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Abstract

Fast Incremental Expectation Maximization was introduced to design Expectation-Maximization (EM) for the large scale learning framework involving finite-sum and possibly non-convex optimization. In this paper, we first recast this iterative algorithm and other incremental EM type algorithms in the *Stochastic Approximation within EM* framework. Then, we provide non asymptotic convergence bounds as a function of the number of examples n and of the maximal number of iterations $K_{\max}$. We propose two strategies for achieving an ϵ-approximate stationary point: either with $K_{\max} = O(n^{2/3}\epsilon^{-1})$ or with $K_{\max} = O(\sqrt{n}\epsilon^{-3/2})$, both strategies relying on a random termination rule before $K_{\max}$ and on a constant step size in the Stochastic Approximation step. Our bounds are explicit and improve over previous results. We provide a complexity bound which scales as $\sqrt{n}$ which improves over the bounds obtained so far; it is at the cost of a larger dependence upon the tolerance ϵ thus making this control relevant for small to medium accuracy with respect to the number of examples n. For the $n^{2/3}$-rate, our bounds show a numerical improvement thanks to a tighter definition of crucial quantities playing a role in the efficiency of the algorithm.

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1 Introduction

Expectation Maximization (EM) is a popular algorithm introduced by Dempster et al. (1977) to solve non linear programming on $\Theta \subseteq \mathbb{R}^d$ when the function F to be minimized is defined through an integral:

$$F(\theta) = -\frac{1}{n} \log \int_{\mathsf{Z}_n} G(z;\theta) \, \mathrm{d}\mu_n(z) \, , \tag{1}$$

for $n \in \mathbb{N} \setminus \{0\}$, a positive function G and a σ-finite positive measure μ_n on a measurable set $(\mathsf{Z}_n, \mathcal{Z}_n)$. EM is a Majorize-Minimization (MM) algorithm which, based on the current value of the iterate θ_{curr}, defines a majorizing function $\theta \mapsto \mathsf{Q}(\theta, \theta_{\mathrm{curr}})$ given, up to an additive constant, by

$$\mathsf{Q}(\theta, \theta_{\mathrm{curr}}) = -\frac{1}{n} \int_{\mathsf{Z}_n} \log G(z,\theta) \; G(z, \theta_{\mathrm{curr}}) \exp(n \, F(\theta_{\mathrm{curr}})) \; \mu_n(\mathrm{d}z) \; ;$$

then, the new point is chosen as the/a minimum of $\mathsf{Q}(\cdot, \theta_{\mathrm{curr}})$. Each iteration of EM is divided into two steps: the definition of the surrogate function is called the E step (expectation step), and its optimization is the M step (maximization or minimization step, depending on the convention). The computation of a function at each iteration can be greedy and even intractable; in many models, $\log G$ has a special form: there exist explicit functions $\phi : \Theta \to \mathbb{R}^q$ and $S : \mathsf{Z} \to \mathbb{R}^q$ such that $n^{-1} \log G(z,\theta) = \langle S(z), \phi(\theta) \rangle$. In these cases, the function Q is defined by a vector $\bar{s}(\theta_{\mathrm{curr}})$, equal to the expectation of the function S with respect to (w.r.t.) the probability distribution $\mathrm{d}\pi_{\theta_{\mathrm{curr}}} \stackrel{\mathrm{def}}{=} G(\cdot; \theta_{\mathrm{curr}}) \exp(n \, F(\theta_{\mathrm{curr}})) \mathrm{d}\mu_n$.

This paper is concerned with the optimization of a function F when $\mathsf{Z}_n = \mathsf{Z}^n$, $S(z) = n^{-1} \sum_{i=1}^n s_i(z_i)$, $\mathrm{d}\mu_n(z) = \otimes_{i=1}^n \mathrm{d}\mu(z_i)$ so that

$$F(\theta) = \frac{1}{n} \sum_{i=1}^n f_i(\theta), \qquad f_i(\theta) \stackrel{\mathrm{def}}{=} -\log \int_{\mathsf{Z}} \exp(\langle s_i(z), \phi(\theta) \rangle) \, \mathrm{d}\mu(z) \; ; \tag{2}$$

it addresses the finite-sum setting in the case n is large so that the computation of the full sum $\bar{s}(\theta_{\mathrm{curr}}) = n^{-1} \sum_{i=1}^n \int s_i \mathrm{d}\pi_{\theta_{\mathrm{curr}}}$ has to be avoided.

This framework is motivated by computational problems in large scale learning when the n available data are modeled as independent; the function f_i stands for a

non-convex loss associated to the example $\#i$ and it can also include a penalty (or a regularization) term. For finite-sum optimization, the existing EM-based algorithms are incremental: at each iteration, a single example or a mini-batch is selected and the E step uses this new information in an iterative updating mechanism of the Q-quantity. The time to convergence of these incremental procedures is always a trade-off between a loss of information since only part of the data are used per iterations, and a quicker exploitation of the new information since the parameter is updated more often without waiting the full scan of the data set.

A pioneering work in this vein is the *incremental EM* by Neal and Hinton (1998): the data set is divided into B blocks and one block is visited per EM iteration chosen through a deterministic cycle or through a random selection. The Q-quantity of *incremental EM* is again a sum over n terms, but each E step consists in updating only one term (or a block) in this sum while the original EM would update the n terms. The role of the size of the blocks is studied for some specific applications (see e.g. Ng and McLachlan (2003) for an application to inference of Gaussian mixture models).

The *online EM* algorithm, proposed by Cappé and Moulines (2009), can be easily adapted to the framework of an incremental processing of a large data set even if originally, is was designed to process a stream of data. It is derived in the case the function G is of the form $\exp(\langle S(z), \phi(\theta)\rangle)$, which in the statistical context when F is the negative normalized log-likelihood of the observations in a latent variable model, means that the complete likelihood G is from the exponential family. In that case, as explained above, the E step of EM is equivalent to the construction of expected sufficient statistics $\bar{s}(\theta_{\text{curr}})$; when this integration is intractable, Delyon et al. (1999) proved that the successive E steps can be replaced with a Stochastic Approximation (SA) procedure targeting the roots of a *mean field* h. Exploiting the parallel between SA and gradient descent algorithms, *online EM* mimes what *Stochastic Gradient Descent* (see e.g. Bottou and Le Cun (2004)) is for finite-sum optimization. Going further in this parallel, Chen et al. (2018) and Karimi et al. (2019c) proposed resp. *Stochastic EM with Variance Reduction (sEM-vr)* and *Fast Incremental EM (FIEM)* as variance reduction techniques within *online EM* as an echo to *Stochastic Variance Reduced Gradient* (SVRG, Johnson and Zhang (2013)) and *Stochastic Averaged Gradient* (SAGA, Defazio et al. (2014)) introduced as variance reduction techniques within *Stochastic Gradient Descent*.

In this paper, we aim to study such incremental EM methods combined with a Stochastic Approximation approach. The first goal of this paper is to cast *online EM*, *incremental EM* and *FIEM* into a framework called hereafter *Stochastic Approximation within EM* approaches; see subsection 2.3. We show that the E step of FIEM can be seen as the combination of a SA update and of a control variate; we propose to optimize the balance between these two quantities yielding to the *optimized FIEM*

algorithm; this new algorithm is numerically explored in section 4.

The second and main objective of this paper, is to derive non asymptotic convergence bounds for FIEM (see section 3).

Following Ghadimi and Lan (2013) (see also Allen-Zhu and Hazan (2016), Reddi et al. (2016), Fang et al. (2018), Zhou et al. (2018) and Karimi et al. (2019c)), we propose to fix a maximal length $K_{\max}$ and terminate a path $\{\theta^k, k \geq 0\}$ of the algorithm at some random time K uniformly sampled in the range $\{0, \ldots, K_{\max} - 1\}$ prior the run and independently of it; our convergence bounds control $\mathbb{E}\left[\|\nabla F(\theta^K)\|^2\right]$ and as a corollary, we discuss how to fix $K_{\max}$ as a function of the sample size n in order to reach an ϵ-approximate stationary point i.e.to find $\hat{\theta}^{K,\epsilon}$ such that $\mathbb{E}\left[\|\nabla F(\hat{\theta}^{K,\epsilon})\|^2\right] \leq \epsilon$. Such a property is sometimes called ϵ-accuracy in expectation (see e.g. (Reddi et al., 2016, Definition 1)).

Karimi et al. (2019c) established that *incremental EM*, which picks at random one example per iteration, reaches ϵ-accuracy by choosing $K_{\max} = O(n\epsilon^{-1})$: even if the algorithm is terminated at a random time K, this random time is chosen as a function of $K_{\max}$ which has to increase linearly with the size n of the data set. Karimi et al. (2019b) and Karimi et al. (2019c) provide the same analysis for *online EM* and *FIEM* showing that for both methods, ϵ-approximate stationarity is reached with $K_{\max} = O(n^{2/3}\epsilon^{-1})$ - here again, with one example picked at random per iteration. For these reasons, *online EM* and *FIEM* are preferable especially when n is large (see section 5 for a numerical illustration). Our major contribution in this paper is to show that for *FIEM*, the rate depends on the choice of some design parameters. By choosing a constant step size sequence in the SA step, depending upon n as $O(n^{-2/3})$, then ϵ-accuracy requires $K_{\max} = O(n^{2/3}\epsilon^{-1})$; we therefore retrieve the conclusions of Karimi et al. (2019c) but provide a value of the step size and a value of the convergence bounds which improve on Karimi et al. (2019c) - as numerically illustrated in section 4. We then prove that such an ϵ-accuracy is possible with $K_{\max} = O(\sqrt{n}\epsilon^{-3/2})$ and another strategy for the definition of the step size. To our best knowledge, this second result is new: it provides a weaker dependence on n but a larger dependence on the tolerance ϵ; the second approach is preferable for small to medium accuracy ϵ w.r.t. the size of the data set n.

2 Stochastic Approximation within EM algorithms for non-convex optimization

Notations. $\langle a, b\rangle$ denotes the standard Euclidean scalar product on $\mathbb{R}^\ell$, for $\ell \geq 1$; and $\|a\|$ the associated norm. For a matrix A, A^T is its transpose. For a smooth function ϕ, $\dot{\phi}$ denotes its gradient.

2.1 A non-convex finite-sum optimization problem

This paper deals with EM-based algorithms to solve

$$\mathrm{Argmin}_{\theta\in\Theta} F(\theta), \qquad F(\theta) \stackrel{\text{def}}{=} \frac{1}{n}\sum_{i=1}^{n} \mathcal{L}_i(\theta) + \mathsf{R}(\theta)\ , \tag{3}$$

where

$$\mathcal{L}_i(\theta) \stackrel{\text{def}}{=} -\log \int_{\mathsf{Z}} \tilde{h}_i(z)\ \exp\left(\langle s_i(z), \phi(\theta)\rangle\right)\ \mu(\mathrm{d}z)\ , \tag{4}$$

under the following assumption:

H 1. *$\Theta \subseteq \mathbb{R}^d$ is a measurable convex subset. $(\mathsf{Z}, \mathcal{Z})$ is a measurable space and μ is a σ-finite positive measure on $\mathcal{Z}$. The functions $\mathsf{R} : \Theta \to \mathbb{R}$, $\phi : \Theta \to \mathbb{R}^q$ and $\tilde{h}_i : \mathsf{Z} \to \mathbb{R}_+$, $s_i : \mathsf{Z} \to \mathbb{R}^q$ for $i \in \{1, \ldots, n\}$ are measurable functions. Finally, for any $\theta \in \Theta$ and $i \in \{1, \ldots, n\}$, $-\infty < \mathcal{L}_i(\theta) < \infty$.*

Under H1, for any $\theta \in \Theta$ and $i \in \{1, \ldots, n\}$, the quantity $p_i(z;\theta)\,\mu(\mathrm{d}z)$ where

$$p_i(z;\theta) \stackrel{\text{def}}{=} \tilde{h}_i(z)\ \exp\left(\langle s_i(z), \phi(\theta)\rangle + \mathcal{L}_i(\theta)\right)\ ,$$

defines a probability distribution on $\mathcal{Z}$. We assume below that

H 2. *For all $\theta \in \Theta$ and $i \in \{1, \ldots, n\}$, the expectation*

$$\bar{s}_i(\theta) \stackrel{\text{def}}{=} \int_{\mathsf{Z}} s_i(z)\ p_i(z;\theta)\mu(\mathrm{d}z)$$

exists and is computationally tractable.

Define

$$\bar{s} \stackrel{\text{def}}{=} \frac{1}{n}\sum_{i=1}^{n} \bar{s}_i\ . \tag{5}$$

The framework defined by (3) and (4) covers many computational learning problems such as empirical risk minimization with non-convex losses: R may include a regularization condition on the parameter θ, $n^{-1}\sum_{i=1}^{n}\mathcal{L}_i$ is the empirical loss and $\mathcal{L}_i$ is the loss function associated to example $\#i$.

Among applications concerned with the form (4) of the loss function, let us cite the normalized negative log-likelihood in a latent variable models (see e.g. Little and Rubin (2002)), when the complete data likelihood is from a curved exponential family (see e.g. Brown (1986); Sundberg (2019) for properties of exponential families): the additive form of the global loss $n^{-1}\sum_{i=1}^{n}\mathcal{L}_i(\theta)$ is the consequence of an independence

assumption on the n observations; in such models, the likelihood of the observation $\#i$ is of the form

$$y \mapsto \int_{\mathsf{Z}} \tilde{h}(y,z)\ \exp(\langle T(y,z), \phi(\theta)\rangle - \psi(\theta))\ \mathrm{d}\mu(z)$$

which corresponds to (4) by setting $\tilde{h}_i(z) \stackrel{\text{def}}{=} \tilde{h}(y,z)$ and $s_i(z) \stackrel{\text{def}}{=} T(y,z)$. The normalizing constant $\exp(-\psi(\theta))$ can be part of the term $\mathsf{R}(\theta)$ in (3).

2.2 A Majorize-Minimization approach based on EM

Given $\theta' \in \Theta$, define the function $\overline{F}(\cdot, \theta') : \Theta \to \mathbb{R}$ by

$$\overline{F}(\theta, \theta') \stackrel{\text{def}}{=} -\left\langle \bar{s}(\theta'), \phi(\theta)\right\rangle + \mathsf{R}(\theta) + \frac{1}{n}\sum_{i=1}^{n} \mathcal{C}_i(\theta')\ ,$$

$$\mathcal{C}_i(\theta') \stackrel{\text{def}}{=} \mathcal{L}_i(\theta') + \left\langle \bar{s}_i(\theta'), \phi(\theta')\right\rangle\ .$$

The following result shows that $\{\overline{F}(\cdot, \theta'), \theta' \in \Theta\}$ is a family of majorizing function of the objective function F from which a Majorize-Minimization approach for solving (3) can be derived under the following assumption:

H 3. *For any $s \in \mathbb{R}^q$, $\mathrm{Argmin}_{\theta\in\Theta}\ (-\langle s, \phi(\theta)\rangle + \mathsf{R}(\theta))$ exists and is unique. It is denoted by $\mathsf{T}(s)$.*

When $\theta \mapsto -\langle s, \phi(\theta)\rangle + \mathsf{R}(\theta)$ is continuous and either Θ is compact or the function is coercive[1] on Θ, then $\mathsf{T}(s)$ exists.

