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AFFINE CONES OVER FANO-MUKAI FOURFOLDS OF GENUS 10 ARE FLEXIBLE

YURI PROKHOROV AND MIKHAIL ZAIDENBERG

ABSTRACT. We show that the affine cones over any Fano–Mukai fourfold of genus 10 are flexible in the sense of [AFK⁺13]. In particular, the automorphism group of such a cone acts highly transitively outside the vertex. Furthermore, any Fano–Mukai fourfold of genus 10, with one exception, admits a covering by open charts isomorphic to \mathbb{A}^4 . Besides, we complete the description of the automorphism groups of these fourfolds. The neutral components of these groups were found in our previous paper [PZ18]; here we finish the description of the discrete parts.

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INTRODUCTION

Our base field is the complex number field \mathbb{C} . We use the notation of [PZ18]. Let V_{18} be a Fano-Mukai fourfold of genus 10 and degree 18 half-anticanonically embedded in \mathbb{P}^{12} . Recall ([KR13], [PZ18, Remark 13.4]) that the moduli space of these fourfolds is one-dimensional. It contains two special members, namely, V_{18}^a with $\text{Aut}^0(V_{18}^a) = \mathbb{G}_a \times \mathbb{G}_m$ and V_{18}^s with $\text{Aut}^0(V) = \text{GL}_2(\mathbb{C})$ [PZ18, Theorem 1.3.a,b]. For the general member $V_{18}^g \notin \{V_{18}^a, V_{18}^s\}$ one has $\text{Aut}^0(V_{18}^g) = \mathbb{G}_m^2$ [PZ18, Theorem 1.3.c].

Let X be an affine variety. Consider the subgroup $\text{SAut}(X)$ of the automorphism group $\text{Aut}(X)$ generated by all the one-parameter unipotent subgroups of $\text{Aut}(X)$. The variety X is called *flexible* if $\text{SAut}(X)$ acts highly transitively on the smooth locus $\text{reg}(X)$, that is, m -transitively for any natural number m [AFK⁺13]. There are many examples of flexible affine varieties, see, e.g., [AZK12, AFK⁺13, MPS18]; studies on this subject are in an active phase. For a projective variety, the flexibility of the affine cones might depend on the choice of an ample polarization. If the affine cone over a smooth projective variety V with Picard number one is flexible, then V is a Fano variety. For a pluri-anticanonical polarization of a Fano variety V , the flexibility of the affine cone X over V is a nontrivial

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new invariant of V . Among del Pezzo surfaces with their pluri-anticanonical polarizations, only the surfaces of degree ≥ 4 have flexible affine cones, see [Per13, PW16]. Moreover, the group $\text{SAut}(X)$ of the affine cone X over $(V, -mK_V)$, $m > 0$, is trivial for any del Pezzo surface V of degree at most 3, see [CPW16, KPZ11]. The affine cones over flag varieties of dimension ≥ 2 are flexible [AZK12]. The secant varieties of the Segre-Veronese varieties provide another class of examples [MPS18]. There are many examples of Fano threefolds with flexible affine cones, see, e.g., [MPS18]. However, we know just two examples of such Fano fourfolds, and one of these is V_{18}^s , see [PZ18, Theorem 14.3]. In the following theorem we extend the latter result to all the Fano-Mukai fourfolds of genus 10.

Theorem A. *Let $V = V_{18}$ be a Fano-Mukai fourfold of genus 10. Then the affine cone over V is flexible for any ample polarization of V .*

The proof exploits the criteria of flexibility of affine cones borrowed from [Per13, Theorem 5] and [MPS18, Theorem 1.4]. These criteria are based on existence of some special open coverings of the underlined projective variety, for instance, a covering by flexible affine charts, or by suitable toric affine varieties, or by the affine spaces, see Theorem 3.6. To apply these criteria, one needs to construct such a covering. To this end, we use Theorem B below.

Any Fano-Mukai fourfold $V = V_{18}$ can be represented, at least in two ways, as an $\text{Aut}^0(V)$ -equivariant compactification of the affine space \mathbb{A}^4 [PZ18, Theorem 1.1]. More precisely, there exists at least two different $\text{Aut}^0(V)$ -invariant hyperplane sections A_1, A_2 of V such that $U_i := V \setminus A_i$ is isomorphic to \mathbb{A}^4 , $i = 1, 2$. Statement (a) of the following theorem is proven in [PZ18, Theorem 13.5(f)].

Theorem B. *For the Fano-Mukai fourfolds $V = V_{18}$ of genus 10 the following hold.*

- (a) *If $\text{Aut}^0(V) = \text{GL}_2(\mathbb{C})$ then V admits a covering by a one-parameter family of Zariski open subsets U_t isomorphic to \mathbb{A}^4 , where each U_t is the complement of a hyperplane section of V . Exactly two of these sets are $\text{Aut}^0(V)$ -invariant;*
- (b) *if $\text{Aut}^0(V) = \mathbb{G}_m^2$ then V is covered by six $\text{Aut}^0(V)$ -invariant subsets $U_i \cong \mathbb{A}^4$ as in (a), $i = 1, \dots, 6$;*
- (c) *if $\text{Aut}^0(V) = \mathbb{G}_a \times \mathbb{G}_m$ then there exist in V four $\text{Aut}^0(V)$ -invariant affine charts $U_i \cong \mathbb{A}^4$, $i = 1, \dots, 4$ such that $V \setminus \bigcup_{i=1}^4 U_i$ is a projective line covered by a one-parameter family of \mathbb{A}^2 -cylinders $U_t \cong \mathbb{A}^2 \times Z_t$, where Z_t is a smooth affine surface, $t \in \mathbb{P}^1$.*

Besides, we complete the description of the automorphism groups $\text{Aut}(V)$ of the Fano-Mukai fourfolds V of genus 10. These groups are completely described in [PZ18, Theorem 1.3] in the exceptional cases $V = V_{18}^s$ and $V = V_{18}^a$, and only partially in the general case $V = V_{18}^g$. Summarizing the results of [PZ18] and the ones of Theorem 6.2 and Corollary 6.3 of the present paper we get the following

Theorem C. *Let V be a Fano-Mukai fourfold V of genus 10. Then the following hold.*

- (i) *$\text{Aut}(V)$ is isomorphic to one of the following groups:*
- $$(0.1) \quad \text{GL}_2(\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}, \quad (\mathbb{G}_a \times \mathbb{G}_m) \rtimes \mathbb{Z}/2\mathbb{Z}, \quad \mathbb{G}_m^2 \rtimes \mathbb{Z}/6\mathbb{Z}, \quad \mathbb{G}_m^2 \rtimes \mathbb{Z}/2\mathbb{Z}.$$
- (ii) *For any one of the first three groups in (0.1) there exists a unique, up to isomorphism, Fano-Mukai fourfold V with $\text{Aut}(V)$ isomorphic to this group. The Fano-Mukai fourfolds V with $\text{Aut}(V)$ being the last group in (0.1) vary in a one-parameter family.*

- (iii) *The generator of the subgroup $\mathbb{Z}/2\mathbb{Z}$ in (0.1) acts on $\text{Aut}^0(V)$ via the automorphism $g \mapsto (g^t)^{-1}$ in the first case and via the inversion otherwise, and acts on V interchanging the cubic cones in the disjoint pairs. The generator of the subgroup $\mathbb{Z}/6\mathbb{Z}$ in (0.1) acts on $\text{Aut}^0(V)$ via an automorphism of order 6. Its action on V induces a cyclic shift of the cycle formed by the six cubic cones on V , cf. Lemma 1.8 (i). a homogeneous fivefold This is a*
- (iv) *Consider the adjoint variety $\Omega \subset \mathbb{P}^{13}$ of the simple algebraic group G_2 . Under the Mukai realization of V as a hyperplane section of the homogeneous fivefold Ω , the group $\text{Aut}(V)$ coincides with the stabilizer of V in G_2 , except in the case where $\text{Aut}^0(V) = \mathbb{G}_a \times \mathbb{G}_m$.*

0.2. The one-dimensional family of Fano-Mukai fourfolds of genus 10 with reductive automorphism groups admits the following description in the framework of Mukai's realization [Muk89]. Consider a parabolic subgroup of the simple complex algebraic group of type G_2 which corresponds to a short root. Then the flag variety $\Omega = G_2/P$ is a Fano fivefold. It is naturally embedded in $\mathbb{P}^{13} = \mathbb{P}\mathfrak{g}_2$, where \mathfrak{g}_2 is the Lie algebra of G_2 .

Let $\mathfrak{h} \subset \mathfrak{g}_2$ be a Cartan subalgebra. Any nonzero element $h \in \mathfrak{h}$ defines the hyperplane section $V(h)$ of Ω by the hyperplane orthogonal to h with respect to the Killing form. If $V(h)$ is smooth, then $V(h)$ is a Fano-Mukai fourfold of genus 10 with a reductive group $\text{Aut}^0(V(h))$. Conversely, any Fano-Mukai fourfold V of genus 10 with a reductive group $\text{Aut}^0(V)$ has the form $V = V(h)$ for some $h \in \mathfrak{h} \setminus \{0\}$. This correspondence between points of an open subset of the projective line $\mathbb{P}\mathfrak{h}$ and Fano-Mukai fourfolds V of genus 10 with a reductive group $\text{Aut}^0(V)$ is not bijective meaning that distinct points could correspond to isomorphic fourfolds. However, this open set surjects onto the moduli space of the fourfolds in question.

Consider the Weyl group $W(G_2)$ acting on \mathfrak{h} . There are three homogeneous semi-invariants $\psi_2, p_3, q_3 \in \mathcal{O}(\mathfrak{h})$ of $W(G_2)$ of degrees 2, 3, and 3, respectively, such that $V(h)$ is singular if and only if $p_3(h) = 0$, $\text{Aut}^0(V(h)) = \text{GL}_2(\mathbb{C})$ if and only if $q_3(h) = 0$, and $\text{Aut}^0(V(h)) = \mathbb{G}_m^2 \rtimes \mathbb{Z}/6\mathbb{Z}$ if and only if $\psi_2(h) = 0$. If $(\psi_2 p_3 q_3)(h) \neq 0$ then $\text{Aut}^0(V(h)) = \mathbb{G}_m^2 \rtimes \mathbb{Z}/2\mathbb{Z}$; see Remark 6.28 for further details. Under the induced action of $W(G_2)$ on $\mathbb{P}\mathfrak{h}$, the points in the same orbit correspond to isomorphic Fano-Mukai fourfolds.

Some other families of Fano varieties demonstrate similar behavior of the automorphism groups. For instance, this concerns the Fano threefolds of degree 22 and Picard number 1 [KP18, Pro90] and the Fano threefolds of degree 28 and Picard number 2 [PCS19, § 9]. Any member of the first family is a compactification of the affine 3-space. It is plausible that the same is true for the second family. It would be interesting to investigate the flexibility of the affine cones over these varieties.

The paper is organized as follows. In Sections 1 and 2 we gather necessary preliminaries, in particular, some results from [KR13, PZ16, PZ18]; cf. also [PZ17]. Based on this, in Section 2 we prove Theorems A and B in the case where $\text{Aut}^0(V) = \mathbb{G}_m^2$. In Section 3 we proceed with the proof of Theorem B in the case $\text{Aut}^0(V) = \mathbb{G}_a \times \mathbb{G}_m$. Besides, we provide a criterion of flexibility related to the existence of a special family of \mathbb{A}^2 -cylinders on V . Such a family of cylinders is constructed in Section 4 for the smooth quadric fourfold Q^4 and for the del Pezzo quintic fourfold W_5 , and in Section 5 for any Fano-Mukai fourfold V of genus 10. This enables us to complete the proofs of Theorems A and B in the remaining case where $\text{Aut}(V) = \mathbb{G}_a \times \mathbb{G}_m$. In the last Section 6 we prove Theorem C.

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1. CUBIC SCROLLS IN THE FANO-MUKAI FOURFOLDS V_{18}

In this section we recall and extend some facts from of [PZ18] used in the sequel. Throughout this paper we let V be a Fano-Mukai fourfold $V = V_{18}$ of genus 10 half-anticanonically embedded in \mathbb{P}^{12} . Following [KR13] we call a *cubic scroll* both a smooth cubic surface scroll and a cone over a rational twisted cubic curve. The latter cones in \mathbb{P}^4 will be called *cubic cones* for short.

1.1. Theorem ([KR13, Propositions 1 and 2 and the proof of Proposition 4]).

- (a) *Let $\Sigma(V)$ be the Hilbert scheme of lines in V . Then $\Sigma(V)$ is isomorphic to the variety $\text{Fl}(\mathbb{P}^2)$ of full flags on \mathbb{P}^2 .*
- (b) *Let $\mathcal{S}(V)$ be the Hilbert scheme of cubic scrolls in V . Then $\mathcal{S}(V)$ is isomorphic to a disjoint union of two projective planes.*

This theorem and the next lemma show that the cubic scrolls in V play the same role as do the planes in a smooth quadric fourfold, cf. [Har92, Lecture 22].

