biot-Savart law in (1.3). We rearrange now the linearized system (1.10) by introducing the scaled density and the parameter  $R$

$$q = \frac{\rho}{\beta\bar{\rho}}, \quad R = \beta g. \tag{1.11}$$

Using that  $v^x = -\partial_y\psi$ , we recover the equations of system (1.2). Due to the (linear) relation between  $\beta$  and  $R$ , our Theorem 1.1 provides an answer to the question of asymptotic stability in the case  $\beta > \frac{1}{4g}$ , or equivalently  $R > \frac{1}{4}$ . This threshold is expected due to the celebrated *Miles-Howard criterion* for (spectral) stability of stratified fluids [20, 28], which precisely requires the *Richardson number*

$$\text{Ric}(y) = -g\frac{\bar{\rho}'(y)}{\bar{\rho}(y)}\frac{1}{|U'(y)|^2} = \frac{R}{|U'(y)|^2}$$

to be greater than  $1/4$ . We point out that, besides stability of the stratification profile, the positivity of  $\beta$  is actually needed to satisfy the criterion  $R = \beta g > \frac{1}{4}$ , which in our strategy turns to be the key condition in order to have a coercive energy functional.

**1.5. Organization of the paper.** The next Section 2 is dedicated to the set-up of the functional-analytic tools needed to study our problem. In Section 3, relying on a dynamical system approach, we prove optimal inviscid damping decay rates of (1.2) for the Couette flow  $U(y) = y$ . Next, we are able to extend the result (up to a small loss in terms of the decay rates) to shears of general profiles, which are close to Couette in a suitable sense. The latter result is obtained by means of a different method, which is based on the use of a properly weighted energy functional, and is presented in Section 4.

## 2. Preliminaries

In this section, we introduce the set of variables, the functional spaces and the operators which will be used in our analysis. First, we define new coordinates and variables, which follow the background shear flow and will be sometimes referred as the “moving frame”. Next, we set the notation for the Fourier transform and the functional spaces. Since the new spatial coordinates as well as partial derivatives depend on derivatives of the shear flow, then the action of differential operators in the moving coordinates has to be established and well-defined. Related results are provided in Proposition 2.1-2.2 at the end of this section.

**2.1. Change of coordinates and decoupling.** It is best to further re-write the system (1.2) by defining the auxiliary variable

$$\theta = \omega - \beta \partial_y \psi = (I - \beta \partial_y \Delta^{-1}) \omega. \quad (2.1)$$

Since the operator  $I - \beta \partial_y \Delta^{-1}$  is well-defined and invertible for any  $\beta \geq 0$ , we can write (1.2) in terms of  $\theta$  and  $q$  alone as

$$\begin{aligned} (\partial_t + U(y) \partial_x) \theta &= -R \partial_x q + (U''(y) - \beta U'(y)) \partial_x \Delta^{-1} (I - \beta \partial_y \Delta^{-1})^{-1} \theta, \\ (\partial_t + U(y) \partial_x) q &= \partial_x \Delta^{-1} (I - \beta \partial_y \Delta^{-1})^{-1} \theta, \end{aligned} \quad (2.2)$$

with initial data

$$\theta(0, x, y) = \theta_{in}(x, y), \quad q(0, x, y) = q_{in}(x, y).$$

Due to the transport nature of (2.2), the convenient set of coordinates is that of a moving frame that follows the background shear. Define

$$X = x - U(y)t, \quad Y = U(y)$$

and the corresponding new unknowns

$$\Theta(t, X, Y) = \theta(t, X + tY, Y), \quad Q(t, X, Y) = q(t, X + tY, Y), \quad \Omega(t, X, Y) = \omega(t, X + tY, Y). \quad (2.3)$$

The differential operators will change accordingly, in analogy with the cases already studied in the incompressible literature [11, 24, 39, 40]. In particular, by defining

$$g(Y) := U'(U^{-1}(Y)), \quad b(Y) := U''(U^{-1}(Y)), \quad (2.4)$$

we obtain the rules

$$\partial_x \rightsquigarrow \partial_X, \quad \partial_y \rightsquigarrow g(Y)(\partial_Y - t \partial_X),$$

and

$$\Delta \rightsquigarrow \Delta_t := \partial_{XX} + g^2(Y)(\partial_Y - t \partial_X)^2 + b(Y)(\partial_Y - t \partial_X). \quad (2.5)$$

As we shall see, if  $g, b$  are small in a way made more precise later (see Proposition 2.1), the operator  $\Delta_t$  is invertible. Similarly (see Proposition 2.2), the operator

$$B_t = (I - \beta g(Y)(\partial_Y - t \partial_X) \Delta_t^{-1})^{-1}. \quad (2.6)$$

is well-defined for any  $\beta$ . Moreover, it follows from (2.1) and (2.3) that

$$\Theta := B_t^{-1} \Omega.$$

In the moving frame, the equations (2.2), written in terms of  $\Theta$  and  $Q$  read

$$\begin{aligned} \partial_t \Theta &= -R \partial_X Q + (b(Y) - \beta g(Y)) \partial_X \Delta_t^{-1} B_t \Theta, \\ \partial_t Q &= \partial_X \Delta_t^{-1} B_t \Theta. \end{aligned} \quad (2.7)$$



It is apparent that the above system decouples in the  $X$ -frequency. Precisely, we adopt the following convention. Given a function  $f = f(X, Y)$ , we can write

$$f(X, Y) = \sum_{k \in \mathbb{Z}} f_k(Y) e^{ikX}, \quad f_k(Y) = \frac{1}{2\pi} \int_{\mathbb{T}} f(X, Y) e^{-ikX} dX, \quad (2.8)$$

where throughout the paper  $k \in \mathbb{Z}$  will denote the  $X$ -Fourier variable. Applying this reasoning to  $\Omega$  and  $Q$ , we can interpret (2.7) as a coupled system of infinitely many equations that read

$$\begin{aligned} \partial_t \Theta_k &= -ikRQ_k + ik(b(Y) - \beta g(Y)) \Delta_t^{-1} B_t \Theta_k, \\ \partial_t Q_k &= ik \Delta_t^{-1} B_t \Theta_k. \end{aligned} \quad (2.9)$$

In the above equations, we identified  $\Delta_t^{-1}$  and  $B_t$  with their  $X$ -Fourier localizations. It is clear that for  $k = 0$  there is nothing to prove, and therefore we will always assume  $k \neq 0$ .

**2.2. Function spaces and operators.** The Fourier transform in  $Y$  of a function  $f = f(X, Y)$  is defined by

$$\widehat{f}(X, \eta) = \int_{\mathbb{R}} f(X, Y) e^{-i\eta Y} dY, \quad \eta \in \mathbb{R}. \quad (2.10)$$

For any  $k \in \mathbb{Z}$  and  $\eta \in \mathbb{R}$ , we define

$$\langle \eta \rangle = \sqrt{1 + \eta^2}, \quad \langle (k, \eta) \rangle = \sqrt{1 + k^2 + \eta^2}.$$

We define the Sobolev space  $H^s$  of order  $s \in \mathbb{R}$  by the scalar product

$$\langle f, h \rangle_s = \langle \widehat{f}, \widehat{h} \rangle_s = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \langle (k, \eta) \rangle^{2s} \widehat{f}_k(\eta) \overline{\widehat{h}_k(\eta)} d\eta$$

and norm

$$\|f\|_s^2 = \|\widehat{f}\|_s^2 = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \langle (k, \eta) \rangle^{2s} |\widehat{f}_k(\eta)|^2 d\eta.$$

By the above notation, we mean that we make tacit use of Plancherel theorem. When  $s = 0$ , the above reduce to the standard  $L^2$  scalar product and norm, and we will use the explicit notation  $\|\cdot\|_{L^2}$  for the norm. An important role will be played by the operator

$$\Delta_L := \partial_{XX} + (\partial_Y - t\partial_X)^2,$$

which is exactly what one obtains from  $\Delta_t$  in (2.5) when setting  $g \equiv 1$  and  $b \equiv 0$ . The Fourier symbol of  $-\Delta_L$  will be denoted by

$$p(t; k, \eta) = k^2 + (\eta - kt)^2, \quad (2.11)$$

along with its time-derivative

$$p'(t; k, \eta) = -2k(\eta - kt).$$

To see  $\Delta_t$  as a perturbation of  $\Delta_L$ , it is convenient to write

$$\Delta_t = [I - T_\varepsilon] \Delta_L, \quad (2.12)$$

with

$$T_\varepsilon = [(g^2 - 1)(\partial_Y - t\partial_X)^2 + b(\partial_Y - t\partial_X)] (-\Delta_L^{-1}). \quad (2.13)$$

In particular,

$$T_\varepsilon = T_\varepsilon^g + T_\varepsilon^b,$$

where

$$T_\varepsilon^g = (g^2 - 1)(\partial_Y - t\partial_X)^2 (-\Delta_L^{-1}), \quad T_\varepsilon^b = b(\partial_Y - t\partial_X) (-\Delta_L^{-1}). \quad (2.14)$$

The following proposition holds true (see also [2]).



PROPOSITION 2.1. *Let  $s \geq 0$  be arbitrarily fixed. There exists a constant  $c_s \geq 1$  with the following property. Assume that*

$$\|g - 1\|_{s+1} + \|b\|_{s+1} \leq \varepsilon,$$

*for a positive  $\varepsilon < 1/c_s$ . Then  $\Delta_t$  is invertible on  $H^s$  and*

$$\Delta_t^{-1} = \Delta_L^{-1} T_L, \quad (2.15)$$

*where*

$$T_L = \sum_{n=0}^{\infty} T_\varepsilon^n = I + T_\varepsilon T_L. \quad (2.16)$$

*The convergence above is in the  $H^s$ -operator norm, and*

$$\|T_L\|_{H^s \rightarrow H^s} \leq \frac{1}{1 - c_s \varepsilon}. \quad (2.17)$$

PROOF. We begin to prove the convergence of the Neumann series (2.16) by obtaining a suitable bound on

$$\|T_\varepsilon f\|_s \leq \|T_\varepsilon^g f\|_s + \|T_\varepsilon^b f\|_s,$$

for any  $f : \mathbb{R} \rightarrow \mathbb{C}$  in  $H^s$ . To estimate the first part, notice that the Fourier symbol of the operator  $(\partial_Y - t\partial_X)^2(-\Delta_L^{-1})$  is uniformly bounded above by 1. Therefore, by Young's convolution inequality and the fact that  $g \in L^\infty$ , we have

$$\|T_\varepsilon^g f\|_s \leq \left\| \widehat{|g^2 - 1|} * \widehat{f} \right\|_s \leq \left\| \langle \cdot \rangle^s \widehat{|g^2 - 1|} \right\|_{L^1} \|\langle \cdot \rangle^s f\|_{L^2} \lesssim \|g - 1\|_{s+1} \|f\|_s,$$

where the last inequality follows from

$$\left\| \langle \cdot \rangle^s \widehat{|g^2 - 1|} \right\|_{L^1} = \int_{\mathbb{R}} \langle \eta \rangle^s \left| \widehat{|g^2 - 1|} \right| d\eta = \int_{\mathbb{R}} \frac{1}{\langle \eta \rangle} \langle \eta \rangle^{s+1} \left| \widehat{|g^2 - 1|} \right| d\eta \lesssim \|g^2 - 1\|_{s+1} \lesssim \|g - 1\|_{s+1}.$$

In the same way,

$$\|T_\varepsilon^b f\|_s \lesssim \|b\|_{s+1} \|f\|_s.$$

Hence, there exists a constant  $c_s$  such that

$$\|T_\varepsilon f\|_s \leq c_s (\|g - 1\|_{s+1} + \|b\|_{s+1}) \|f\|_s \leq c_s \varepsilon \|f\|_s. \quad (2.18)$$

If  $\varepsilon$  is chosen such that  $c_s \varepsilon < 1$ , the Neumann series (2.16) converges. In particular,

$$T_L = [I - T_\varepsilon]^{-1},$$

so that from (2.12) we also obtain the relation (2.1), and the proof is over.  $\square$

We also need to describe the action of  $B_t$  on  $H^s$ . Its definition relies on the operator

$$B_L := (I - \beta(\partial_Y - t\partial_X)\Delta_L^{-1})^{-1}, \quad (2.19)$$

which is precisely  $B_t$  in the Couette case where  $g \equiv 1$  and  $b \equiv 0$  (compare with (2.6)). More precisely, adopting the same notation as for the expansion of  $\Delta_t^{-1}$  in (2.15)-(2.16), we prove in the following proposition that we can write

$$B_t = T_B B_L,$$

where  $T_B$  is a proper Neumann series (see Proposition 2.2). Notice that the operator  $B_L$  is a Fourier multiplier, and

$$B_L^{-1} = 1 + \frac{i\beta(\eta - kt)}{p},$$

so everything is well-defined. It is now an easy task to show that, for any fixed frequency, the following bound holds true

$$\frac{1}{\sqrt{1+\beta^2}} \leq |B_L(t; k, \eta)| \leq 1. \quad (2.20)$$

For further reference, we also need to introduce

$$B_\varepsilon := B_L(\beta(g-1)(\partial_Y - t\partial_X)\Delta_L^{-1} + \beta g(\partial_Y - t\partial_X)\Delta_L^{-1}T_\varepsilon T_L). \quad (2.21)$$

The action of the above operators on  $H^s$  is detailed in the proposition below.