Proposition 1. *Assume H1 and H2.*

1. *For any $i \in \{1, \cdots, n\}$ and $\theta' \in \Theta$, $\mathcal{L}_i(\cdot) \leq -\langle \bar{s}_i(\theta'), \phi(\cdot)\rangle + \mathcal{C}_i(\theta')$.*
2. *For any $\theta' \in \Theta$, $F \leq \overline{F}(\cdot, \theta')$, and $\overline{F}(\theta', \theta') = F(\theta')$.*
3. *Assume also H3. Given $\theta^0 \in \Theta$, the sequence defined by $\theta^{k+1} \stackrel{\text{def}}{=} \mathsf{T} \circ \bar{s}(\theta^k)$ for any $k \geq 0$, satisfies $F(\theta^{k+1}) \leq F(\theta^k)$.*

The proof is provided for completeness in subsubsection 6.1.1.

The algorithm described by item 3 of Proposition 1, is the EM algorithm: upon noting that (3) is equivalent to the maximization of

$$\theta \mapsto \log \int_{\mathsf{Z}^n} \left(\prod_{i=1}^{n} \tilde{h}_i(z_i)\right) \exp\left(\left\langle \sum_{i=1}^{n} s_i(z_i), \phi(\theta)\right\rangle\right) \mu(\mathrm{d}z_1)\ldots\mu(\mathrm{d}z_n) - n\mathsf{R}(\theta)\ ,$$

[1] for any $A > 0$, there exist $\rho, B > 0$ such that if $\theta \in \Theta$ and either $\|\theta\| \geq B$ or $\mathrm{d}(\theta, \Theta^{\mathrm{c}}) \leq \rho$, then $|-\langle s, \phi(\theta)\rangle + \mathsf{R}(\theta)| \geq A$.

the E step of the EM algorithm would compute the auxiliary quantity

$$\begin{aligned}\mathsf{Q}(\theta, \tau^k) &\stackrel{\text{def}}{=} \int_{\mathsf{Z}^n} \left\langle \sum_{i=1}^n s_i(z_i), \phi(\theta) \right\rangle \prod_{i=1}^n p_i(z_i; \tau^k)\, \mu(\mathrm{d}z_i) - n\mathsf{R}(\theta) \\ &= n \left\langle \bar{s}(\tau^k), \phi(\theta) \right\rangle - n\mathsf{R}(\theta) \ ,\end{aligned}$$

given the current parameter τ^k; and then the M step would update the parameter by setting $\tau^{k+1} \in \mathrm{Argmax}_{\theta \in \Theta} \mathsf{Q}(\theta, \tau^k)$. It is easily seen that this mechanism is equal to $\tau^{k+1} = \mathsf{T} \circ \bar{s}(\tau^k)$.

The map T defined in H3 is therefore the maximization map of the M step in EM. We assume that T is explicit even if the optimization may be constrained.

The assumption that T is defined for any $s \in \mathbb{R}^q$ may be restrictive for some applications. When deriving theoretical analysis of EM-based algorithms, it is sometimes assumed that T is defined on a convex subset (and sometimes also compact) $\mathcal{S} \subseteq \mathbb{R}^q$ (see e.g. (Delyon et al., 1999, Assumption M5), (Fort and Moulines, 2003, Assumption M2), (Kuhn and Lavielle, 2004, Theorem 1), (Cappé and Moulines, 2009, Assumption 1), (Allassonnière et al., 2010, Theorem 1), (Le Corff and Fort, 2013, Section 4.1), (Karimi et al., 2019c, Assumption H4)). While in many applications, it is difficult to prove that the argument of T remains in $\mathcal{S}$ - and often is not even discussed - it is often observed that a smart implementation - such as a convenient initialization - may make the conditions to be satisfied numerically (see e.g. Donnet and Samson (2007), Cappé and Moulines (2009)). For the theoretical results derived hereafter, we assume H3 which is the easiest way to cover the EM-based algorithms studied here; it is out of the scope of this paper to address a more general case.

Starting from the current point θ^k, the iterative scheme $\theta^{k+1} = \mathsf{T} \circ \bar{s}(\theta^k)$ first computes a point in $\bar{s}(\Theta)$ through the expectation $\bar{s}$, and then apply the map T to obtain the new iterate θ^{k+1}. It can therefore be described in the $\bar{s}(\Theta)$-space, a space sometimes called the *expectation space*, being equivalently defined as follows: define $\{\bar{s}^k,\ k \in \mathbb{N}\}$ by $\bar{s}^0 \in \mathbb{R}^q$ and for any $k \geq 0$, and is

$$\bar{s}^{k+1} \stackrel{\text{def}}{=} \bar{s} \circ \mathsf{T}(\bar{s}^k) \ . \tag{6}$$

This approach in the expectation space comes up more naturally in the derivation of incremental algorithms designed for the large scale learning setting; it will be adopted throughout this paper.

Before deriving incremental EM-based methods, we conclude this section by a discussion on the limiting points of the iterative method (6). Sufficient conditions for the characterization of the limit points of any instance $\{\bar{s}^k, k \geq 0\}$ as the critical points of $F \circ \mathsf{T}$, for the convergence of the functional along the sequence $\{F \circ \mathsf{T}(\bar{s}^k), k \geq 0\}$, or

for the convergence of the iterates $\{\bar{s}^k, k \geq 0\}$ towards the critical points of $F \circ \mathsf{T}$ exist in the literature (see e.g. Wu (1983); Lange (1995); Delyon et al. (1999) in the EM context and Zangwill (1967); Csiszár and Tusnády (1984); Gunawardana and Byrne (2005); Parisi et al. (2019) for general iterative Majorize-Minimization algorithms). Proposition 2 characterizes the fixed points of $\mathsf{T} \circ \bar{s}$ and of $\bar{s} \circ \mathsf{T}$ under a set of conditions which will be adopted for the convergence analysis in Section 3.

H 4. *1. The functions ϕ and R are continuously differentiable on Θ^v where $\Theta^v \stackrel{\text{def}}{=} \Theta$ if Θ is open, or Θ^v is a neighborhood of Θ otherwise. T is continuously differentiable on $\mathbb{R}^q$.*

2. The function F is continuously differentiable on Θ^v and for any $\theta \in \Theta$,

$$\dot{F}(\theta) = -\left(\dot{\phi}(\theta)\right)^T \bar{s}(\theta) + \dot{\mathsf{R}}(\theta) \; .$$

3. For any $s \in \mathbb{R}^q$, $B(s) \stackrel{\text{def}}{=} (\phi \stackrel{\cdot}{\circ} \mathsf{T})(s)$ is a symmetric $q \times q$ matrix with positive minimal eigenvalue.

Under H1 to H4-item 1 and the assumption that Θ and $\phi(\Theta)$ are open subsets of resp. $\mathbb{R}^d$ and $\mathbb{R}^q$, then Lemma 8 in subsubsection 6.1.3 shows that H4-item 2 holds and the functions $\mathcal{L}_i$ are continuously differentiable on Θ for all $i \in \{1, \ldots, n\}$.

Under H1, H3 and the assumptions that *(i)* T is continuously differentiable on $\mathbb{R}^q$ and *(ii)* for any $s \in \mathbb{R}^q$, $\tau \mapsto \mathsf{Q}(s, \tau) \stackrel{\text{def}}{=} -\langle s, \phi(\tau) \rangle + \mathsf{R}(\tau)$ is twice continuously differentiable on Θ^v (defined in H4-item 1), then for any $s \in \mathbb{R}^q$, $\partial^2_\tau \mathsf{Q}(s, \mathsf{T}(s))$ is positive-definite and

$$B(s) = \left(\dot{\mathsf{T}}(s)\right)^T \partial^2_\tau \mathsf{Q}(s, \mathsf{T}(s)) \left(\dot{\mathsf{T}}(s)\right) \; ;$$

see Lemma 9 in subsubsection 6.1.3. Therefore, $B(s)$ is a symmetric matrix and if $\text{rank}(\dot{\mathsf{T}}(s)) = q = q \wedge d$, its minimal eigenvalue is positive.

Proposition 2. *Assume H1, H2 and H3. Define the measurable functions $V : \mathbb{R}^q \to \mathbb{R}$ and $h : \mathbb{R}^q \to \mathbb{R}^q$ by*

$$V(s) \stackrel{\text{def}}{=} F \circ \mathsf{T}(s) \; , \qquad h(s) \stackrel{\text{def}}{=} \bar{s} \circ \mathsf{T}(s) - s \; .$$

1. If $s^\star$ is a fixed point of $\bar{s} \circ \mathsf{T}$, then $\theta^\star \stackrel{\text{def}}{=} \mathsf{T}(s^\star)$ is a fixed point of $\mathsf{T} \circ \bar{s}$. Conversely, if $\theta^\star$ is a fixed point of $\mathsf{T} \circ \bar{s}$ then $s^\star \stackrel{\text{def}}{=} \bar{s}(\theta^\star)$ is a fixed point of $\bar{s} \circ \mathsf{T}$.

2. Assume in addition H4. Then for all $s \in \mathbb{R}^q$, $\dot{V}(s) = -B(s)\, h(s)$; and the zeros of h are the critical points of V.

The proof is in subsubsection 6.1.2. As a conclusion, the EM algorithm summarized in algorithm 1, is designed to converge to the zeros of

$$s \mapsto h(s) \stackrel{\text{def}}{=} \bar{s} \circ \mathsf{T}(s) - s \;, \tag{7}$$

which, for some models, are the critical points of $F \circ \mathsf{T}$.

However, the computational cost per iteration of EM is proportional to the number n of examples, since it requires the computation of $\bar{s}$ i.e. a sum over n terms. It is therefore intractable in the large scale learning framework. We review in subsection 2.3 few alternatives based on an incremental approach, and proposed in the literature to overcome this intractability.

Data: $K_{\max} \in \mathbb{N}$, $\bar{s}^0 \in \mathbb{R}^q$
Result: The EM sequence: $\bar{s}^k, k = 0, \ldots, K_{\max}$
for $k = 0, \ldots, K_{\max} - 1$ **do**
  $\bar{s}^{k+1} = \bar{s} \circ \mathsf{T}(\bar{s}^k)$

Algorithm 1: The EM algorithm in the expectation space

2.3 Stochastic Approximation within EM approaches

It was proposed in Delyon et al. (1999) to overcome the intractability of the expectation $\bar{s}$ by substituting the induction $\bar{s}^{k+1} = \bar{s} \circ \mathsf{T}(\bar{s}^k)$ with the definition of a random sequence $\{\widehat{S}^k, k \geq 0\}$ satisfying

$$\widehat{S}^{k+1} = \widehat{S}^k + \gamma_{k+1} \left(s^{k+1} - \widehat{S}^k \right) \;, \tag{8}$$

where $\{\gamma_k, k \geq 1\}$ is a $[0,1]$-valued deterministic positive sequence of *step sizes* (also called *learning rates*) chosen by the user and s^{k+1} is a Monte Carlo approximation of $\bar{s} \circ \mathsf{T}(\widehat{S}^k)$; the definition of s^{k+1} is such that the updating rule (8) is a Stochastic Approximation (SA) algorithm designed to target the zeros of the mean field $h(s)$ (see (7)); see e.g. Benveniste et al. (1990); Borkar (2008) for a general review on SA. Many stochastic perturbations of EM can be described by (8): let us cite for example the Stochastic EM Celeux and Diebolt (1985), or the Monte Carlo EM (introduced by Wei and Tanner (1990) and studied by Fort and Moulines (2003)) which corresponds to $\gamma_{k+1} = 1$.

We review below some stochastic perturbations of EM, recently introduced to overcome the large scale learning intractability. The originality of this review is to recast these algorithms in a SA framework. For this purpose, the key observation is the equality

$$h(s) = \mathbb{E}\left[\bar{s}_I \circ \mathsf{T}(s) - s + V\right] \tag{9}$$

where I is a uniform random variable on $\{1,\dots,n\}$ and V is a zero-mean random vector. Such an expression gives insights for the definition of SA schemes, including the combination with a variance reduction techniques through an adequate choice of V (see e.g. (Glasserman, 2004, Section 4.1.) for an introduction to control variates). Since I is finitely sampled, the mean field h is from the finite-sum family of functions.

2.3.1 Online EM

A first natural idea is given by algorithm 2: at iteration $(k+1)$, s^{k+1} is to sample at random an example $\#I_{k+1} \in \{1,\dots,n\}$ and to compute the expectation $\bar{s}_{I_{k+1}} \circ \mathsf{T}(\widehat{S}^k)$. Each iteration only requires the computation of one expectation $\bar{s}_i$. This algorithm,

Data: $K_{\max} \in \mathbb{N}$, $\widehat{S}^0 \in \mathbb{R}^q$, $\gamma_k \in (0,\infty)$ for $k = 1,\dots,K_{\max}$
Result: The Online EM sequence: $\widehat{S}^k, k = 0,\dots,K_{\max}$
1 **for** $k = 0,\dots,K_{\max}-1$ **do**
2 Sample I_{k+1} uniformly on $\{1,\dots,n\}$;
3 $\widehat{S}^{k+1} = \widehat{S}^k + \gamma_{k+1}\left(\bar{s}_{I_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k\right)$.

Algorithm 2: The Online EM algorithm

hereafter called `Online EM`, corresponds to a SA scheme: $\widehat{S}^{k+1} - \widehat{S}^k = \gamma_{k+1} H_{k+1}$ with $H_{k+1} \overset{\text{def}}{=} \bar{s}_{I_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k$ satisfying $\mathbb{E}\left[H_{k+1}|\mathcal{F}_k\right] = h(\widehat{S}^k)$; the filtration $\mathcal{F}_k$ is defined by $\mathcal{F}_k \overset{\text{def}}{=} \sigma(\widehat{S}^0, I_1, \cdots, I_k)$. It is a natural extension of the `online EM` by Cappé and Moulines (2009).

Different variants were proposed: instead of sampling a single observation among a batch of size n (or incorporating a single new observation in a data stream), a mini-batch of examples can be used: line 3 would get into

$$\widehat{S}^{k+1} = \widehat{S}^k + \gamma_{k+1}\left(N^{-1}\sum_{i\in\mathcal{I}_{k+1}} \bar{s}_i \circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k\right)$$

where $\mathcal{I}_{k+1}$ is a set of integers of cardinal N, sampled uniformly and with replacement in $\{1,\dots,n\}$. Convergence of the iterates in the long-time behavior ($K_{\max} \to \infty$) for `online EM` is addressed in Cappé and Moulines (2009); similar convergence results in the mini-batch case for the ML estimation of exponential family mixture models were recently established by Nguyen et al. (2020). Karimi et al. (2019a) and Kuhn et al. (2019) also proposed an asymptotic convergence result, in the mini-batch case for the ML estimation in latent variable models from the exponential family, combined with

a Monte Carlo approximation of the expectations $\bar{s}_i$. Finally, non asymptotic error rates are derived in Karimi et al. (2019b).

Note that an algorithm close to `Online EM` is proposed in Nowlan (1991): it corresponds to an update of the statistic of the form $\widehat{S}^{k+1} = \gamma \widehat{S}^k + \bar{s}_{I_{k+1}} \circ \mathsf{T}(\widehat{S}^k)$. A convenient choice of γ seems to favor an exponential forgetting of out-of-date statistics; and convergence to the same limiting value of EM is observed - when convergence is observed which is not guaranteed.