1.2. Lemma ([PZ18, Proposition 9.6]). *Let \mathcal{S}_1 and \mathcal{S}_2 be the connected components of $\mathcal{S}(V)$. Then for any $S_i \in \mathcal{S}_i$ and $S_j \in \mathcal{S}_j$, $i, j \in \{1, 2\}$ we have the following relation: $S_i \cdot S_j = \delta_{i,j}$ in $H^*(V, \mathbb{Z})$.*

1.3. Lemma ([PZ18, Lemma 9.2, Corollary 4.5.2]).

- (i) *For any cubic scroll S on V there exists a unique hyperplane section A_S of V such that $\text{Sing}(A_S) = S$. This hyperplane section coincides with the union of lines on V meeting S , and any line contained in A_S meets S .*
- (ii) *For any point $P \in A_S \setminus S$ there is exactly one line $l \subset A_S$ passing through P and meeting S . Such a line meets S in a single point.*
- (iii) *If $S \subset V$ is a cubic cone, then $V \setminus A_S \cong \mathbb{A}^4$.*

1.4. Lemma ([PZ18, Proposition 8.2]). *Let $v \in V$ be a point. There exists a divisor $\mathcal{B} \subset V$ such that the following hold:*

- (i) *for any point $v \in V \setminus \mathcal{B}$ there are exactly three lines passing through v ;*
- (ii) *for $v \in \mathcal{B}$ the number of lines passing through v either is infinite, or equals 2 or 1;*
- (iii) *if the number of lines passing through v is infinite, then the union of these lines is a cubic cone S with vertex v , and $S \subset \mathcal{B}$.*

1.5. Lemma ([PZ18, Lemmas 9.3, 9.4, 9.9(a), Corollaries 9.7.3, 9.7.4, 9.10.1]).

- (i) *Any line on V can be contained in at most finite number of cubic scrolls, and in at most two cubic cones;*
- (ii) *two cubic cones from different components of $\mathcal{S}(V)$ either are disjoint, or share a unique common ruling. Two cubic cones from the same component \mathcal{S}_i meet transversally in a single point different from their vertices;*
- (iii) *if two cubic cones S and S' are disjoint, then $v_S \notin A_{S'}$, where v_S stands for the vertex of S ;*
- (iv) *if two cubic cones S and S' share a common ruling l , then $S \cup S'$ coincides with the union of lines on V meeting l .*

1.6. Lemma ([PZ18, Lemma 9.7.2, Corollary 9.7.3(i)]). *In the notation of Theorem 1.1(a) the Hilbert scheme of lines $\Sigma(V) = \text{Fl}(\mathbb{P}^2)$ is a divisor of type $(1, 1)$ on $\mathbb{P}^2 \times (\mathbb{P}^2)^\vee$. Let*

$$(1.7) \quad \text{pr}_1 : \Sigma(V) = \text{Fl}(\mathbb{P}^2) \longrightarrow \mathbb{P}^2 \quad \text{and} \quad \text{pr}_2 : \Sigma(V) = \text{Fl}(\mathbb{P}^2) \longrightarrow (\mathbb{P}^2)^\vee$$

be the natural projections. Any line l on V is the common ruling of exactly two cubic scrolls, say, $S_1(l) \in \mathcal{S}_1$ and $S_2(l) \in \mathcal{S}_2$. Hence, there are well-defined morphisms

$$\mathrm{pr}_i: \Sigma(V) \longrightarrow \mathcal{S}_i = \mathbb{P}^2, \quad l \longmapsto S_i(l), \quad i \in \{1, 2\}$$

which coincide with the ones in (1.7). The fiber of pr_i over $S \in \mathcal{S}_i$ is the line on $\Sigma(V)$ which parameterizes the rulings of S .

Recall that the Fano-Mukai fourfolds $V = V_{18}$ are classified in three types according to the group $\mathrm{Aut}^0(V)$, which can be isomorphic to one of the following groups

$$\mathbb{G}_m^2, \quad \mathbb{G}_a \times \mathbb{G}_m, \quad \mathrm{GL}_2(\mathbb{C}).$$

This classification reflects the geometry of V , namely, the number of cubic cones on V .

1.8. Lemma. *Let $\mathcal{S}_i \cong \mathbb{P}^2$ be a connected component of $\mathcal{S}(V)$. Then the following hold.*

- (i) *If $\mathrm{Aut}^0(V) = \mathbb{G}_m^2$ then \mathcal{S}_i contains exactly 3 cubic cones for $i = 1, 2$. The six cubic cones in V form a cycle so that the neighbors have a common ruling and belong to different components of $\mathcal{S}(V)$, and the pairs of opposite vertices of the cycle correspond to the pairs of disjoint cubic cones.*
- (ii) *If $\mathrm{Aut}^0(V) = \mathbb{G}_a \times \mathbb{G}_m$ then \mathcal{S}_i contains exactly 2 cubic cones for $i = 1, 2$. The four cubic cones in V form a chain, that is, the neighbors have a common ruling and belong to different components of $\mathcal{S}(V)$, and the pair of extremal vertices corresponds to the unique pair of disjoint cones.*
- (iii) *If $\mathrm{Aut}^0(V) = \mathrm{GL}_2(\mathbb{C})$ then the subfamily of cubic cones in $\mathcal{S}_i \cong \mathbb{P}^2$ consists of a projective line and an isolated point.*

Proof. By Lemma 1.6 the variety $\Lambda(S) \subset \Sigma(V)$ of rulings of a cubic scroll $S \subset V$ is the fiber of one of the projections pr_i , so it is a line under the Segre embedding $\Sigma(V) \subset \mathbb{P}^2 \times (\mathbb{P}^2)^\vee \hookrightarrow \mathbb{P}^8$, and any line on $\Sigma(V)$ appears in this way. A line l on V is called a *splitting line* if the union of lines on V meeting l splits into a union of two cubic scrolls. Assuming $\mathrm{Aut}^0(V) \neq \mathrm{GL}_2(\mathbb{C})$ the subvariety $\Sigma_s(V) \subset \Sigma(V)$ of splitting lines is a del Pezzo sextic, which admits two birational contractions to \mathbb{P}^2 [PZ18, Proposition 10.2]. The scroll S on V is a cubic cone exactly when the line $\Lambda(S)$ lies on $\Sigma_s(V)$ [PZ18, Proposition 9.10]. The surface $\Sigma_s(V)$ is smooth in case (i), and has a unique node in case (ii) [PZ18, Proposition 10.2]. It is well known that a smooth sextic del Pezzo surface contains exactly 6 lines, and these are arranged in a cycle. If $\Sigma_s(V)$ has a singularity of type A_1 , then it contains exactly 4 lines, and these are arranged in a chain.

In the both cases, a pair of intersecting lines on $\Sigma_s(V)$ corresponds to a pair of cubic cones on V sharing a common ruling. By Lemma 1.5(iv), such cones belong to distinct components of $\mathcal{S}(V)$. Thus, two neighbors of the same cubic cone belong to the same component \mathcal{S}_i and meet transversally in a unique point, and two cubic cones separated by two others belong to distinct components of $\mathcal{S}(V)$ and are disjoint, see Lemma 1.5(ii). This gives (i) and (ii). See [PZ18, Corollary 10.3.2] for (iii). \square

1.9. Corollary ([PZ18, Lemmas 12.2 and 12.8.1]). *If $\mathrm{Aut}^0(V) = \mathbb{G}_m^2$ or $\mathbb{G}_a \times \mathbb{G}_m$ then any cubic cone in V is $\mathrm{Aut}^0(V)$ -invariant. If $\mathrm{Aut}^0(V) = \mathrm{GL}_2(\mathbb{C})$ then \mathcal{S}_i contains exactly one $\mathrm{Aut}^0(V)$ -invariant cubic cone.*

1.10. Lemma. *Let $\mathcal{S}_1 \subset \mathcal{S}(V)$ be a connected component. Then the set $\bigcap_{S \in \mathcal{S}_1} A_S$ coincides with the union of vertices of cubic cones in \mathcal{S}_1 . In particular, $\bigcap_{S \in \mathcal{S}(V)} A_S = \emptyset$.*

Proof. Let $v \in V$ be the vertex of a cubic cone $S_v \in \mathcal{S}_1$. By Lemma 1.2 one has $S \cap S_v \neq \emptyset$ for any $S \in \mathcal{S}_1$, and so, $v \in \bigcap_{S \in \mathcal{S}_1} A_S$ due to Lemma 1.3.

Conversely, let $v \in \bigcap_{S \in \mathcal{S}_1} A_S$. Assume to the contrary that the lines on V passing through v form a finite set, say, $\{l_1, \dots, l_k\}$. By Lemmas 1.4(iii) and 1.5(i), the number of lines on V passing through a general point of l_i is finite. Hence, the family $\Sigma(V; l_1, \dots, l_k)$ of lines in V meeting $\bigcup_{i=1}^k l_i$ is one-dimensional, and again by Lemma 1.5(i), any line in this family is contained in a finite number of cubic scrolls. It follows that the subfamily $\mathcal{S}_1(l_1, \dots, l_k)$ of cubic scrolls from \mathcal{S}_1 meeting $\bigcup_{i=1}^k l_i$ is one-dimensional too. Since $\dim \mathcal{S}_1 = 2$, see Theorem 1.1(b), one has $\mathcal{S}_1(l_1, \dots, l_k) \neq \mathcal{S}_1$, and the general cubic scroll $S \in \mathcal{S}_1$ does not meet $\bigcup_{i=1}^k l_i$. This implies $v \notin A_S$, a contradiction.

Thus, any point $v \in \bigcap_{S \in \mathcal{S}_1} A_S$ is the vertex of a cubic cone, say, S_v . Then one has $S_v \cap S \neq \emptyset$ for any $S \in \mathcal{S}_1$. Assuming $S_v \notin \mathcal{S}_1$ it follows from Lemma 1.2 that for any $S \in \mathcal{S}_1$ the intersection $S_v \cap S$ contains a curve. Then any line in S_v meets S , and so, $S_v \subset \bigcap_{S \in \mathcal{S}_1} A_S$. By the preceding, any point of S_v is a vertex of a cubic cone in V . This contradicts Lemma 1.5(i) and proves the first assertion. The second assertion follows from the first since two distinct cubic cones cannot share the same vertex, see Lemma 1.4(iii). \square

For instance, in the case $\text{Aut}^0(V) = \text{GL}_2(\mathbb{C})$ the intersection $\bigcap_{S \in \mathcal{S}_i} A_S$ is the twisted cubic Γ_i , $i = 1, 2$ as in Lemma 2.2 below, see [PZ18, Theorem 13.5(b)].

2. $\text{Aut}^0(V)$ -ACTION ON THE FANO-MUKAI FOURFOLD V OF GENUS 10

Generalities.

- 2.1. Lemma.** (i) *Under the induced $\text{Aut}^0(V)$ -action on the Hilbert scheme of lines $\Sigma(V)$, the stabilizer of the general point is trivial.*
(ii) *Under the induced $\text{Aut}^0(V)$ -action on the Hilbert scheme of cubic scrolls $\mathcal{S}(V)$, the stabilizer of the general point is finite.*

Proof. Statement (i) follows from the fact that through the general point of V pass at least two lines, see Lemma 1.4(i), and two lines meet in a single point. In turn, (ii) follows from the fact that through the general line l on V pass a finite number of cubic scrolls, and l is a component of the intersection of these scrolls, see Lemma 1.5(i). \square

2.2. Lemma. *Let $S_1, S_2 \subset V$ be disjoint cubic cones, and let $\Gamma_1 = S_1 \cap A_{S_2}$, $\Gamma_2 = S_2 \cap A_{S_1}$. Then the following holds.*

- (i) Γ_1 and Γ_2 are rational twisted cubic curves;
- (ii) there exists a one-parameter family of lines l_t in V joining Γ_1 and Γ_2 ;
- (iii) any line in V passing through a point $\gamma \in \Gamma_i$ is either a ruling of S_i , or a unique member of the family l_t ;
- (iv) $D := \bigcup_{t \in \mathbb{P}^1} l_t$ is a rational normal sextic scroll contained in $A_{S_1} \cap A_{S_2}$;
- (v) If S_1 and S_2 are $\text{Aut}^0(V)$ -invariant then D and the curves Γ_i , $i = 1, 2$, are as well, and the stabilizer of the general point of D in $\text{Aut}^0(V)$ is finite.

Proof. Since $S_1 \cap S_2 = \emptyset$ one has $v_i \notin A_{S_j}$ for $i, j = 1, 2$, $i \neq j$, see Lemma 1.5(iii). This yields (i). By Lemma 1.3(i)–(ii) for any $\gamma \in \Gamma_1$ there exists a unique line $l_\gamma \subset A_{S_1} \cap A_{S_2}$ joining γ and S_2 . This line l_γ meets $\Gamma_2 = S_2 \cap A_{S_1}$. This shows (ii) and the inclusion $D \subset A_{S_1} \cap A_{S_2}$, where D is as in (iv). Statement (iii) follows from Lemma 1.4(i)–(iii). For the first assertion in (iv) see, e.g., [Har92, Example 8.17]. The first assertion in (v) is immediate. To prove the second, we assume $\text{Aut}^0(V)$ to be abelian. In the case $\text{Aut}^0(V) = \text{GL}_2(\mathbb{C})$ one can either restrict to the maximal torus of $\text{GL}_2(\mathbb{C})$, or simply deduce the result from [PZ18, Theorem 13.5]. Suppose G is a one-parameter subgroup of $\text{Aut}^0(V)$ acting trivially on D . Then G fixes the general line meeting D . So, there is

a two-dimensional subvariety $\Sigma' \subset \Sigma(V)$ parameterizing G -invariant lines on V . By the description of $\Sigma(V)$, see Theorem 1.1(a) and Lemma 1.6, the group G acts trivially on $\Sigma(V)$ and on V , a contradiction. \square

2.3. Lemma. *Let S be an $\text{Aut}^0(V)$ -invariant cubic cone in V , and let G_0 be a two-dimensional connected abelian subgroup of $\text{Aut}^0(V)$. Then G_0 acts on S with an open orbit.*

Proof. We repeat the argument from the proof of the last statement in Lemma 2.2(v). By Lemmas 1.3(ii) and 1.4(ii)–(iii) though the general point P of S passes a unique line different from the ruling of S through P . Hence, the family of lines on V meeting S is two-dimensional. If the G_0 -action on S does not have an open orbit, then all the lines in this family have a common one-dimensional stabilizer G . As before, this stabilizer acts trivially on $\Sigma(V)$ and on V , a contradiction. \square

The case $\text{Aut}^0(V) = \mathbb{G}_m^2$. In this subsection we prove Theorems B and A in the case $\text{Aut}(V) = \mathbb{G}_m^2$, that is, we let $V = V_{18}^g$.

