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A Set-Valued Approach to Multiple Source Evidence

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Abstract. This short note studies a multiple source extension of categorical mass functions in the sense of Shafer’s evidence theory. Each subset of possible worlds is associated with a subset of information sources and represents a tentative description of what is known. Analogs of belief, plausibility, commonality functions, valued in terms of subsets of agents or sources, are defined, replacing summation by set union. Set-valued plausibility is nothing but set-valued possibility because it is union-decomposable with respect to the union of events. In a special case where each source refers to a single information item, set-valued belief functions decompose with respect to intersection and are thus multiple source necessity-like function. Connections with Belnap epistemic truth-values for handling multiple source inconsistent information are shown. A formal counterpart of Dempster rule of combination is defined and discussed as to its merits for information fusion.

1 Introduction

Qualitative settings (e.g., [4,6]) often refer to approaches where maximum and minimum operations replace sum-product structures. But other kinds of qualitative frameworks exist, in particular where set functions become set-valued rather than being numerical or taking their values on an ordinal scale. An example of such a set-valued counterpart is given in possibility theory by the semantical counterpart of multiple agent logic [2].

In this short paper, we study a set-valued framework that seems to mimic Shafer’s evidence theory [7]. We first introduce a multiple source extension of a categorical basic assignment and its intended meaning, from which set-valued belief, plausibility, commonality functions are defined. Actually, these set-valued functions are quite close to the ones of set-valued possibility theory. Then the interest of this setting in information fusion is advocated, and a counterpart of Dempster rule of combination can be defined. Lastly, directions for further research are briefly mentioned.

2 An Evidential Set-Valued Framework

In Shafer’s evidence theory, information is represented by a mass function m , originally called ‘basic probability assignment’, from the power set of a frame of

discernment Ω to the unit interval. Then the mass function is supposed to be such that $\forall A \subseteq \Omega, m(A) \geq 0$ and $\sum_{A \subseteq \Omega} m(A) = 1$ [7].

Set-Valued Mass Function. In the set-valued counterpart we propose here, we start with a set of sources of information (that could be agents) named ALL. In the most elementary setting, each source a proposes a certain view of the real world in the form of a non-empty set $\Gamma(a) = E$ of Ω . Then, the analog of a basic assignment function associates to each set $E \in 2^\Omega$ the subset of sources $\mathcal{M}(E) \subseteq \text{ALL}$ such that $\Gamma(a) = E$. As usual, the *focal sets* E are non-empty subsets of Ω , such that $\mathcal{M}(E) \neq \emptyset$, forming a collection \mathcal{F} . Total ignorance is represented by $\mathcal{M}(\Omega) = \text{ALL}$ and $\forall E \neq \Omega, \mathcal{M}(E) = \emptyset$, i.e., no source provide information. We assume no inconsistent source, i.e. $\mathcal{M}(\emptyset) = \emptyset$. Since each source in ALL is supposed provide some information, we should have

$$\bigcup_{E \in \mathcal{F}} \mathcal{M}(E) = \text{ALL}$$

This understanding supposes that the piece of information provided by each source is precisely known. Namely,

$$\forall E_1, E_2 \in \mathcal{F}, \text{ if } E_1 \neq E_2 \text{ then } \mathcal{M}(E_1) \cap \mathcal{M}(E_2) = \emptyset \quad (*)$$

This assumption is not compulsory for the definitions given in the following. In fact, we may extend the setting by assigning to each source a a collection $\tilde{\Gamma}(a)$ of subsets of Ω . Intuitively, it means that the receiver is unsure about what message the source conveys. In this case $\mathcal{M}(E)$ is the set of sources that *possibly* forward information item E (i.e., $\mathcal{M}(E) = \{a : E \in \tilde{\Gamma}(a)\}$). In that case, it is a higher-order model of information source, more similar to a usual mass function. Conversely, $\tilde{\Gamma}(a) = \{E : a \in \mathcal{M}(E)\}$.

Moreover, it is worth mentioning that, provided that condition (*) holds, a regular probability assignment m can be associated in a natural way with a set-valued basic assignment by replacing the subset of sources by their cardinality, namely $m_{\mathcal{M}}(A) = \frac{|\mathcal{M}(A)|}{|\text{ALL}|}$ since then $\sum_A m_{\mathcal{M}}(A) = 1$.