2.3.2 The incremental EM algorithm

The `Incremental EM (iEM)` algorithm is described by algorithm 3. Lines 4 to 7 are

Data: $K_{\max} \in \mathbb{N}$, $\widehat{S}^0 \in \mathbb{R}^q$, $\gamma_k \in (0, \infty)$ for $k = 1, \ldots, K_{\max}$
Result: The iEM sequence: $\widehat{S}^k, k = 0, \ldots, K_{\max}$

1 $\mathsf{S}_{0,i} = \bar{s}_i \circ \mathsf{T}(\widehat{S}^0)$ for all $i = 1, \ldots, n$;
2 $\widetilde{S}^0 = n^{-1} \sum_{i=1}^n \mathsf{S}_{0,i}$;
3 **for** $k = 0, \ldots, K_{\max} - 1$ **do**
4 $\quad I_{k+1} \sim \mathcal{U}(\{1, \ldots, n\})$;
5 $\quad \mathsf{S}_{k+1,i} = \mathsf{S}_{k,i}$ for $i \neq I_{k+1}$;
6 $\quad \mathsf{S}_{k+1,I_{k+1}} = \bar{s}_{I_{k+1}} \circ \mathsf{T}(\widehat{S}^k)$;
7 $\quad \widetilde{S}^{k+1} = \widetilde{S}^k + n^{-1} \left(\mathsf{S}_{k+1,I_{k+1}} - \mathsf{S}_{k,I_{k+1}} \right)$;
8 $\quad \widehat{S}^{k+1} = \widehat{S}^k + \gamma_{k+1} (\widetilde{S}^{k+1} - \widehat{S}^k)$

Algorithm 3: The iEM algorithm

a recursive computation of

$$\widetilde{S}^{k+1} = n^{-1} \sum_{i=1}^{n} \mathsf{S}_{k+1,i} \ , \tag{10}$$

where for $k \geq 0$,

$$\mathsf{S}_{k+1,i} \overset{\text{def}}{=} \begin{cases} \bar{s}_{I_{k+1}} \circ \mathsf{T}(\widehat{S}^k) & \text{if } i = I_{k+1} \ , \\ \mathsf{S}_{k,i} & \text{otherwise} \ . \end{cases} \tag{11}$$

It avoids the implementation of a sum over n terms. The sequence $\{\widehat{S}^k, k \geq 0\}$ is not a SA sequence but the bivariate sequence $\{(\widehat{S}^k, \mathsf{S}_{k,\cdot}), k \geq 0\}$ is: we have $\mathsf{S}_{k+1,\cdot} - \mathsf{S}_{k,\cdot} =$

$n^{-1}\, H_{k+1}^{(1)}$ and $\widehat{S}^{k+1} - \widehat{S}^k = \gamma_{k+1} H_{k+1}^{(2)}$ where

$$\mathbb{E}\left[H_{k+1}^{(1)} | \mathcal{F}_k\right] = \begin{bmatrix} \bar{s}_1 \circ \mathsf{T}(\widehat{S}^k) \\ \cdots \\ \bar{s}_n \circ \mathsf{T}(\widehat{S}^k) \end{bmatrix} - \mathsf{S}_{k,\cdot} \ ,$$

$$\mathbb{E}\left[H_{k+1}^{(2)} | \mathcal{F}_k\right] = h(\widehat{S}^k) + (n^{-1} - 1)\left(\bar{s} \circ \mathsf{T}(\widehat{S}^k) - \widetilde{S}^k\right) \ ;$$

here again, $\mathcal{F}_k \stackrel{\text{def}}{=} \sigma(\widehat{S}^0, I_1, \ldots, I_k)$. If there exists $(s^\star, \mathsf{S}_{\star,\cdot})$ such that $\lim_k (\widehat{S}^k, \mathsf{S}_{k,\cdot}) = (s^\star, \mathsf{S}_{\star,\cdot})$, then it may be seen (with no rigorous proof) that it satisfies $n^{-1} \sum_{i=1}^n \mathsf{S}_{\star,i} = \bar{s} \circ \mathsf{T}(s^\star) = s^\star$. This observation and the following equality obtained from lines 5 to 8 of algorithm 3

$$\widehat{S}^{k+1} = \widehat{S}^k + \frac{\gamma_{k+1}}{n}\left\{\bar{s}_{I_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k + \widetilde{S}^k - \mathsf{S}_{k,I_{k+1}} + (n-1)\left(\widetilde{S}^k - \widehat{S}^k\right)\right\} \ ,$$

show that *(i)* the update mechanism for $\{\widehat{S}^k, k \geq 0\}$ is of the form (9) with a random variable V correlated to I whose conditional expectation is $(n-1)\left(\widetilde{S}^k - \widehat{S}^k\right)$; *(ii)* if convergence holds, a convergence of $\{\widehat{S}^k, k \geq 0\}$ to a fixed point of h is expected and the conditional expectation of V vanishes to zero.

algorithm 3 generalizes the original incremental EM proposed by Neal and Hinton (1998), which corresponds to the case $\gamma_{k+1} = 1$ and to a deterministic visit to the successive examples. With $\gamma_{k+1} = 1$ for any $k \geq 0$, we have $\widehat{S}^k = \widetilde{S}^k = n^{-1} \sum_{i=1}^n \mathsf{S}_{k,i}$. algorithm 3 can be adapted in order to use a mini-batch of examples per iteration: the data set is divided into B blocks prior running `iEM`: per iteration, the examples of only a block are processed for the update of $\widehat{S}^k$ (see line 6) and along iterations, either the blocks are visited in turn or they are chosen randomly through a mechanism possibly depending on the fluctuations of the current iterate. The efficiency of `iEM` is therefore a trade-off between the size of the block which is related to the computational cost of a full scan of the data, and the fewer number of total scans required for convergence since `iEM` exploits information more quickly. Ng and McLachlan (2003) provide a numerical analysis of the role of B when `iEM` is applied to fitting a normal mixture model with fixed number of components. Gunawardana and Byrne (2005) provides sufficient conditions for the convergence to stationary points of F in the case the data set is processed through B blocks visited according to a deterministic cycling.

When $\gamma_{k+1} = 1$ for any $k \geq 0$ and the examples are chosen randomly at each iteration, the sequence $\{\widehat{S}^k, k \geq 0\}$ is the same as the one given by a Majorize-Minorization algorithm based on the inequality, at iteration $(k+1)$

$$F(\theta) \leq \frac{1}{n} \sum_{i=1}^n \left\{ - \left\langle \bar{s}_i(\theta^{k,i}), \phi(\theta) \right\rangle + \mathsf{R}(\theta) + \mathcal{C}_i(\theta^{k,i}) \right\} \ ,$$

where $\theta^{k,i} \overset{\text{def}}{=} \mathsf{T}(\widehat{S}^k)$ if $i = I_{k+1}$ and $\theta^{k,i} \overset{\text{def}}{=} \theta^{k-1,i}$ otherwise (see subsection 2.2 and Proposition 1 for the definition of $\mathcal{C}_i$ and the properties of these surrogate functions). This point of view and its link with *Minimization by Incremental Surrogate Optimization* (MISO, introduced by Mairal (2015)) is observed by Karimi et al. (2019c). Sufficient conditions for the asymptotic convergence of the functional and of the iterates, in the non-convex case, can be deduced from the convergence analysis of MISO (Mairal, 2015, Proposition 2.5); it is also addressed in Karimi et al. (2019a). Karimi et al. (2019c) provide non asymptotic convergence rates for `iEM` (case $\gamma_{k+1} = 1$). Asymptotic convergence analysis of `iEM` (with $\gamma_{k+1} = 1$) is also addressed by Srivastava et al. (2019) in the case of a asynchronous distributed implementation of the algorithm.

In algorithm 3, the computational cost of each iteration is the draw of one example $\#I_{k+1}$, the computation of the expectation $\bar{s}_{I_{k+1}}(\widehat{S}^k)$ and the update (and storage) of an auxiliary quantity $\mathsf{S}_{k+1,\cdot} \in \mathbb{R}^{qn}$; the initialization step also requires the computation of a sum over the n examples. Observe that the $K_{\max}$ integers I_k can be sampled before running the algorithm, so the space cost for the storage of $\mathsf{S}_{k,\cdot}$ can be reduced to $q(n \wedge K_{\max})$.

2.3.3 The Fast Incremental EM algorithm

The `Fast Incremental EM (FIEM)`, introduced by Karimi et al. (2019c), is defined by algorithm 4. Each iteration selects two examples independently, say $\#I_{k+1}$ and

Data: $K_{\max} \in \mathbb{N}$, $\widehat{S}^0 \in \mathbb{R}^q$, $\gamma_k \in (0, \infty)$ for $k = 1, \ldots, K_{\max}$
Result: The FIEM sequence: $\widehat{S}^k, k = 0, \ldots, K_{\max}$

$\mathsf{S}_{0,i} = \bar{s}_i \circ \mathsf{T}(\widehat{S}^0)$ for all $i = 1, \ldots, n$;
$\widetilde{S}^0 = n^{-1} \sum_{i=1}^n \mathsf{S}_{0,i}$;
for $k = 0, \ldots, K_{\max} - 1$ **do**
  $I_{k+1} \sim \mathcal{U}(\{1, \ldots, n\})$;
  $\mathsf{S}_{k+1,i} = \mathsf{S}_{k,i}$ for $i \neq I_{k+1}$;
  $\mathsf{S}_{k+1,I_{k+1}} = \bar{s}_{I_{k+1}} \circ \mathsf{T}(\widehat{S}^k)$;
  $\widetilde{S}^{k+1} = \widetilde{S}^k + n^{-1} \left(\mathsf{S}_{k+1,I_{k+1}} - \mathsf{S}_{k,I_{k+1}} \right)$;
  $J_{k+1} \sim \mathcal{U}(\{1, \ldots, n\})$;
  $\widehat{S}^{k+1} = \widehat{S}^k + \gamma_{k+1}(\bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k + \widetilde{S}^{k+1} - \mathsf{S}_{k+1,J_{k+1}})$

Algorithm 4: The Fast Incremental EM (FIEM) algorithm

$\#J_{k+1}$, and computes the expectations $\bar{s}_{I_{k+1}}(\widehat{S}^k)$ and $\bar{s}_{J_{k+1}}(\widehat{S}^k)$; as in `iEM`, `FIEM` computes the sum $\widetilde{S}^{k+1} = n^{-1} \sum_{i=1}^n \mathsf{S}_{k+1,i}$ (see (11)) in a recursive way, avoiding a

sum over n terms at each iteration (see line 4 to line 7 of algorithm 4). Then, this auxiliary quantity is used in the update mechanism of the sequence (see line 9). Here again, the sequence $\{\widehat{S}^k, k \geq 0\}$ is not a SA sequence, but $\{(\widehat{S}^k, \mathsf{S}_{k,\cdot}), k \geq 0\}$ is. We have $\mathsf{S}_{k+1,\cdot} - \mathsf{S}_{k,\cdot} = n^{-1} H^{(1)}_{k+1}$ and $\widehat{S}^{k+1} - \widehat{S}^k = \gamma_{k+1} H^{(2)}_{k+1}$ where

$$\mathbb{E}\left[H^{(1)}_{k+1} | \mathcal{F}_k\right] = \begin{bmatrix} \bar{s}_1 \circ \mathsf{T}(\widehat{S}^k) \\ \cdots \\ \bar{s}_n \circ \mathsf{T}(\widehat{S}^k) \end{bmatrix} - \mathsf{S}_{k,\cdot} \ , \quad \mathbb{E}\left[H^{(2)}_{k+1} | \mathcal{F}_{k+1/2}\right] \qquad = h(\widehat{S}^k) \ ;$$

we set $\mathcal{F}_k \overset{\text{def}}{=} \sigma(\widehat{S}^0, I_1, J_1, \ldots, J_{k-1}, I_k, J_k)$ and $\mathcal{F}_{k+1/2} \overset{\text{def}}{=} \sigma(\mathcal{F}_k \cup \{I_{k+1}\})$. It is easily seen that if there exists $(s^\star, \mathsf{S}_{\star,\cdot})$ such that $\lim_k (\widehat{S}^k, \mathsf{S}_{k,\cdot}) = (s^\star, \mathsf{S}_{\star,\cdot})$, then it satisfies $n^{-1} \sum_{i=1}^n \mathsf{S}_{\star,i} = \bar{s} \circ \mathsf{T}(s^\star) = s^\star$. This observation shows that *(i)* the update mechanism for $\{\widehat{S}^k, k \geq 0\}$ is of the form (9) with a random variable V, conditionally centered, and correlated to I; *(ii)* if convergence holds, a convergence of $\{\widehat{S}^k, k \geq 0\}$ to a fixed point of h is expected.

The introduction of such a variable V can be seen as a variance reduction technique, inherited from the Stochastic Averaged Gradient (SAGA, by Defazio et al. (2014)) which was proposed to improve convergence properties of incremental stochastic gradient methods. A similar idea is developed in Chen et al. (2018), a paper which adapts Stochastic Variance Reduced Gradient (SVRG, Johnson and Zhang (2013)) to incremental EM algorithms: their motivation is to improve on `online EM` (see subsubsection 2.3.1) which surpasses EM in the burn-in phase but is penalized by the large variance when approximating the E step in the convergence phase.

To our best knowledge, the convergence analyses of `FIEM` are only given in Karimi et al. (2019c): non asymptotic convergence rates for `FIEM` are derived. The novel theoretical contribution of our paper, detailed in section 3, is to complement and improve on these results.

On the computational side, each iteration of `FIEM` requires two draws from $\{1, \ldots, n\}$ and two computations of an expectation of the form $\bar{s}_i(\theta)$; as for `iEM`, there is a space complexity through the storage of the auxiliary quantity $\mathsf{S}_{k,\cdot}$ - its size being proportional to $q(2K_{\max} \wedge n)$ (in some specific situations, see the comment in (Schmidt et al., 2017, Section 4.1), the size can be reduced). The initialization step also requires the computation of a sum over the n examples.

2.3.4 An optimized FIEM algorithm, opt-FIEM

From (9), line 9 of algorithm 4 and the control variate technique, we explore here the idea to modify the original `FIEM` as follows (compare to line 9 in algorithm 4)

$$\widehat{S}^{k+1} = \widehat{S}^k + \gamma_{k+1} \left(\bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k + \lambda_{k+1} \left(\widetilde{S}^{k+1} - \mathsf{S}_{k+1,J_{k+1}} \right) \right) \qquad (12)$$

where $\lambda_{k+1} \in \mathbb{R}$ is chosen in order to minimize the conditional fluctuation

$$\gamma_{k+1}^{-2}\ \mathbb{E}\left[\|\widehat{S}^{k+1} - \widehat{S}^k\|^2 | \mathcal{F}_{k+1/2}\right]\ .$$

Upon noting that $\mathbb{E}\left[\widehat{S}^{k+1} - \widehat{S}^k | \mathcal{F}_{k+1/2}\right] = \gamma_{k+1} h(\widehat{S}^k)$, it is easily seen that equivalently, λ_{k+1} is chosen as the minimum of the conditional variance

$$\mathbb{E}\left[\|\gamma_{k+1}^{-1}\left(\widehat{S}^{k+1} - \widehat{S}^k\right) - h(\widehat{S}^k)\|^2 | \mathcal{F}_{k+1/2}\right]\ .$$

We will refer to this technique as the optimized FIEM (`opt-FIEM`) below. Observe that `Online EM` corresponds to the choice $\lambda_{k+1} = 0$ for any $k \geq 0$ (see algorithm 2); and `FIEM` corresponds to the choice $\lambda_{k+1} = 1$ for any $k \geq 0$ (see algorithm 4).