2.4. The Fano-Mukai fourfold V with $\text{Aut}^0(V) = \mathbb{G}_m^2$ contains exactly six cubic cones, see Lemma 1.8(i). By Lemma 1.4(iii), any cubic cone $S \subset V$ coincides with the union of lines in V passing through its vertex, is $\text{Aut}^0(V)$ -invariant, and its vertex is fixed under $\text{Aut}^0(V)$. By Lemma 1.4(iii), the vertices of distinct cubic cones are distinct. Using Lemma 6.14 we deduce the following corollary.

2.5. Corollary. *Let $\text{Aut}^0(V) = \mathbb{G}_m^2$. Then the vertices v_i of the cubic cones S_i , $i = 1, \dots, 6$, are the only fixed points of the torus $T = \text{Aut}^0(V)$ acting on V .*

The next corollary yields Theorem B(b).

2.6. Corollary. *One has $\bigcap_{i=1}^6 A_{S_i} = \emptyset$.*

Proof. Assume the contrary holds. Then by the Borel fixed point theorem, the intersection $\bigcap_{i=1}^6 A_{S_i}$ contains a fixed point of the torus $\text{Aut}^0(V) = \mathbb{G}_m^2$. By Corollary 2.5 this point is the vertex of a cubic cone, say $S_1 \in \mathcal{S}_1$. By Lemma 1.5(iii), S_1 meets any of the remaining cones S_i , $i = 2, \dots, 6$. In particular, S_1 meets the cones, say S_2 , S_4 , and S_6 , which belong to the other component \mathcal{S}_2 of $\mathcal{S}(V)$. Then $S_1 \cdot S_i = 0$ for $i = 2, 4, 6$, see Lemma 1.2. By Lemma 1.5(ii), S_1 shares a common ruling with S_2 , S_4 , and S_6 . These three rulings of S_1 are $\text{Aut}^0(V)$ -invariant and pairwise distinct, see [PZ18, Corollary 9.7.3(ii)]. It follows that the $\text{Aut}^0(V)$ -action on the base of S_1 is trivial, and so, any ruling of S_1 is invariant. However, this contradicts the fact that the torus \mathbb{G}_m^2 acts on S_1 with an open orbit, see Lemma 2.3. \square

2.7. Remark. An alternative argument is as follows. By Lemma 1.8 one has $S_1 \cap S_4 = \emptyset$, and so, $v_{S_1} \notin A_{S_4}$ due to Lemma 1.5(iii). This gives again a contradiction.

Proof of Theorem A in the case $\text{Aut}(V) = \mathbb{G}_m^2$. Recall that the affine space \mathbb{A}^4 is a flexible variety, see, e.g., [KZ99, Lemma 5.5]. By Proposition 3.1, V is covered by the flexible Zariski open subsets $U_i = V \setminus A_{S_i} \cong \mathbb{A}^4$, $i = 1, \dots, 6$. Thus, the criterion of Theorem 3.6(ii) applies and gives the result. \square

The case $\text{Aut}^0(V) = \mathbb{G}_a \times \mathbb{G}_m$. In this subsection we let $V = V_{18}^a$.

2.8. Lemma. Any effective action of $\mathbb{G}_a \times \mathbb{G}_m$ on \mathbb{P}^2 can be given in suitable coordinates by the matrices

$$(2.9) \quad \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & \mu \\ 0 & 0 & 1 \end{pmatrix} \quad \text{where } \lambda \in \mathbb{C}^*, \mu \in \mathbb{C}.$$

This action has exactly two invariant lines $J_1, J_2 \subset \mathbb{P}^2$ and exactly two fixed points $P_0 \in J_1 \cap J_2$ and $P_1 \in J_2 \setminus J_1$, see Fig. 2.A.

Proof. Since the group $\mathbb{G}_a \times \mathbb{G}_m$ is abelian and acts effectively on \mathbb{P}^2 , (2.9) is the only possibility for the Jordan normal form of its elements modulo scalar matrices. \square

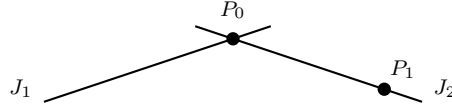


FIGURE 2.A.

2.10. Proposition. For $V = V_{18}^a$ the following assertions hold.

- (i) The action of $\text{Aut}^0(V) = \mathbb{G}_a \times \mathbb{G}_m$ on each component $\mathcal{S}_i \cong \mathbb{P}^2$ of $\mathcal{S}(V)$, $i = 1, 2$ is given by matrices (2.9), and the action on $\Sigma(V) \cong \text{Fl}(\mathbb{P}^2)$ is the induced one.
- (ii) The subfamily \mathcal{S}_1^a of \mathbb{G}_a -invariant cubic scrolls corresponds to the line J_2 on $\mathcal{S}_1 = \mathbb{P}^2$, and the subfamily \mathcal{S}_1^m of \mathbb{G}_m -invariant cubic scrolls corresponds to J_1 , see Fig. 2.A.
- (iii) There are exactly three $\text{Aut}^0(V)$ -invariant lines $l_{i,i+1}$, $i \in \{1, 2, 3\}$ and exactly four $\text{Aut}^0(V)$ -invariant cubic scrolls S_i , $i = 1, \dots, 4$ on V . With a suitable enumeration, $l_{i,i+1}$ is a common ruling of S_i and S_{i+1} , while S_i and S_j have no common ruling if $j - i \neq \pm 1$. Furthermore, S_i and S_j belong to the same connected component of $\mathcal{S}(V)$ if and only if $j \equiv i \pmod{2}$.¹
- (iv) Any cubic scroll S_i in (iii) is a cubic cone, and any cubic cone on V coincides with one of the S_i 's.
- (v) There are exactly four $\text{Aut}^0(V)$ -fixed points v_1, \dots, v_4 on V . These points are the vertices of cubic cones S_1, \dots, S_4 . Furthermore, one has
$$v_2 = l_{1,2} \cap l_{2,3} = S_1 \cap S_3, \quad v_3 = l_{2,3} \cap l_{3,4} = S_2 \cap S_4, \quad S_1 \cap S_4 = \emptyset.$$
The three lines $l_{i,i+1}$ form a chain.
- (vi) There are exactly two families of \mathbb{G}_a -invariant lines on V . These are the families of rulings of S_2 and S_3 .
- (vii) There are exactly three families of \mathbb{G}_m -invariant lines on V . These are the families of rulings of S_1, S_4 , and of the smooth sextic scroll $D \subset V$, see 2.11.

Collecting the information from Proposition 2.10 we see that the configuration of the cones S_i looks as the one on Fig 2.B.

2.11. Notation. There is exactly one pair of disjoint cubic cones on V , namely, (S_1, S_4) . We let $D \subset V$ be the smooth sextic scroll as in Lemma 2.2(iv), that is, the union of lines on V joining the corresponding points of the rational twisted cubic curves $\Gamma_1 = S_1 \cap A_{S_4}$ and $\Gamma_4 = S_4 \cap A_{S_1}$. Notice that the curves Γ_1 and Γ_4 , as well as the surface D are $\text{Aut}^0(V)$ -invariant.

¹This gives an alternative proof of Lemma 1.8(ii).

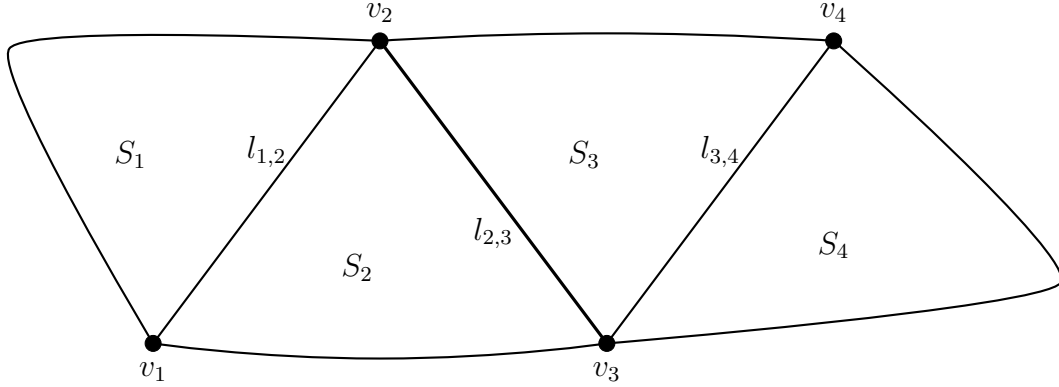


FIGURE 2.B.

Proof of Proposition 2.10. Assertions (i) and (ii) are immediate, see Lemma 2.8. Assertion (iii) follows from Lemmas 1.6 and 2.8.

(iv). Any cubic cone on V is $\text{Aut}^0(V)$ -invariant, hence, is contained in $\{S_1, \dots, S_4\}$, see Corollary 1.9. We have to show only that any invariant cubic scroll is a cubic cone. Actually, this follows from Lemma 1.8(ii), however, we provide an alternative proof.

Suppose to the contrary S_2 is smooth, and so $S_2 \cong \mathbb{F}_1$. Then its exceptional section is an invariant line, hence, it coincides with $l_{3,4}$. By [PZ18, Proposition 9.5] the corresponding connected component of $\mathcal{S}(V)$ contains a cubic cone, which must coincide with S_4 due to (iii). Let l be the ruling of S_2 passing through the vertex v_4 of S_4 . Then l is a common ruling of S_2 and S_4 . This contradicts (iii). Thus, S_2 is a cubic cone. By symmetry, S_3 is a cubic cone too.

Suppose further S_1 is smooth. If S_1 contains the vertex v_3 of S_3 , then S_1 and S_3 have a common ruling, a contradiction. Thus, $v_3 \notin S_1$, and so, S_1 does not contain the invariant lines $l_{2,3}$ and $l_{3,4}$. It follows that the ruling $l_{1,2}$ of S_1 is the only $\text{Aut}^0(V)$ -invariant line on S_1 , which coincides then with the exceptional section of S_1 , a contradiction. Thus, S_1 is a cubic cone, and by symmetry, the same holds for S_4 .

(v) Notice that the vertices of the cubic cones in V are fixed by $\text{Aut}^0(V)$, because these cones are invariant. Let v be a fixed point of the $\text{Aut}^0(V)$ -action on V different from the vertices of cubic cones. According to Lemma 1.4(iii), the number of lines on V passing through v is finite. Hence each of these lines is $\text{Aut}^0(V)$ -invariant. By (iii) such a line coincides with the common ruling $l_{i,i+1}$ of a pair (S_i, S_{i+1}) , $i \in \{1, 2, 3\}$. However, any line $l_{i,i+1}$ contains exactly two $\text{Aut}^0(V)$ -fixed points, namely, the vertices v_i and v_{i+1} . The remaining statements are immediate.

(vi) and (vii). By Theorem 1.1(a) we have $\Sigma(V) \cong \text{Fl}(\mathbb{P}^2)$. By Lemma 2.8 the group $\text{Aut}^0(V) \cong \mathbb{G}_a \times \mathbb{G}_m$ acts on $\Sigma(V) \cong \text{Fl}(\mathbb{P}^2)$ via (2.9). Looking at Fig.2.A one can select all the lines on V with one-dimensional stabilizers. There are exactly five such families of lines; they correspond to the following five families of flags on \mathbb{P}^2 :

- (P_0, l) , where l runs over the pencil of lines through P_0 ;
- (P_1, l) , where l runs over the pencil of lines through P_1 ;
- (P, J_2) , where P runs over J_2 ;
- (P, J_1) , where P runs over J_1 ;
- (P, l) , where P runs over J_1 and l passes through P_1 .

The last family corresponds to the family of rulings of D , and the other four correspond to the families of rulings of the cubic cones S_1, \dots, S_4 . \square

3. AFFINE 4-SPACES IN V_{18}^a AND FLEXIBILITY OF AFFINE CONES

Affine 4-spaces in V_{18}^a . In this subsection we analyze affine charts isomorphic to \mathbb{A}^4 on the Fano-Mukai fourfold $V = V_{18}^a$ of genus 10 with $\text{Aut}^0(V) = \mathbb{G}_a \times \mathbb{G}_m$, and provide a modified criterion of flexibility of affine cones over projective varieties in terms of existence of certain cylinders.

The following proposition proves the first part of Theorem B(c); the second part will be proven in Proposition 5.3.

3.1. Proposition. *In the notation of Proposition 2.10 one has $\bigcap_{j=1}^4 A_{S_j} = l_{2,3}$.*

The proof is done below. We need the following auxiliary facts. Let $Y = \bigcap_{j=1}^4 A_{S_j}$. Clearly, Y is $\text{Aut}^0(V)$ -invariant.