**PROPOSITION 2.2.** *Let  $s \geq 0$  and  $\beta \geq 0$  be arbitrarily fixed. Assume that*

$$\|g-1\|_{s+1} + \|b\|_{s+1} \leq \varepsilon, \quad (2.22)$$

*for a positive  $\varepsilon \in (0, \varepsilon_0]$ . Then we have*

$$\|B_\varepsilon\|_{H^s \rightarrow H^s} \lesssim \varepsilon. \quad (2.23)$$

*The operator  $B_t$  on  $H^s$  is well-defined and explicitly given by*

$$B_t = (I - B_\varepsilon)^{-1}B_L = T_B B_L, \quad (2.24)$$

*where*

$$T_B = \sum_{n=0}^{\infty} B_\varepsilon^n = I + B_\varepsilon T_B. \quad (2.25)$$

*Moreover,*

$$\|T_B\|_{H^s \rightarrow H^s} \leq 2. \quad (2.26)$$

**PROOF OF PROPOSITION 2.2.** The expansion of  $B_t$  in (2.6) in terms of the Neumann series (2.24) follows from the definition of  $\Delta_t^{-1}$  in (2.15)-(2.16) and the identity

$$\begin{aligned} B_t &= (I - \beta g(\partial_Y - t\partial_X)\Delta_t^{-1})^{-1} \\ &= (I - \beta g(\partial_Y - t\partial_X)(\Delta_L^{-1} + \Delta_L^{-1}T_\varepsilon T_L))^{-1} \\ &= (I - \beta(g-1)(\partial_Y - t\partial_X)\Delta_L^{-1} - \beta(\partial_Y - t\partial_X)\Delta_L^{-1} - \beta g(\partial_Y - t\partial_X)\Delta_L^{-1}T_\varepsilon T_L)^{-1} \\ &= (B_L^{-1}(1 - B_\varepsilon))^{-1} = (1 - B_\varepsilon)^{-1}B_L = T_B B_L, \end{aligned}$$

where  $B_L, B_\varepsilon$  are in (2.19) and (2.21). The above expansion is rigorous provided that  $(I - B_\varepsilon)$  is invertible. The expression of  $B_\varepsilon$  in (2.21) is rearranged here as follows

$$B_\varepsilon = B_L(\beta(g-1)(\partial_Y - t\partial_X)\Delta_L^{-1} + \beta(g-1)(\partial_Y - t\partial_X)\Delta_L^{-1}T_\varepsilon T_L + \beta(\partial_Y - t\partial_X)\Delta_L^{-1}T_\varepsilon T_L).$$

We use the upper bound on the multiplier  $B_L$  in (2.20) and the fact that  $|\eta - kt|p^{-1} \leq p^{-1/2} \leq 1$  to obtain

$$\begin{aligned} \|B_\varepsilon f\|_s &\leq \beta \left\| \widehat{|g-1|} * |\eta - kt|p^{-1} \widehat{|f|} \right\|_s + \beta \left\| \widehat{|g-1|} * |\eta - kt|p^{-1} \widehat{|T_\varepsilon T_L f|} \right\|_s \\ &\quad + \beta \left\| |\eta - kt|p^{-1} \widehat{|T_\varepsilon T_L f|} \right\|_s \\ &\leq \beta \left\| \widehat{|g-1|} * \widehat{|f|} \right\|_s + \beta \left\| \widehat{|g-1|} * \widehat{|T_\varepsilon T_L f|} \right\|_s + \beta \|T_\varepsilon T_L f\|_s \\ &\lesssim \beta \left\| \langle \cdot \rangle^s \widehat{|g-1|} \right\|_{L^1} \|f\|_s + \left\| \langle \cdot \rangle^s \widehat{|g-1|} \right\|_{L^1} \|T_\varepsilon T_L f\|_s + \beta \|T_\varepsilon T_L f\|_s \\ &\lesssim \beta \|g-1\|_{s+1} \|f\|_s + \beta(1 + \|g-1\|_{s+1}) \|T_\varepsilon T_L f\|_s, \end{aligned}$$

where we applied the Young convolution inequality exactly as in the proof of Proposition 2.1. It remains to deal with the last term above. Thus we first appeal to (2.18) and after we employ (2.17), so that

$$\|T_\varepsilon T_L f\|_s \leq c_s(\|g-1\|_{s+1} + \|b\|_{s+1}) \|T_L f\|_s \leq \frac{c_s}{1 - c_s \varepsilon_0} (\|g-1\|_{s+1} + \|b\|_{s+1}) \|f\|_s,$$

which yields (2.23) thanks to the smallness assumptions (2.22), where  $\varepsilon_0 \geq \varepsilon$  satisfies

$$\beta \varepsilon_0 \left( 1 + \frac{4c_s}{1 - c_s \varepsilon_0} \right) < 1,$$

so that  $T_B$  in (2.25) is well-defined and  $(I - B_\varepsilon)$  is invertible. The bound (2.26) directly follows from its definition (2.25) and the bound on  $B_\varepsilon$ . The proof is over.  $\square$

**2.3. Notation and conventions.** We highlight the following conventions, which will be used throughout the paper.

- As already pointed out before,  $p'(t; k, \eta)$  stands for the time-derivative of the Fourier symbol  $p(t; k, \eta)$ . This notation is systematically applied to the Fourier symbols of the weights involved in our  $H^s$ -weighted norm.
- We occasionally identify operators with their symbols in order to avoid additional notations (this is for instance the case of the operator  $\Delta_t^{-1}$ , as pointed out in (2.9) and the lines below).
- We drop the subscript  $k$  related to the  $X$ -Fourier-localization of the variables defined in (2.8) and we also identify functions with their  $Y$ -Fourier transform in (2.10) when there is no confusion. The adoption of this convention is usually pointed out at the beginning of the proofs.
- We use the notation  $f_1 \lesssim f_2$  if there exists a constant  $C = C(R, \beta, s)$  such that  $f_1 \leq C f_2$ , where  $C$  is independent of  $k, \varepsilon$  and  $\beta$ . We denote  $f_1 \approx f_2$  if  $f_1 \lesssim f_2$  and  $f_2 \lesssim f_1$ .

### 3. The Couette flow

We begin our analysis with the Couette flow, when  $U(y) = y$ . In this case, in (2.4) we have  $g \equiv 1$  and  $b \equiv 0$ , so that  $B_t$  in (2.6) coincides with  $B_L$  in (2.19), so that system (2.9) takes the simpler form

$$\begin{aligned} \partial_t \Theta_k &= -ikRQ_k - ik\beta \Delta_L^{-1} B_L \Theta_k, \\ \partial_t Q_k &= ik\Delta_L^{-1} \Theta_k + ik\Delta_L^{-1} (B_L - 1) \Theta_k, \end{aligned} \tag{3.1}$$

where

$$\Theta_k = B_L^{-1} \Omega_k.$$

We remark again that  $B_L$  is just a Fourier multiplier, and hence commutes with  $\Delta_L^{-1}$ . One can explicitly write its symbol and the symbol of its inverse operator, while the lower and upper bounds were already provided in (2.20).

System (3.1) is a non-autonomous dynamical system for each fixed frequency  $(k, \eta)$ . Although the proof carried out in the next Section 4 for more general shears applies in this case as well, we consider a different point of view, and argue *point-wise* in both  $k$  and  $\eta$ . From now on, we slightly abuse notation and identify  $\Theta_k$  and  $Q_k$  with their  $Y$ -Fourier transforms  $\hat{\Theta}_k$  and  $\hat{Q}_k$ . Our result reads as follows.

**THEOREM 3.1.** *Let  $\beta \geq 0$  and  $k \neq 0$ . The solution to (3.1) satisfies the uniform bounds*

$$|p^{-\frac{1}{4}} \Theta_k(t)|^2 + |p^{\frac{1}{4}} Q_k(t)|^2 \approx |(k^2 + \eta^2)^{-\frac{1}{4}} \Theta_k(0)|^2 + |(k^2 + \eta^2)^{\frac{1}{4}} Q_k(0)|^2, \quad \forall t \geq 0, \tag{3.2}$$

*point-wise in  $\eta \in \mathbb{R}$ . In particular, thanks to (2.20),*

$$|p^{-\frac{1}{4}} \Omega_k(t)|^2 + |p^{\frac{1}{4}} Q_k(t)|^2 \approx |(k^2 + \eta^2)^{-\frac{1}{4}} \Omega_k(0)|^2 + |(k^2 + \eta^2)^{\frac{1}{4}} Q_k(0)|^2, \quad \forall t \geq 0, \tag{3.3}$$

*point-wise in  $\eta \in \mathbb{R}$ .*

The pre-factors  $p^{\pm\frac{1}{4}}$  in (3.3) appear in view of a natural symmetrization of system (3.1) that we carry out in the next Section 3.1. Now we show that the proof of the main Theorem 1.1 in the case the Couette flow (namely,  $\varepsilon = 0$ ) follows from (3.2). Recall that from (1.3) the velocity field (in the moving frame) is given by

$$\mathbf{V}_k = (V_k^x, V_k^y) = (-(\partial_Y - t\partial_X)\Delta_L^{-1}\Omega_k, \partial_X\Delta_L^{-1}\Omega_k).$$

Thus, using that  $|\eta - kt|^2 p^{-1} \leq 1$ , from (3.3) we deduce

$$\begin{aligned} \|V_k^x(t)\|_{L^2}^2 &= \int_{\mathbb{R}} \frac{|\eta - kt|^2}{p^{\frac{3}{2}}\langle(k, \eta)\rangle} |p^{-\frac{1}{4}}\Omega_k(t)|^2 \langle(k, \eta)\rangle d\eta \leq \int_{\mathbb{R}} \frac{1}{p^{\frac{1}{2}}\langle(k, \eta)\rangle} |p^{-\frac{1}{4}}\Omega_k(t)|^2 \langle(k, \eta)\rangle d\eta \\ &\lesssim \frac{1}{\langle kt \rangle} [\|\Omega_k(0)\|_{L^2}^2 + \|Q_k(0)\|_1^2]. \end{aligned}$$

Similarly,

$$\|V_k^y(t)\|_{L^2}^2 \lesssim \frac{1}{\langle kt \rangle^3} [\|\Omega_k(0)\|_{\frac{1}{4}}^2 + \|Q_k(0)\|_{\frac{5}{4}}^2],$$

and

$$\|Q_k(t)\|_{L^2}^2 \lesssim \frac{1}{\langle kt \rangle} [\|\Omega_k(0)\|_{L^2}^2 + \|Q_k(0)\|_1^2].$$

Thus, Theorem 1.1 is proven.

REMARK 3.2 (On the Corollary 1.2). Just by looking at the lower bound in (3.3), we have

$$|p^{-\frac{1}{4}}\Omega_k(t)|^2 \gtrsim |(k^2 + \eta^2)^{-\frac{1}{4}}\Omega_k(0)|^2 + |(k^2 + \eta^2)^{\frac{1}{4}}Q_k(0)|^2 := \Xi_k(0).$$

The latter estimate shows that

$$|\Omega_k(t)| \gtrsim (k^2 + (\eta - kt)^2)^{\frac{1}{4}} \Xi_k(0), \quad (3.4)$$

which in particular implies that the vorticity experiences a *transient decay* for  $t < \eta/k$  after which it begins to grow. This is somehow the contrary of what happens in the case of the Couette flow with constant density and shares analogies with the compressible case, see [2]. Clearly, (3.4) also implies a Lyapunov instability of the vorticity in  $L^2$ , in fact we have

$$\|\Omega(t)\|_{L^2} \gtrsim \langle t \rangle^{1/2} \|\Xi(0)\|_{\frac{1}{2}}.$$

Anyhow, even in the presence of instability for the vorticity, we still have the inviscid damping for the velocity field. This proves Corollary 1.2.