Upon noting that, given two random variables U, V such that $\mathbb{E}[\|V\|^2] > 0$, the function $\lambda \mapsto \mathbb{E}\left[\|U + \lambda V\|^2\right]$ reaches its minimum at a unique point given by $\lambda_\star \overset{\text{def}}{=} -\mathbb{E}\left[U^T V\right] / \mathbb{E}\left[\|V\|^2\right]$, the optimal choice for λ_{k+1} is given by (remember that conditionally to $\mathcal{F}_{k+1/2}$, $\widetilde{S}^{k+1} - \mathsf{S}_{k+1,J_{k+1}}$ is centered),

$$\lambda_{k+1}^\star \overset{\text{def}}{=} -\frac{\text{Tr Cov}\left(\bar{s}_J \circ \mathsf{T}(\widehat{S}^k), \widetilde{S}^{k+1} - \mathsf{S}_{k+1,J} | \mathcal{F}_{k+1/2}\right)}{\text{Tr Var}\left(\widetilde{S}^{k+1} - \mathsf{S}_{k+1,J} | \mathcal{F}_{k+1/2}\right)} \tag{13}$$

where J is a uniform random variable on $\{1, \ldots, n\}$, independent of $\mathcal{F}_{k+1/2}$, Tr denotes the trace of a matrix, and Cov, Var are resp. the covariance and variance matrices. With this optimal value, it holds from (12)

$$\begin{aligned}
&\gamma_{k+1}^{-2}\ \mathbb{E}\left[\|\widehat{S}^{k+1} - \widehat{S}^k\|^2 | \mathcal{F}_{k+1/2}\right] \\
&\quad = \text{Tr Var}\left(\bar{s}_J \circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k | \mathcal{F}_{k+1/2}\right) \cdots \\
&\qquad \times \left(1 - \text{Corr}^2\left(\bar{s}_J \circ \mathsf{T}(\widehat{S}^k), \widetilde{S}^{k+1} - \mathsf{S}_{k+1,J} | \mathcal{F}_{k+1/2}\right)\right)\ ,
\end{aligned} \tag{14}$$

where $\text{Corr}(U, V) \overset{\text{def}}{=} \text{TrCov}(U, V) / \{\text{TrVar}(U)\ \text{TrVar}(V)\}^{1/2}$. If the `opt-FIEM` algorithm $\{(\widehat{S}^k, \mathsf{S}_{k,\cdot}), k \geq 0\}$ converges to $(s^\star, \mathsf{S}_{\star,\cdot})$, we have $n^{-1} \sum_{i=1}^n \mathsf{S}_{\star,i} = s^\star = \bar{s} \circ \mathsf{T}(s^\star)$ and $\mathsf{S}_{\star,i} = \bar{s}_i \circ \mathsf{T}(s^\star)$ (see subsubsection 2.3.2 and subsubsection 2.3.3 for a similar discussion) thus showing that asymptotically, $\lambda_k^\star \approx 1$ (which implies that the correlation is 1 in (14)). This value is the value proposed in the original `FIEM`: therefore, asymptotically `opt-FIEM` and `FIEM` should be equivalent and `opt-FIEM` should have a better behavior in the transient phase. We will compare numerically `Online EM`, `FIEM` and `opt-FIEM` in section 4.

Upon noting that

$$\lambda_{k+1}^{\star} = -\frac{n^{-1}\sum_{j=1}^{n}\left\langle \bar{s}_j \circ \mathsf{T}(\widehat{S}^k), \widetilde{S}^{k+1} - \mathsf{S}_{k+1,j}\right\rangle}{n^{-1}\sum_{j=1}^{n}\|\widetilde{S}^{k+1} - \mathsf{S}_{k+1,j}\|^2} ,$$
$$= -\frac{n^{-1}\sum_{j=1}^{n}\left\langle \bar{s}_j \circ \mathsf{T}(\widehat{S}^k), \widetilde{S}^{k+1} - \mathsf{S}_{k+1,j}\right\rangle}{n^{-1}\sum_{j=1}^{n}\|\mathsf{S}_{k+1,j}\|^2 - \|\widetilde{S}^{k+1}\|^2} .$$

the computational cost of $\lambda_{k+1}^{\star}$ is proportional to n: it is therefore an intractable quantity in the large scale learning setting considered in this paper. A numerical approximation has to be designed: for example, a Monte Carlo approximation of the numerator; and a recursive approximation (along the iterations k) of the denominator, miming the same idea as the recursive computation of the sum $\widetilde{S}^k = n^{-1}\sum_{i=1}^{n}\mathsf{S}_{k,i}$ in `online-EM` and `FIEM`.

3 Non asymptotic convergence bounds

3.1 A general result

In this non-convex setting, convergence is characterized by the behavior of the gradient of the objective function along the path, or in the EM setting, by a "distance" of the path to the set of the roots of h. Under our assumptions, both quantities are related as stated in 3.

The convergence bounds are obtained by strengthening H4 with the following assumptions

H5. *1. There exist $0 < v_{\min} \leq v_{max} < \infty$ such that for all $s \in \mathbb{R}^q$, the spectrum of $B(s)$ is in $[v_{\min}, v_{\max}]$; $B(s)$ is defined in H4.*

2. For any $i \in \{1, \ldots, n\}$, $\bar{s}_i \circ \mathsf{T}$ is globally Lipschitz on $\mathbb{R}^q$ with constant L_i.

3. The function $s \mapsto \dot{V}(s) = -B(s)h(s)$ is globally Lipschitz on $\mathbb{R}^q$ with constant $L_{\dot{V}}$.

Finding a point $\hat{\theta}^{\epsilon}$ such that $F(\hat{\theta}^{\epsilon}) - \min F \leq \epsilon$ is NP-hard in the non-convex setting (see Murty and Kabadi (1987)). Hence, in non-convex deterministic optimization of a smooth function F, convergence is often characterized by the quantity $\inf_{1\leq k\leq K_{\max}}\|\nabla F(\theta^k)\|$ along a path of length $K_{\max}$; in non-convex stochastic optimization, the quantity $\inf_{1\leq k\leq K_{\max}}\mathbb{E}\left[\|\nabla F(\theta^k)\|^2\right]$ is sometimes considered when the expectation is w.r.t. the randomness introduced to replace intractable quantities with oracles. Nevertheless, in many frameworks such as the finite-sum optimization one

we are interested in, such a criterion can not be used to define a termination rule for the algorithm since ∇F is intractable. Given a maximal number of iterations $K_{\max}$, and a random variable K taking values in $\{0, \dots, K_{\max} - 1\}$, define

$$\begin{aligned}
\mathsf{E_0} &\stackrel{\text{def}}{=} \frac{1}{v_{\min}^2} \mathbb{E}\left[\|\dot{V}(\widehat{S}^K)\|^2 \right] , \\
\mathsf{E_1} &\stackrel{\text{def}}{=} \mathbb{E}\left[\|h(\widehat{S}^K)\|^2 \right] , \\
\mathsf{E_2} &\stackrel{\text{def}}{=} \mathbb{E}\left[\|\widetilde{S}^{K+1} - \bar{s} \circ \mathsf{T}(\widehat{S}^K)\|^2 \right] ,
\end{aligned}$$

where K is a random termination rule, chosen independently of the path: upper bounds of these quantities provide insights on the behavior of FIEM when stopped at a random time. Except in subsection 3.4, below K is the uniform r.v. on $\{0, \dots, K_{\max} - 1\}$.

The quantities $\mathsf{E_0}$ and $\mathsf{E_1}$ are classical in the literature: they stand for a way to measure resp. a distance to a stationary point of the objective $V = F \circ \mathsf{T}$, and a distance to the fixed points of EM. $\mathsf{E_2}$ is specific to FIEM: it quantifies how far the control variate $\widetilde{S}^{k+1}$ is from the intractable mean $\bar{s} \circ \mathsf{T}(\widehat{S}^k)$ (see (10) for the definition of $\widetilde{S}^{k+1}$).

From Proposition 2-item 2, we trivially have (the proof is omitted)

Proposition 3. *Assume H1, H2, H3, H4 and H5-item 1. Then for any $s \in \mathbb{R}^q$, $\left\langle h(s), \dot{V}(s) \right\rangle \leq -v_{\min} \|h(s)\|^2$ and $\mathsf{E_0} \leq \mathsf{E_1}$.*

Theorem 4 is a general result for the control of quantities of the form

$$\sum_{k=0}^{K_{\max}-1} \alpha_k \mathbb{E}\left[\|h(\widehat{S}^k)\|^2 \right] + \sum_{k=0}^{K_{\max}-1} \delta_k \mathbb{E}\left[\|\widetilde{S}^{k+1} - \bar{s} \circ \mathsf{T}(\widehat{S}^k)\|^2 \right]$$

where $\alpha_k \in \mathbb{R}$ and $\delta_k > 0$. In subsection 3.2 and subsection 3.3, we discuss how to choose the step sizes $\{\gamma_k, k \geq 1\}$ such that for any $k \in \{0, \dots, K_{\max} - 1\}$, α_k is non-negative and such that $A_{K_{\max}} \stackrel{\text{def}}{=} \sum_{k=0}^{K_{\max}-1} \alpha_k$ is positive. We then deduce from Theorem 4 an upper bound for

$$\sum_{k=0}^{K_{\max}-1} \frac{\alpha_k}{A_{K_{\max}}} \mathbb{E}\left[\|h(\widehat{S}^k)\|^2 \right] + \sum_{k=0}^{K_{\max}-1} \frac{\delta_k}{A_{K_{\max}}} \mathbb{E}\left[\|\widetilde{S}^{k+1} - \bar{s} \circ \mathsf{T}(\widehat{S}^k)\|^2 \right] \tag{15}$$

such that the larger $A_{K_{\max}}$ is, the better the bound is. (15) is then used to obtain upper bounds on $\mathsf{E_1}$ and $\mathsf{E_2}$; which provides also an upper bound on $\mathsf{E_0}$ by Proposition 3.

Theorem 4. *Assume H1, H2, H3, H4 and H5. Define* $L^2 \stackrel{\text{def}}{=} n^{-1}\sum_{i=1}^n L_i^2$.

Let $K_{\max}$ *be a positive integer,* $\{\gamma_k,\ k\in\mathbb{N}\}$ *be a sequence of positive step sizes and* $\widehat{S}^0 \in \mathbb{R}^q$. *Consider the FIEM sequence* $\{\widehat{S}^k,\ k\in\mathbb{N}\}$ *given by algorithm 4. Set* $\Delta V \stackrel{\text{def}}{=} \mathbb{E}\left[V(\widehat{S}^0)\right] - \mathbb{E}\left[V(\widehat{S}^{K_{\max}})\right]$.

For any positive numbers $\beta_1,\ldots,\beta_{K_{\max}-1}$, *we have*

$$\sum_{k=0}^{K_{\max}-1} \alpha_k \quad \mathbb{E}\left[\|h(\widehat{S}^k)\|^2\right] \ + \ \sum_{k=0}^{K_{\max}-1} \delta_k \mathbb{E}\left[\|\widetilde{S}^{k+1} - \bar{s}\circ\mathsf{T}(\widehat{S}^k)\|^2\right] \quad \leq \quad \Delta V \quad ,$$

where for any $k=0,\ldots,K_{\max}-1$,

$$\alpha_k \stackrel{\text{def}}{=} \gamma_{k+1} v_{min} - \gamma_{k+1}^2\left(1+\Lambda_k L^2\right)\frac{L_{\dot{V}}}{2}, \quad \delta_k \stackrel{\text{def}}{=} \gamma_{k+1}^2\left(1+\frac{\Lambda_k\beta_{k+1}L^2}{(1+\beta_{k+1})}\right)\frac{L_{\dot{V}}}{2}\ ,$$

with $\Lambda_{K_{\max}-1}=0$ *and for* $k=0,\ldots,K_{\max}-2$,

$$\Lambda_k \stackrel{\text{def}}{=} \left(1+\frac{1}{\beta_{k+1}}\right)\sum_{j=k+1}^{K_{\max}-1}\gamma_{j+1}^2\prod_{\ell=k+2}^{j}\left(1-\frac{1}{n}+\beta_\ell+\gamma_\ell^2 L^2\right)\ .$$

Proof. The detailed proof is provided in Section 6.2; let us give here a sketch of proof. Define H_{k+1} such that $\widehat{S}^{k+1} = \widehat{S}^k + \gamma_{k+1}H_{k+1}$. V is regular enough so that

$$V(\widehat{S}^{k+1}) - V(\widehat{S}^k) - \gamma_{k+1}\left\langle H_{k+1}, \dot{V}(\widehat{S}^k)\right\rangle \leq \gamma_{k+1}^2\frac{L_{\dot{V}}}{2}\|H_{k+1}\|^2\ .$$

Then, the next step is to prove that

$$\begin{aligned}\mathbb{E}\left[V(\widehat{S}^{k+1})\right] - \mathbb{E}\left[V(\widehat{S}^k)\right] + \gamma_{k+1}\left(v_{\min} - \gamma_{k+1}\frac{L_{\dot{V}}}{2}\right)\mathbb{E}\left[\|h(\widehat{S}^k)\|^2\right] \\ \leq \gamma_{k+1}^2\frac{L_{\dot{V}}}{2}\mathbb{E}\left[\|H_{k+1} - \mathbb{E}\left[H_{k+1}|\mathcal{F}_{k+1/2}\right]\|^2\right]\ ,\end{aligned}$$

which, by summation from $k=0$ to $k=K_{\max}-1$, yields

$$\begin{aligned}\sum_{k=0}^{K_{\max}-1}\gamma_{k+1}\left(v_{\min}-\gamma_{k+1}\frac{L_{\dot{V}}}{2}\right)\mathbb{E}\left[\|h(\widehat{S}^k)\|^2\right] \leq \mathbb{E}\left[V(\widehat{S}^0)\right] - \mathbb{E}\left[V(\widehat{S}^K_{\max})\right] \\ + \frac{L_{\dot{V}}}{2}\sum_{k=0}^{K_{\max}-1}\gamma_{k+1}^2\mathbb{E}\left[\|H_{k+1} - \mathbb{E}\left[H_{k+1}|\mathcal{F}_{k+1/2}\right]\|^2\right]\ .\end{aligned}$$

The most technical part is to prove that the last term on the RHS is upper bounded by

$$\frac{L_{\dot{V}}}{2}\sum_{k=0}^{K_{\max}-1}\gamma_{k+1}^2L^2\ \Big\{\Lambda_k\mathbb{E}\left[\|h(\widehat{S}^k)\|^2\right] \\ -\left(1+(1+\beta_{k+1}^{-1})^{-1}\Lambda_k\right)\ \mathbb{E}\left[\|\widetilde{S}^{k+1}-\bar{s}\circ\mathsf{T}(\widehat{S}^k)\|^2\right]\Big\}\ .$$

This concludes the proof. □

3.2 A uniform random stopping rule for a $n^{2/3}$-complexity

The main result of this section establishes that by choosing a constant stepsize sequence and a termination rule K sampled uniformly on $\{0,\cdots,K_{\max}-1\}$, an ϵ-approximate stationary point can be reached before $K_{\max}=O(n^{2/3}\epsilon^{-1}L_{\dot{V}}^{1/3}L^{2/3})$ iterations. In the Stochastic Gradient Descent literature, complexity is evaluated in terms of *Incremental First-order Oracle* introduced by Agarwal and Bottou (2015), that is, roughly speaking, number of calls to an oracle which returns a pair $(f_i(x),\nabla f_i(x))$. In our case, the equivalent cost is the number of computations of an expectation $\bar{s}_i(\theta)$ - see H2. $K_{\max}$ iterations of FIEM calls $2K_{\max}$ computations of such expectations. As a consequence, the complexity analyses consist in discussing how $K_{\max}$ has to be chosen as a function of n and ϵ.