3.2. Claim. *One has $l_{2,3} \subset Y$.*

Proof. The line $l_{2,3}$ intersects all the S_i . Hence, $l_{2,3} \subset A_{S_i}$ for all i . □

3.3. Claim. *$S_1 \cap Y = \{v_2\}$ and $S_4 \cap Y = \{v_3\}$.*

Proof. We have $v_1 \notin A_{S_4}$ because $S_1 \cap S_4 = \emptyset$. Hence $S_1 \cap A_{S_4}$ is a smooth irreducible hyperplane section of the cone S_1 . Since $v_1 \in A_{S_2}$, the intersection $S_1 \cap A_{S_2}$ is a singular hyperplane section of S_1 . Therefore, the intersection $S_1 \cap Y$ is a finite set, whose points are fixed by $\text{Aut}^0(V)$. Then by Proposition 2.10(v) and Claim 3.2 we have $S_1 \cap Y = \{v_2\}$. □

3.4. Claim. *$S_2 \cap Y = S_3 \cap Y = l_{2,3}$.*

Proof. We have $v_2 \in S_2 \cap Y$ and $S_2 \not\subset Y$ because $l_{1,2} \not\subset Y$ by Claim 3.3. Since $S_2 \cap Y = S_2 \cap \langle Y \rangle$ the intersection $S_2 \cap Y$ consists of a finite number of rulings of S_2 . These rulings are $\text{Aut}^0(V)$ -invariant. Since $l_{1,2} \not\subset Y$ by Claim 3.3 the only possibility is $S_2 \cap Y = l_{2,3}$, see Proposition 2.10(iii). □

Proof of Proposition 3.1. Assume there is a point $P \in Y \setminus l_{2,3}$. By Claims 3.3 and 3.4 $P \notin S_j$ for all j . By Lemma 1.3(ii) for any $j = 1, \dots, 4$ through P passes a unique line $l_j \subset A_{S_j}$ meeting S_j . Since there are at most three lines on V passing through P , one has $l_i = l_j$ for some $i \neq j$. Set $l := l_i = l_j$. Thus, one has $P \in l \subset A_{S_i} \cap A_{S_j}$. By Lemma 1.4(iii), V contains no plane. Hence, the set of lines contained in the surface $A_{S_i} \cap A_{S_j}$ has dimension at most one, cf. [KPS18, Lemma A.1.1]. Since $l \not\subset S_j$ for all j , the line l cannot be $\text{Aut}^0(V)$ -invariant, see Proposition 2.10(iii). Therefore, the stabilizer G of l under the $\text{Aut}^0(V)$ -action on $\Sigma(V)$ is one-dimensional. Since $l \not\subset S_j$ for all j , due to Proposition 2.10(vi)-(vii), l is a ruling of D , and so, is \mathbb{G}_m -invariant. The rulings l and $l_{2,3}$ of D are disjoint, because $P \notin l_{2,3}$ by our choice. It follows from Claim 3.4 that $l \cap S_2 = l \cap S_3 = \emptyset$. So, we have $\{i, j\} = \{1, 4\}$, that is, l meets S_1 and S_4 in \mathbb{G}_m -fixed points. Besides, there exists a unique line $l_2 \neq l$ in V joining P and S_2 . If the stabilizer of P in G is finite, then $l = \overline{G \cdot P} \subset Y$. In particular, the intersection point P_1 of l and S_1 lies on Y . By Claim 3.3 one has $P_1 = v_2$, contrary to the fact that $l \cap S_2 = \emptyset$. Hence, P is fixed by \mathbb{G}_m , and then the line l_2 is \mathbb{G}_m -invariant. Since l_2 is not a ruling of D or of one of the S_i , we get a contradiction with Proposition 2.10(vii). □

Criteria of flexibility of affine cones. To formulate the flexibility criteria, we need the following notions.

3.5. Definition ([Per13, Definitions 3–4]). An open covering $(U_i)_{i \in I}$ of a projective variety V by the \mathbb{A}^1 -cylinders $U_i \cong \mathbb{A}^1 \times Z_i$ is called *transversal* if it does not admit any proper invariant subset. A subset $Y \subset V$ is proper if it is nonempty and different from V . It

is called *invariant* with respect to this covering if for any cylinder $U_i \rightarrow Z_i$, $i \in I$, the intersection $Y \cap U_i$ is covered by the fibers of $U_i \rightarrow Z_i$.

3.6. Theorem. *Let (V, H) be a polarized smooth projective variety. Then the affine cone over (V, H) is flexible if one of the following holds:*

- (i) ([Per13, Theorem 5]) *V admits a transversal covering by a family of \mathbb{A}^1 -cylinders $U_i = V \setminus \text{Supp}(D_i) \cong \mathbb{A}^1 \times Z_i$, where Z_i is a smooth affine variety, $i \in I$;*
- (ii) ([MPS18, Theorem 1.4]) *V admits a covering by a family of flexible Zariski open subsets $U_i = V \setminus \text{Supp}(D_i)$, $i \in I$,*

where in the both cases the D_i are effective \mathbb{Q} -divisors on V with $D_i \sim_{\mathbb{Q}} H$, $i \in I$.

For instance, the affine cone over (V, H) is flexible provided one can find an open covering $\{U_i\}_{i \in I}$ of V by toric affine varieties $U_i = V \setminus \text{Supp}(D_i)$ with no torus factor, that is, non-decomposable as a product $U_i = W_i \times (\mathbb{A}^1 \setminus \{0\})$. Indeed, any such variety U_i is flexible [AZK12, Theorem 2.1]. In the simplest case where $\text{Pic}(V) = \mathbb{Z}$ and V admits an open covering by the affine spaces, the affine cone X over (V, H) is flexible whatever is an ample polarization H of V . According to Theorem B(a),(b), such a covering exists for any Fano-Mukai fourfold V_{18} with a reductive automorphism group. In the remaining case of $V = V_{18}^a$ with $\text{Aut}^0(V) = \mathbb{G}_a \times \mathbb{G}_m$ we ignore whether V admits a covering by flexible affine charts, cf. Proposition 3.1. Hence, we cannot apply the criterion of Theorem 3.6(ii) in this case. Instead, we will apply the following version, which mixes the two criteria of Theorem 3.6.

Let (V, H) be a polarized projective variety, and let A be an effective divisor on V . We say that an open set $U = V \setminus \text{Supp}(A)$ is *polar* if $A \sim_{\mathbb{Q}} H$.

3.7. Proposition. *Let (V, H) be a smooth projective variety of dimension $n \geq 3$ with an ample polarization. Suppose V possesses a family of flexible polar \mathbb{A}^1 -cylinders $U_i = V \setminus \text{Supp}(A_i) \cong \mathbb{A}^1 \times Z_i$, $i \in I$, where Z_i is an affine variety of dimension $n - 1$. Assume further that*

- *$D := \bigcap_{i \in I} A_i$ is a subvariety of V of dimension $m \leq n - 2$;*
- *through any point of V pass at most k components of D ;*
- *any point $P \in D$ is contained either in a principal open polar \mathbb{A}^{m+1} -cylinder $U_P \cong \mathbb{A}^{m+1} \times Z_P$ in V , or in $k + 1$ principal polar open \mathbb{A}^m -cylinders $U_{P,j} \cong \mathbb{A}^m \times Z_{P,j}$ in V , $j = 1, \dots, k + 1$, where Z_P and $Z_{P,j}$ are affine varieties, and for any two cylinders $U_{P,j}$ and $U_{P,j'}$ with $j \neq j'$ the \mathbb{A}^m -fibers through P of the natural projections $U_{P,j} \rightarrow Z_{P,j}$ and $U_{P,j'} \rightarrow Z_{P,j'}$ meet properly, that is, the dimension of their intersection is smaller than m .*

Then the affine cone over (V, H) is flexible.

Proof. We use the criterion of Theorem 3.6(i), that is, we show the existence of a transversal covering of V by polar open \mathbb{A}^1 -cylinders. For such a covering, we take the union of the collections $\{U_i\}$, $\{U_P\}$, and $\{U_{P,j}\}$, where each member is endowed with all possible structures of an \mathbb{A}^1 -cylinder.

Let $Y \subset V$ be a nonempty subset invariant with respect to the above covering of V by \mathbb{A}^1 -cylinders, see Definition 3.5. We claim that if $Y \cap U_i \neq \emptyset$ for some $i \in I$, then $Y \supset U_i$. Indeed, let $P \in Y \cap U_i$, and let l be the ruling of the \mathbb{A}^1 -cylinder U_i passing through P . Then $l \subset Y$ because Y is invariant. Since U_i is flexible, for any point $P' \in U_i$ different from P one can find an automorphism $\alpha \in \text{SAut}(U_i)$ such that $\alpha(P) = P'$ and $\alpha^{-1}(P') \in l$. Then $\alpha(l)$ is a ruling of a new \mathbb{A}^1 -cylinder structure on U_i . Since $P, P' \in \alpha(l)$ and $P \in Y$, where Y is invariant, then also $P' \in \alpha(l) \subset Y$. Hence, one has $U_i \subset Y$, as claimed.

It follows that $Y \cap U_j \neq \emptyset$ for any $j \in I$. Thereby, one has $Y \supset \bigcup_{i \in I} U_i = V \setminus D$. Due to our assumptions, for any point $P \in D$ one can choose either a principal open \mathbb{A}^{m+1} -cylinder U_P , or a principal open \mathbb{A}^m -cylinder $U_{P,j}$ such that the \mathbb{A}^m -fiber passing through P of the projection $U_{P,j} \rightarrow Z_{P,j}$ is not contained in D . Then for a suitable \mathbb{A}^1 -cylinder structure on U_P or $U_{P,j}$, respectively, the ruling $l \cong \mathbb{A}^1$ passing through P is not contained in D , and so, meets $V \setminus D \subset Y$. Then one has $P \in l \subset Y$. Since this holds for any $P \in D$, one has $D \subset Y$, and so, $Y = V$, that is, Y is not a proper subset of V .

The latter argument shows as well that a nonempty invariant subset $Y \subset V$ cannot be contained in D . Thus, the criterion of Theorem 3.6(i) applies and yields the result. \square

Using Proposition 3.1 we deduce such a corollary.

3.8. Corollary. *The affine cones over the Fano-Mukai fourfold V with $\text{Aut}^0(V) = \mathbb{G}_a \times \mathbb{G}_m$ are flexible provided any point P of the common ruling $l_{2,3}$ of the cubic cones S_2 and S_3 on V is contained in a principal open \mathbb{A}^2 -cylinder U_P in V .*

In Proposition 5.3 we construct such an open covering of $l_{2,3}$ in V by principal open \mathbb{A}^2 -cylinders. Combining with Corollary 3.8 this gives a proof of Theorem A in the remaining case $\text{Aut}^0(V) = \mathbb{G}_a \times \mathbb{G}_m$.

4. \mathbb{A}^2 -CYLINDERS IN SMOOTH QUADRIC FOURFOLDS AND IN THE DEL PEZZO QUINTIC FOURFOLD

The next lemma on the existence of an \mathbb{A}^2 -cylinder will be used the proof of Proposition 4.5 below.

4.1. Lemma. *Let $Q \subset \mathbb{P}^5$ be a smooth quadric, and let Q', Q^* be distinct hyperplane sections of Q . Let Q_1, \dots, Q_k be the members of the pencil $\langle Q', Q^* \rangle$ generated by Q' and Q^* which have singularities outside $Q' \cap Q^*$, and let P_i be the unique singular point of Q_i . Given a point $P \in Q \setminus (Q' \cup Q^* \cup \{P_1, \dots, P_k\})$ there exists a principal affine open subset $U = U_P \subset Q \setminus (Q' \cup Q^*)$ such that*

- (i) $P \in U$;
- (ii) $U \cong \mathbb{A}^2 \times Z$, where Z is an affine surface.

Proof. Pick a general point $P^\bullet \in Q' \cap Q^*$, and let $\mathbf{T}_{P^\bullet}Q \subset \mathbb{P}^5$ be the embedded tangent space to Q at P^\bullet . The projection with center P^\bullet defines an isomorphism

$$Q \setminus Q^\bullet \cong \mathbb{P}^4 \setminus \mathbb{P}^3 \cong \mathbb{A}^4, \quad \text{where } Q^\bullet := Q \cap \mathbf{T}_{P^\bullet}Q.$$

The quadric cone Q^\bullet with vertex P^\bullet coincides with the union of lines on Q passing through P^\bullet . If the quadric cone $\Delta_P(Q) = Q \cap \mathbf{T}_P Q$ with vertex $P \in Q \setminus (Q' \cap Q^*)$ contains $Q' \cap Q^*$, then $\Delta_P(Q)$ coincides with a member Q_i of the pencil $\langle Q', Q^* \rangle$, which has the singular point $P = P_i$ for some $i \in \{1, \dots, k\}$. However, the latter is excluded by our assumption. So, $Q' \cap Q^* \not\subset \Delta_P(Q)$. Hence, for the general point $P^\bullet \in Q' \cap Q^*$ the line joining P and P^\bullet is not contained in Q . The latter implies $P \notin Q^\bullet$.

The images of $Q' \setminus Q^\bullet$ and $Q^* \setminus Q^\bullet$ in $\mathbb{A}^4 = \mathbb{P}^4 \setminus \mathbb{P}^3$ under the projection with center P^\bullet is a pair of affine hyperplanes with nonempty intersection. Thus, we obtain

$$(4.2) \quad P \in Q \setminus (Q^\bullet \cup Q' \cup Q^*) \cong \mathbb{A}^2 \times (\mathbb{A}^1 \setminus \{\text{a point}\}) \times (\mathbb{A}^1 \setminus \{\text{a point}\}). \quad \square$$

Recall that a smooth del Pezzo quintic fourfold $W = W_5 \subset \mathbb{P}^7$ is unique up to isomorphism [Fuj81]. This variety is quasihomogeneous, more precisely, the automorphism group $\text{Aut}(W)$ has the open orbit $W \setminus R \cong \mathbb{A}^4$ in W , where R is the hyperplane section of W covered by the lines on W which meet the unique $\sigma_{2,2}$ -plane $\Xi \subset W$, see [PZ16,

Section 4]. The planes on W different from Ξ form a one-parameter family, and their union coincides with R ; we call them Π -planes.