**3.1. Symmetric variables.** Since (3.1) decouples in the  $X$ -Fourier variable, we fix a nonzero integer  $k$  and define the auxiliary variables

$$Z_1 := p^{-\frac{1}{4}}\Theta_k, \quad Z_2 := p^{\frac{1}{4}}i\sqrt{R}Q_k. \quad (3.5)$$

Taking into account (2.24), we then find

$$\partial_t \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4}\frac{p'}{p} & -k\sqrt{R}p^{-\frac{1}{2}} \\ k\sqrt{R}p^{-\frac{1}{2}} & \frac{1}{4}\frac{p'}{p} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} + \begin{pmatrix} \beta\frac{ik}{p}B_L & 0 \\ k\sqrt{R}p^{-\frac{1}{2}}(B_L - 1) & 0 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}. \quad (3.6)$$

The first matrix in (3.6) is what determines most of the behavior of the system, while the second one is considered as a remainder (i.e. time-integrable). To exploit this structure, we define the point-wise energy functional as

$$E(t) = \frac{1}{2} \left[ |Z_1(t)|^2 + |Z_2(t)|^2 + \frac{1}{2k\sqrt{R}} \operatorname{Re} \left( p' p^{-\frac{1}{2}} Z_1(t) \overline{Z_2(t)} \right) \right]. \quad (3.7)$$

The last term is simply coming from the ratio between the diagonal and non-diagonal entries in the first matrix in (3.6). Now using that  $|p'|p^{-\frac{1}{2}} \leq 2|k|$ , we deduce that

$$\frac{1}{2|k|\sqrt{R}} \left| p' p^{-\frac{1}{2}} Z_1 \overline{Z_2} \right| \leq \frac{1}{\sqrt{R}} |Z_1| |Z_2| \leq \frac{1}{2\sqrt{R}} (|Z_1|^2 + |Z_2|^2).$$

As a consequence, the functional  $E$  is coercive whenever  $R > 1/4$ , namely

$$\frac{1}{2} \left( 1 - \frac{1}{2\sqrt{R}} \right) [|Z_1|^2 + |Z_2|^2] \leq E \leq \frac{1}{2} \left( 1 + \frac{1}{2\sqrt{R}} \right) [|Z_1|^2 + |Z_2|^2]. \quad (3.8)$$

We now compute the time-derivative of  $E$ . We first notice that

$$\frac{1}{2} \frac{d}{dt} |Z_1|^2 = \operatorname{Re}(\partial_t Z_1 \overline{Z_1}) = -\frac{1}{4} \frac{p'}{p} |Z_1|^2 - \beta \frac{k}{p} \operatorname{Im}(B_L) |Z_1|^2 - k\sqrt{R} p^{-\frac{1}{2}} \operatorname{Re}(Z_1 \overline{Z_2}).$$

Similarly,

$$\frac{1}{2} \frac{d}{dt} |Z_2|^2 = \frac{1}{4} \frac{p'}{p} |Z_2|^2 + k\sqrt{R} p^{-\frac{1}{2}} \operatorname{Re}(Z_1 \overline{Z_2}) + k\sqrt{R} p^{-\frac{1}{2}} \operatorname{Re}((B_L - 1) Z_1 \overline{Z_2}).$$

Regarding the last term, we find that

$$\begin{aligned} \frac{d}{dt} \operatorname{Re} \left( p' p^{-\frac{1}{2}} Z_1 \overline{Z_2} \right) &= \left( p' p^{-\frac{1}{2}} \right)' \operatorname{Re} (Z_1 \overline{Z_2}) + \beta k \frac{p'}{p^{\frac{3}{2}}} \operatorname{Re} (i B_L Z_1 \overline{Z_2}) \\ &\quad - k\sqrt{R} \frac{p'}{p} [|Z_2|^2 - |Z_1|^2] + k\sqrt{R} \frac{p'}{p} \operatorname{Re}(B_L - 1) |Z_1|^2. \end{aligned}$$

Therefore, the energy function  $E$  satisfies the equation

$$\frac{d}{dt} E = \mathcal{C}.$$

where

$$\begin{aligned} \mathcal{C} &:= -\beta \frac{k}{p} \operatorname{Im}(B_L) |Z_1|^2 + k\sqrt{R} p^{-\frac{1}{2}} \operatorname{Re}((B_L - 1) Z_1 \overline{Z_2}) + \frac{1}{4k\sqrt{R}} \left( p' p^{-\frac{1}{2}} \right)' \operatorname{Re} (Z_1 \overline{Z_2}) \\ &\quad + \frac{\beta}{4\sqrt{R}} \frac{p'}{p^{\frac{3}{2}}} \operatorname{Re} (i B_L Z_1 \overline{Z_2}) + \frac{1}{4} \frac{p'}{p} \operatorname{Re}(B_L - 1) |Z_1|^2. \end{aligned} \quad (3.9)$$

We now proceed to show that  $\mathcal{C}$  has good time-integrability, and therefore we can close a good Grönwall estimate.

**3.2. Proof of Theorem 3.1.** The goal of this section is to properly estimate the error terms in the above (3.9). We will use the following properties of  $B_L$ .

$$\begin{aligned} |\operatorname{Im}(B_L)| &= \beta |\eta - kt| p^{-\frac{1}{2}} \leq \beta p^{-\frac{1}{2}}, \\ |\operatorname{Re}(B_L - 1)| &\leq \beta^2 |\eta - kt|^2 p^{-2} \leq \beta^2 p^{-1}, \end{aligned} \quad (3.10)$$

$$|B_L - 1| \leq \beta(1 + \beta) p^{-\frac{1}{2}}. \quad (3.11)$$

This yields

$$\left| \beta \frac{k}{p} \operatorname{Im}(B_L) |Z_1|^2 \right| \leq \beta \frac{|k|}{p} |Z_1|^2.$$

Similarly, since  $|p'p^{-1/2}| \leq 2|k|$  and  $|B_L| \leq 1$ , one can check that

$$\left| \frac{\beta}{4\sqrt{R}} \frac{p'}{p^{3/2}} \operatorname{Re}(iB_L Z_1 \overline{Z_2}) \right| \leq \frac{\beta|k|}{4p\sqrt{R}} [|Z_1|^2 + |Z_2|^2] \leq \beta \frac{|k|}{p} [|Z_1(t)|^2 + |Z_2(t)|^2].$$

Appealing to (3.10), we also have the following bound for the second term of  $\mathcal{C}$ ,

$$|k| \sqrt{R} p^{-1/2} |\operatorname{Re}((B_L - 1)Z_1 \overline{Z_2})| \leq \beta^2 \sqrt{R} \frac{|k|}{p^{3/2}} [|Z_1|^2 + |Z_2|^2].$$

For the last term of  $\mathcal{C}$ , since  $|p'p^{-1/2}| \leq 2|k|$ , by using (3.10) we have

$$\frac{1}{4} \frac{|p'|}{p} |\operatorname{Re}(B_L - 1)| |Z_1|^2 \leq \beta^2 \frac{|k|}{p^{3/2}} |Z_1|^2.$$

Lastly, since  $0 < (p'p^{-1/2})' = 2|k|^4/p^{3/2} \leq 2|k|^3/p$ , it is not hard to conclude that

$$|\mathcal{C}| \lesssim \frac{\sqrt{R}(1 + \beta^2)|k|^2}{p} [|Z_1(t)|^2 + |Z_2(t)|^2]. \quad (3.12)$$

Thus, from (3.8) and (3.12) we deduce that

$$-\frac{1}{2\sqrt{R}-1} \frac{|k|^2}{p} E \lesssim \frac{d}{dt} E \lesssim \frac{1}{2\sqrt{R}-1} \frac{|k|^2}{p} E. \quad (3.13)$$

Now, since

$$\int_0^\infty \frac{|k|^2}{p} dt \leq \frac{\pi}{2},$$

we can apply the Grönwall lemma to (3.13) and obtain

$$E(t) \approx E(0).$$

This translates immediately into (3.2) thanks to (3.8), thereby concluding the proof of Theorem 3.1.

#### 4. Shears close to Couette

This section deals with the full system (2.9). Using the expressions of  $\Delta_t^{-1}$  and  $B_t$  in (2.15)-(2.16) and (2.24)-(2.25), the system reads

$$\partial_t \Theta_k = -ikRQ_k + ik(b(Y) - \beta g(Y)) \Delta_t^{-1} B_t \Theta_k, \quad (4.1)$$

$$\partial_t Q_k = ik\Delta_L^{-1} \Theta_k + ik\Delta_L^{-1} (B_L - 1) \Theta_k + ik\Delta_L^{-1} T_\varepsilon T_L B_L \Theta_k + ik\Delta_t^{-1} B_\varepsilon B_t \Theta_k. \quad (4.2)$$

We have expanded the right-hand side above to highlight the similarities with the Couette flow (3.1). In the first equation (4.1), the second term in the right-hand side is treated as an error term, just like in the previous section. In the second equation (4.2), we have expanded the operator  $\Delta_t^{-1}$  and  $B_t$  in order to extract the Couette-like structure, while all the other terms are treated as errors. Shear flows near Couette are still stable in  $H^s$ , as stated in the following result.

**THEOREM 4.1.** *Let  $R > 1/4$ ,  $\beta \geq 0$  and  $s \geq 0$  be fixed. There exist  $\varepsilon_0 \in (0, 1)$  with the following property. If  $\varepsilon \in (0, \varepsilon_0]$  and*

$$\|g - 1\|_{s+5} + \|b\|_{s+4} \leq \varepsilon,$$

*then for every  $k \neq 0$  the solution to (4.1)-(4.2) satisfies the uniform  $H^s$ -bound*

$$\left\| p^{-\frac{1}{4}-\delta} \Theta_k(t) \right\|_s^2 + \left\| p^{\frac{1}{4}-\delta} Q_k(t) \right\|_s^2 \lesssim \|\Theta_k(0)\|_{s-\frac{1}{2}}^2 + \|Q_k(0)\|_{s+\frac{1}{2}}^2, \quad \forall t \geq 0, \quad (4.3)$$

*where  $\delta = \sqrt{\varepsilon}$  and  $p$  is given by (2.11).*

The bound on the vorticity follows directly.

COROLLARY 4.2. *Under the same assumptions of Theorem 4.1, for  $k \neq 0$  the following inequality holds true*

$$\left\| p^{-\frac{1}{4}-\delta} \Omega_k(t) \right\|_s^2 + \left\| p^{\frac{1}{4}-\delta} Q_k(t) \right\|_s^2 \lesssim \left\| \Omega_k(0) \right\|_{s-\frac{1}{2}}^2 + \left\| Q_k(0) \right\|_{s+\frac{1}{2}}^2, \quad \forall t \geq 0.$$

Appealing to Theorem 4.1 and Corollary 4.2, which will be proved throughout the paper, we first prove Theorem 1.1.