For $\lambda\in(0,1)$, $C>0$ and n such that $n^{-1/3}<\lambda/C$, define

$$f_n(C,\lambda)\stackrel{\text{def}}{=}\left(\frac{1}{n^{2/3}}+\frac{C}{\lambda-Cn^{-1/3}}\left(\frac{1}{n}+\frac{1}{1-\lambda}\right)\right)\ . \tag{16}$$

Proposition 5 (application of Theorem 4)**.** *Let $\mu\in(0,1)$. Choose $\lambda\in(0,1)$ and $C\in(0,+\infty)$ such that*

$$\sqrt{C}f_n(C,\lambda)=2\mu v_{\min}\frac{L}{L_{\dot{V}}}\ . \tag{17}$$

Let $\{\widehat{S}^k,k\in\mathbb{N}\}$ be the FIEM sequence given by algorithm 4 run with the constant step size

$$\gamma_\ell=\gamma_{\text{FGM}}\stackrel{\text{def}}{=}\frac{\sqrt{C}}{n^{2/3}L}=\frac{2\mu v_{\min}}{f_n(C,\lambda)\,n^{2/3}L_{\dot{V}}}\ . \tag{18}$$

For any $n>(C/\lambda)^3$ and $K_{\max}\geq 1$, we have

$$\mathsf{E}_1+\frac{\mu}{(1-\mu)f_n(C,\lambda)\,n^{2/3}}\mathsf{E}_2\leq\frac{n^{2/3}}{K_{\max}}\frac{L_{\dot{V}}\,f_n(C,\lambda)}{2\mu(1-\mu)v_{\min}^2}\Delta V\ , \tag{19}$$

where the errors E_i are defined with a random variable K sampled uniformly on $\{0,\cdots,K_{\max}-1\}$.

The proof of Proposition 5 is given in subsubsection 6.2.2. A first suggestion to solve the equation (17) is to choose $\lambda = C$ and $C \in (0,1)$ such that $\sqrt{C} f_n(C,C) = 2\mu v_{\min} L / L_{\dot{V}}$. This equation possesses an unique solution in $(0,1)$ which is upper bounded by C^+ given by

$$C^+ \stackrel{\text{def}}{=} \frac{\sqrt{1 + 16\mu v_{\min}^2 L^2 L_{\dot{V}}^{-2}} - 1}{4\mu v_{\min} L L_{\dot{V}}^{-1}}$$

The consequence is that, given $\varepsilon \in (0,1)$, by setting

$$M \stackrel{\text{def}}{=} \frac{L_{\dot{V}}}{\mu(1-\mu) v_{\min}^2} f_2(C^+, C^+) \; ,$$

we have

$$K_{\max} = M \; n^{2/3} \varepsilon^{-1} \Longrightarrow \mathsf{E}_1 + \frac{L_{\dot{V}}}{2L} \frac{\sqrt{C}}{v_{\min} n^{2/3}} \mathsf{E}_2 \leq \varepsilon \, \Delta V \; ;$$

see subsubsection 6.2.2 for a detailed proof of this comment.

Another suggestion is to exploit how (16) behaves when $n \to +\infty$; we prove in subsubsection 6.2.2 again that, there exists $N_\star$ depending only upon $L, L_{\dot{V}}, v_{\min}$ such that for any $n \geq N_\star$,

$$\mathsf{E}_1 + \frac{4}{3} \frac{C}{n^{2/3}} \left(\frac{L_{\dot{V}}}{L v_{\min}} \right)^{4/3} \mathsf{E}_2 \leq \frac{n^{2/3}}{K_{\max}} \frac{8}{3} \frac{L}{v_{\min}} \left(\frac{L_{\dot{V}}}{L v_{\min}} \right)^{1/3} \Delta V \; ,$$

by choosing $C \leftarrow 0.25 \left(v_{\min} L / L_{\dot{V}} \right)^{2/3}$ in the definition of the step size γ_{FGM}.

The conclusions of Proposition 5 confirm and improve previous results in the literature: (Karimi et al., 2019c, Theorem 2) proves that for FIEM run with the constant size

$$\gamma_{\text{K}} \stackrel{\text{def}}{=} \frac{v_{min}}{\max(6, 1 + 4 v_{\min}) \; \max(L_{\dot{V}}, L_1, \ldots, L_n) \; n^{2/3}} \; , \tag{20}$$

it holds

$$\mathsf{E}_1 \leq \frac{n^{2/3}}{K_{\max}} \frac{(\max(6, 1 + 4 v_{\min}))^2 \; \max(L_{\dot{V}}, L_1, \ldots, L_n)}{v_{\min}^2} \; \Delta V \; . \tag{21}$$

We improve on this result by first showing that the RHS in (19) controls a larger quantity than E_1. In addition, numerical explorations (see e.g. section 4) show that our step size γ_{FGM} is larger than the step size γ_{K} thus providing a more aggressive step size which may have a beneficial effect on the numerical implementation. It also shows that Proposition 5 provides a tighter control of convergence. In both contributions however, the step size depends on n as $O(1/n^{2/3})$ and, the explicit control increases at the rate $n^{2/3}$ and decreases at the rate $1/K_{\max}$. The rate of the step size is the same as what is observed for Stochastic Gradient Descent (see e.g. Allen-Zhu and Hazan (2016)).

3.3 A uniform random stopping rule for a $\sqrt{n}$-complexity

In subsubsection 6.2.3, we prove the following control obtained, here again, along a FIEM path run with a constant stepsize sequence and stopped at a random time K sampled uniformly on $\{0, \cdots, K_{\max}-1\}$: an ϵ-stationary point can be reached before $K_{\max} = O(\sqrt{n}\epsilon^{-3/2})$ iterations.

Define

$$\tilde{f}_n(C,\lambda) \stackrel{\text{def}}{=} \frac{1}{(nK_{\max})^{1/3}} + C\left(\frac{1}{n} + \frac{1}{1-\lambda}\right) . \tag{22}$$

Proposition 6 (application of Theorem 4)**.** *Let $\mu \in (0,1)$. Choose $\lambda \in (0,1)$ and $C > 0$ such that*

$$\sqrt{C}\tilde{f}_n(C,\lambda) = 2\mu v_{\min}\frac{L}{L_{\dot{V}}} . \tag{23}$$

Let $\{\widehat{S}^k, k \in \mathbb{N}\}$ be the FIEM sequence given by algorithm 4 run with the constant step size

$$\gamma_\ell = \tilde{\gamma}_{\text{FGM}} \stackrel{\text{def}}{=} \frac{\sqrt{C}}{n^{1/3}K_{\max}^{1/3}L} = \frac{2\mu v_{\min}}{L_{\dot{V}}\tilde{f}_n(C,\lambda)\, n^{1/3}K_{\max}^{1/3}} . \tag{24}$$

Then for any $n, K_{\max} \geq 1$ such that $n^{1/3}K_{\max}^{-2/3} \leq \lambda/C$, we have

$$\mathsf{E_1} + \frac{\mu}{(1-\mu)\tilde{f}_n(C,\lambda)}\frac{1}{(nK_{\max})^{1/3}}\mathsf{E_2} \leq \frac{n^{1/3}}{K_{\max}^{2/3}}\frac{L_{\dot{V}}\,\tilde{f}_n(C,\lambda)}{2\mu(1-\mu)v_{\min}^2}\,\Delta V ,$$

where the errors E_i are defined with a random variable K sampled uniformly on $\{0, \cdots, K_{\max}-1\}$.

The proof of Proposition 6 is given in subsubsection 6.2.3. From this upper bound, it can be shown (see subsubsection 6.2.3) that for any $\tau > 0$, there exists $M > 0$ depending upon $L, L_{\dot{V}}, v_{\min}, \mu$ and τ such that for any $\varepsilon > 0$,

$$K_{\max} \geq \left(\sqrt{n}\tau^{3/2}\right) \vee \left(M\sqrt{n}\varepsilon^{-3/2}\right) \Longrightarrow \frac{n^{1/3}}{K_{\max}^{2/3}}\frac{L_{\dot{V}}\,\tilde{f}_n(\lambda\tau,\lambda)}{2\mu(1-\mu)v_{\min}^2} \leq \varepsilon .$$

To our best knowledge, this is the first result in the literature which establishes a non asymptotic control for FIEM at such a rate: the upper bound is an increasing function of n at the rate $n^{1/3}$ and a decreasing function of $K_{\max}$ at the rate $K_{\max}^{-2/3}$.

As a corollary of Proposition 5 and Proposition 6, we have two upper bounds of the errors $\mathsf{E}_1, \mathsf{E}_2$: the first one is $O(n^{2/3}K_{\max}^{-1})$ and the second one is $O(n^{1/3}K_{\max}^{-2/3})$. Given a tolerance $\varepsilon > 0$, the first or second strategy will be chosen depending on how $\sqrt{n}\varepsilon^{-3/2}$ and $n^{2/3}\varepsilon^{-1}$ compare: for any $A > 0$, $\sqrt{n}\epsilon^{-3/2} < An^{2/3}\epsilon^{-1}$ iff $n^{-1/3} < \epsilon A^2$.

When $K_{\max} = A\sqrt{n}\epsilon^{-3/2}$, then $\tilde{\gamma}_{\mathrm{FGM}} = \sqrt{C}\sqrt{\epsilon}/(LA^{1/3}\sqrt{n})$. In the case $\sqrt{n}\epsilon^{-3/2} < \tilde{A}n^{2/3}\epsilon^{-1}$, we have $\tilde{\gamma}_{\mathrm{FGM}} > \sqrt{C}/(LA^{1/3}\tilde{A}n^{2/3})$ thus showing that the step size is lower bounded by $O(n^{-2/3})$ (see γ_{FGM} in Proposition 5). We have $\tilde{\gamma}_{\mathrm{FGM}} \propto 1/\sqrt{n}$ when $K_{\max} \propto \sqrt{n}$: the result of Proposition 6 is obtained with a slower step size (seen as a function of n) than what was required in Proposition 5.

We now discuss a choice for the pair (C, λ) which exploits how (22) behaves when $n \to +\infty$; we prove in subsubsection 6.2.3 that for any $\tau > 0$, there exists $N_\star$ depending only upon $L, L_{\dot{V}}, v_{\min}, \tau$ such that for any $N_\star \leq n \leq \tau^3 K_{\max}^2$,

$$\mathsf{E}_1 + \frac{2^{4/3}(1-\lambda_\star)^{-1/3}\mu^2}{\tilde{f}_n^2(\lambda_\star\tau, \lambda_\star)} \left(\frac{Lv_{\min}}{L_{\dot{V}}}\right)^{2/3} \frac{1}{(nK_{\max})^{1/3}} \mathsf{E}_2 \leq \frac{n^{1/3}}{K_{\max}^{2/3}} \frac{4}{3} \left(\frac{2L^2 L_{\dot{V}}}{v_{\min}^4}\right)^{1/3} (1-\lambda_\star)^{-1/3} \, \Delta V \, ,$$

where $\lambda_\star$ is the unique solution of $(v_{\min}L)^2 \tau^3 (1-\lambda_\star)^2 = (2L_{\dot{V}})^2 \lambda_\star^3$.

3.4 A non-uniform random stopping rule

Given a distribution $p_0, \ldots, p_{K_{\max}-1}$ for the r.v. K, we show how to fix the step sizes $\gamma_1, \ldots, \gamma_{K_{\max}}$ in order to deduce from Theorem 4 a control of the errors E_1 and E_2. For $\lambda \in (0,1)$, $C > 0$ and $n > (C/\lambda)^3$, define the function $F_{n,C,\lambda}$

$$F_{n,C,\lambda} : x \mapsto \frac{L_{\dot{V}}}{2L^2 n^{2/3}} x \left(v_{\min} \frac{2L}{L_{\dot{V}}} - x f_n(C,\lambda) \right) \, ,$$

where f_n is defined by (16). $F_{n,C,\lambda}$ is positive, increasing and continuous on $\left(0, v_{\min}L/(L_{\dot{V}} f_n(C,\lambda))\right]$.

Proposition 7 (application of Theorem 4)**.** *Let K be a $\{0, \ldots, K_{\max}-1\}$-valued random variable with positive weights $p_0, \ldots, p_{K_{\max}-1}$. Choose $\lambda \in (0,1)$ and $C > 0$ such that*

$$\sqrt{C}\, f_n(C,\lambda) = v_{\min} \frac{L}{L_{\dot{V}}} \, . \tag{25}$$

For any $n > (C/\lambda)^3$ and $K_{\max} \geq 1$, we have

$$\mathsf{E}_1 + \frac{L_{\dot{V}}^2}{v_{\min}^2} n^{2/3} \max_k p_k \, f_n(C,\lambda) \sum_{k=0}^{K_{\max}-1} \gamma_{k+1}^2 \mathbb{E}\left[\|\widetilde{S}^{k+1} - \bar{s} \circ \mathsf{T}(\widehat{S}^k)\|^2 \right] \leq n^{2/3} \, \max_k p_k \, \frac{2L_{\dot{V}} \, f_n(C,\lambda)}{v_{\min}^2} \, \Delta V \, ,$$

where the FIEM sequence $\{\widehat{S}^k, k \in \mathbb{N}\}$ *is obtained with*

$$\gamma_{k+1} = \frac{1}{n^{2/3}L}\ F^{-1}_{n,C,\lambda}\left(\frac{p_k}{\max_\ell p_\ell}\frac{v^2_{\min}}{2L_{\dot V} f_n(C,\lambda)}\frac{1}{n^{2/3}}\right)\ .$$

The proof of Proposition 7 is in subsubsection 6.2.4.

Since $\sum_k p_k = 1$, we have $\max_k p_k \geq 1/K_{\max}$ thus showing that among the distributions $\{p_j, 0 \leq j \leq K_{\max}-1\}$, the term $\max_k p_k$ is minimal with the uniform distribution. In that case, the results of Proposition 7 can be compared to the results of Proposition 5: both RHS are increasing functions of n at the rate $n^{2/3}$; both are decreasing functions of $K_{\max}$ at the rate $1/K_{\max}$; the constants C, λ solving the equality in (17) in the case $\mu = 1/2$ are the same as the constants C, λ solving (25): as a consequence,

$$\frac{2L_{\dot V} f_n(C,\lambda)}{v^2_{\min}} = \frac{L_{\dot V} f_n(C,\lambda)}{2\mu(1-\mu)v^2_{\min}}\ , \qquad \mu = 1/2.$$

Finally, when $k \mapsto p_k$ is constant, the step sizes given by Proposition 7 are constant as in Proposition 5; and they are equal since

$$F^{-1}_{n,C,\lambda}\left(\frac{v^2_{\min}n^{-2/3}}{2L_{\dot V} f_n(C,\lambda)}\right) = \sqrt{C} = \frac{v_{\min}L}{L_{\dot V} f_n(C,\lambda)}\ .$$

Hence Proposition 7 and Proposition 5 are the same when $p_k = 1/K_{\max}$ for any k.