4.3. Proposition. *Let $W = W_5 \subset \mathbb{P}^7$ be the del Pezzo quintic fourfold. Then the following hold.*

- (i) ([PZ16, Corollary 2.6]) *The Hilbert scheme $\Sigma(W)$ of lines on W is smooth, irreducible of dimension $\dim \Sigma(W) = 4$. For any point $P \in W$ the Hilbert scheme $\Sigma(W; P) \subset \Sigma(W)$ of lines passing through P has pure dimension 1.*
- (ii) ([PZ16, Proposition 4.11.iv]) *For any line $l \subset W$ there exists a unique hyperplane section B_l of W with $\text{Sing}(B_l) \supset l$. This B_l is the union of lines meeting l .*
- (iii) *Given a point $P \in W$, let Δ_P be the union of lines in W passing through P . If $P \in W \setminus R$, then Δ_P is a cubic cone. If $P \in R \setminus \Xi$, then Δ_P is the union of a plane Π_P passing through P and a quadric cone Δ'_P with vertex P .*
- (iv) *Let $B \subset W$ be a hyperplane section whose singular locus is two-dimensional. Then $B = R$ and $\text{Sing}(B) = \Xi$.*
- (v) *Let $B \subset W$ be a hyperplane section whose singular locus is one-dimensional. Then $B = B_l$ for some line l .*
- (vi) *Let $B \subset W$ be a hyperplane section, and let $C \subset B$ be an irreducible curve. Assume B contains a two-dimensional family of lines meeting C . Then one of the following holds:*
 - (a) *C is contained in a plane on B ;*
 - (b) *$C = l$ is a line, and $B = B_l$;*
 - (c) *$B = R$.*

Proof. (iii) By (i) the universal family of lines $\mathfrak{L}(W) \subset \Sigma(W) \times W$ is smooth, and the natural projection $s : \mathfrak{L}(W) \rightarrow W$ is a flat morphism of relative dimension one. Its fiber $s^{-1}(P)$ is isomorphic to the base of the cone Δ_P . Let $P \in W \setminus R$. Since $W \setminus R$ is the open orbit of $\text{Aut}(W)$, see, e.g., [PZ18, (5.5.5)], the fiber $s^{-1}(P)$ is smooth in this case. Let H be a general hyperplane section of W passing through P . By Bertini's theorem, H is smooth, and by the adjunction formula, H is a del Pezzo threefold of degree 5. It is well known, see, e.g., [Isk80, Ch. 2, § 1.6] or [KPS18, Corollary 5.1.5], that through a general point of H pass exactly three lines. This implies $\deg \Delta_P = 3$. On the other hand, one has $\Delta_P = \mathbf{T}_P W \cap W$. Since W is intersection of quadrics, Δ_P cannot be a cone over a plane cubic. It follows that Δ_P is a cubic cone.

Let further $P \in R \setminus \Xi$. By [PVdV99, Theorem 6.9], $\text{Aut}(W)$ acts transitively on $R \setminus \Xi$. We have $\Delta_P \not\supset \Xi$, and Δ_P contains a Π -plane Π_P passing through P . Such a plane is unique because no two planes on W meet outside Ξ . By the flatness of s we have $\deg \Delta_P = 3$. There are lines on W which are not contained in R and meet $R \setminus \Xi$, and one of these lines passes through P . Therefore, one has $\Delta_P \neq 3\Pi_P$. Then the only possibility is $\Delta_P = \Pi_P + \Delta'_P$, where Δ'_P is a quadric cone.

(iv) Let Z_2 be an irreducible component of $\text{Sing}(B)$ of dimension two. Choose general hyperplane sections H_1 and H_2 of W , and let $C = B \cap H_1 \cap H_2$. By Bertini's theorem, C is an irreducible curve with $\text{Sing}(C) = Z_2 \cap H_1 \cap H_2$. By the adjunction formula one has $p_a(C) = 1$. Hence $\deg Z_2 = 1$, i.e. Z_2 is a plane. Since B contains any line meeting Z_2 we have $B = R$, and then $\text{Sing}(B) = \Xi$.

(v) Let Z_1 be the union of one-dimensional irreducible components of $\text{Sing}(B)$. Since $Z_1 \subset \text{Sing}(B)$ and B is a hyperplane section, B contains any line meeting Z_1 . Hence, B is the union of lines meeting Z_1 , see (i). By (iv) one has $B \neq R$. If Z_1 is a line, then $B = B_{Z_1}$ by (ii). Thus, we may assume Z_1 to be a curve of degree $d > 1$. Consider the general hyperplane section $H \subset B$. By Bertini's theorem, H has exactly d singular

points, and these are the points of $H \cap Z_1$. By the adjunction formula, $-K_H$ is the class of a hyperplane section of H . Hence, H is a normal Gorenstein del Pezzo quintic surface.

By [CT88, Proposition 8.5] we have $d = 2$, that is, H has exactly two singular points, say, P_1 and P_2 , and Z_1 is a conic. Again by [CT88, Proposition 8.5], H contains the line joining P_1 and P_2 . Hence, B contains the linear span $\langle Z_1 \rangle$. Since R is the union of planes contained in W , we have $Z_1 \subset \langle Z_1 \rangle \subset R$. By (iv) one has $\langle Z_1 \rangle \neq \Xi$. So, $\langle Z_1 \rangle = \Pi$ is a Π -plane on R , where $\Pi \subset B$. Take a general line $l \subset \Pi$, and let $Z_1 \cap l = \{P_1, P_2\}$. Since Ξ meets $l \subset \text{Sing}(B_l)$ one has $\Xi \subset B_l$. Likewise, since Ξ meets $Z_1 \subset \text{Sing}(B)$ one has $\Xi \subset B$. Besides, the quadric cones Δ'_{P_1} and Δ'_{P_2} as in (iii) are contained in both B and B_l . Thus, we obtain

$$B \cap B_l \supset \Pi \cup \Xi \cup \Delta'_{P_1} \cup \Delta'_{P_2}.$$

Assuming $B \neq B_l$ the latter contradicts the fact that $\deg(B \cap B_l) = \deg W = 5$.

(vi) We may suppose that $B \neq R$ and the singular locus of B has dimension ≤ 1 , see (iv). By assumption, $C \subset B$ is an irreducible curve, and there exists an irreducible two-dimensional family of lines $\Sigma(B, C) \subset \Sigma(B)$ on B meeting C . Let $r : \mathfrak{L}(C, B) \rightarrow \Sigma(C, B)$ be the universal family, and let $s : \mathfrak{L}(C, B) \rightarrow B$ be the natural projection. If $s(\mathfrak{L}(C, B)) \neq B$, then $s(\mathfrak{L}(C, B))$ is a plane, see, e.g., [KPS18, Lemma A.1.1], and so, C is contained in a plane on B . Assume further $s(\mathfrak{L}(C, B)) = B$, and so, s is a generically étale morphism. We claim that the general line from $\Sigma(C, B)$ meets the singular locus of B . The argument below is well-known, see e.g. [Isk80, Ch. 3, Prop. 1.3] or [KPS18, Lemma 2.2.6], and we repeat it in brief for the sake of completeness. Suppose to the contrary that the general line l from $\Sigma(C, B)$ lies in the smooth locus of B . Using the fact that the restriction of s to $r^{-1}([l])$ is an isomorphism, we may identify l with $r^{-1}([l])$. For the normal bundles of l we have

$$(4.4) \quad \mathcal{N}_{l/\mathfrak{L}(C, B)} = \mathcal{O}_l \oplus \mathcal{O}_l, \quad \mathcal{N}_{l/B} = \mathcal{O}_l(a) \oplus \mathcal{O}_l(-a), \quad a \geq 0.$$

Over the point $l \cap C$ the map s is not an isomorphism. Hence the differential

$$ds : \mathcal{N}_{l/\mathfrak{L}(C, B)} \longrightarrow \mathcal{N}_{l/B}$$

is not an isomorphism either. From (4.4) we see that ds degenerates along l . This means that s is not generically étale, a contradiction.

Thus, the general line from $\Sigma(C, B)$ meets $\text{Sing}(B)$. Since B is not a cone, by our assumption we have $\dim \text{Sing}(B) = 1$. Due to (v) there is a line l_0 on W such that $B = B_{l_0}$. If $l_0 = C$, then we are done. Otherwise, the lines on B passing through the general point $P \in C$ meet $l_0 = \text{Sing}(B)$. The union of these lines is the cone with vertex P over l_0 , that is, a plane. It follows that B is swept out by planes. Since any plane on W is contained in R , we conclude that $B = R$, contrary to our assumption. \square

4.5. Proposition. *Let $B \subset W$ be a hyperplane section. For any point $P \in W \setminus B$ there exists a principal affine open subset $U_P \subset W \setminus B$ such that*

- (i) $P \in U_P$;
- (ii) $U_P \cong \mathbb{A}^2 \times Z_P$, where Z_P is an affine surface.

Proof. If $B = R$ then $W \setminus B \cong \mathbb{A}^4$ [PZ18, Corollary 2.2.2], and the assertion follows. So, we assume in the sequel $B \neq R$. We apply the following construction from [Fuj81] (see also [PZ16, Proposition 4.11]). Fix a line $l \subset W$ not contained in R . There is the Sarkisov link

$$(4.6) \quad \begin{array}{ccc} & \tilde{W} & \\ \rho_W \swarrow & & \searrow \varphi_Q \\ W & \overset{\theta_l}{\dashrightarrow} & Q \end{array}$$

where $Q \subset \mathbb{P}^5$ is a smooth quadric, θ_l is induced by the linear projection with center l , ρ_W is the blowup of l , and φ_Q is the blowup of a smooth cubic scroll $\Lambda = \Lambda_3 \subset Q$. Furthermore, φ_Q sends the ρ_W -exceptional divisor $E_W \subset \tilde{W}$ onto the quadric cone $Q^* = Q \cap \langle \Lambda \rangle$, while the φ_Q -exceptional divisor $\tilde{B}_l \subset \tilde{W}$ is the proper transform of $B_l \subset W$. Let $Q' := \theta_l(B)$.

Suppose our line $l \subset W$ satisfies the following conditions:

- (A) $l \subset B$, $l \not\subset R$, $P \notin B_l$, and
- (B) $\theta_l(P) \notin \{P_1, \dots, P_k\}$, where P_1, \dots, P_k have the same meaning as in Lemma 4.1.

We have an isomorphism

$$W \setminus (B \cup B_l) \cong Q \setminus (Q' \cup Q^*).$$

By Lemma 4.1 there exists a principal affine open subset $\tilde{U}_P \subset Q \setminus (Q' \cup Q^*)$ such that

- (i) $\theta(P) \in \tilde{U}_P$;
- (ii) $\tilde{U}_P \cong \mathbb{A}^2 \times Z_P$, where Z_P is an affine surface.

Then $U_P = \theta^{-1}(\tilde{U}_P) \subset W$ verifies (i)–(ii) of Proposition 4.5.

It remain to show the existence of a line $l \subset W$ satisfying (A) and (B).

Consider the union Δ_P of lines on W passing through P . By Proposition 4.3(iii), Δ_P is a (possibly reducible) cubic cone with vertex P on W of pure dimension two. Recall that $\Delta_P = (\mathbf{T}_P W \cap W)_{\text{red}}$, because W is an intersection of quadrics.

Let $C = \Delta_P \cap B$, and let $\Sigma(B, C)$ be the Hilbert scheme of lines in B which meet C , or, which is equivalent, which meet Δ_P . If $\dim \Sigma(B, C) = 2$ then by Proposition 4.3 (vi), either the lines from $\Sigma(B, C)$ sweep out a plane, say, D on B , or $C = l$ is a line and $B = B_l$. However, the latter case is impossible. Indeed, B_l being singular along $l = C$, any ruling of the cone Δ_P meets $C \subset \text{Sing}(B_l)$, hence is contained in $B_l = B$. Then also $P \in B$, which is a contradiction.

Assume further $\dim \Sigma(B, C) = 1$. Then the lines from $\Sigma(B, C)$ sweep out a surface scroll D on B . Thus, the latter holds whatever is the dimension of $\Sigma(B, C)$.

It follows from Proposition 4.3(i) that the threefold B is covered by lines on W . Since $B \neq R$ by our assumption, there is a point $P' \in B \setminus (D \cup R)$. Any line l through P' on B does not lie on $D \cup R$, and so, does not belong to $\Sigma(B, C)$. Hence, $\Delta_P \cap l = \emptyset$ for the general line l on B . Since B_l is the union of lines on W meeting l , see Proposition 4.3 (ii), we deduce $P \notin B_l$. Thus, the general line l on B satisfies (A).

To show (B), we use the notation from the proof of (A). By the preceding, one has $\Delta_P \cap l = \emptyset$. Hence, the projection θ_l with center l is regular in a neighborhood of Δ_P . It follows that $\Delta_P^Q := \theta_l(\Delta_P)$ is a cone with vertex $P_Q := \theta_l(P)$. Suppose to the contrary that $P_Q = P_i$ for some $i \in \{1, \dots, k\}$. Then P_Q is the vertex of a quadric cone Q_i over $Q' \cap Q^*$, see Lemma 4.1. Clearly, one has $\Delta_P^Q \subset Q_i$. From $l \not\subset D$ we deduce $C \not\subset B_l$.

On the other hand, we have $\theta_l(B_l) = \Lambda$, $Q^* = Q \cap \langle \Lambda \rangle$, and

$$\theta_l(C) = \theta_l(\Delta_P \cap B) \subset \theta_l(\Delta_P) \cap \theta_l(B) = \Delta_P^Q \cap Q' \subset Q_i \cap Q' = Q^* \cap Q' \subset \langle \Lambda \rangle \cap Q.$$

From these inclusions we deduce $B_l = \theta_l^{-1}(\Lambda) \subset \theta_l^{-1}(\langle \Lambda \rangle \cap Q)$. The latter inclusion is an equality, since the both sets are hyperplane section of W . Thus, one has $C \subset B_l$. This contradiction proves (B). \square

5. \mathbb{A}^2 -CYLINDERS IN V_{18} AND FLEXIBILITY OF AFFINE CONES OVER V_{18}^a

In this section we finish the proofs of Theorems A and B in the case where $\text{Aut}^0(V) = \mathbb{G}_a \times \mathbb{G}_m$. The assertion of Theorem A follows immediately in this case by the flexibility criterion of Corollary 3.8 due to the following result.