The Four Set-Valued Set Functions. The multiple source counterpart of a numerical belief function $Bel(A) = \sum_{\emptyset \neq B \subseteq A} m(B)$ is $\mathcal{BEL}(A)$, the set of all sources that support A according to the information they convey:

$$\mathcal{BEL}(A) = \bigcup_{E \text{ s.t. } \emptyset \neq E \subseteq A} \mathcal{M}(E)$$

The idea of attaching to propositions sets of sources that support them was already discussed in [1]. Note that \mathcal{BEL} will not change if one adds in $\tilde{\Gamma}(a)$ a set E' containing some set $E \in \tilde{\Gamma}(a)$. So we can modify $\mathcal{M}(E)$ by restricting, for each agent $a \in \text{All}$ to minimal subsets for inclusion in $\tilde{\Gamma}(a)$, which leaves $\mathcal{BEL}(A)$ unchanged.

The counterpart $\mathcal{PL}(A)$ of plausibility is the set of sources providing information not inconsistent with A . Namely, we have (without assuming $\mathcal{M}(\emptyset) = \emptyset$):

$$\mathcal{PL}(A) = \bigcup_{E \cap A \neq \emptyset} \mathcal{M}(E).$$

The subset of sources that find A plausible includes the subset of sources that are certain about A , i.e., $\mathcal{PL}(A) \supseteq \mathcal{BEL}(A)$. $\mathcal{PL}(A)$ does not exhibit the usual duality with respect to \mathcal{BEL} , that is, $\mathcal{PL}(A) \neq \text{ALL} \setminus (\mathcal{BEL}(\bar{A}) \cup \mathcal{M}(\emptyset))$ where $\bar{A} = \Omega \setminus A$, except if condition (*) holds. Indeed, if $E_1 \neq E_2 \in \tilde{\Gamma}(a)$, then there is an event A such that $a \in \mathcal{PL}(A) \cap \mathcal{BEL}(\bar{A})$ (e.g. $E_2 \cap A \neq \emptyset$, while $E_1 \subseteq \bar{A}$). These set-valued set-functions are monotonic in the sense that if $A \subseteq B$, then $\mathcal{BEL}(A) \subseteq \mathcal{BEL}(B)$ and $\mathcal{PL}(A) \subseteq \mathcal{PL}(B)$.

The counterpart of commonality $Q(A) = \sum_{E \supseteq A} m(E)$ is the set of sources whose information is possibly not more specific than A :

$$Q(A) = \bigcup_{E \supseteq A} \mathcal{M}(E).$$

If $\mathcal{M}(\Omega) = \emptyset$, letting $\overline{\mathcal{M}}(E) = \mathcal{M}(\bar{E})$, we can see that $\mathcal{Q}_{\mathcal{M}}(A) = \mathcal{BEL}_{\overline{\mathcal{M}}}(\bar{A})$. The dual is $\tilde{\mathcal{O}}(A) = \bigcup_{E \cup A \neq \Omega} \mathcal{M}(E)$, generally not equal to $\text{ALL} \setminus \mathcal{Q}(\bar{A})$, except if condition (*) holds. \mathcal{Q} and $\tilde{\mathcal{O}}$ are anti-monotonic with respect to set inclusion.

Non Unicity of the Function \mathcal{M} in the General Case. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\text{ALL} = \{a, b\}$. The table below shows four functions \mathcal{M}_0 , \mathcal{M}_1 , \mathcal{M}_2 and \mathcal{M}_3 , the associated belief functions of the first three \mathcal{BEL}_0 , \mathcal{BEL}_1 and \mathcal{BEL}_2 being equal.