As already commented in subsection 3.2, if we choose $C = \lambda$, then (25) gets into

$$\sqrt{C}\ \left(\frac{1}{n^{2/3}} + \frac{1}{1-n^{-1/3}}\left(\frac{1}{n} + \frac{1}{1-C}\right)\right) = \frac{v_{\min}L}{L_{\dot V}}\ .$$

There exists an unique solution $C^\star$, which is upper bounded by a quantity which only depends upon $L, L_{\dot V}, v_{\min}$; hence, so $f_n(C^\star, C^\star)$ is and the control of E_i given in Proposition 7 has the same behavior in $n, K_{\max}$ as $n^{2/3}\max_k p_k$.

If we choose $\lambda = 1/2$, the constant C satisfies (see subsubsection 6.2.4)

$$C \leq \left(\frac{v_{\min}L}{4L_{\dot V}}\right)^{2/3}\ ,$$

and the non asymptotic control given by Proposition 7 is available for $8n > (v_{\min}L/L_{\dot V})^2$.

4 A toy example

In this section, we consider a very simple optimization problem which does not require the incremental EM machinery to be solved [2]

[2]The numerical applications are developed in MATLAB by the first author of the paper. They will be publicly available from her webpage https://perso.math.univ-toulouse.fr/gfort/ by July 1st

4.1 Description

n $\mathbb{R}^y$-valued observations are modeled as the realization of n vectors $Y_i \in \mathbb{R}^y$ whose distribution is described as follows: conditionally to $(Z_1, \ldots, Z_n)$, the r.v. are independent with distribution $Y_i \sim \mathcal{N}_y(AZ_i, \mathrm{I}_y)$ where $A \in \mathbb{R}^{y \times p}$ is a deterministic matrix and I_y denotes the $y \times y$ identity matrix; $(Z_1, \ldots, Z_n)$ are i.i.d. under the distribution $\mathcal{N}_p(X\theta, \mathrm{I}_p)$, where $\theta \in \Theta \stackrel{\text{def}}{=} \mathbb{R}^q$ and $X \in \mathbb{R}^{p \times q}$ is a deterministic matrix. Here, X and A are known, and θ is unknown; the objective is estimate θ, as a solution of a (possibly) penalized maximum likelihood estimator, with penalty term $\rho(\theta) \stackrel{\text{def}}{=} \upsilon\|\theta\|^2/2$ for some $\upsilon \geq 0$. If $\upsilon = 0$, it is assumed that the rank of X and AX is resp. $q = q \wedge y$ and $p = p \wedge y$. In this model, the r.v. $(Y_1, \ldots, Y_n)$ are i.i.d. with distribution $\mathcal{N}_y(AX\theta; \mathrm{I}_y + AA^T)$. The minimum of the function $\theta \mapsto F(\theta) \stackrel{\text{def}}{=} -n^{-1} \log g(Y_{1:n}; \theta) + \rho(\theta)$, where $g(Y_{1:n}; \cdot)$ denotes the likelihood of the vector $(Y_1, \ldots, Y_n)$, is unique and is given by

$$\theta_\star \stackrel{\text{def}}{=} \left(\upsilon \mathrm{I}_q + X^T A^T \left(\mathrm{I}_y + AA^T\right)^{-1} AX\right)^{-1} X^T A^T \left(\mathrm{I}_y + AA^T\right)^{-1} \overline{Y}_n,$$

$$\overline{Y}_n \stackrel{\text{def}}{=} \frac{1}{n} \sum_{i=1}^{n} Y_i.$$

Nevertheless, using the above description of the distribution of Y_i, this optimization problem can be cast into the general framework described in Section 2.1. The loss function (see (4)) is the normalized negative log-likelihood of the distribution of Y_i and is of the form (4) with

$$\phi(\theta) \stackrel{\text{def}}{=} \theta, \quad \mathsf{R}(\theta) \stackrel{\text{def}}{=} \frac{1}{2}\theta^T (X^T X + \upsilon \mathrm{I}_q)\theta, \quad s_i(z) \stackrel{\text{def}}{=} X^T z.$$

Under the stated assumptions on X, the function $\theta \mapsto -\langle s, \phi(\theta)\rangle + R(\theta)$ is defined on $\mathbb{R}^q$ and for any $s \in \mathbb{R}^q$, it possesses an unique minimum given by

$$\mathsf{T}(s) \stackrel{\text{def}}{=} (\upsilon \mathrm{I}_q + X^T X)^{-1} s.$$

Define

$$\Pi_1 \stackrel{\text{def}}{=} X^T (\mathrm{I}_p + A^T A)^{-1} A^T \in \mathbb{R}^{q \times y}\,,$$

$$\Pi_2 \stackrel{\text{def}}{=} X^T (\mathrm{I}_p + A^T A)^{-1} X (\upsilon \mathrm{I}_q + X^T X)^{-1} \in \mathbb{R}^{q \times q}\,.$$

The a posteriori distribution $p_i(\cdot, \theta)\mathrm{d}\mu$ of the latent variable Z_i given the observation Y_i is a Gaussian distribution

$$\mathcal{N}_p\left((\mathrm{I}_p + A^T A)^{-1}(A^T Y_i + X\theta), (\mathrm{I}_p + A^T A)^{-1}\right),$$

2020; and are also available upon request until this free access.

so that for all $i \in \{1, \ldots, n\}$,

$$\bar{s}_i(\theta) \stackrel{\text{def}}{=} X^T(\mathrm{I}_p + A^T A)^{-1}(A^T Y_i + X\theta) = \Pi_1 Y_i + X^T(\mathrm{I}_p + A^T A)^{-1} X\theta \in \mathbb{R}^q,$$
$$\bar{s}_i \circ \mathsf{T}(s) = \Pi_1 Y_i + \Pi_2 s.$$

Therefore, the assumptions H1, H2, H3 and H4-item 1,item 2 are satisfied. Since $\phi \circ \mathsf{T}(s) = \mathsf{T}(s)$ then $B(s) = (\upsilon \mathrm{I}_q + X^T X)^{-1}$ for any $s \in \mathbb{R}^q$, H4-item 3 and H5-item 1 hold with

$$\upsilon_{\min} \stackrel{\text{def}}{=} \frac{1}{\upsilon + \max\_\mathrm{eig}(X^T X)}, \qquad \upsilon_{\max} \stackrel{\text{def}}{=} \frac{1}{\upsilon + \min\_\mathrm{eig}(X^T X)};$$

here, max_eig and min_eig denotes resp. the maximum and the minimum of the eigenvalues. $\bar{s}_i \circ \mathsf{T}(s) = \Pi_1 Y_i + \Pi_2 s$ thus showing that H5-item 2 holds with the same constant $L_i = L$ for all i. Finally, $s \mapsto B^T(s)\,(\bar{s} \circ \mathsf{T}(s) - s)$ is globally Lipschitz with constant

$$L_{\dot{V}} \stackrel{\text{def}}{=} \max \left| \mathrm{eig}\left((\upsilon \mathrm{I}_q + X^T X)^{-1} (\Pi_2 - \mathrm{I}_q) \right) \right|;$$

here eig denotes the eigenvalues. This concludes the proof of H5-item 3.

4.2 The algorithms

Given the current value $\widehat{S}^k$, one iteration of `EM`, `Online EM`, `FIEM` and `opt-FIEM` are given by algorithm 5 and algorithm 6.

All the algorithms (except EM) require $K_{\max}$ random draws from $\{1, \ldots, n\}$ per run of length $K_{\max}$ iterations; `FIEM` and `opt-FIEM` require $2 \times K_{\max}$ draws. For a fair comparison of the algorithms along one run, one vector of integers is sampled prior the runs and is common to all the algorithms. Such a protocol allows to compare the strategies by "freezing" the randomness due to the random choice of the examples, and to really explain the different behaviors only by the values of the design parameters (the step size, for example) or by the updating scheme specific to each algorithm.

All the paths, whatever the algorithms, are started at the same value $\widehat{S}^0$.

Data: $\widehat{S}^k \in \mathbb{R}^q$, Π_1, Π_2 and $\overline{Y}_n$
Result: $\widehat{S}^{k+1}_{\mathrm{EM}}$
1 $\widehat{S}^{k+1}_{\mathrm{EM}} = \Pi_1 \overline{Y}_n + \Pi_2 \widehat{S}^k$

Algorithm 5: Toy example: one iteration of `EM`.

Data: $\widehat{S}^k \in \mathbb{R}^q$, $\mathsf{S} \in \mathbb{R}^{qn}$, $\widetilde{S} \in \mathbb{R}^q$; a step size $\gamma_{k+1} \in (0,1]$ and a coefficient λ_{k+1}; the matrices Π_1, Π_2; the examples $Y_1, \cdots, Y_n$

Result: $\widehat{S}^{k+1}_{\text{FIEM}}$

Sample independently $I_{k+1}, J_{k+1} \sim \mathcal{U}(\{1, \cdots, n\})$;

Store $s = \mathsf{S}_{I_{k+1}}$;

Update $\mathsf{S}_{I_{k+1}} = \Pi_1 Y_{I_{k+1}} + \Pi_2 \widehat{S}^k$;

Update $\widetilde{S} = \widetilde{S} + n^{-1}(\mathsf{S}_{I_{k+1}} - s)$;

Update $\widehat{S}^{k+1}_{\text{FIEM}} = \widehat{S}^k + \gamma_{k+1}\left(\Pi_1 Y_{J_{k+1}} + \Pi_2 \widehat{S}^k - \widehat{S}^k + \lambda_{k+1}\left\{\widetilde{S} - \mathsf{S}_{J_{k+1}}\right\}\right)$

Algorithm 6: Toy example: one iteration of `Online EM` ($\lambda_{k+1} = 0$), `FIEM` ($\lambda_{k+1} = 1$) and `opt-FIEM`.

4.3 Numerical analysis

We choose $Y_i \in \mathbb{R}^{15}$, $Z_i \in \mathbb{R}^{10}$ and $\theta_{\text{true}} \in \mathbb{R}^{20}$. The entries of the matrix A (resp. X) are obtained as a stationary Gaussian auto-regressive process: the first column is sampled from $\sqrt{1-\rho^2}\,\mathcal{N}_{15}(0;\mathrm{I})$ (resp. from $\sqrt{1-\tilde{\rho}^2}\,\mathcal{N}_{10}(0;\mathrm{I})$) with $\rho = 0.8$ (resp. $\tilde{\rho} = 0.9$). θ_{true} is sparse with 40% of the components set to zero; and the other ones are sampled uniformly in $[-5, 5]$.

The regularization parameter υ is set to 0.1.

`FIEM`: the step sizes and the non asymptotic controls. The first analysis is to compare the non asymptotic bounds and the constant step sizes provided by Proposition 5, Proposition 6 and (Karimi et al., 2019c, Theorem 2) (see also (20) and (21)): the bounds are of the form

$$\frac{n^{\mathsf{a}}}{K_{\max}^{\mathsf{b}}}\ \mathcal{B}\ \Delta V\ ;$$

the numerical results below correspond to $\Delta V = 1$ and are obtained with a data set of size $n = 1e6$. Figure 1 shows the value of the constant C solving (17) when λ is successively set to $\{0.25, 0.5, 0.75\}$ and as a function of $\mu \in (0.01, 0.9)$. Figure 2 shows the same analysis for the constant C solving (23). Figure 3 and Figure 4 display the quantity $\mathcal{B}$ as a function of μ and when (λ, C) is fixed to $\lambda \in \{0.25, 0.5, 0.75\}$ and C solves resp. (17) and (23). The role of λ looks quite negligible; the bound $\mathcal{B}$ seems to be optimal with $\mu \approx 0.25$. Note that the constants C and $\mathcal{B}$ given by Proposition 6 also depends on $K_{\max}$: the results displayed here correspond to $K_{\max} = n$ but we observed that the plots are the same with $K_{\max} = 1e2\,n$ and $K_{\max} = 1e3\,n$ (remember that $n = 1e6$).

Figure 5 displays the step sizes as a function of $\mu \in (0.01, 0.9)$, when $\lambda = 1/2$ and for different strategies of $K_{\max}$: $K_{\max} \in \{n, 1e2\,n, 1e3\,n\}$. Figure 6 displays the

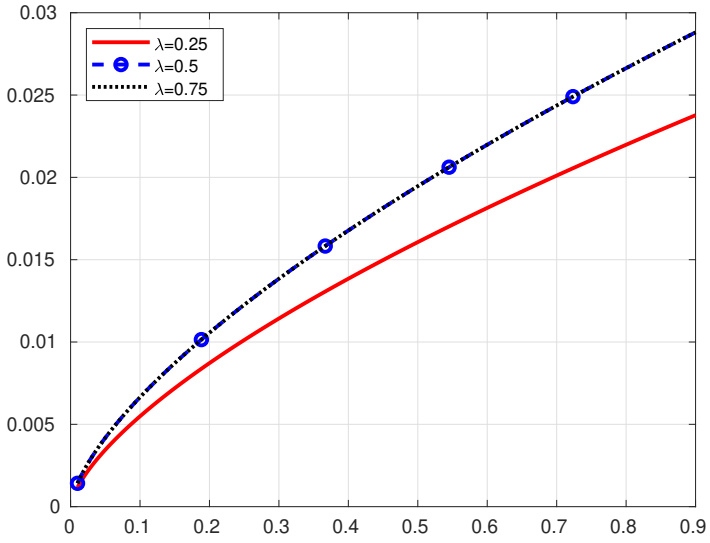

Figure 1: For $\lambda \in \{0.25, 0.5, 0.75\}$ and $\mu \in (0.01, 0.9)$, evolution of the constant C solving (17)

quantity $n^{\mathsf{a}} K_{\max}^{-\mathsf{b}} \mathcal{B}$. `Case 1` (resp. `Case 2`) corresponds to the definition given in Proposition 5 (resp. Proposition 6). For `Case 1` and `Karimi et al`, $(\mathsf{a}, \mathsf{b}) = (2/3, 1)$ and for `Case 2`, $(\mathsf{a}, \mathsf{b}) = (1/3, 2/3)$. The first conclusion is that our results improve on those by Karimi et al. (2019c): we provide a larger step size (improved by a factor up to 55, with the strategy `Case 1`, $\mu = 0.25$, $\lambda = 0.5$) and a tighter bound (reduced by a factor up to 235, with the strategy `Case 1`, $\mu = 0.25$, $\lambda = 0.5$). The second conclusion is about the comparison of Proposition 5 and Proposition 6: as already commented (see subsection 3.3), the first strategy is preferable when the tolerance level ϵ is small (w.r.t. $n^{-1/3}$).