5.1. Let S be a cubic scroll on $V = V_{18}$. Then either S is smooth and isomorphic to the Hirzebruch surface \mathbb{F}_1 , or S is a cubic cone. By virtue of [PZ18, Proposition 3.1], in both cases the linear projection

$$\theta_S: V = V_{18} \subset \mathbb{P}^{12} \dashrightarrow \mathbb{P}^7$$

with center $\langle S \rangle = \mathbb{P}^4$ restricted to V yields a Sarkisov link

$$(5.2) \quad \begin{array}{ccc} & \tilde{V} & \\ \rho_V \swarrow & & \searrow \varphi_W \\ V & \xrightarrow{\theta_S} & W \end{array}$$

where $W = W_5 \subset \mathbb{P}^7$ is the del Pezzo quintic fourfold, ρ_V is the blowup of S , and φ_W is the blowup of a smooth rational quintic scroll $F = F_5 \subset W$ isomorphic to \mathbb{F}_1 . Furthermore, φ_W sends the ρ_V -exceptional divisor $E_V \subset \tilde{V}$ onto the hyperplane section $B_F := W \cap \langle F \rangle$ of W , while the φ_W -exceptional divisor $\tilde{A}_S \subset \tilde{V}$ is the proper transform of A_S . We have

$$V \setminus A_S \cong W \setminus B_F.$$

If S is a cubic cone, then $B_F = R$ is the hyperplane section of W singular along the Ξ -plane, and $V \setminus A_S \cong W \setminus R \cong \mathbb{A}^4$.

5.3. Proposition. *Let $V = V_{18}$ be a Fano-Mukai fourfold of genus 10. Then for any point $P \in V$ there exists a principal affine open subset $U_P \subset V$ such that*

- (i) $P \in U_P$;
- (ii) $U_P \cong \mathbb{A}^2 \times Z_P$, where Z_P is an affine surface.

Proof. Let $S \subset V$ be a cubic scroll, and let A_S be the hyperplane section of V with $\text{Sing}(A_S) = S$, see Lemma 1.3. The general cubic scroll $S \in \mathcal{S}(V)$ is smooth, that is, not a cubic cone, and $P \notin A_S$ by Lemma 1.10. By Lemma 1.10, for the general $S \in \mathcal{S}(V)$ the map θ_S in (5.2) is well defined at P , and $\theta_S(P) \notin B_F$. Now the assertion follows from Proposition 4.5. \square

6. ON THE AUTOMORPHISM GROUPS OF FANO-MUKAI FOURFOLDS OF GENUS 10

It is known [PZ18, Theorem 1.3.a,b] that

$$\text{Aut}(V_{18}^s) = \text{GL}_2(\mathbb{C}) \rtimes (\mathbb{Z}/2\mathbb{Z}) \quad \text{and} \quad \text{Aut}(V_{18}^a) = (\mathbb{G}_a \times \mathbb{G}_m) \rtimes (\mathbb{Z}/2\mathbb{Z}),$$

where the generator of $\mathbb{Z}/2\mathbb{Z}$ acts on $\text{Aut}^0(V)$ via the involution $g \mapsto (g^t)^{-1}$ in the first case and $g \mapsto g^{-1}$ in the second. In the general case we have $\text{Aut}^0(V) = \mathbb{G}_m^2$ and

$$(6.1) \quad \text{Aut}(V_{18}^g) \subset \mathbb{G}_m^2 \rtimes (\mathbb{Z}/6\mathbb{Z}),$$

see [PZ18, Theorem 1.3.c and Lemma 11.4]. The following Theorem 6.2 and Corollary 6.3 complete these results and provide a proof of Theorem C from the Introduction.

- 6.2. Theorem.** (i) *There exists a unique, up to isomorphism, Fano-Mukai fourfold V^r of genus 10 with $\text{Aut}(V^r) = \mathbb{G}_m^2 \rtimes (\mathbb{Z}/6\mathbb{Z})$, where each generator of $\mathbb{Z}/6\mathbb{Z}$ acts on \mathbb{G}_m^2 via an automorphism of order 6.*
- (ii) *For any Fano-Mukai fourfold V of genus 10 with $\text{Aut}^0(V) = \mathbb{G}_m^2$ non-isomorphic to V^r one has $\text{Aut}(V) = \mathbb{G}_m^2 \rtimes (\mathbb{Z}/2\mathbb{Z})$, where the generator of the factor $\mathbb{Z}/2\mathbb{Z}$ acts on \mathbb{G}_m^2 via $g \mapsto g^{-1}$.*

6.3. Corollary. *For any Fano-Mukai fourfold V of genus 10, $\text{Aut}(V) \setminus \text{Aut}^0(V)$ contains an element of order two which interchanges the members in each pair of disjoint cubic cones on V and acts on $\text{Aut}^0(V)$ via the involution $g \mapsto (g^t)^{-1}$. Furthermore, if $\text{Aut}^0(X)$ is reductive then $\text{Aut}(V) = \text{Stab}_{\text{Ad}(G_2)}(V)$.*

6.4. **Remark.** The latter conclusion does not hold in the exceptional case $V = V_{18}^a$, where $\text{Aut}(V) = (\mathbb{G}_a \times \mathbb{G}_m) \rtimes (\mathbb{Z}/2\mathbb{Z})$. Indeed, the generator of the factor $\mathbb{Z}/2\mathbb{Z}$ admits no extension to an element of $\text{Ad}(G_2)$, see [PZ18, Remark 15.5].

The proofs of Theorem 6.2 and Corollary 6.3 are done at the end of this section. Let us introduce the following notation.

6.5. **Notation.** Consider the Lie algebra \mathfrak{g}_2 of the simple complex algebraic group G_2 , the projective space $\mathbb{P}\mathfrak{g}_2 = \mathbb{P}^{13}$, and the projectivized adjoint representation $\text{Ad}(G_2)$ on $\mathbb{P}\mathfrak{g}_2$. The adjoint variety $\Omega = G_2/P \subset \mathbb{P}^{13}$ is the projectivized minimal nilpotent orbit of G_2 , see [Muk89]. Fix a maximal torus $T \subset P$ of G_2 , and let $\mathfrak{h} = \text{Lie}(T)$ be the corresponding Cartan subalgebra of \mathfrak{h} . Choose a basis of simple roots (α_1, α_2) of the root system $\Delta \subset \mathfrak{h}^\vee$ with $\alpha_1 \in \Delta_s$ and $\alpha_2 \in \Delta_\ell$, where Δ_ℓ and Δ_s stand for the subsets of long and short roots, respectively. We identify \mathfrak{g}_2 with the dual vector space $(\mathfrak{g}_2)^\vee$ via the duality on \mathfrak{g}_2 defined by the Killing form. Under this identification, the $\text{Ad}(G_2)$ -orbits coincide with the corresponding $\text{Ad}^*(G_2)$ -orbits [CMG93, Section 1.3]. Consider the root space decomposition

$$\mathfrak{g}_2 = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta_\ell} \mathfrak{g}_\alpha \right) \oplus \left(\bigoplus_{\alpha \in \Delta_s} \mathfrak{g}_\alpha \right)$$

where $\mathfrak{g}_\alpha = \mathbb{C}e_\alpha$ for $\alpha \in \Delta$ is the root subspace generated by the root vector e_α . One can choose for P the parabolic subgroup of G_2 with Lie algebra

$$\mathfrak{p} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \right) \oplus \mathfrak{g}_{-\alpha_1} = \mathfrak{b} \oplus \mathfrak{g}_{-\alpha_1},$$

where \mathfrak{b} is the corresponding Borel subalgebra of \mathfrak{g}_2 .

6.6. *Mukai presentation revisited.* Recall [Muk89] that any Fano-Mukai fourfold of genus 10 can be presented as a hyperplane section $V(h) = \Omega \cap \mathbb{P}h^\perp$ for a nonzero element $h \in \mathfrak{g}_2 = \text{Lie}(G_2)$, where h^\perp stands for the orthogonal complement of $\mathbb{C}h$ in \mathfrak{g}_2 with respect to the Killing form.

The following two propositions will be important in the proof of Theorem 6.2.

6.7. **Proposition.** *For a nonzero element $h \in \mathfrak{g}_2$ the following hold.*

- (i) *If h is semisimple then $\text{Aut}^0(V(h))$ is reductive;*
- (ii) *if $\text{Aut}^0(V(h))$ is reductive and $V(h)$ is smooth then h is semisimple;*
- (iii) *for any Fano-Mukai fourfold V with reductive group $\text{Aut}^0(X)$ there exist $h \in \mathfrak{h} \setminus \{0\}$ and $g \in G_2$ such that $V = \text{Ad}(g)(V(h))$.*

The proof is done in 6.11 below.

Consider the family \mathfrak{H} of hyperplane sections $V(h)$ of Ω with $h \in \mathfrak{h} \setminus \{0\}$ parameterized by the projective line $\mathbb{P}\mathfrak{h} \subset \mathbb{P}^{13}$.

6.8. **Corollary.** (i) *Any Fano-Mukai fourfold V of genus 10 with reductive group $\text{Aut}^0(X)$ is isomorphic to a member of \mathfrak{H} .*

- (ii) *\mathfrak{H} contains exactly three singular members, exactly three smooth members with $\text{Aut}^0(V) = \text{GL}_2(\mathbb{C})$, and exactly two smooth members with $\text{Aut}(V) = T \rtimes (\mathbb{Z}/6\mathbb{Z})$. For any other member $V(h)$ of \mathfrak{H} one has $\text{Aut}(V(h)) = T \rtimes (\mathbb{Z}/2\mathbb{Z})$.*

Proof. (i) follows from Proposition 6.7, and (ii) follows from Theorem 6.2 and Corollary 6.19 below. \square

For the proof of Proposition 6.7 we need the following facts; see [Tev05, Theorem 8.25].

- 6.9. Lemma.** (i) *The fivefold Ω is the unique closed projectivized nilpotent orbit of the $\text{Ad}(G_2)$ -action on $\mathbb{P}\mathfrak{g}_2 = \mathbb{P}^{13}$. This orbit contains the six T -fixed points $\mathbb{P}\mathfrak{g}_\alpha$ corresponding to the long roots $\alpha \in \Delta_\ell$.*
- (ii) *The dual projective variety D_ℓ of Ω is an irreducible hypersurface in \mathbb{P}^{13} given by a homogeneous polynomial δ_ℓ of degree six on \mathfrak{g}_2 such that $\delta_\ell|_{\mathfrak{h}} = \prod_{\alpha \in \Delta_\ell} \alpha$.*
- (iii) *For a short root $\alpha \in \Delta_s$ the orbit $\Omega_s = \text{Ad}(G_2) \cdot \mathbb{P}\mathfrak{g}_\alpha \subset \mathbb{P}^{13}$ does not depend on the choice of $\alpha \in \Delta_s$.*
- (iv) *The dual projective variety D_s of the orbit closure $\overline{\Omega_s}$ is an irreducible hypersurface in \mathbb{P}^{13} given by a homogeneous polynomial δ_s of degree six on \mathfrak{g}_2 such that $\delta_s|_{\mathfrak{h}} = \prod_{\alpha \in \Delta_s} \alpha$.*

Any orbit of the projectivized adjoint representation is the image of an orbit of the adjoint representation. We keep the terminology “a regular orbit”, “a semisimple orbit”, “a nilpotent orbit”, etc. for the images in \mathbb{P}^{13} of the $\text{Ad}(G_2)$ -orbits in \mathfrak{g}_2 consisting of regular, semisimple, nilpotent, etc. elements, respectively.

6.10. Lemma. *For the adjoint representation $\text{Ad}(G_2)$ on $\mathbb{P}\mathfrak{g}_2 = \mathbb{P}^{13}$ the following hold.*

- (i) ([CMG93, Corollary 2.1.13]) *The complement $\mathbb{P}\mathfrak{g}_2 \setminus (D_\ell \cup D_s)$ is the projectivization of the set of regular semisimple elements of \mathfrak{g}_2 , and $D_\ell \cap D_s$ is the projectivization of the nilpotent cone of \mathfrak{g}_2 ;*
- (ii) ([KR13, Lemma 1]) *$D_s \setminus D_\ell$ is the union of two orbits, namely,*
- *a (non-closed) non-semisimple regular orbit Ω^a of dimension 12, and*
 - *a closed semisimple subregular orbit Ω^{sr} of dimension 10;*
- (iii) ([KR13, Lemma 1]) *let $\mathcal{L} = \langle D_\ell, D_s \rangle$ be the pencil of sextic hypersurfaces in \mathbb{P}^{13} generated by D_ℓ and D_s . Then for any member D_t of \mathcal{L} different from D_ℓ and D_s the complement $D_t \setminus D_\ell$ is a regular semisimple orbit of $\text{Ad}(G_2)$;*
- (iv) *both Ω and Ω_s are nilpotent orbits contained in $D_\ell \cap D_s$.*

Proof. (iv) Both Ω and Ω_s are orbits of root subspaces consisting of nilpotent elements, see Lemma 6.9(i), (iii). The inclusion $\Omega \cup \Omega_s \subset D_\ell \cap D_s$ follows from (i). \square

6.11. Proof of Proposition 6.7. (i) If h is semisimple then the centralizer of h in \mathfrak{g}_2 has dimension 2 if h is regular and 4 otherwise; in the latter case h is subregular, see [PZ18, Proposition 7.7]. In both cases, $\text{Aut}^0(V)$ is a reductive group of rank 2².