PROOF OF THEOREM 1.1. The components of the velocity field in the coordinates driven by the flow read

$$V_k^x = -g(Y)(\partial_Y - t\partial_X)\Delta_t^{-1}\Omega_k, \quad V_k^y = \partial_X\Delta_t^{-1}\Omega_k.$$

In order to prove (1.5), since  $\Delta_t^{-1} = \Delta_L^{-1}T_L = \Delta_L^{-1} + \Delta_t^{-1}T_\varepsilon T_L$  as in (2.15), we write

$$\begin{aligned} \|Q_k(t)\|_{L^2} + \|V_k^x(t)\|_{L^2} &\lesssim \left\| p^{-\frac{1}{4}+\delta} \left( p^{\frac{1}{4}-\delta} Q_k(t) \right) \right\|_{L^2} + \left\| \eta - kt \left| p^{-1} p^{\frac{1}{4}+\delta} \left( p^{-\frac{1}{4}-\delta} T_L \Omega_k(t) \right) \right\|_{L^2} \right. \\ &\lesssim \frac{1}{\langle kt \rangle^{\frac{1}{2}-2\delta}} \left( \left\| p^{\frac{1}{4}-\delta} Q_k(t) \right\|_{\frac{1}{2}-2\delta} + \left\| p^{-\frac{1}{4}-\delta} T_L \Omega_k(t) \right\|_{\frac{1}{2}-2\delta} \right). \end{aligned} \quad (4.4)$$

We need to control the last term in the right-hand side of the previous equation. Thanks to (2.16) we know that  $T_L = I + T_\varepsilon T_L$ , where  $T_\varepsilon$  is defined in (2.13). Therefore

$$\left\| p^{-\frac{1}{4}-\delta} T_L \Omega_k(t) \right\|_{\frac{1}{2}-2\delta} \leq \left\| p^{-\frac{1}{4}-\delta} \Omega_k(t) \right\|_{\frac{1}{2}-2\delta} + \left\| p^{-\frac{1}{4}-\delta} T_\varepsilon T_L \Omega_k(t) \right\|_{\frac{1}{2}-2\delta} \quad (4.5)$$

Now we use inequality (4.19) in Lemma 4.6, which gives

$$p^{-\frac{1}{4}-\delta}(t; k, \eta) \lesssim \langle \eta - \xi \rangle^{\frac{1}{2}+2\delta} p^{-\frac{1}{4}-\delta}(t; k, \xi),$$

to exchange frequencies in the last term of the right-hand side of (4.5), so that, using also  $|p'| \leq 2|k|\sqrt{p}$ , we obtain

$$\begin{aligned} \left\| p^{-\frac{1}{4}-\delta} T_\varepsilon T_L \Omega_k \right\|_{\frac{1}{2}-2\delta} &= \left\| p^{-\frac{1}{4}-\delta} \left( \widehat{g^2 - 1} * \frac{|p'|^2}{4k^2 p} \widehat{T_L \Omega_k} \right) \right\|_{\frac{1}{2}-2\delta} + \left\| p^{-\frac{1}{4}-\delta} \left( \widehat{b} * \frac{p'}{2kp} \widehat{T_L \Omega_k} \right) \right\|_{\frac{1}{2}-2\delta} \\ &\lesssim \left\| \langle \xi \rangle^{\frac{1}{2}+2\delta+s} |\widehat{g^2 - 1}| * p^{-\frac{1}{4}-\delta} \langle (k, \eta) \rangle^{\frac{1}{2}-2\delta} \widehat{T_L \Omega_k} \right\|_{L^2} \\ &\quad + \left\| \langle \xi \rangle^{\frac{1}{2}+2\delta+s} \widehat{b} * p^{-\frac{1}{4}-\delta} \langle (k, \eta) \rangle^{\frac{1}{2}-2\delta} \widehat{T_L \Omega_k} \right\|_{L^2}. \end{aligned}$$

Thanks to Young convolution inequality, by combining the previous bound with (4.5) we infer

$$\left\| p^{-\frac{1}{4}-\delta} T_L \Omega_k \right\|_{\frac{1}{2}-2\delta} \lesssim \left\| p^{-\frac{1}{4}-\delta} \Omega_k \right\|_{\frac{1}{2}-2\delta} + \varepsilon \left\| p^{-\frac{1}{4}-\delta} T_L \Omega_k \right\|_{\frac{1}{2}-2\delta},$$

namely

$$\left\| p^{-\frac{1}{4}-\delta} T_L \Omega_k \right\|_{\frac{1}{2}-2\delta} \lesssim \left\| p^{-\frac{1}{4}-\delta} \Omega_k \right\|_{\frac{1}{2}-2\delta}.$$

This implies, using Corollary 4.2, that (4.4) gives

$$\begin{aligned} \|Q_k(t)\|_{L^2} + \|V_k^x(t)\|_{L^2} &\lesssim \frac{1}{\langle kt \rangle^{\frac{1}{2}-2\delta}} \left( \left\| p^{\frac{1}{4}-\delta} Q_k(t) \right\|_{\frac{1}{2}-2\delta} + \left\| p^{-\frac{1}{4}-\delta} T_L \Omega_k(t) \right\|_{\frac{1}{2}-2\delta} \right) \\ &\lesssim \frac{1}{\langle kt \rangle^{\frac{1}{2}-2\delta}} (\|\Omega_k(0)\|_{-2\delta} + \|Q_k(0)\|_{1-2\delta}). \end{aligned}$$

Applying the same reasoning again, we get

$$\|V_k^y(t)\|_{L^2} \lesssim \frac{1}{\langle kt \rangle^{\frac{3}{2}-2\delta}} \left[ \|\Omega_k(0)\|_{\frac{1}{4}-2\delta} + \|Q_k(0)\|_{\frac{5}{4}-2\delta} \right].$$

Hence, by summing up in  $k$  and defining  $\delta_\varepsilon = 2\delta = 2\sqrt{\varepsilon}$ , the proof of Theorem 1.1 is concluded.  $\square$



**4.1. Symmetric variables and the energy functional.** Since (4.1)-(4.2) decouples in the  $X$ -Fourier variable, we fix a nonzero integer  $k$  and define the auxiliary variables

$$Z_1 := m^{-1}p^{-\frac{1}{4}}\Theta_k, \quad Z_2 := m^{-1}p^{\frac{1}{4}}i\sqrt{R}Q_k, \quad (4.6)$$

with  $m = m(t; k, \eta)$  a positive weight, that will be specified later, such that  $m' > 0$ . The choice made in (4.6), up to the weight  $m$  is exactly the one of the Couette case, see (3.5). In fact, one immediately sees that  $Z_1, Z_2$  satisfy

$$\partial_t \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4}\frac{p'}{p} & -k\sqrt{R}p^{-\frac{1}{2}} \\ k\sqrt{R}p^{-\frac{1}{2}} & \frac{1}{4}\frac{p'}{p} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} - \frac{m'}{m} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} + \mathcal{R}(t) \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix},$$

where  $\mathcal{R}(t)$  will be treated as a perturbation being indeed a matrix of remainders. The error terms which are hidden (up to now) in  $\mathcal{R}(t)$  need to be controlled by means of the artificial dissipation introduced by the weight  $m$ .

Given a real number  $s \geq 0$ , we define the energy functional as

$$E_s(t) = \frac{1}{2} \left[ \|Z_1(t)\|_s^2 + \|Z_2(t)\|_s^2 + \frac{1}{2k\sqrt{R}} \operatorname{Re} \langle p'p^{-\frac{1}{2}}Z_1(t), Z_2(t) \rangle_s \right],$$

which is exactly the one used for the Couette case, see (3.7), when integrated in the variable  $\eta$ .

Now using that  $|p'|p^{-\frac{1}{2}} \leq 2|k|$ , we deduce that

$$\frac{1}{2|k|\sqrt{R}} \left| \langle p'p^{-\frac{1}{2}}Z_1, Z_2 \rangle_s \right| \leq \frac{1}{\sqrt{R}} \|Z_1\|_s \|Z_2\|_s \leq \frac{1}{2\sqrt{R}} (\|Z_1\|_s^2 + \|Z_2\|_s^2).$$

As a consequence, the functional is coercive whenever  $R > 1/4$ , namely

$$\frac{1}{2} \left( 1 - \frac{1}{2\sqrt{R}} \right) [\|Z_1\|_s^2 + \|Z_2\|_s^2] \leq E_s \leq \frac{1}{2} \left( 1 + \frac{1}{2\sqrt{R}} \right) [\|Z_1\|_s^2 + \|Z_2\|_s^2].$$

The rest of the paper aims at proving that  $t \rightarrow E_s(t)$  is non-increasing.

**LEMMA 4.3.** *The functional  $E_s(t)$  satisfies*

$$\frac{d}{dt} E_s + \left( 1 - \frac{1}{2\sqrt{R}} \right) \left[ \left\| \sqrt{\frac{m'}{m}} Z_1 \right\|_s^2 + \left\| \sqrt{\frac{m'}{m}} Z_2 \right\|_s^2 \right] \leq \frac{1}{4|k|\sqrt{R}} \left| \langle (p'p^{-\frac{1}{2}})' Z_1, Z_2 \rangle_s \right| + \sum_{i=1}^8 \mathcal{R}_i, \quad (4.7)$$

where the error terms  $\mathcal{R}_i$  are

$$\begin{aligned}
\mathcal{R}_1 &= |k| \left| \langle Z_1, m^{-1} p^{-\frac{1}{4}} ((b - \beta g) \Delta_L^{-1} T_L T_B B_L \Theta) \rangle_s \right|, \\
\mathcal{R}_2 &= |k| \sqrt{R} \left| \langle p^{-\frac{1}{2}} (B_L - 1) Z_1, Z_2 \rangle_s \right|, \\
\mathcal{R}_3 &= |k| \sqrt{R} \left| \langle m^{-1} p^{-\frac{3}{4}} T_\varepsilon T_L B_L \Theta, Z_2 \rangle_s \right|, \\
\mathcal{R}_4 &= |k| \sqrt{R} \left| \langle m^{-1} p^{-\frac{3}{4}} T_L B_\varepsilon T_B B_L \Theta, Z_2 \rangle_s \right|, \\
\mathcal{R}_5 &= \frac{1}{4\sqrt{R}} \left| \langle p' p^{-\frac{3}{4}} m^{-1} ((b - \beta g) \Delta_L^{-1} T_L T_B B_L \Theta), Z_2 \rangle_s \right|, \\
\mathcal{R}_6 &= \frac{1}{4} \left| \langle Z_1, \frac{p'}{p} (B_L - 1) Z_1 \rangle_s \right|, \\
\mathcal{R}_7 &= \frac{1}{4} \left| \langle \frac{p'}{p} Z_1, m^{-1} p^{-\frac{1}{4}} T_\varepsilon T_L B_L \Theta \rangle_s \right|, \\
\mathcal{R}_8 &= \frac{1}{4} \left| \langle \frac{p'}{p} Z_1, m^{-1} p^{-\frac{1}{4}} T_L B_\varepsilon T_B B_L \Theta \rangle_s \right|.
\end{aligned}$$

PROOF. Taking the time-derivative of the functional, from  $Z_1$  we have

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|Z_1\|_s^2 &= - \left\| \sqrt{\frac{m'}{m}} Z_1 \right\|_s^2 - \frac{1}{4} \langle Z_1, \frac{p'}{p} Z_1 \rangle - k\sqrt{R} \operatorname{Re} \langle Z_1, p^{-\frac{1}{2}} Z_2 \rangle_s \\
&\quad + \operatorname{Re} \langle Z_1, i k m^{-1} p^{-\frac{1}{4}} ((b - \beta g) \Delta_L^{-1} T_L T_B B_L \Theta) \rangle_s.
\end{aligned} \tag{4.8}$$

About  $Z_2$ ,

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|Z_2\|_s^2 &= - \left\| \sqrt{\frac{m'}{m}} Z_2 \right\|_s^2 + \frac{1}{4} \langle Z_2, \frac{p'}{p} Z_2 \rangle_s + k\sqrt{R} \operatorname{Re} \langle p^{-\frac{1}{2}} Z_1, Z_2 \rangle_s \\
&\quad + k\sqrt{R} \operatorname{Re} \langle p^{-\frac{1}{2}} (B_L - 1) Z_1, Z_2 \rangle_s + k\sqrt{R} \operatorname{Re} \langle m^{-1} p^{-\frac{3}{4}} T_\varepsilon T_L B_L \Theta, Z_2 \rangle_s \\
&\quad + k\sqrt{R} \operatorname{Re} \langle m^{-1} p^{-\frac{3}{4}} T_L B_\varepsilon T_B B_L \Theta, Z_2 \rangle_s.
\end{aligned} \tag{4.9}$$

For the mixed term we have

$$\begin{aligned}
\frac{d}{dt} \operatorname{Re} \langle p' p^{-\frac{1}{2}} Z_1, Z_2 \rangle_s &= \operatorname{Re} \left( \left( p' p^{-\frac{1}{2}} \right)' Z_1, Z_2 \right)_s - 2 \langle p' p^{-\frac{1}{2}} m' m^{-1} Z_1, Z_2 \rangle_s - k\sqrt{R} \langle Z_2, \frac{p'}{p} Z_2 \rangle_s \\
&\quad + \operatorname{Re} \langle i k p' p^{-\frac{3}{4}} m^{-1} ((b - \beta g) \Delta_L^{-1} T_L B_t \Theta), Z_2 \rangle_s \\
&\quad + k\sqrt{R} \langle Z_1, \frac{p'}{p} Z_1 \rangle_s \\
&\quad + k\sqrt{R} \operatorname{Re} \langle Z_1, \frac{p'}{p} (B_L - 1) Z_1 \rangle_s + k\sqrt{R} \operatorname{Re} \langle \frac{p'}{p} Z_1, m^{-1} p^{-\frac{1}{4}} T_\varepsilon T_L B_L \Theta \rangle_s \\
&\quad + k\sqrt{R} \operatorname{Re} \langle \frac{p'}{p} Z_1, m^{-1} p^{-\frac{1}{4}} T_L B_\varepsilon T_B B_L \Theta \rangle_s.
\end{aligned} \tag{4.10}$$