	$\{\omega_1\}$	$\{\omega_2\}$	$\{\omega_3\}$	$\{\omega_1, \omega_2\}$	$\{\omega_1, \omega_3\}$	$\{\omega_2, \omega_3\}$	$\{\omega_1, \omega_2, \omega_3\}$
\mathcal{M}_0	$\{a\}$	\emptyset	$\{b\}$	\emptyset	\emptyset	\emptyset	\emptyset
\mathcal{M}_1	$\{a\}$	\emptyset	$\{b\}$	\emptyset	\emptyset	\emptyset	$\{a, b\}$
\mathcal{M}_2	$\{a\}$	\emptyset	$\{b\}$	$\{a\}$	$\{b\}$	$\{b\}$	\emptyset
\mathcal{M}_3	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	$\{a, b\}$
$\mathcal{BEL}_0 = \mathcal{BEL}_1 = \mathcal{BEL}_2$	$\{a\}$	\emptyset	$\{b\}$	$\{a\}$	$\{a, b\}$	$\{b\}$	$\{a, b\}$
\mathcal{BEL}_3	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	\emptyset	$\{a, b\}$
\mathcal{PL}_0	$\{a\}$	\emptyset	$\{b\}$	$\{a\}$	$\{a, b\}$	$\{b\}$	$\{a, b\}$
$\mathcal{PL}_1 = \mathcal{PL}_3$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$
\mathcal{PL}_2	$\{a, b\}$	$\{a, b\}$	$\{b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$

\mathcal{M}_0 describes the case when each source a and b supply precise information. But \mathcal{M}_1 and \mathcal{M}_2 violate condition (*) on mass functions. According to \mathcal{M}_1 , information provided by sources is either precise or vacuous, while for \mathcal{M}_2 , the receiver has a much more blurred view of source information. It is not always possible to recover \mathcal{M} from \mathcal{BEL} if some focal sets are nested as for \mathcal{M}_1 and \mathcal{M}_2 that yield the same \mathcal{BEL} . But note that plausibility functions differ. Actually given a basic assignment \mathcal{M} and the belief function \mathcal{BEL} , then the same belief function is obtained by restricting each set $\Gamma(a)$ to its minimal elements for inclusion. When, $\forall a \in \text{All}$, no two focal sets in $\tilde{\Gamma}(a)$ are nested (in particular if condition (*) holds), it can be shown that $\mathcal{M}(E) = \mathcal{BEL}(E)$ if $\mathcal{BEL}(E) \neq \emptyset$ and $\mathcal{BEL}(E \setminus \{\omega\}) = \emptyset, \forall \omega \in E$. This is like focal sets of qualitative capacities [4].

Link with Belnap Epistemic Truth-Values. Belnap logic [3] deals with a set of conflicting sources and assigns to any atomic proposition B an epistemic truth-value in $\{\mathbf{T}, \mathbf{F}, \mathbf{U}, \mathbf{C}\}$, where \mathbf{T} means that B is supported by one source and negated by none, \mathbf{F} means that B is negated by one source and supported by none, \mathbf{U} means that B receives no support, and \mathbf{C} means that B and its negation receive support from some source. It is clear that these assignments can be generalised to any proposition B in terms of set-valued belief functions: \mathbf{T} if $\mathcal{BEL}(B) \neq \emptyset$ and $\mathcal{BEL}(\bar{B}) = \emptyset$, \mathbf{F} if $\mathcal{BEL}(B) = \emptyset$ and $\mathcal{BEL}(\bar{B}) \neq \emptyset$, \mathbf{U} if $\mathcal{BEL}(B) = \mathcal{BEL}(\bar{B}) = \emptyset$, \mathbf{C} if $\mathcal{BEL}(B) \neq \emptyset$ and $\mathcal{BEL}(\bar{B}) \neq \emptyset$.

Behavior with Respect to Combinations of Events. It is worth noticing that if condition (*) holds,

$$\mathcal{BEL}(A \cap B) = \mathcal{BEL}(A) \cap \mathcal{BEL}(B) \quad (1)$$

Indeed, $\forall a \in All$, $a \in \mathcal{BEL}(A \cap B)$ if and only if $\Gamma(a) \subseteq A \cap B$ if and only if $a \in \mathcal{BEL}(A) \cap \mathcal{BEL}(B)$. However, in general (if some agent is associated to several focal subsets), this equality does not hold, and we only have inclusion due to monotonicity. This is very similar to the graded qualitative capacity setting [4, 6] where the union in \mathcal{BEL} is replaced by the max, and only $Bel_q(A \cap B) \leq \min(Bel_q(A), Bel_q(B))$ holds in general, since the maximum over focal sets in $A \cap B$ may be smaller than the minimum of the maxima over A and over B .