(ii) The hyperplane section $V(h)$ of Ω is smooth if and only if $\mathbb{P}(\mathbb{C}h) \in \mathbb{P}^{13}$ is not a point of the dual variety D_ℓ of Ω . If $V(h)$ is smooth and h is not semisimple, then h is regular non-semisimple by Lemma 6.10(i)–(ii). In the latter case $\text{Aut}^0(V(h)) = \mathbb{G}_a \times \mathbb{G}_m$ has rank 1, see [PZ18, Lemma 12.8.1].

(iii) By 6.6 one has $V = V(\tilde{h})$ for some $\tilde{h} \in \mathfrak{g}_2$. By our assumptions, $V = V(\tilde{h})$ is smooth and $\text{rk}(\text{Aut}(V)) = 2$. Hence, \tilde{h} is semisimple due to (ii). So, \tilde{h} is contained in a Cartan subalgebra, say, $\tilde{\mathfrak{h}}$ of \mathfrak{g}_2 . The Cartan subalgebras \mathfrak{h} and $\tilde{\mathfrak{h}}$ being conjugated one has $\tilde{\mathfrak{h}} = \text{Ad}(g)(\mathfrak{h})$ for some $g \in G_2$. Letting $h = (\text{Ad}(g))^{-1}(\tilde{h}) \in \mathfrak{h}$ one gets $V = V(\tilde{h}) = \text{Ad}(g)(V(h))$. \square

6.12. The fixed points of the torus. The hyperplane section $V(h)$ is T -invariant if and only if the vector line $\mathbb{C}h$ is, if and only if

$$(6.13) \quad h \in \mathfrak{h} \cup \left(\bigcup_{\alpha \in \Delta_\ell} \mathfrak{g}_\alpha \right) \cup \left(\bigcup_{\alpha \in \Delta_s} \mathfrak{g}_\alpha \right),$$

²This group coincides with the stabilizer of V in G_2 , see [PZ18, Theorem 1.3].

see [Hum72, Section 8]. The line $\mathbb{C}h$ is T -invariant if and only if its projectivization $\mathbb{P}(\mathbb{C}h)$ is a fixed point of the induced T -action on $\mathbb{P}\mathfrak{g}_2 = \mathbb{P}^{13}$. Using (6.13) and Lemma 6.9 we deduce the following facts.

6.14. Lemma. *The fixed points of the torus T acting on \mathbb{P}^{13} are the points of the projective line $\mathbb{P}\mathfrak{h}$ and the twelve isolated points $\mathbb{P}\mathfrak{g}_\alpha$, where $\alpha \in \Delta_\ell \cup \Delta_s$. The six fixed points $\mathbb{P}\mathfrak{g}_\alpha$ which correspond to the long roots $\alpha \in \Delta_\ell$ are the only fixed points of T on Ω . The six fixed points $\mathbb{P}\mathfrak{g}_\alpha$ which correspond to the short roots $\alpha \in \Delta_s$ lie on the orbit Ω_s .*

For any root $\alpha \in \Delta$ we let \mathfrak{h}_α be the vector line in \mathfrak{h} given by the equation $\alpha = 0$. Let

$$\mathfrak{h}^0 = \mathfrak{h} \setminus \left\{ \prod_{\alpha \in \Delta} \alpha = 0 \right\} = \mathfrak{h} \setminus \bigcup_{\alpha \in \Delta} \mathfrak{h}_\alpha.$$

6.15. Corollary.

- (i) For any root $\alpha \in \Delta$ the fourfold $V(e_\alpha)$ is singular;
- (ii) for $h \in \mathfrak{h} \setminus \{0\}$ the fourfold $V(h)$ is smooth if and only if $h \in \mathfrak{h} \setminus D_\ell = \mathfrak{h} \setminus \bigcup_{\alpha \in \Delta_\ell} \mathfrak{h}_\alpha$;
- (iii) $h \in \mathfrak{h}^0 \implies V(h)$ is smooth and $\text{Aut}^0(V(h)) = \text{Stab}_{G_2}^0(h) = T$;
- (iv) $h \in \bigcup_{\alpha \in \Delta_s} \mathfrak{h}_\alpha \implies V(h)$ is smooth and $\text{Aut}^0(V(h)) = \text{Stab}_{G_2}^0(h) = \text{GL}_2(\mathbb{C}) \supset T$;
- (v) $[h] \in \Omega^a \iff V(h)$ is smooth and $\text{Aut}^0(V(h)) \cong \mathbb{G}_a \times \mathbb{G}_m$.

Proof. Statements (i) and (ii) follow from Lemma 6.9(ii). Statements (iii)-(v) follow from [PZ18, Corollaries 7.8.3 and 13.2.1]. \square

6.16. Remarks. 1. We have $\bigcup_{\alpha \in \Delta_s} \mathbb{P}\mathfrak{h}_\alpha = \mathbb{P}\mathfrak{h} \cap D_s \subset \Omega^{\text{sr}}$.

2. The Cartan subalgebra \mathfrak{h} is $N_{G_2}(T)$ -invariant, and the Weyl group $N_{G_2}(T)/T = W(G_2) = \mathfrak{D}_6$ acts on $\mathbb{P}\mathfrak{h}$. The center $z(W(G_2))$ acts trivially on $\mathbb{P}\mathfrak{h}$. The $N_{G_2}(T)$ -action on \mathfrak{h} induces an effective \mathfrak{S}_3 -action on $\mathbb{P}\mathfrak{h}$, where $\mathfrak{S}_3 = \mathfrak{D}_6/z(\mathfrak{D}_6)$ stands for the symmetric group of rank three. The quotient $\mathbb{P}\mathfrak{h}/\mathfrak{S}_3$ parameterizes the semisimple orbits of $\text{Ad}(G_2)$ in $\mathbb{P}\mathfrak{g}_2$, cf. [CMG93, Theorem 2.2.4]. The line $\mathbb{P}\mathfrak{h}$ does not meet the nilpotent orbits Ω and Ω_s .

6.17. Lemma. *Consider the rational function Φ on \mathbb{P}^{13} whose fibers are the members of the pencil \mathcal{L} generated by D_ℓ and D_s , and let \mathfrak{S}_3 stand for the symmetric group of order 6. The morphism $\varphi = \Phi|_{\mathbb{P}\mathfrak{h}}: \mathbb{P}\mathfrak{h} \cong \mathbb{P}^1 \rightarrow \mathbb{P}^1$ of degree 6 is a Galois \mathfrak{S}_3 -covering given by the homogeneous \mathfrak{S}_3 -invariant rational function on $\mathbb{P}\mathfrak{h}$,*

$$(6.18) \quad \varphi = \left(\prod_{\alpha \in \Delta_\ell^+} \alpha^2 \right) / \left(\prod_{\alpha \in \Delta_s^+} \alpha^2 \right) \quad \text{where} \quad \Delta_\ell^+ = \Delta_\ell \cap \Delta^+ \quad \text{and} \quad \Delta_s^+ = \Delta_s \cap \Delta^+.$$

The ramification locus of φ in $\mathbb{P}\mathfrak{h}$ consists of the following 8 points:

- the six points $\mathbb{P}\mathfrak{h}_\alpha$ given by $\prod_{\alpha \in \Delta} \alpha = 0$; the morphism $\varphi: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is ramified with order two in each of these points;
- the two points which correspond to the one-dimensional eigenspaces of the rotation subgroup $\mathbb{Z}/6\mathbb{Z}$ of \mathfrak{D}_6 ; the morphism φ is ramified with order 3 in each of these points.

The first group of six points $(\mathbb{P}\mathfrak{h}_\alpha)_{\alpha \in \Delta}$ consists of two orbits of length three of the \mathfrak{S}_3 -action on $\mathbb{P}\mathfrak{h}$, and the remaining two points form a single \mathfrak{S}_3 -orbit.

Proof. Any line \mathfrak{h}_α with $\alpha \in \Delta$ is the mirror of a reflexion from $W(g_2) = \mathfrak{D}_6$ acting on \mathfrak{h} . This action is free off \mathfrak{h}_α , and the quotient morphism $\mathfrak{h} \rightarrow \mathfrak{h}/\mathfrak{D}_6 = \mathbb{A}^2$ is ramified to order 2 along \mathfrak{h}_α . The subgroup of rotations $\mathfrak{R} = \mathbb{Z}/6\mathbb{Z} \subset \mathfrak{D}_6$ acts on the set of

mirrors with two orbits, which are the triples of lines \mathfrak{h}_α which correspond to the short and the long roots, respectively. The two eigenspaces of \mathfrak{R} in \mathfrak{h} are components of the ramification divisor with ramification of order 3 along each of them. Any reflection from \mathfrak{D}_6 with mirror \mathfrak{h}_α , where α is a short root, interchanges these eigenspaces. This proves the lemma. \square

Let $s, l \in \mathbb{P}\mathfrak{h}/\mathfrak{S}_3 \cong \mathbb{P}^1$ denote the points which correspond, respectively, to the two \mathfrak{S}_3 -orbits of length 3, and let $r \in \mathbb{P}\mathfrak{h}/\mathfrak{S}_3$ correspond to the unique \mathfrak{S}_3 -orbit of length 2. The following corollary is immediate.

6.19. Corollary. *The projective line $\mathbb{P}\mathfrak{h} \subset \mathbb{P}^{13}$ meets any sextic D_t from the pencil $\mathcal{L} = \langle D_\ell, D_s \rangle$ different from D_ℓ , D_s , and D_r in six distinct points outside D_ℓ . It meets D_ℓ , resp., D_s in the triple $\{\mathbb{P}\mathfrak{h}_\alpha\}_{\alpha \in \Delta_\ell}$, resp., $\{\mathbb{P}\mathfrak{h}_\alpha\}_{\alpha \in \Delta_s}$ with simple tangency in each of these six points, and meets D_r in two points with local intersection indices 3 in each of these points.*

6.20. Lemma. *Let $S \subset V$ be a cubic cone. If $\text{Aut}^0(V) = \mathbb{G}_m^2$ then $\text{Aut}(V, S) = \text{Aut}^0(V)$.*

Proof. One has $\text{Aut}^0(V) = \text{Aut}^0(V, S)$. The Sarkisov link (5.2) transforms (V, S) into a pair (W, F) , where $F \subset R$ is a smooth rational normal quintic scroll (see 5.1), and induces an isomorphism $\text{Aut}(V, S) \cong \text{Aut}(W, F)$. So, $\text{Aut}(V, S)$ acts effectively on W via this isomorphism. Its action on F is effective too, since $\text{Aut}(V, S)$ is reductive in our case, and the hyperplane section $R = W \cap \langle F \rangle$ is singular. Since any line on R is contained in a plane, the rulings of F are lines in Π -planes, and each ruling is contained in a unique Π -plane. Moreover, $J := F \cap \Xi$ is a section of F , where $\Xi = \text{Sing}(R)$ is the Ξ -plane. Since F is an intersection of quadrics, J is a smooth conic.

Let S^\vee be the unique cubic cone on V disjoint with S . Clearly, $\text{Aut}(V, S)$ leaves S^\vee invariant. When acting on W it leaves invariant the image S_W^\vee of S^\vee in W . Since the linear projection $\theta : V \dashrightarrow W$ is regular near S^\vee , the image $\theta(S^\vee)$ is a cubic cone with vertex $P \in W \setminus R$. This cone meets R in a rational twisted cubic curve Ψ , see Proposition 4.3(iii). For any Π -plane in R one has $\Psi \cdot \Pi = 1$.

Recall the following facts.

- 6.21. Claim.**
- (i) *Any Π -plane meets the plane Ξ along a tangent line to a common smooth conic $\Upsilon \subset \Xi$;*
 - (ii) *R , Ξ , and Υ are $\text{Aut}(W)$ -invariant;*
 - (iii) *([PZ18, Proposition 6.3]) F , Ψ , and J are $\text{Aut}(V, S)$ -invariant;*
 - (iv) *([PZ18, Corollary 6.3.1]) the conics J and Υ share common tangents at two distinct points P_1 and P_2 , while no further line in Ξ is tangent to both J and Υ ;*
 - (v) *([PZ18, Lemma 6.2]) one has $\text{Aut}(\Xi, \Upsilon, J) = \mathbb{G}_m \rtimes (\mathbb{Z}/2\mathbb{Z})$, where the factor $\mathbb{Z}/2\mathbb{Z}$ is generated by an involution κ interchanging the points P_1 and P_2 , and the \mathbb{G}_m -factor fixes these points.*

6.22. Claim. *$\text{Aut}(V, S)$ acting on J fixes the points P_1 and P_2 .*

Proof of Claim 6.22. Assume to the contrary that some $\alpha \in \text{Aut}(V, S)$ interchanges P_1 and P_2 . It is easily seen that $\alpha|_\Upsilon$ has exactly two fixed points, say, γ_1 and γ_2 . For $i = 1, 2$ the unique Π -plane Π_i passing through γ_i is invariant under α . The same is true for the tangent line $l_i = T_{\gamma_i}\Upsilon = \Pi_i \cap \Xi$ and the unordered pair $\{r_i, s_i\}$ of distinct intersection points of l_i with the conic J , see Claim 6.21(iv) and Fig. 6.C. The intersection points $q_i \in \Pi_i \cap \Psi$, $i = 1, 2$, are distinct and fixed by α . Since α fixes q_i and preserves F , the ruling Λ_i of F through q_i is α -invariant. The intersection point, say, r_i of $\Lambda_i \cap \Xi \subset l_i \cap J$

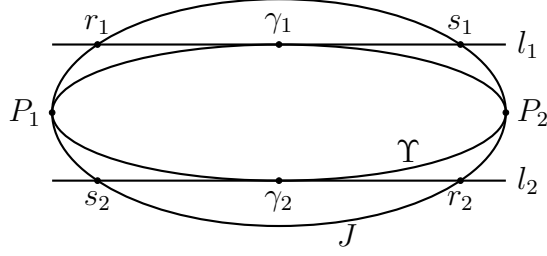


FIGURE 6.C.

is fixed by α , see [PZ18, Proposition 6.3(a) and its proof]. Then α fixes as well the second intersection point s_i of $l_i \cap J$. Thus, $l_1 \cup l_2$ meets J in the points r_1, s_1, r_2, s_2 fixed by α . Since at least three of these points are distinct, it follows that $\alpha|_J = \text{id}_J$, a contradiction. \square

According to Claim 6.22, $\text{Aut}(V, S)$ fixes the intersection point $P_1 \in J \cap \Upsilon$ and leaves invariant the ruling Λ_{P_1} of F through P_1 and the tangent line $T_{P_1}J \subset \Xi$. Hence, the reductive group $\text{Aut}(V, S)$ acting effectively on the tangent plane $T_{P_1}F$ is diagonalizable, and so, abelian.