Now, notice that the sum of the last term on the right-hand side of (4.8) and the last term on the right-hand side of (4.9) is zero. Next, the last term of (4.10) multiplied by  $1/(4k\sqrt{R})$  and the second term in the right-hand side of (4.9) balance each other, while the sum of (4.11) multiplied by  $1/(4k\sqrt{R})$  and the second term in the right-hand side of (4.8) vanishes. Thus, we end up with

$$\begin{aligned} \frac{d}{dt}E_s(t) + \left\| \sqrt{\frac{m'}{m}} Z_1 \right\|_s^2 + \left\| \sqrt{\frac{m'}{m}} Z_2 \right\|_s^2 \\ \leq \frac{1}{4|k|\sqrt{R}} \left| \left\langle \left( p' p^{-\frac{1}{2}} \right)' Z_1, Z_2 \right\rangle_s \right| + \frac{1}{2|k|\sqrt{R}} \left| \left\langle p' p^{-\frac{1}{2}} m' m^{-1} Z_1, Z_2 \right\rangle_s \right| + \sum_{i=1}^8 \mathcal{R}_i. \end{aligned}$$

This way, using that  $|p' p^{-\frac{1}{2}}| \leq 2|k|$ , we get

$$\frac{1}{2|k|\sqrt{R}} \left| \left\langle p' p^{-\frac{1}{2}} m' m^{-1} Z_1, Z_2 \right\rangle_s \right| \leq \frac{1}{2\sqrt{R}} \left( \left\| \sqrt{\frac{m'}{m}} Z_1 \right\|_s^2 + \left\| \sqrt{\frac{m'}{m}} Z_2 \right\|_s^2 \right),$$

and the proof is concluded.  $\square$

**4.2. Choice of weights and their properties.** The crucial observation for a proper choice of the weights is that we need an additional (uniformly bounded) weight to control the error terms generated by the various  $\mathcal{R}_i$ 's. More precisely, as it will be clarified later on, the time-derivative of this new weight has to be comparable with  $|p'|/p$ . This implies that the decay rates proven in Theorem 1.1 have a small  $\delta$ -correction compared to those in Couette. For these reasons, the weight  $m$  in the definition of  $Z_1$  and  $Z_2$  in (4.6) now takes the form

$$m = m_1 w^\delta, \quad (4.12)$$

where we recall that  $\delta = \sqrt{\epsilon}$ . The weight  $w$  encodes the decay correction, so that

$$\frac{w'}{w} = \frac{1}{4} \frac{|p'|}{p}, \quad w|_{t=0} = 1. \quad (4.13)$$

The choice of the weight  $m_1$ ,

$$\frac{m'_1}{m_1} = C_\beta \frac{|k|^2}{p}, \quad m_1|_{t=0} = 1, \quad (4.14)$$

where the constant  $C_\beta$  is given by

$$C_\beta = 256\sqrt{R} \left( \frac{2\sqrt{R}}{2\sqrt{R}-1} \right) (1 + \beta^2), \quad (4.15)$$

plays a role in controlling the right-hand side of (4.7), it has indeed the same form of (3.12) in the Couette case. It is worth pointing out that  $C_\beta$  blows up as  $R \rightarrow 1/4$ . Notice that

$$\frac{m'}{m} = \delta \frac{w'}{w} + \frac{m'_1}{m_1}. \quad (4.16)$$

Explicitly, for  $w$  we have

$$w(t; k, \eta) = \begin{cases} \left( \frac{k^2 + \eta^2}{p(t; k, \eta)} \right)^{\frac{1}{4}}, & t < \frac{\eta}{k}, \\ \left( \frac{(k^2 + \eta^2)p(t; k, \eta)}{k^4} \right)^{\frac{1}{4}}, & t \geq \frac{\eta}{k}, \end{cases} \quad (4.17)$$

while

$$m_1(t; k, \eta) = \exp \left[ C_\beta \left( \arctan \left( \frac{\eta}{k} - t \right) - \arctan \left( \frac{\eta}{k} \right) \right) \right]. \quad (4.18)$$

In particular, the weight  $m_1$  is uniformly bounded.

REMARK 4.4. The weight  $m_1$  is standard in the case of incompressible Euler/Navier-Stokes with constant density [6, 39, 40], while the weight  $w$  has been used in the compressible case in [2].

The goal of the remaining part of this section is to prove the following proposition.

**PROPOSITION 4.5.** *Let  $R > 1/4$ ,  $\beta \geq 0$  and  $s \geq 0$  be fixed. There exist  $\varepsilon_0 \in (0, 1)$  with the following property. If  $\varepsilon \in (0, \varepsilon_0]$  and*

$$\|g - 1\|_{s+5} + \|b\|_{s+4} \leq \varepsilon,$$

*then*

$$\frac{d}{dt}E_s + \frac{1}{4} \left(1 - \frac{1}{2\sqrt{R}}\right) \left[ \left\| \sqrt{\frac{m'}{m}} Z_1 \right\|_s^2 + \left\| \sqrt{\frac{m'}{m}} Z_2 \right\|_s^2 \right] \leq 0,$$

*for every  $t \geq 0$ .*

Theorem 4.1 follows from the above proposition (see Section 4.4). The proof of this proposition is a consequence of the properties of the chosen weights (see Lemmas 4.6 and 4.8 below), and the estimates on the various error terms, postponed in the next Section 4.3. We begin by computing how much it costs to exchange weights in the various convolutions appearing in the error terms.

**LEMMA 4.6.** *Let  $k \neq 0$  and  $t \geq 0$  be fixed. For any  $\eta, \xi \in \mathbb{R}$  we have*

$$p^{-1}(t; k, \eta) \lesssim \langle \eta - \xi \rangle^2 p^{-1}(t; k, \xi), \quad (4.19)$$

$$\frac{|p'|}{p}(t; k, \eta) \lesssim \langle \eta - \xi \rangle^2 \frac{|p'|}{p}(t; k, \eta) + |k| \langle \eta - \xi \rangle^3 p^{-1}(t; k, \eta), \quad (4.20)$$

$$m^{-1}(t; k, \eta) \lesssim \langle \eta - \xi \rangle^\delta m^{-1}(t; k, \xi). \quad (4.21)$$

**PROOF.** In the course of the proof, we omit the dependency of all the quantities on  $k$  and  $t$ . Starting with (4.19), the inequality to prove is equivalent to

$$\left\langle \frac{\xi}{k} - t \right\rangle^2 \lesssim \left\langle \frac{\eta - \xi}{k} \right\rangle^2 \left\langle \frac{\eta}{k} - t \right\rangle^2.$$

Choosing  $a = \frac{\xi}{k} - t$  and  $b = \frac{\eta}{k} - t$ , this follows from the general inequality

$$\langle a \rangle \lesssim \langle a - b \rangle \langle b \rangle.$$

Turning to (4.20), we write

$$\begin{aligned} \frac{|p'|}{p}(\eta) &= \frac{2|\frac{\eta}{k} - t|}{\langle \frac{\eta}{k} - t \rangle^2} \leq \frac{2|\frac{\xi}{k} - t| + 2|\frac{\xi}{k} - \frac{\eta}{k}|}{\langle \frac{\eta}{k} - t \rangle^2} \\ &\leq \left[ 2|\frac{\xi}{k} - t| + 2|\frac{\xi}{k} - \frac{\eta}{k}| \right] \left\langle \frac{\eta}{k} - \frac{\xi}{k} \right\rangle^2 \frac{1}{\langle \frac{\xi}{k} - t \rangle^2} \\ &\leq \left[ 2|\frac{\xi}{k} - t| + 2|\frac{\xi}{k} - \frac{\eta}{k}| \right] \langle \eta - \xi \rangle^2 k^2 p^{-1}(\xi) \\ &\lesssim \frac{|p'|}{p}(\xi) \langle \eta - \xi \rangle^2 + |k| \langle \eta - \xi \rangle^3 p^{-1}(\xi). \end{aligned}$$

We now prove (4.21). Recalling that

$$m^{-1} = (w^\delta m_1)^{-1},$$

where  $w, m_1$  are is by (4.17), (4.18) and, since  $m_1$  is uniformly bounded, then

$$(m_1)^{-1}(\eta) \lesssim (m_1)^{-1}(\xi).$$

Thus we deal with  $w^{-\delta}$ . We know from (4.19) that

$$p^{-1}(\xi) \lesssim \langle \eta - \xi \rangle^2 p^{-1}(\eta).$$

Therefore

$$p(\eta) \lesssim \langle \eta - \xi \rangle^2 p(\xi).$$

Notice that

$$\frac{p(\eta)}{k^2 + \eta^2} \lesssim \langle \eta - \xi \rangle^4 \frac{p(\xi)}{k^2 + \xi^2}.$$

Recalling the expression of  $w$  in (4.17) for  $t < \frac{\eta}{k}$ , it follows that

$$w^{-\delta}(\eta) \leq \langle \eta - \xi \rangle^\delta w^{-\delta}(\xi).$$

The complementing case  $t \geq \frac{\eta}{k}$  is analogous, and the proof is over.  $\square$

REMARK 4.7. It is worth pointing out that the constant value which is implicitly involved in the right-hand side of (4.21) (through the notation “ $\lesssim$ ”) is exactly  $e^{2C_\beta}$ , up to additional constants (independent of  $\varepsilon, \beta$ ). This exponential term  $e^{2C_\beta}$  comes directly from the explicit expression of the weight  $m_1$  in (4.18).

We also need to compute how much it costs to commute  $T_L, T_\varepsilon, T_B, B_\varepsilon$  with the weights.

LEMMA 4.8. *Let  $R > 1/4$ ,  $\beta \geq 0$  and  $s \geq 0$  be fixed. There exist  $\varepsilon_0 \in (0, 1)$  with the following property. If  $\varepsilon \in (0, \varepsilon_0]$  and*

$$\|g - 1\|_{s+5} + \|b\|_{s+4} \leq \varepsilon, \quad (4.22)$$

*then, for every smooth function  $f$  and every  $t \geq 0$ , the following estimates hold true*

$$\left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} T_\varepsilon f \right\|_s \lesssim \varepsilon \left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} f \right\|_s, \quad (4.23)$$

$$\left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} B_\varepsilon f \right\|_s \lesssim \varepsilon \left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} f \right\|_s, \quad (4.24)$$

and

$$\left\| \sqrt{\frac{w'}{w}} p^{-\frac{1}{4}} m^{-1} T_\varepsilon f \right\|_s \lesssim \varepsilon \left( \left\| \sqrt{\frac{w'}{w}} p^{-\frac{1}{4}} m^{-1} f \right\|_s + \left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} f \right\|_s \right), \quad (4.25)$$

$$\left\| \sqrt{\frac{w'}{w}} p^{-\frac{1}{4}} m^{-1} B_\varepsilon f \right\|_s \lesssim \varepsilon \left( \left\| \sqrt{\frac{w'}{w}} p^{-\frac{1}{4}} m^{-1} f \right\|_s + \left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} f \right\|_s \right). \quad (4.26)$$

In addition, the following inequalities hold

$$\left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} T_L f \right\|_s \leq 2 \left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} f \right\|_s, \quad (4.27)$$

$$\left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} T_B f \right\|_s \leq 2 \left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} f \right\|_s, \quad (4.28)$$

and

$$\left\| \sqrt{\frac{w'}{w}} p^{-\frac{1}{4}} m^{-1} T_L f \right\|_s \leq 2 \left( \left\| \sqrt{\frac{w'}{w}} p^{-\frac{1}{4}} m^{-1} f \right\|_s + \left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} f \right\|_s \right), \quad (4.29)$$

$$\left\| \sqrt{\frac{w'}{w}} p^{-\frac{1}{4}} m^{-1} T_B f \right\|_s \leq 2 \left( \left\| \sqrt{\frac{w'}{w}} p^{-\frac{1}{4}} m^{-1} f \right\|_s + \left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} f \right\|_s \right). \quad (4.30)$$