A situation when (1) holds without satisfying condition (*) is when the focal sets are *singletons*. It means that information provided by sources is precise, but each source can propose several information items. Then, $\forall A, \mathcal{BEL}(A) = \mathcal{PL}(A) = \bigcup_{\omega \in A} \mathcal{M}(\{\omega\})$. If, on top, condition (*) holds, we are in a probabilistic-like situation (one Dirac function per agent), and the numerical mass function $m_{\mathcal{M}}$ induced by counting the proportion of sources supporting a focal set is a probability distribution.

In contrast with (1), we always have

$$\mathcal{PL}(A \cup B) = \mathcal{PL}(A) \cup \mathcal{PL}(B).$$

In particular, $\forall a \in All$, $a \in \mathcal{PL}(A \cup B)$ if and only if $\exists E \in \tilde{\Gamma}(a)$, $E \cap (A \cup B) \neq \emptyset$ if and only if $E \cap A \neq \emptyset$ or $E \cap B \neq \emptyset$, if and only if $a \in \mathcal{PL}(A)$ or $a \in \mathcal{PL}(B)$. Then, \mathcal{PL} can be directly expressed from a set-valued distribution $\pi_{\mathcal{F}}$, from Ω to 2^{ALL} , namely the contour function $\pi_{\mathcal{F}}(\omega) = \mathcal{PL}(\{\omega\}) = \bigcup_{E \ni \omega} \mathcal{M}(E)$ as

$$\mathcal{PL}(A) = \bigcup_{E \cap A \neq \emptyset} \mathcal{M}(E) = \bigcup_{\omega \in A} \pi_{\mathcal{F}}(\omega)$$

where $\pi_{\mathcal{F}}(\omega) = \{a : \exists E, \omega \in E \in \tilde{\Gamma}(a)\}$. Thus, \mathcal{PL} being U-decomposable is nothing but a set-valued possibility measure \mathbf{II} . Its dual is $\mathbf{N}(A) = \mathcal{PL}(\bar{A})$ is a multiple-agent necessity measure that satisfies \cap -decomposability (1), hence generally different from \mathcal{BEL} . Function \mathbf{N} is thus induced by a mass function $\mathcal{M}_{\mathcal{PL}}$ obeying condition (*), and associated to a set-valued mapping $\Gamma_{\mathcal{PL}}$ such

that $\Gamma_{\mathcal{P}\mathcal{L}}(a) = \bigcup\{E \in \tilde{I}(a)\}$ (see mass function \mathcal{M}_3 in the previous table). It can be checked that $\omega \in \Gamma_{\mathcal{P}\mathcal{L}}(a)$ if and only if $a \in \pi_{\mathcal{F}}(\omega)$. In other words, each agent a has epistemic state $\Gamma_{\mathcal{P}\mathcal{L}}(a)$. This is the setting of multiple agent logic [2] briefly recalled below.

In this logic, constraints of the form $\mathbf{N}(A) \supseteq \mathcal{S}$ where $\mathcal{S} \subseteq \text{ALL}$ express that according to at least all sources in \mathcal{S} the true state of the world is in A . Clearly, $\mathbf{N}(A) \supseteq \mathcal{S}$ is the semantical counterpart of a possibilistic logic-like formula (A, \mathcal{S}) . This formula is associated with the multiple agent *ma*-distribution:

$$\forall \omega \in \Omega, \pi_{\{(A, \mathcal{S})\}}(\omega) = \begin{cases} \text{ALL} & \text{if } \omega \in A \\ \overline{\mathcal{S}} & \text{if } \omega \in \overline{A}. \end{cases}$$

Indeed the complement of \mathcal{S} is the maximal subset of sources that may find possible that the real world be outside A (since for all sources in \mathcal{S} the true state of the world is in A). More generally, the *ma*-distribution π_{Γ} semantically associated to a set of *ma*-formulas $\mathcal{K} = \{(A_i, \mathcal{S}_i), i = 1, m\}$ is given by

$$\pi_{\mathcal{K}}(\omega) = \begin{cases} \text{ALL} & \text{if } \forall (A_i, \mathcal{S}_i) \in \Gamma, \omega \in A_i \\ \bigcap \{\overline{\mathcal{S}_i} : (A_i, \mathcal{S}_i) \in \Gamma, \omega \in \overline{A_i}\} & \text{otherwise.} \end{cases}$$

Finally, taking advantage of the connection between $\mathcal{B}\mathcal{E}\mathcal{L}$ and \mathcal{Q} , it is easy to see that the property $\mathcal{Q}(A \cup B) = \mathcal{Q}(A) \cap \mathcal{Q}(B)$ holds under condition (*), while $\overline{\mathcal{O}}(A \cap B) = \overline{\mathcal{O}}(A) \cup \overline{\mathcal{O}}(B)$ always holds.