Comparison of `Online EM`, `FIEM` and `opt-FIEM`. The algorithms are run with the same constant step size given by (18) when C solves (17) with $\mu = 0.25$ and $\lambda = 0.5$. The size of the data set is $n = 1e3$ and the maximal number of iterations is $K_{\max} = 20\, n$. Since the non asymptotic convergence bounds are essentially based on the control of $\gamma_{k+1}^{-2}\, \mathbb{E}\left[\|\widehat{S}^{k+1} - \widehat{S}^{k}\|^2\right]$ (see the sketch of proof of Theorem 4 in section 3), we first compare the algorithms through this criterion: the expectation is approximated by a Monte Carlo sum over $1e3$ independent runs. The second criterion for comparison is a distance of the iterates to the unique solution $\theta_\star$ via the expectation

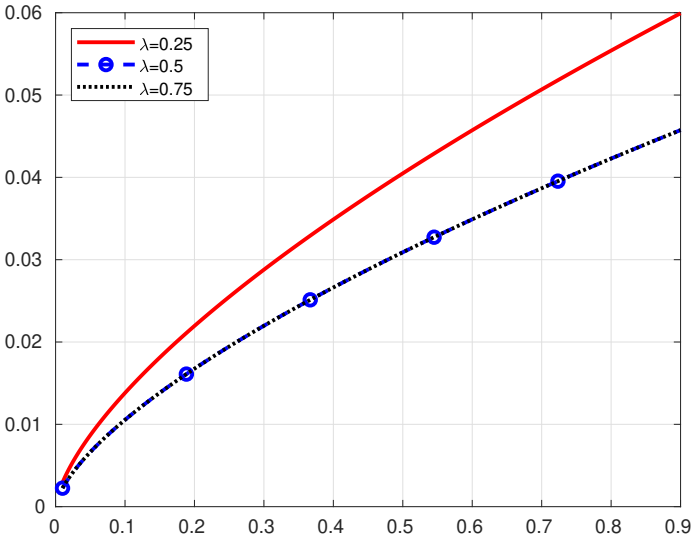

Figure 2: For $\lambda \in \{0.25, 0.5, 0.75\}$ and $\mu \in (0.01, 0.9)$, evolution of the constant C solving (23)

$\mathbb{E}\left[\|\theta^k - \theta_\star\|\right]$ and the standard deviation $\mathrm{std}\left(\|\theta^k - \theta_\star\|\right)$ again approximated by a Monte Carlo sum over the same $1e3$ independent runs.

Figure 7 displays the evolution of $k \mapsto \lambda^\star_{k+1}$, the optimal coefficient given by (13); in this toy example, it is computed explicitly. We have $\lambda^\star_{k+1} \approx 1$ for large iteration index k: `FIEM` and `opt-FIEM` are expected to be equivalent in the convergence phase. The ratio of the expectations $\mathbb{E}\left[\|\theta^k_{\text{opt\_FIEM}} - \theta_\star\|\right] / \mathbb{E}\left[\|\theta^k_{\text{alg}} - \theta_\star\|\right]$ and of the standard deviations $\mathrm{std}(\|\theta^k_{\text{opt\_FIEM}} - \theta_\star\|)/\mathrm{std}(\|\theta^k_{\text{alg}} - \theta_\star\|)$ are displayed on Figure 8 when alg is `FIEM` and `Online EM`. They are shown as a function of $k \in \{1e2, 5e2, 1e3, 1.5e3, \ldots, 6e3, 7e3, \ldots, 20e3\}$. The plot illustrates that if `opt-FIEM` and `FIEM` are equivalent in expectation, `opt-FIEM` surpasses `FIEM` in the transient phase by reducing the variance up to 22%. It also shows that `Online EM` has a really poor behavior w.r.t. `opt-FIEM` (and therefore also with `FIEM`) in the convergence phase, `Online EM` reduces the variability of `opt-FIEM` up to 18% in the transient phase, but `opt-FIEM` provides a drastic variability reduction in the first iterations. Since we advocate to stop `FIEM` at a random time K sampled in the range $\{0, \ldots, K_{\max} - 1\}$, `opt-FIEM` gives insights on how to improve the behavior of incremental EM algorithms in the transient phase. Figure 9 shows $k \mapsto \gamma_{k+1}^{-2}\mathbb{E}\left[\|\widehat{S}^{k+1} - \widehat{S}^k\|^2\right]$ for the

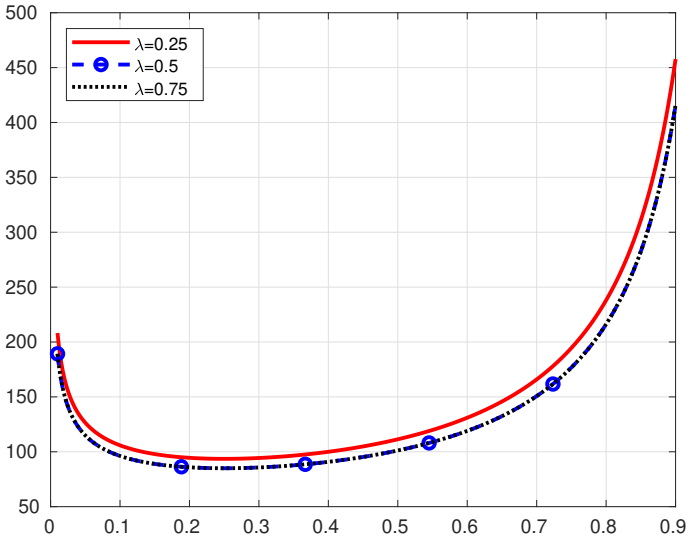

Figure 3: For $\lambda \in \{0.25, 0.5, 0.75\}$ and $\mu \in (0.01, 0.9)$, evolution of the quantity $\mathcal{B}$ given by Proposition 5

three algorithms, in the transient phase $k \in [1.5e3, 5e3]$. The plot illustrates again that `opt-FIEM` improves on `FIEM` during this phase of the algorithm; and improves drastically on `Online EM`.

5 Mixture of Gaussian distributions

In this section [3], `FIEM` is applied to solve the Maximum Likelihood inference in a mixture of L Gaussian distributions centered at μ_ℓ and sharing the same covariance matrix Σ (see Frühwirth-Schnatter et al. (2019) for a recent review on mixture models): given n $\mathbb{R}^p$-valued observations $y_1, \ldots, y_n$, find a point $\hat{\theta}_n^{\mathrm{ML}} \in \Theta$ satisfying $F(\hat{\theta}_n^{\mathrm{ML}}) \stackrel{\mathrm{def}}{=} \mathsf{R}(\hat{\theta}_n^{\mathrm{ML}}) + n^{-1} \sum_{i=1}^n \mathcal{L}_i(\hat{\theta}_n^{\mathrm{ML}}) \leq F(\theta)$ for any $\theta \in \Theta$ where $\theta \stackrel{\mathrm{def}}{=} (\alpha_1, \ldots, \alpha_L, \mu_1, \ldots, \mu_L, \Sigma)$,

$$\Theta \stackrel{\mathrm{def}}{=} \left\{ \alpha_\ell \geq 0, \sum_{\ell=1}^L \alpha_\ell = 1 \right\} \times \mathbb{R}^{pL} \times (\mathcal{M}_p^+) \subseteq \mathbb{R}^{L+pL+(p\times p)} \; ;$$

[3] The numerical applications are developed in MATLAB by the first author of the paper. They will be publicly available from her webpage https://perso.math.univ-toulouse.fr/gfort/ by July 1st 2020; and are also available upon request until this free access.

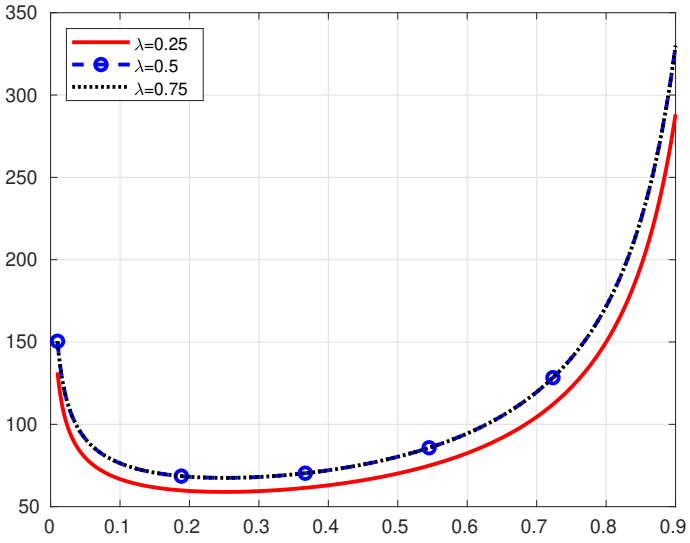

Figure 4: For $\lambda \in \{0.25, 0.5, 0.75\}$ and $\mu \in (0.01, 0.9)$, evolution of the quantity $\mathcal{B}$ given by Proposition 6

$\mathcal{M}_p^+$ denotes the invertible $p \times p$ covariance matrices. In addition,

$$\mathsf{R}(\theta) + \frac{1}{n}\sum_{i=1}^{n} \mathcal{L}_i(\theta) = -\frac{1}{n}\sum_{i=1}^{n} \log \sum_{\ell=1}^{L} \alpha_\ell \, \mathcal{N}_p(\mu_\ell, \Sigma)[y_i] \;,$$

where we set (the term $p\log(2\pi)/2$ is omitted)

$$R(\theta) \stackrel{\text{def}}{=} \frac{1}{2}\log\det(\Sigma) + \frac{1}{2n}\sum_{i=1}^{n} y_i^T \Sigma^{-1} y_i = \frac{1}{2}\left(\left\langle \Sigma^{-1}, \frac{1}{n}\sum_{i=1}^{n} y_i y_i^T \right\rangle - \log\det(\Sigma^{-1}) \right) \;.$$

In this example, we have $\mathcal{L}_i(\theta) = -\log \sum_{z=1}^{L} \exp(\langle s_i(z), \phi(\theta) \rangle)$ with

$$s_i(z) = \mathsf{A}_{y_i} \begin{bmatrix} \mathbb{1}_{z=1} \\ \cdots \\ \mathbb{1}_{z=L} \end{bmatrix} \in \mathbb{R}^{L+pL} \;, \qquad \mathsf{A}_y \stackrel{\text{def}}{=} \begin{bmatrix} \mathrm{I}_L \\ \mathrm{I}_L \otimes y \end{bmatrix} \;;$$

$\otimes$ stands for the Kronecker product. We use the MNIST dataset [4]. The data are pre-processed as in Nguyen et al. (2020): the training set contains $n = 6e4$ images of

[4] available at http://yann.lecun.com/exdb/mnist/

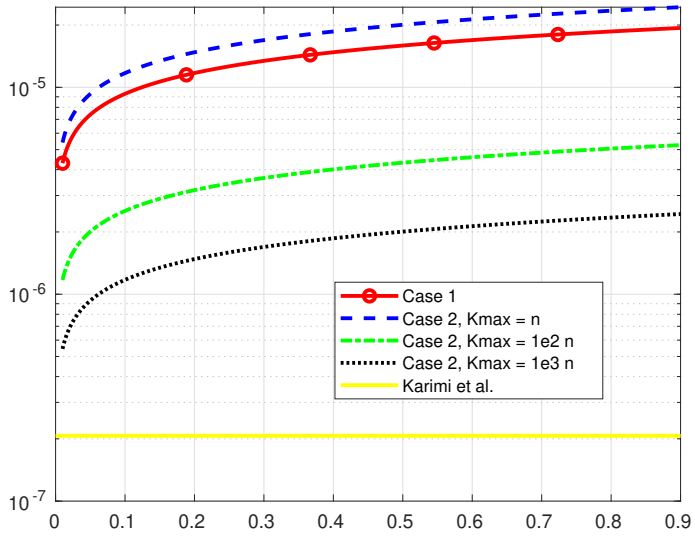

Figure 5: Value of the constant step size given by Karimi et al., Proposition 5 (`Case 1`) and Proposition 6 (`Case 2`). The step size is shown as a function of $\mu \in (0.01, 0.9)$. In `Case 2`, different strategies for $K_{\max}$ are considered.

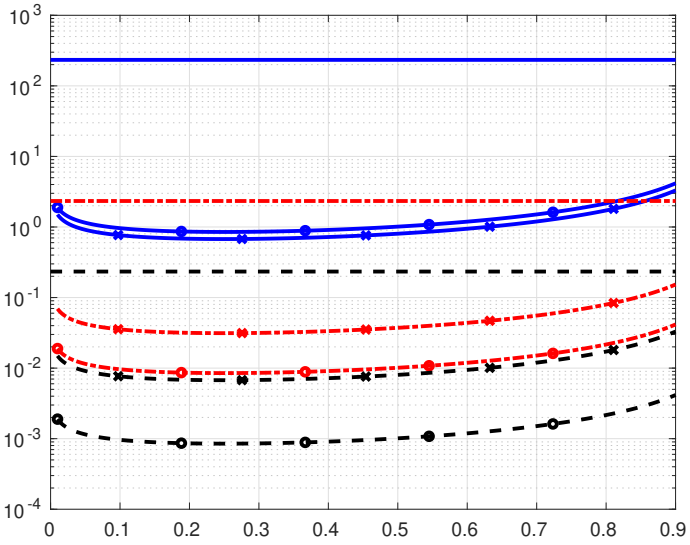

Figure 6: Value of the control $n^{\mathsf{a}} K_{\max}^{-\mathsf{b}} \mathcal{B}$ given by Proposition 5 (`Case 1`, with a circle), Proposition 6 (`Case 2`, with a cross) and Karimi et al. (no markers). The control is displayed as a function of $\mu \in (0.01, 0.9)$ and for different values of $K_{\max}$: $K_{\max} = n$ (solid line), $K_{\max} = 1e2\, n$ (dash-dot line) and $K_{\max} = 1e3\, n$ (dashed line).

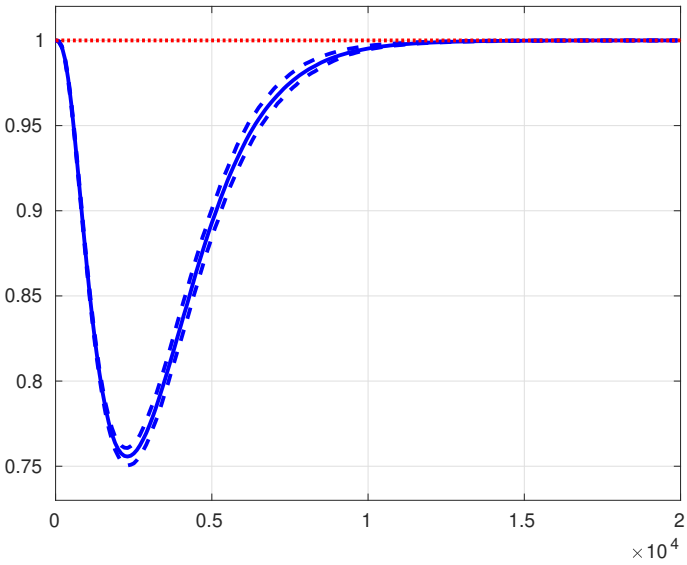

Figure 7: The coefficient $\lambda_k^\star$ (see (13)) as a function of the number of iterations k; it is a random variable, and the solid line is the mean value (the dashed lines are resp. the quantiles 0.25 and 0.75) over $1e3$ independent paths.

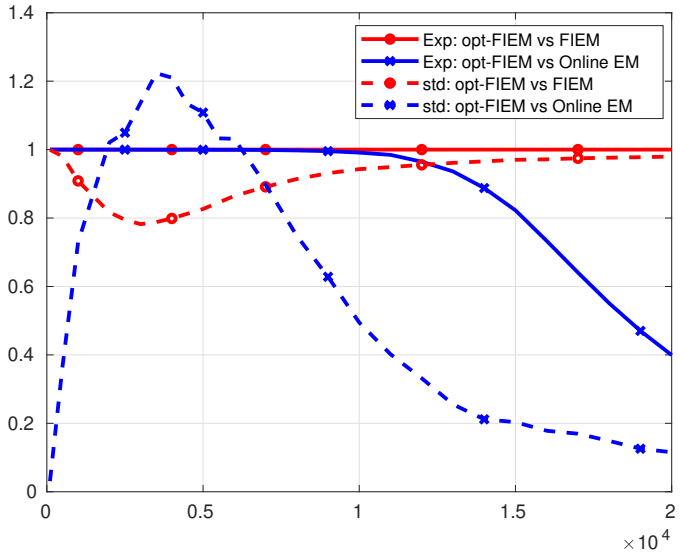

Figure 8: For $k \in \{1e2, 5e2, 1e3, 1.5e3, \ldots, 6e3, 7e3, \ldots, 20e3\}$, ratio of the expectations (Exp) $\mathbb{E}\left[\|\theta^k_{\text{opt}-\text{FIEM}} - \theta_\star\|\right] / \mathbb{E}\left[\|\theta^k_{\text{Alg}} - \theta_\star\|\right]$ when Alg is FIEM (solid line with circle) and then `Online EM` (solid line with cross); and the standard deviations (std) $\text{std}(\|\theta^k_{\text{opt}_\text{F}\text{IEM}} - \theta_\star\|)/\text{std}(\|\theta^k_{\text{FIEM}} - \theta_\star\|)$ when Alg is `FIEM` (dashed line with circle) and then `Online EM` (dashed line with cross). The expectations and standard deviations are approximated by a Monte Carlo sum over $1e3$ independent runs.