PROOF. We start with  $T_L, T_\varepsilon$ . In order to prove (4.23), we first use the operators  $T_\varepsilon^g$  and  $T_\varepsilon^b$  defined in (2.14) and the identity

$$T_\varepsilon = T_\varepsilon^g + T_\varepsilon^b$$

to obtain the bound

$$\left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} T_\varepsilon f \right\|_s \leq \left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} T_\varepsilon^g f \right\|_s + \left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} T_\varepsilon^b f \right\|_s. \quad (4.31)$$

Now, by Lemma 4.6 we find that

$$\left( \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} \right) (\eta) \lesssim \langle \xi - \eta \rangle^{\frac{3}{2} + \delta} \left( \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} \right) (\xi), \quad (4.32)$$

where we recall that the hidden constant in the right-hand side of the previous bound (4.32) is proportional to  $e^{2C_\beta}$ , see Remark 4.7. Therefore,

$$\begin{aligned} \left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} T_\varepsilon^g f \right\|_s &\leq \left\| \langle (k, \cdot) \rangle^s \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} |\widehat{g^2 - 1}| * |\widehat{f}| \right\|_s \\ &\lesssim \left\| \langle \cdot \rangle^{s + \frac{3}{2} + \delta} |\widehat{g^2 - 1}| * \langle (k, \cdot) \rangle^s \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} |\widehat{f}| \right\|_s \\ &\lesssim \|g^2 - 1\|_{s + \frac{5}{2} + \delta} \left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} f \right\|_s. \end{aligned}$$

In a similar fashion,

$$\left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} T_\varepsilon^b f \right\|_s \lesssim \|b\|_{s + \frac{5}{2} + \delta} \left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} f \right\|_s.$$

Plugging the above two bounds in (4.31) and using (4.22) we obtain (4.23). Looking at (4.25), we have again that

$$\left\| \sqrt{\frac{w'}{w}} p^{-\frac{1}{4}} m^{-1} T_\varepsilon f \right\|_s \leq \left\| \sqrt{\frac{w'}{w}} p^{-\frac{1}{4}} m^{-1} T_\varepsilon^g f \right\|_s + \left\| \sqrt{\frac{w'}{w}} p^{-\frac{1}{4}} m^{-1} T_\varepsilon^b f \right\|_s.$$

As above, Lemma 4.6 now gives us

$$\left( \sqrt{\frac{w'}{w}} p^{-\frac{1}{4}} m^{-1} \right) (\eta) \lesssim \langle \eta - \xi \rangle^{\frac{3}{2} + \delta} \left( \sqrt{\frac{w'}{w}} p^{-\frac{1}{4}} m^{-1} \right) (\xi) + \langle \eta - \xi \rangle^{2 + \delta} \left( \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} \right) (\xi).$$

Therefore,

$$\begin{aligned} \left\| \sqrt{\frac{w'}{w}} p^{-\frac{1}{4}} m^{-1} T_\varepsilon^g f \right\|_s &\leq \left\| \langle (k, \cdot) \rangle^s \sqrt{\frac{w'}{w}} p^{-\frac{1}{4}} m^{-1} |\widehat{g^2 - 1}| * |\widehat{f}| \right\|_s \\ &\lesssim \left\| \langle \cdot \rangle^{s + \frac{3}{2} + \delta} |\widehat{g^2 - 1}| * \langle (k, \cdot) \rangle^s \sqrt{\frac{w'}{w}} p^{-\frac{1}{4}} m^{-1} |\widehat{f}| \right\|_s \\ &\quad + \left\| \langle \cdot \rangle^{s + 2 + \delta} |\widehat{g^2 - 1}| * \langle (k, \cdot) \rangle^s \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} |\widehat{f}| \right\|_s \\ &\lesssim \|g^2 - 1\|_{s+3+\delta} \left[ \left\| \sqrt{\frac{w'}{w}} p^{-\frac{1}{4}} m^{-1} f \right\|_s + \left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} f \right\|_s \right]. \end{aligned}$$

The part with  $T_\varepsilon^b$  is similar, so that

$$\left\| \sqrt{\frac{w'}{w}} p^{-\frac{1}{4}} m^{-1} T_\varepsilon^b f \right\|_s \lesssim \|b\|_{s+3+\delta} \left[ \left\| \sqrt{\frac{w'}{w}} p^{-\frac{1}{4}} m^{-1} f \right\|_s + \left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} f \right\|_s \right].$$

By combining the previous two bounds we prove (4.25). About (4.27), since  $T_L = I + T_\varepsilon T_L$ , see (2.16), by using (4.23) we get

$$\left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} T_L f \right\|_s \leq \left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} f \right\|_s + C\varepsilon_0 \left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} T_L f \right\|_s,$$

where  $C$  is properly chosen so that the constant value which is hidden in the right-hand side of (4.23) is compensated. Therefore, for  $\varepsilon_0$  small enough we prove (4.27). To deal with (4.29), we combine (2.16) with (4.25) to get

$$\begin{aligned} \left\| \sqrt{\frac{w'}{w}} p^{-\frac{1}{4}} m^{-1} T_L f \right\|_s &\leq \left\| \sqrt{\frac{w'}{w}} p^{-\frac{1}{4}} m^{-1} f \right\|_s + C\varepsilon_0 \left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} T_L f \right\|_s \\ &\quad + C\varepsilon_0 \left\| \sqrt{\frac{w'}{w}} p^{-\frac{1}{4}} m^{-1} T_L f \right\|_s. \end{aligned}$$

For  $\varepsilon_0$  small, we can absorb the last term using the left-hand side. Applying (4.27) we end up with (4.29).



Now we deal with  $B_\varepsilon$  and  $T_B$ , starting with (4.24). From the definition of  $B_\varepsilon$  in (2.21), since  $|B_L| \leq 1$  as in (2.20), we get

$$\begin{aligned} \left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} B_\varepsilon f \right\|_s &\lesssim \left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} \left( \widehat{g-1} * \frac{p'}{2k} p^{-1} \widehat{f} \right) \right\|_s \\ &\quad + \left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} \frac{p'}{2k} p^{-1} \widehat{T_\varepsilon T_L f} \right\|_s \\ &\quad + \left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} \left( \widehat{g-1} * \frac{p'}{2k} p^{-1} \widehat{T_\varepsilon T_L f} \right) \right\|_s. \end{aligned} \quad (4.33)$$

Considering the term in the right-hand side of (4.33), by using (4.32) we get

$$\begin{aligned} \left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} \left( \widehat{g-1} * \frac{p'}{2k} p^{-1} \widehat{f} \right) \right\|_s &\lesssim \left\| \langle \cdot \rangle^{s+\frac{3}{2}+\delta} |\widehat{g-1}| * \langle (k, \cdot) \rangle \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} \widehat{f} \right\|_s \\ &\lesssim \|g-1\|_{s+\frac{5}{2}+\delta} \left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} f \right\|_s. \end{aligned} \quad (4.34)$$

Thanks to (4.23) and (4.27) we obtain

$$\left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} \frac{p'}{2k} p^{-1} \widehat{T_\varepsilon T_L f} \right\|_s \lesssim \varepsilon \left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} f \right\|_s. \quad (4.35)$$

Proceeding as done to get (4.34), by using the previous bound we also have

$$\left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} \left( \widehat{g-1} * \frac{p'}{2k} p^{-1} \widehat{T_\varepsilon T_L f} \right) \right\|_s \lesssim \varepsilon \left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} f \right\|_s. \quad (4.36)$$

Therefore, combining (4.33) with (4.34), (4.35) and (4.36) we have that (4.24) holds true.

From (2.21) and the upper bound on  $B_L$  in (2.20), it is easy to check that (4.26) follows by arguing as already done to obtain (4.25). Turning to (4.28), by using (2.24) we have

$$\left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} T_B f \right\|_s \leq \left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} f \right\|_s + \left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} B_\varepsilon T_B f \right\|_s,$$

therefore, since  $|B_L| \leq 1$ , by using (4.24) we get

$$\left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} T_B f \right\|_s \leq \left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} f \right\|_s + C\varepsilon_0 \left\| \sqrt{\frac{m'_1}{m_1}} p^{-\frac{1}{4}} m^{-1} T_B f \right\|_s,$$

which proves (4.28) choosing  $\varepsilon_0$  small enough. The proof of (4.30) is similar.  $\square$

**4.3. Proof of Proposition 4.5.** The starting point is the differential inequality (4.7). Since  $(p'p^{-\frac{1}{2}})' = 2|k|^4/p^{\frac{3}{2}} \leq 2|k|^3/p$ , we have that

$$\frac{1}{4|k|\sqrt{R}} \left| \langle (p'p^{-\frac{1}{2}})' Z_1, Z_2 \rangle_s \right| \leq \frac{1}{2C_\beta} \left( \left\| \sqrt{\frac{m'_1}{m_1}} Z_1 \right\|_s^2 + \left\| \sqrt{\frac{m'_1}{m_1}} Z_2 \right\|_s^2 \right),$$

where we recall  $C_\beta$  is given in (4.15).

It remains to deal with the  $\mathcal{R}_i$ 's. The idea is to absorb the estimates of these terms by using the second (positive) term in the left-hand side of (4.7). These computations are developed below.

**Estimate on  $\mathcal{R}_1$ .** We want to show that

$$\mathcal{R}_1 = |k| \left| \langle Z_1, m^{-1} p^{-\frac{1}{4}} ((b - \beta g) \Delta_L^{-1} T_L T_B B_L \Theta) \rangle_s \right| \leq \frac{1}{32} \left( 1 - \frac{1}{2\sqrt{R}} \right) \left\| \sqrt{\frac{m'_1}{m_1}} Z_1 \right\|_s^2. \quad (4.37)$$

By using that  $m'_1/m_1 = C_\beta |k|^2/p$  we have

$$\mathcal{R}_1 \leq \langle |Z_1|, m^{-1} p^{-\frac{1}{4}} \left( (|\widehat{b}| + \beta |\widehat{g} - 1|) * |k| p^{-1} |T_L \widehat{T_B B_L} \Theta| \right) \rangle_s \quad (4.38)$$

$$+ \beta \langle |Z_1|, m^{-1} p^{-\frac{1}{4}} |k| p^{-1} |T_L \widehat{T_B B_L} \Theta| \rangle_s \quad (4.39)$$

$$\leq \frac{1}{C_\beta |k|} \langle |Z_1|, m^{-1} p^{-\frac{1}{4}} \left( (|\widehat{b}| + \beta |\widehat{g} - 1|) * m'_1 m_1^{-1} |T_L \widehat{T_B B_L} \Theta| \right) \rangle_s \quad (4.40)$$

$$+ \frac{\beta}{C_\beta |k|} \langle |Z_1|, m^{-1} p^{-\frac{1}{4}} m'_1 m_1^{-1} |T_L \widehat{T_B B_L} \Theta| \rangle_s$$

$$:= \mathcal{R}_1^1 + \mathcal{R}_1^2.$$

Appealing to Lemma 4.6, since  $\sqrt{m'_1/m_1} = \sqrt{C_\beta} |k|/\sqrt{p}$  and  $\langle (k, \eta) \rangle^s \lesssim \langle \eta - \xi \rangle^s \langle (k, \xi) \rangle^s$ , then

$$\left( \langle (k, \eta) \rangle^s m^{-1} p^{-\frac{1}{4}} \right) (\eta) \sqrt{\frac{m'_1}{m_1}} (\xi) \lesssim \langle \eta - \xi \rangle^{s+\frac{3}{2}+\delta} \left( \langle (k, \xi) \rangle^s m^{-1} p^{-\frac{1}{4}} \right) (\xi) \sqrt{\frac{m'_1}{m_1}} (\eta).$$

Plugging in (4.40) the previous inequality and using Cauchy-Schwarz,

$$\mathcal{R}_1^1 \lesssim \left\| \sqrt{\frac{m'_1}{m_1}} Z_1 \right\|_s \left\| \langle \cdot \rangle^{s+\frac{3}{2}+\delta} (|\widehat{b}| + \beta |\widehat{g} - 1|) * \langle (k, \cdot) \rangle^s \sqrt{\frac{m'_1}{m_1}} m^{-1} p^{-\frac{1}{4}} |T_L \widehat{T_B B_L} \Theta| \right\|_{L^2}.$$

Applying Young's convolution inequality we have

$$\mathcal{R}_1^1 \lesssim \left( \|b\|_{s+\frac{5}{2}+\delta} + \|g - 1\|_{s+\frac{5}{2}+\delta} \right) \left\| \sqrt{\frac{m'_1}{m_1}} Z_1 \right\|_s \left\| \sqrt{\frac{m'_1}{m_1}} m^{-1} p^{-\frac{1}{4}} T_L T_B B_L \Theta \right\|_s.$$

Using (4.27) and (4.28), since  $|B_L| \leq 1$ , see (2.20), we bound the last term as follows

$$\left\| \sqrt{\frac{m'_1}{m_1}} m^{-1} p^{-\frac{1}{4}} T_L T_B B_L \Theta \right\|_s \leq 2 \left\| \sqrt{\frac{m'_1}{m_1}} m^{-1} p^{-\frac{1}{4}} T_B B_L \Theta \right\|_s \leq 4 \left\| \sqrt{\frac{m'_1}{m_1}} Z_1 \right\|_s. \quad (4.41)$$

Thus we get

$$\mathcal{R}_1^1 \lesssim \varepsilon \left\| \sqrt{\frac{m'_1}{m_1}} Z_1 \right\|_s^2. \quad (4.42)$$

To bound  $\mathcal{R}_1^2$ , thanks to (4.41) we get

$$\mathcal{R}_1^2 \leq \frac{4\beta}{C_\beta} \left\| \sqrt{\frac{m'_1}{m_1}} Z_1 \right\|_s^2, \quad (4.43)$$

therefore, by the choice of  $C_\beta$ , see (4.15), combining (4.42) with (4.43) and choosing  $\varepsilon_0$  small enough, the proof of (4.37) is over.