3 Fusion Rules

Dempster rule of combination is central in the theory of evidence. Then, the combination of two mass functions can be viewed as a (normalized) intersection of the two random sets represented by these two basic probability assignments. A kind of counterpart of this rule in the set-valued setting is as follows: $\forall E \neq \emptyset$:

$$(\mathcal{M}_1 \otimes \mathcal{M}_2)(E) = \mathcal{M}_{12}(E) = \bigcup_{A, B \text{ s.t. } E=F \cap G} \mathcal{M}_1(F) \cup \mathcal{M}_2(G)$$

It is especially meaningful to combine two disjoint sets of sources and refine their information. It makes no sense to use $\mathcal{M}_1(F) \cap \mathcal{M}_2(G)$ in the above expression if $\text{ALL} \cap \text{ALL}' = \emptyset$, $\mathcal{M}_1(F) \subseteq \text{ALL}$, $\mathcal{M}_2(G) \subseteq \text{ALL}'$ since then $\mathcal{M}_1(F) \cap \mathcal{M}_2(G) = \emptyset$. In the case where $\text{ALL} = \text{ALL}'$, it means that we have two versions of the information provided by each source, and information items are combined conjunctively. For instance, consider two sources a and b with epistemic states $\Gamma_1(a) = E_a$ and $\Gamma_2(b) = E_b$. The fusion rule gives $\Gamma_{12}(a) = \Gamma_{12}(b) = E_a \cap E_b$, i.e., $\mathcal{M}_{12}(E_a \cap E_b) = \{a, b\}$, and $\mathcal{M}_{12}(E) = \emptyset$ otherwise. Just as Dempster rule of combination amounts to performing the product of commonality functions [7], the above rule corresponds to the union of the set-valued commonality functions, as shown below.

$$\begin{aligned}
\mathcal{Q}_{12}(D) &= \bigcup_{C \supseteq D} \mathcal{M}_{12}(C) = \bigcup_{C \supseteq D} \bigcup_{A, B \text{ s.t. } C=A \cap B} \mathcal{M}_1(A) \cup \mathcal{M}_2(B) \\
&= \bigcup_{A, B \text{ s.t. } A \cap B \supseteq D} \mathcal{M}_1(A) \cup \mathcal{M}_2(B) = \bigcup_{A, B \text{ s.t. } A \supseteq D, B \supseteq D} \mathcal{M}_1(A) \cup \mathcal{M}_2(B) \\
&= \left(\bigcup_{A \supseteq D} \mathcal{M}_1(A) \right) \cup \left(\bigcup_{B \supseteq D} \mathcal{M}_2(B) \right) = \mathcal{Q}_1(D) \cup \mathcal{Q}_2(D)
\end{aligned}$$

Hence, the combination rule is commutative, associative and idempotent. Other combination rules make sense such as the generalized disjunction, replacing $A \cap B$ by $A \cup B$ in the fusion rule. This rule boils down to the union of the set-valued belief functions $\mathcal{BEL}_1(A) \cup \mathcal{BEL}_2(A), \forall A \subseteq \Omega$.

4 Concluding Remarks

There are several lines for further research. First, this set-valued setting has been presented in terms of knowledge, but it might apply as well to preference modeling. Besides, the approach could be generalized by allowing the use of weighted subsets of sources leading to fuzzy set-valued set functions. Moreover, one may follow ideas expressed in [5, 8] advocating the interest of a bipolar view of information that can be represented in evidence or in possibility theory, for introducing a similar bipolar setting in set-valued evidence theory. Lastly, the set of sources that are the values of the set functions may be understood in terms of arguments, following an idea already suggested in [1] in a different setting.

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