The group $\text{Aut}(V, S)$ being reductive, there are natural embeddings

$$\text{Aut}^0(V) = \text{Aut}(V, S) \subset \text{Aut}(S, \Gamma)$$

where $\Gamma = A_{S^\vee} \cap S$ is a rational twisted cubic curve. The torus $\text{Aut}^0(V) \cong \mathbb{G}_m^2$ acts on Γ with two fixed points, say, v' and v'' ; these are the vertices of cubic cones in V which are the neighbors of S in the cycle (S_1, \dots, S_6) . We have $\text{Aut}(\Gamma, \{v', v''\}) \cong \mathbb{G}_m \rtimes \mathbb{Z}/2\mathbb{Z}$, where the generator of $\mathbb{Z}/2\mathbb{Z}$ interchanges v' and v'' and acts on \mathbb{G}_m via the inversion. The image of the abelian group $\text{Aut}(V, S)$ under the restriction homomorphism $\text{Aut}(V, S) \rightarrow \text{Aut}(\Gamma, \{v', v''\})$ coincides with the neutral component \mathbb{G}_m of $\text{Aut}(\Gamma, \{v', v''\})$ (that is, it fixes the points v' and v''), and the kernel is isomorphic to \mathbb{G}_m too. It follows that $\text{Aut}(V, S) = \text{Aut}^0(V) \cong \mathbb{G}_m^2$. \square

6.23. Corollary. *Let $\text{Aut}^0(V) = \mathbb{G}_m^2$, and let S_1, \dots, S_6 be the cyclically enumerated cubic cones in V , see Lemma 1.8(i). Then the quotient $\text{Aut}(V)/\text{Aut}^0(V) \subset \mathbb{Z}/6\mathbb{Z}$ acts effectively on the cycle (S_1, \dots, S_6) via cyclic shifts.*

6.24. Remark. Due to [PZ18, Theorem 1.3.3] the exact sequence

$$1 \rightarrow \text{Aut}^0(V) \rightarrow \text{Aut}(V) \rightarrow \text{Aut}(V)/\text{Aut}^0(V) \rightarrow 1$$

splits, and so, one has $\text{Aut}(V) \subset T \rtimes (\mathbb{Z}/6\mathbb{Z})$, cf. (6.1). The element $\tau \in \mathbb{Z}/6\mathbb{Z}$ of order two acts on the Cartan subalgebra \mathfrak{h} via $-\text{id}_{\mathfrak{h}}$. Hence, τ acts on T via the inversion. Likewise, any generator of the factor $\mathbb{Z}/6\mathbb{Z}$ acts on the torus T via an automorphism of order six.

6.25. Lemma. *For any Fano-Mukai fourfold $V = V_{18}^g$ with $\text{Aut}^0(V) \supset \mathbb{G}_m^2$ one has $\text{Aut}(V) \supset \mathbb{G}_m^2 \rtimes (\mathbb{Z}/2\mathbb{Z})$, where the generator of $\mathbb{Z}/2\mathbb{Z}$*

- *extends to an element of $\text{Ad}(G_2)$ acting on \mathfrak{g}_2 ;*
- *interchanges the members in each pair of disjoint cubic cones on V ;*
- *acts on the torus \mathbb{G}_m^2 via the inversion.*

Proof. We may suppose $\text{Aut}^0(V) \supset T$. There is an exact sequence

$$1 \rightarrow \text{Ad}(G_2) = \text{Aut}^0(\mathfrak{g}_2) \rightarrow \text{Aut}(\mathfrak{g}_2) \rightarrow \text{Aut}(\Gamma_2) \rightarrow 1,$$

where Γ_2 stands for the Dynkin graph of G_2 , see [FH04, Proposition D40]. Since Γ_2 admits no nontrivial automorphism and the center of G_2 is trivial, one has $\text{Aut}(\mathfrak{g}_2) = \text{Aut}^0(\mathfrak{g}_2) = \text{Ad}(G_2) \cong G_2$. It follows that

$$(6.26) \quad \text{Stab}_{\text{Aut}(\mathfrak{g}_2)}(\mathfrak{h}) = \text{Stab}_{G_2}(T) = N_{G_2}(T) \cong T \rtimes W(G_2),$$

see [AH17, Theorem A] for the latter splitting.

Let $\tau \in z(W(G_2)) \cong \mathbb{Z}/2\mathbb{Z}$ be the central element of order two. By [Hum72, Proposition 14.3], τ can be extended to an element $\tilde{\tau} \in \text{Stab}_{\text{Aut}(\mathfrak{g}_2)}(\mathfrak{h}) = N_{G_2}(T)$ of order two which preserves the Killing form and sends e_α to $-e_{-\alpha}$ for $\alpha \in \{\alpha_1, \alpha_2\}$. It is easily seen that $\tilde{\tau}(e_\alpha) \in \{\pm e_{-\alpha}\}$ for any $\alpha \in \Delta$. For the induced action of $\tilde{\tau}$ on \mathbb{P}^{13} one has $\tilde{\tau}(\mathbb{P}\mathfrak{g}_\alpha) = \mathbb{P}\mathfrak{g}_{-\alpha}$ for any $\alpha \in \Delta$.

The subgroup of $N_{G_2}(T)$ generated by T and $\tilde{\tau}$ is isomorphic to $T \rtimes (\mathbb{Z}/2\mathbb{Z})$. Since τ acts trivially on $\mathbb{P}\mathfrak{h}$ and Ω is invariant under the action of $N_{G_2}(T)$ on \mathfrak{g}_2 , any fourfold $V(h)$ with $h \in \mathfrak{h}$ is $(T \rtimes (\mathbb{Z}/2\mathbb{Z}))$ -invariant, as well as the set of the six T -fixed points on $V(h)$. Notice that $\tilde{\tau}$ interchanges the T -fixed points $\mathbb{P}\mathfrak{g}_\alpha$ and $\mathbb{P}\mathfrak{g}_{-\alpha}$ in Ω for $\alpha \in \Delta_\ell$. Hence, the involution $\tilde{\tau}$ acts effectively on $V(h)$ interchanging the members in each of the three pairs of cubic cones in $V(h)$. In particular, $\text{Aut}(V(h)) \supset T \rtimes (\mathbb{Z}/2\mathbb{Z})$. Recall, see Lemma 1.8(i), that the cubic cones on $V(h)$ are arranged in a cycle, where each pair of opposite vertices corresponds to a pair of disjoint cubic cones. By Corollary 6.23 the group $\text{Aut}(V(h))$ acts on this cycle via rotations. Therefore, the involution $\tilde{\tau}$ interchanges the members within each pair of disjoint cubic cones. See Remark 6.24 for the last statement. \square

Proof of Theorem 6.2. According to Proposition 6.7(iii) we may suppose $\text{Aut}^0(V) = T$ and $V = V(h)$ for some $h \in \mathfrak{h}^0$.

Let $l \subset \mathfrak{h}$ be the vector line spanned by h . If $\mathbb{P}l \in \mathbb{P}\mathfrak{h} \setminus (D_\ell \cup D_s \cup D_r)$ then the orbit of the point $\mathbb{P}l$ under the \mathfrak{S}_3 -action on $\mathbb{P}\mathfrak{h}$ has length six, see Corollary 6.19. In this case the stabilizer $\text{Aut}(V(h))$ of $V(h)$ in $N_{G_2}(T)$ coincides with $T \rtimes \langle \tilde{\tau} \rangle = T \rtimes (\mathbb{Z}/2\mathbb{Z})$, as needed in (ii). It remains to investigate the group $\text{Aut}(V(h))$ for $[h]$ from the exceptional \mathfrak{S}_3 -orbit $\mathbb{P}\mathfrak{h} \cap D_r = \{r_1, r_2\} = \varphi^{-1}(r)$, see Corollaries 6.15 and 6.19.

We have $r_i = \mathbb{P}l_i$, where $l_1, l_2 \subset \mathfrak{h}$ are the proper subspaces of the subgroup of rotations $\mathfrak{R} \cong \mathbb{Z}/6\mathbb{Z}$ of $W(G_2) = \mathfrak{D}_6$. It is easily seen that the elements of order two in \mathfrak{S}_3 interchange the points r_1 and r_2 , while the elements of order three fix these points. Hence, the stabilizer of r_i in \mathfrak{S}_3 is the cyclic subgroup \mathfrak{R}' of order three, $i = 1, 2$, and the preimage of \mathfrak{R}' in \mathfrak{D}_6 , that is, the stabilizer of the line l_i in $W(R_2)$ is the cyclic subgroup \mathfrak{R} of order six.

Let σ be a generator of the cyclic subgroup $\mathfrak{R} \subset \mathfrak{D}_6$, and let $\tilde{\sigma} \in N_{G_2}(T)$ be a lift of σ of order six. The line l_i is invariant under the action of $\tilde{\sigma}$. Choose a nonzero vector $h_i \in l_i$ for $i = 1, 2$. Then $V(h_i)$ is invariant under the action of $T \rtimes \langle \tilde{\sigma} \rangle$. By Corollary 6.23, $\tilde{\sigma}$ acts on the cycle (S_1, \dots, S_6) of cubic cones on $V(h_i)$ via a cyclic shift of order six. It follows due to (6.1), (6.26), and Lemma 6.25 that

$$(6.27) \quad \text{Aut}(V(h_i)) = \text{Stab}_{G_2}(V(h_i)) = N_{G_2}(T) \cong T \rtimes (\mathbb{Z}/6\mathbb{Z}).$$

By [KR13, Lemma 1], r_1 and r_2 lie on the same $\text{Ad}(G_2)$ -orbit $\Omega^r = D_r \setminus D_\ell$. Therefore, there is an isomorphism $V(h_1) \cong V(h_2)$. Letting $V^r = V(h_1)$ we obtain (i). \square

6.28. Remark. By Chevalley's restriction theorem the restriction map gives an isomorphism of graded algebras

$$(6.29) \quad \mathbb{C}[\mathfrak{g}_2]^{\text{Ad}(G_2)} \cong \mathbb{C}[\mathfrak{h}]^{W(G_2)} = \mathbb{C}[\psi_2, \psi_6],$$

where ψ_2 and ψ_6 are homogeneous forms in two variables of degree 2 and 6, respectively. Let ϕ_2 and ϕ_6 be the corresponding homogeneous G_2 -invariant forms on \mathfrak{g}_2 . The graded

piece $(\mathbb{C}[\mathfrak{g}_2]^{\text{Ad}(G_2)})_6 = \langle \phi_2^3, \phi_6 \rangle$ is spanned also by the pair (δ_ℓ, δ_s) of irreducible G_2 -invariants defined in Lemma 6.9. Hence, the pencils $\langle \phi_2^3, \phi_6 \rangle$ and $\mathcal{L} = \langle D_\ell, D_s \rangle$ in \mathbb{P}^{13} coincide. Any member D_t of \mathcal{L} is irreducible being a G_2 -orbit closure, see Lemma 6.10. If $Q \subset \mathbb{P}^{13}$ is the G_2 -invariant quadric defined by ϕ_2 , then $3Q$ belongs to \mathcal{L} . In fact, we have $D_r = 3Q$. Indeed, the three distinguished $W(G_2)$ -invariants $\psi_2 = \phi_2|_{\mathfrak{h}}$, $\delta_\ell|_{\mathfrak{h}}$, and $\delta_s|_{\mathfrak{h}}$ correspond to the three orbits of the $W(G_2)$ -action on $\mathbb{P}\mathfrak{h}$ with nontrivial stabilizers, namely, $Q \cap \mathbb{P}\mathfrak{h} = D_r \cap \mathbb{P}\mathfrak{h}$ of length two and $D_\ell \cap \mathbb{P}\mathfrak{h}, D_s \cap \mathbb{P}\mathfrak{h}$ of length three, see Lemma 6.17. In the notation of 0.2 in the Introduction one has $p_3^2 = \delta_\ell|_{\mathfrak{h}}$ and $q_3^2 = \delta_s|_{\mathfrak{h}}$, where $p_3 = \prod_{\alpha \in \Delta_t^+} \alpha$ and $q_3 = \prod_{\alpha \in \Delta_s^+} \alpha$, see (6.18). Besides, one has $\{\psi_2 = 0\} = l_1 \cup l_2$.

Proof of Corollary 6.3. In the exceptional cases where $\text{Aut}^0(V) \neq \mathbb{G}_m^2$ the first statement holds by [PZ18, Theorem 1.3(i)–(ii)]. In the case $\text{Aut}^0(V) = \mathbb{G}_m^2$ this statement follows from Lemma 6.25. The second statement follows from Lemma 6.25 and (6.27). \square

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