**Estimate on  $\mathcal{R}_2$ .** We prove that

$$\mathcal{R}_2 = |k|\sqrt{R} \left| \langle p^{-\frac{1}{2}}(B_L - 1)Z_1, Z_2 \rangle_s \right| \leq \frac{1}{32} \left( 1 - \frac{1}{2\sqrt{R}} \right) \left( \left\| \sqrt{\frac{m'_1}{m_1}} Z_1 \right\|_s^2 + \left\| \sqrt{\frac{m'_1}{m_1}} Z_2 \right\|_s^2 \right). \quad (4.44)$$

Recall the bound given in (3.11), namely

$$|B_L - 1| \leq \beta(1 + \beta)p^{-\frac{1}{2}},$$

from the definition of  $m_1$ , see (4.14), we get

$$\mathcal{R}_2 \leq \frac{\sqrt{R}\beta(1 + \beta)}{C_\beta} \left\| \sqrt{\frac{m'_1}{m_1}} Z_1 \right\|_s \left\| \sqrt{\frac{m'_1}{m_1}} Z_2 \right\|_s.$$

Hence (4.44) follows from the definition of  $C_\beta$ , see (4.15).

**Estimate on  $\mathcal{R}_3$ .** We aim at proving that

$$\mathcal{R}_3 = |k|\sqrt{R} \left| \langle m^{-1}p^{-\frac{3}{4}}T_\varepsilon T_L B_L \Theta, Z_2 \rangle_s \right| \leq \frac{1}{32} \left( 1 - \frac{1}{2\sqrt{R}} \right) \left( \left\| \sqrt{\frac{m'}{m}} Z_1 \right\|_s^2 + \left\| \sqrt{\frac{m'}{m}} Z_2 \right\|_s^2 \right). \quad (4.45)$$

First of all, notice that the following inequality holds true

$$|k|p^{-\frac{1}{2}}(t; k, \eta) \lesssim (|p'|p^{-1})(t; k, \eta) + |k|^2 p^{-1}(t; k, \eta) \lesssim \frac{w'}{w}(t; k, \eta) + \frac{m'_1}{m_1}(t; k, \eta).$$

Therefore, we get

$$\mathcal{R}_3 \lesssim \left\| \sqrt{\frac{w'}{w}} m^{-1} p^{-\frac{1}{4}} T_\varepsilon T_L B_L \Theta \right\|_s \left\| \sqrt{\frac{w'}{w}} Z_2 \right\|_s + \left\| \sqrt{\frac{m'_1}{m_1}} m^{-1} p^{-\frac{1}{4}} T_\varepsilon T_L B_L \Theta \right\|_s \left\| \sqrt{\frac{m'_1}{m_1}} Z_2 \right\|_s. \quad (4.46)$$

By applying (4.23), (4.25) followed by (4.27), (4.29), since  $|B_L| \leq 1$  we infer

$$\mathcal{R}_3 \lesssim \varepsilon \left( \left\| \sqrt{\frac{w'}{w}} Z_1 \right\|_s \left\| \sqrt{\frac{w'}{w}} Z_2 \right\|_s + \left\| \sqrt{\frac{m'_1}{m_1}} Z_1 \right\|_s \left\| \sqrt{\frac{w'}{w}} Z_2 \right\|_s + \left\| \sqrt{\frac{m'_1}{m_1}} Z_1 \right\|_s \left\| \sqrt{\frac{m'_1}{m_1}} Z_2 \right\|_s \right).$$

Finally, by the identity in (4.16) we have

$$\mathcal{R}_3 \lesssim \sqrt{\varepsilon} \left( \left\| \sqrt{\frac{m'}{m}} Z_1 \right\|_s^2 + \left\| \sqrt{\frac{m'}{m}} Z_2 \right\|_s^2 \right), \quad (4.47)$$

hence by choosing  $\varepsilon_0$  sufficiently small we get (4.45).

**REMARK 4.9.** Notice that in the previous bound (4.47), it would have been sufficient to choose  $\delta = C_0(\beta, R, s)\varepsilon$  for a proper constant  $C_0$  big enough.

**Estimate on  $\mathcal{R}_4$ .** The goal is again to show that

$$\mathcal{R}_4 = |k|\sqrt{R} \left| \langle m^{-1}p^{-\frac{3}{4}}T_L B_\varepsilon T_B B_L \Theta, Z_2 \rangle_s \right| \leq \frac{1}{32} \left( 1 - \frac{1}{2\sqrt{R}} \right) \left( \left\| \sqrt{\frac{m'}{m}} Z_1 \right\|_s^2 + \left\| \sqrt{\frac{m'}{m}} Z_2 \right\|_s^2 \right). \quad (4.48)$$

Applying the reasoning that for  $\mathcal{R}_3$  gives (4.46), here we get

$$\mathcal{R}_4 \lesssim \left\| \sqrt{\frac{w'}{w}} m^{-1} p^{-\frac{1}{4}} T_L B_\varepsilon T_L B_L \Theta \right\|_s \left\| \sqrt{\frac{w'}{w}} Z_2 \right\|_s + \left\| \sqrt{\frac{m'_1}{m_1}} m^{-1} p^{-\frac{1}{4}} T_L B_\varepsilon T_L B_L \Theta \right\|_s \left\| \sqrt{\frac{m'_1}{m_1}} Z_2 \right\|_s.$$

Therefore, by applying first (4.29), (4.24) and again (4.29), since  $|B_L| \leq 1$  and (4.16) holds true, we get

$$\mathcal{R}_4 \lesssim \sqrt{\varepsilon} \left( \left\| \sqrt{\frac{m'}{m}} Z_1 \right\|_s^2 + \left\| \sqrt{\frac{m'}{m}} Z_2 \right\|_s^2 \right),$$

The proof of (4.48) is over.

**Estimate on  $\mathcal{R}_5$ .** We show that

$$\begin{aligned} \mathcal{R}_5 &= \frac{1}{4\sqrt{R}} \left| \langle p' p^{-\frac{3}{4}} m^{-1} ((b - \beta g) \Delta_L^{-1} T_L T_B B_L \Theta), Z_2 \rangle_s \right| \\ &\leq \frac{1}{32} \left( 1 - \frac{1}{2\sqrt{R}} \right) \left( \left\| \sqrt{\frac{m'_1}{m_1}} Z_1 \right\|_s^2 + \left\| \sqrt{\frac{m'_1}{m_1}} Z_2 \right\|_s^2 \right). \end{aligned} \quad (4.49)$$

Since  $|p'| \leq 2|k|p^{\frac{1}{2}}$ , by using (2.15) we get

$$\begin{aligned} \mathcal{R}_5 &\leq \frac{1}{2\sqrt{R}} \langle m^{-1} p^{-\frac{1}{4}} ((|\widehat{b}| + \beta|\widehat{g} - 1|) * |k|p^{-1} |T_L \widehat{T_B B_L} \Theta|), |Z_2| \rangle_s \\ &\quad + \frac{\beta}{2\sqrt{R}} \langle m^{-1} p^{-\frac{1}{4}} |k|p^{-1} |T_L \widehat{T_B B_L} \Theta|, |Z_2| \rangle_s. \end{aligned}$$

Up to the constant  $2\sqrt{R}$ , the previous bound has the structure of (4.38)-(4.39) with  $Z_1$  replaced by  $Z_2$ . Thus we repeat the computations performed for  $\mathcal{R}_1$  to obtain

$$\mathcal{R}_5 \leq \frac{1}{C_\beta} \frac{2\beta + C\varepsilon}{\sqrt{R}} \left\| \sqrt{\frac{m'_1}{m_1}} Z_1 \right\|_s \left\| \sqrt{\frac{m'_1}{m_1}} Z_2 \right\|_s,$$

which implies (4.49) thanks to the choice of  $C_\beta$ , see (4.15).

**Estimate on  $\mathcal{R}_6$ .** Since  $|p'| \leq |k|p^{\frac{1}{2}}$ , analogously to what was done for the term  $\mathcal{R}_2$  we get

$$\mathcal{R}_6 = \frac{1}{4} \left| \langle Z_1, \frac{p'}{p} (B_L - 1) Z_1 \rangle_s \right| \leq \frac{1}{32} \left( 1 - \frac{1}{2\sqrt{R}} \right) \left\| \sqrt{\frac{m'_1}{m_1}} Z_1 \right\|_s^2.$$

**Estimate on  $\mathcal{R}_7$ .** We want to prove that

$$\mathcal{R}_7 = \frac{1}{4} \left| \left\langle \frac{p'}{p} Z_1, m^{-1} p^{-\frac{1}{4}} T_\varepsilon T_L B_L \Theta \right\rangle_s \right| \leq \frac{1}{32} \left( 1 - \frac{1}{2\sqrt{R}} \right) \left\| \sqrt{\frac{m'_1}{m_1}} Z_1 \right\|_s^2. \quad (4.50)$$

Recall the definition of  $w'/w$ , see (4.13) to have that

$$\mathcal{R}_7 \leq \left\| \sqrt{\frac{w'}{w}} m^{-1} p^{-\frac{1}{4}} T_\varepsilon T_L B_L \Theta \right\|_s \left\| \sqrt{\frac{w'}{w}} Z_1 \right\|_s.$$

Consequently, by using (4.23), (4.29) with the fact that  $|B_L| \leq 1$ , we get

$$\mathcal{R}_7 \lesssim \varepsilon \left( \left\| \sqrt{\frac{w'}{w}} Z_1 \right\|_s^2 + \left\| \sqrt{\frac{m'_1}{m_1}} Z_1 \right\|_s^2 \right) \lesssim \sqrt{\varepsilon} \left\| \sqrt{\frac{m'}{m}} Z_1 \right\|_s^2,$$

where in the last line we have used (4.16), hence (4.50) follows upon choosing  $\varepsilon_0$  sufficiently small.

**Estimate on  $\mathcal{R}_8$ .** The last step is to prove that

$$\mathcal{R}_8 = \frac{1}{4} \left| \left\langle \frac{p'}{p} Z_1, m^{-1} p^{-\frac{1}{4}} T_L B_\varepsilon T_B B_L \Theta \right\rangle_s \right| \leq \frac{1}{32} \left( 1 - \frac{1}{2\sqrt{R}} \right) \left\| \sqrt{\frac{m'}{m}} Z_1 \right\|_s^2. \quad (4.51)$$

Using (4.29) and (4.24) we obtain that

$$\mathcal{R}_8 \lesssim \varepsilon \left( \left\| \sqrt{\frac{w'}{w}} m^{-1} p^{-\frac{1}{4}} T_B B_L \Theta \right\|_s + \left\| \sqrt{\frac{m'_1}{m_1}} m^{-1} p^{-\frac{1}{4}} T_B B_L \Theta \right\|_s \right) \left\| \sqrt{\frac{w'}{w}} Z_1 \right\|_s.$$

Then, thanks to (4.30), (4.28) and  $|B_L| \leq 1$ , we get we get

$$\mathcal{R}_8 \lesssim \varepsilon \left( \left\| \sqrt{\frac{w'}{w}} Z_1 \right\|_s^2 + \left\| \sqrt{\frac{m'_1}{m_1}} Z_1 \right\|_s^2 \right) \lesssim \sqrt{\varepsilon} \left\| \sqrt{\frac{m'}{m}} Z_1 \right\|_s^2,$$

where in the last line we have used (4.16). This proves (4.51).

#### 4.4. Proof of Theorem 4.1 and Corollary 4.2.

**PROOF OF THEOREM 4.1.** By means of Proposition 4.5, we know that

$$\|Z_1(t)\|_s^2 + \|Z_2(t)\|_s^2 \lesssim \|Z_1(0)\|_s^2 + \|Z_2(0)\|_s^2.$$

Then, by the definition of  $m$  in (4.12), we have that

$$p^{-\delta} \lesssim m^{-1},$$

where the constant values of the latter inequality depends on the uniform bound on  $m_1$ . Recalling the definition of  $Z_1$  and  $Z_2$ , see (4.6), we end up with (4.3), and the proof of Theorem 4.1 is over.  $\square$

**PROOF OF COROLLARY 4.2.** It is enough to prove that

$$\left\| p^{-\frac{1}{4}-\delta} \Omega(t) \right\|_s \lesssim \left\| p^{-\frac{1}{4}-\delta} \Theta(t) \right\|_s, \quad (4.52)$$

$$\|\Theta(0)\|_s \lesssim \|\Omega(0)\|_s. \quad (4.53)$$

Since  $\Omega = B_t \Theta = T_B B_L \Theta$ , thanks to (2.25) and  $|B_L| \leq 1$  we have

$$\left\| p^{-\frac{1}{4}-\delta} \Omega(t) \right\|_s \leq \left\| p^{-\frac{1}{4}-\delta} \Theta(t) \right\|_s + \left\| p^{-\frac{1}{4}-\delta} B_\varepsilon T_B B_L \Omega(t) \right\|_s. \quad (4.54)$$

To treat the second term, we argue as in (4.24) and (4.28) and commute  $p^{-\frac{1}{4}-\delta}$  with  $B_\varepsilon T_B$ . We therefore obtain

$$\left\| p^{-\frac{1}{4}-\delta} B_\varepsilon T_B B_L \Omega(t) \right\|_s \lesssim \varepsilon \left\| p^{-\frac{1}{4}-\delta} \Omega(t) \right\|_s.$$

As  $\varepsilon_0$  is small enough, we can absorb the last term in the left-hand side of (4.54), hence obtaining (4.52). The proof of (4.53) simply comes from the fact that  $\Theta(0) = (I - \beta \partial_y \Delta^{-1}) \Omega(0)$ .  $\square$

**Acknowledgements.** The work of MCZ was partially supported by the Royal Society through a University Research Fellowship (URF\R1\191492). This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation program Grant agreement No 637653, project BLOC “Mathematical Study of Boundary Layers in Oceanic Motion. RB was partially supported by the SingFlows project, grant ANR-18-CE40-0027 of the French National Research Agency (ANR) and from the GNAMPA group of INdAM (GNAMPA project 2019, INdAM, Italy). MD was partially supported from the GNAMPA group of INdAM (GNAMPA project 2019, INdAM, Italy).

## References

- [1] S. Alinhac, *The null condition for quasilinear wave equations in two space dimensions I*, Invent. Math. **145** (2001), no. 3, 597–618.
- [2] P. Antonelli, M. Dolce, and P. Marcati, *Linear stability analysis for 2D shear flows near Couette in the isentropic Compressible Euler equations*, arXiv e-prints (Mar. 2020), available at [2003.01694](https://arxiv.org/abs/2003.01694).
- [3] J. Bedrossian, M. Coti Zelati, and V. Vicol, *Vortex axisymmetrization, inviscid damping, and vorticity depletion in the linearized 2D Euler equations*, Ann. PDE **5** (2019), no. 1, Paper No. 4, 192.
- [4] J. Bedrossian, P. Germain, and N. Masmoudi, *Stability of the couette flow at high reynolds numbers in two dimensions and three dimensions*, Bulletin of the American Mathematical Society (2018).
- [5] J. Bedrossian and N. Masmoudi, *Inviscid damping and the asymptotic stability of planar shear flows in the 2D Euler equations*, Publ. Math. Inst. Hautes Études Sci. **122** (2015), 195–300.
- [6] J. Bedrossian, V. Vicol, and F. Wang, *The Sobolev stability threshold for 2D shear flows near Couette*, J. Nonlinear Sci. **28** (2018), no. 6, 2051–2075.
- [7] R. Bianchini, A.-L. Dalibard, and L. Saint-Raymond, *Near-critical reflection of internal waves*, to appear in Anal. PDE, ArXiv e-prints (Feb. 2019), available at [1902.06669](https://arxiv.org/abs/1902.06669).
- [8] R. Camassa and C. Viotti, *Transient dynamics by continuous-spectrum perturbations in stratified shear flows*, Journal of Fluid Mechanics **717** (2013), R5.
- [9] A. Castro, D. Córdoba, and D. Lear, *On the asymptotic stability of stratified solutions for the 2D Boussinesq equations with a velocity damping term*, arXiv e-prints (May 2018), available at [1805.05179](https://arxiv.org/abs/1805.05179).
- [10] F. Charve, *Global well-posedness and asymptotics for a geophysical fluid system*, Comm. Partial Differential Equations **29** (2004), no. 11–12, 1919–1940.
- [11] M. Coti Zelati and C. Zillinger, *On degenerate circular and shear flows: the point vortex and power law circular flows*, Comm. Partial Differential Equations **44** (2019), no. 2, 110–155.
- [12] R. Danchin and L. He, *The Oberbeck-Boussinesq approximation in critical spaces*, Asymptot. Anal. **84** (2013), no. 1–2, 61–102.
- [13] Y. Deng and C. Zillinger, *Echo Chains as a Linear Mechanism: Norm Inflation, Modified Exponents and Asymptotics*, arXiv e-prints (Oct. 2019), available at [1910.12914](https://arxiv.org/abs/1910.12914).
- [14] B. Desjardins, D. Lannes, and J.-C. Saut, *Normal mode decomposition and dispersive and nonlinear mixing in stratified fluids*, arXiv e-prints (Dec. 2019), available at [1912.05346](https://arxiv.org/abs/1912.05346).
- [15] P. Drazin, *Introduction to hydrodynamic instability*, Cambridge Texts in Applied Mathematics (2002).
- [16] P. G. Drazin and W. H. Reid, *Hydrodynamic stability*, Cambridge University Press, Cambridge, 1981.
- [17] T. Gallay, *Stability of vortices in ideal fluids: the legacy of kelvin and rayleigh* **10** (2018), 42–59. Hyperbolic Problems: Theory, Numerics, Applications, Proceedings of HYP2018.
- [18] E. Grenier, T. T. Nguyen, F. Rousset, and A. Soffer, *Linear inviscid damping and enhanced viscous dissipation of shear flows by using the conjugate operator method*, J. Funct. Anal. **278** (2020), no. 3, 108339, 27.
- [19] R. J. Hartman, *Wave propagation in a stratified shear flow*, J. Fluid Mech. **71** (1975), 89–104.
- [20] L. N. Howard, *A note on the existence of certain viscous flows*, J. Math. and Phys. **40** (1961), 172–176.
- [21] A. Ionescu and H. Jia, *Axi-symmetrization near point vortex solutions for the 2D Euler equation*, ArXiv e-prints (Apr. 2019), available at [1904.09170](https://arxiv.org/abs/1904.09170).
- [22] A. D. Ionescu and H. Jia, *Inviscid Damping Near the Couette Flow in a Channel*, Comm. Math. Phys. **374** (2020), no. 3, 2015–2096.
- [23] A. D. Ionescu and H. Jia, *Nonlinear inviscid damping near monotonic shear flows*, arXiv e-prints (Jan. 2020), available at [2001.03087](https://arxiv.org/abs/2001.03087).
- [24] H. Jia, *Linear Inviscid Damping in Gevrey Spaces*, Arch. Ration. Mech. Anal. **235** (2020), no. 2, 1327–1355.
- [25] L. Kelvin, *Stability of fluid motion: rectilinear motion of viscous fluid between two parallel plates*, Phil. Mag. **24** (1887), no. 5, 188–196.
- [26] A. Majda, *Introduction to PDEs and waves for the atmosphere and ocean*, Courant Lecture Notes in Mathematics, vol. 9, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.

- [27] N. Masmoudi and W. Zhao, *Nonlinear inviscid damping for a class of monotone shear flows in finite channel*, arXiv e-prints (Jan. 2020), available at [2001.08564](#).
- [28] J. W. Miles, *On the stability of heterogeneous shear flows*, J. Fluid Mech. **10** (1961), 496–508.
- [29] C. Mouhot and C. Villani, *On Landau damping*, Acta Math. **207** (2011), no. 1, 29–201.
- [30] W. Orr, *The stability or instability of steady motions of a perfect liquid and of a viscous liquid, Part I: a perfect liquid*, Proc. Royal Irish Acad. Sec. A: Math. Phys. Sci. **27** (1907), 9–68.
- [31] M. Rieutord, *Fluid dynamics: An introduction*, Graduate Texts in Physics, Springer, Cham, 2015.
- [32] P. J. Schmid and D. S. Henningson, *Stability and transition in shear flows*, Applied Mathematical Sciences, vol. 142, Springer-Verlag, New York, 2001. MR1801992 (2001j:76039)
- [33] D. Wei, Z. Zhang, and W. Zhao, *Linear inviscid damping for a class of monotone shear flow in Sobolev spaces*, Comm. Pure Appl. Math. **71** (2018), no. 4, 617–687.
- [34] D. Wei, Z. Zhang, and W. Zhao, *Linear inviscid damping and vorticity depletion for shear flows*, Ann. PDE **5** (2019), no. 1, Art. 3, 101.
- [35] D. Wei, Z. Zhang, and W. Zhao, *Linear inviscid damping and enhanced dissipation for the Kolmogorov flow*, Adv. Math. **362** (2020), 106963, 103.
- [36] D. Wei, Z. Zhang, and H. Zhu, *Linear Inviscid Damping for the  $\beta$ -Plane Equation*, Comm. Math. Phys. **375** (2020), no. 1, 127–174.
- [37] A. M. Yaglom, *Hydrodynamic instability and transition to turbulence* (U Frisch, ed.), Vol. 100, Springer, 2012.
- [38] J. Yang and Z. Lin, *Linear inviscid damping for Couette flow in stratified fluid*, J. Math. Fluid Mech. **20** (2018), no. 2, 445–472.
- [39] C. Zillinger, *Linear inviscid damping for monotone shear flows in a finite periodic channel, boundary effects, blow-up and critical Sobolev regularity*, Arch. Ration. Mech. Anal. **221** (2016), no. 3, 1449–1509.
- [40] C. Zillinger, *Linear inviscid damping for monotone shear flows*, Trans. Amer. Math. Soc. **369** (2017), no. 12, 8799–8855.
- [41] C. Zillinger, *On circular flows: linear stability and damping*, J. Differential Equations **263** (2017), no. 11, 7856–7899.
- [42] C. Zillinger, *Linear Inviscid Damping in Sobolev and Gevrey Spaces*, arXiv e-prints (Nov. 2019), available at [1911.00880](#).
- [43] C. Zillinger, *On Enhanced Dissipation for the Boussinesq Equations*, arXiv e-prints (Apr. 2020), available at [2004.08125](#).

IAC, CONSIGLIO NAZIONALE DELLE RICERCHE, 00185 ROME, ITALY

*E-mail address:* [r.bianchini@iac.cnr.it](mailto:r.bianchini@iac.cnr.it)

DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE LONDON, LONDON, SW7 2AZ, UK

*E-mail address:* [m.coti-zelati@imperial.ac.uk](mailto:m.coti-zelati@imperial.ac.uk)

GSSI - GRAN SASSO SCIENCE INSTITUTE, 67100, L'AQUILA, ITALY

*E-mail address:* [michele.dolce@gssi.it](mailto:michele.dolce@gssi.it)