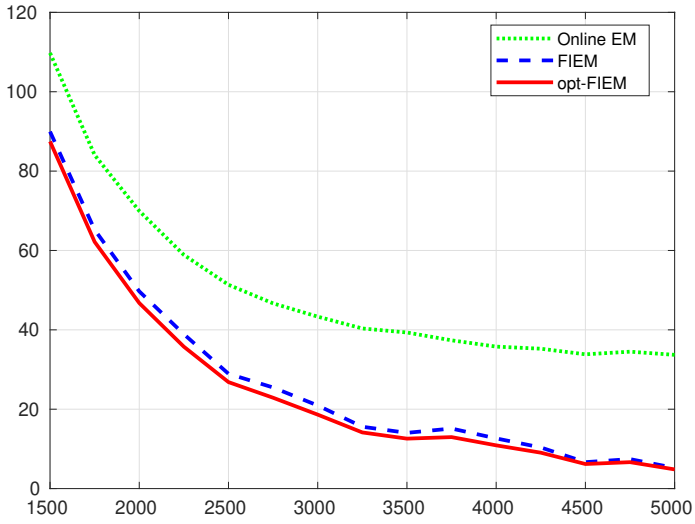

Figure 9: Monte Carlo approximation (over $1e3$ independent runs) of $k \mapsto \gamma_{k+1}^{-2}\mathbb{E}\left[\|\widehat{S}^{k+1} - \widehat{S}^{k}\|^2\right]$ for `Online EM`, `FIEM` and `opt-FIEM`.

size 28×28; among these 784 pixels, 67 are non informative since they are constant over all the pictures so they are removed yielding to n observations of length 717; each feature is centered and standardized (among the n observations) and a PCA of the associated 717×717 covariance matrix is applied in order to summarize the features by the first $p = 20$ principal components. In the numerical applications, we fix $L = 12$ components in the mixture.

The maximization step $\hat{\theta} = \mathsf{T}(s)$ is given by (see the *supplementary material* for the detailed computations)

$$\begin{aligned}
\widehat{\alpha_\ell} &\overset{\text{def}}{=} \frac{s_\ell}{\sum_{u=1}^{L} s_u}\,, \ell = 1, \ldots, L\,, \\
\widehat{\mu_\ell} &\overset{\text{def}}{=} \frac{s_{L+(\ell-1)p+1:\ell p}}{s_\ell}\,, \ell = 1, \ldots, L\,, \\
\widehat{\Sigma} &\overset{\text{def}}{=} \frac{1}{n}\sum_{i=1}^{n} y_i y_i^T - \sum_{\ell=1}^{L} s_\ell\, \widehat{\mu}_\ell\, \widehat{\mu}_\ell^T\,,
\end{aligned}$$

where $s \overset{\text{def}}{=} (s_1, \ldots, s_{L+pL})$. Since we want $\hat{\theta} \in \Theta$, T is defined at least on $\mathcal{S} \subset \mathbb{R}^{L+pL}$

(see the *supplementary material*):

$$\mathcal{S} \quad \stackrel{\text{def}}{=} \quad \left\{ n^{-1} \sum_{i=1}^{n} \mathsf{A}_{y_i} \rho_i : \rho_i = (\rho_{i,1}, \ldots, \rho_{i,L}), \rho_{i,\ell} \geq 0, \sum_{\ell=1}^{L} \rho_{i,\ell} = 1 \right\} \quad .$$

This model is used to go beyond the theoretical framework adopted in this paper. The first extension concerns the domain of T: H3 assumes that T is defined on $\mathbb{R}^q$ (here, $q = L + pL$) while the above description shows that it is not always true. This gap between theory and application is classical for mixture of Gaussian distributions (see the comments in subsection 2.2); while $\rho_{\ell,i}$ may be a signed quantity or while we may have $\sum_{\ell=1}^{L} \rho_{i,\ell} \neq 1$ for the considered algorithms (see *supplementary material* for a detailed derivation), numerically we always obtained input quantities $\widehat{S}^k$ which were in $\mathcal{S}$.

The second extension concerns the use of mini-batches at each iteration of incremental EM algorithms: instead of sampling one example per iteration (see e.g. line 2, line 4 and line 4, line 8 resp. in algorithm 2, algorithm 3 and algorithm 4) a mini-batch of size b is used - sampled at random from the n available examples, possibly with replacement. While, as usual in the literature, the theoretical analyses are derived in the case of a single example (b = 1), the numerical applications use b > 1; we do the same here. The *supplementary material* provides a description of `iEM`, `Online EM` and `FIEM` in the case b > 1, when applied to the current application of inference in a mixture of Gaussian distributions.

`EM`, `iEM`, `Online EM` and `FIEM` are compared when used to solve the above Maximum Likelihood inference problem. All the paths of these algorithms are started from the same point $\theta^0 \in \Theta$ defined by the randomization scheme described in (Kwedlo, 2015, section 4); we then set $\widehat{S}^0 \stackrel{\text{def}}{=} n^{-1} \sum_{i=1}^{n} \bar{s}_i(\theta^0)$; the normalized log-likelihood $-F(\theta^0)$ is equal to -58.3097 (equivalently, the unnormalized log-likelihood is $-3.4986\,\text{e+6}$). Note that, as mentioned below, the evaluation of the log-likelihood does not include the constant $+p \log(2\pi)/2$.

Each iteration of `iEM`, `Online EM` (resp. `FIEM`) calls a mini-batch of b = 100 examples (resp. 2b = 200 examples) sampled uniformly from $\{1, \ldots, n\}$ with replacement; for a fair comparison of the paths produced by these algorithms, the same seed is used.

The paths are seen as cycles of *epochs*, an epoch being defined as the processing of n examples: for `EM`, an epoch is one iteration; for `iEM` and `Online EM`, an epoch is n/b iterations; for `FIEM`, an epoch is $n/(2\text{b})$ iterations. Below, the paths are run until $100\,n$ examples are processed, which means 100 iterations or epochs for `EM`, and $100\,n$/b iterations (or 100 epochs) for both `iEM` and `Online EM`. Instead of a pure `FIEM` algorithm, we implement `h-FIEM`, an hybrid algorithm obtained by first

running kswith epochs of `Online EM` and then switching to epochs of `FIEM`: we choose kswitch $= 6$ so that `h-FIEM` processes $100\,n$ examples after $6\,n/\mathrm{b}$ iterations (or 6 epochs) of `Online EM` and $94\,n/(2\mathrm{b})$ iterations (or 94 epochs) of `FIEM`. The use of `h-FIEM` is to explicitly illustrate the variance reduction of the `FIEM` iterations when compared to the `Online EM` ones.

`iEM` is run with the constant step size $\gamma_{k+1} = 1$; `Online EM` and `FIEM` are run with $\gamma_{k+1} = 5\,\mathrm{e}{-3}$.

Figure 10 and Figure 11 display the normalized log-likelihood along a path of `EM`, `iEM`, `Online EM` and h-FIEM, resp. for the first epochs (from 1 to 25) and by discarding the first ones (from 15 to 100). The first conclusion is that the incremental methods forget the initial value far more rapidly than `EM`, which is the consequence of the incremental processing of the observations which allow many updates of the parameter θ^k (or equivalently, of the statistic $\widehat{S}^k$) before the call to n examples (which is equivalent to the learning cost of one iteration of `EM`). The second conclusion is that the incremental EM-based methods perform a better maximization of the normalized log-likelihood $-F$. Finally, `Online EM` and `h-FIEM` are better than `iEM`: the log-likelihood converges resp. to $-1.9094\,\mathrm{e}{+6}$, $-1.9080\,\mathrm{e}{+6}$ and $-1.9100\,\mathrm{e}{+6}$ (the plot displays the normalized log-likelihood); and it is clear that `h-FIEM` reduces the variability of the `Online EM` path. The same conclusions are drawn from different runs; the *supplementary material* provides a similar plots when the curves are the average over 10 independent paths; Table 1 reports the mean value and the standard deviation of the log-likelihood over these 10 runs.

A fluctuation of 1% (resp. 1 ‰) around the optimal normalized log-likelihood corresponds to a lower bound of -32.1184 (resp. -31.8322): for `EM` such an accuracy is reached after 12 iterations (resp. is never reached); for `iEM`, it is reached after 11 epochs (resp. is never reached); for `Online EM`, after 4 epochs (resp. 23 epochs); for `h-FIEM`, after 4 epochs (resp. 34 epochs). An accuracy of 1 ‱ is never reached by `Online EM` and is reached after 36 epochs for `h-FIEM`.

Figure 12 shows the estimation of the $L = 12$ weights α_ℓ along a path of length 100 epochs. The comparison of `Online EM` (bottom left) and `h-FIEM` (bottom right) shows that `h-FIEM` acts as a variability reduction technique along the path, without slowing down the convergence rate. Figure 13 displays the limiting value of these paths i.e. the estimate of the weights $\alpha_1, \ldots, \alpha_L$ defined as the value of the parameter at the end of 100 epochs; the weights are sorted in descending order. `Online EM` and `h-FIEM` provide similar estimates.

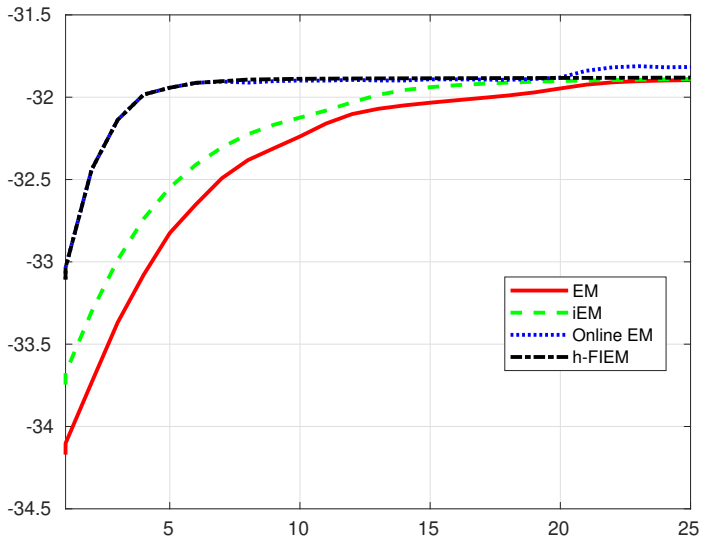

Figure 10: Evolution of the normalized log-likelihood along one path of length 100 epochs: only the epochs 1 to 25 are displayed. All the paths start from the same value at time $t = 0$, with a normalized log-likelihood equal to -58.31.

	#1	#15	#25	#50	#100
EM	−3.4102e+1	−3.2033e+1	−3.1896e+1	−3.1890e+1	−3.1889e+1
	-	-	-	-	-
iEM	−3.3672e+1	−3.1982e+1	−3.1869e+1	−3.1843e+1	−3.1827e+1
	(4.90e−3)	(5.33e−2)	(1.87e−2)	(1.87e−2)	(1.38e−2)
Online EM	**-3.2999**e+1	**-3.1872**e+1	**-3.1828**e+1	−3.1823e+1	−3.1823e+1
	(2.67e−2)	(5.14e−2)	(4.67e−2)	(4.68e−2)	(4.50e−2)
h-FIEM	−3.2999e+1	−3.1900e+1	−3.1853e+1	**-3.1806**e+1	**-3.1804**e+1
	(2.67e−2)	(5.31e−2)	(6.94e−2)	(5.18e−2)	(5.25e−2)

Table 1: Normalized log-likelihood along a `EM`, `iEM`, `Online EM` and `h-FIEM` path, at epoch $\#1, 15, 25, 50, 100$. The value is the average over 10 independent runs (the standard deviation is in parenthesis). The log-likelihood is obtained by multiplying by $n = 6\text{e}{+4}$.

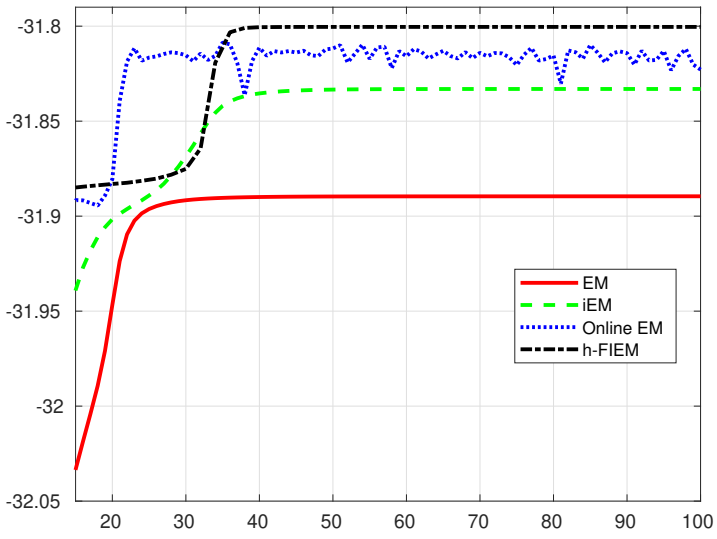

Figure 11: Evolution of the normalized log-likelihood along one path of length 100 epochs: the first 14 epochs are discarded. All the paths start from the same value at time $t = 0$, with a normalized log-likelihood equal to -58.31.

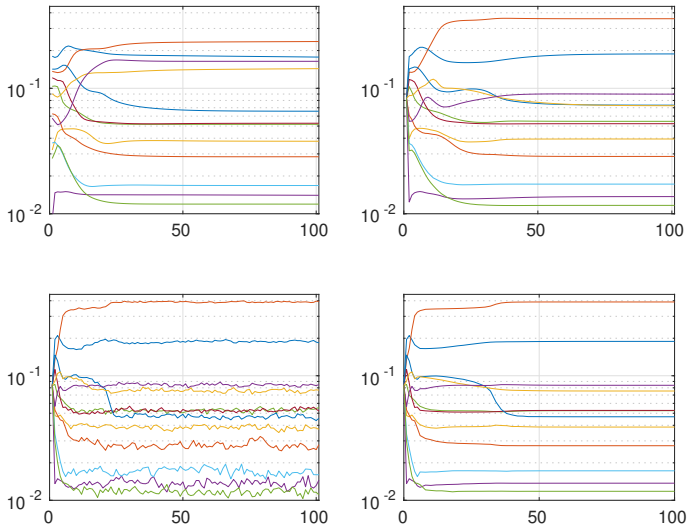

Figure 12: Evolution of the $L = 12$ weights along one path of length 100 epochs. All the paths start from the same value at time $t = 0$. EM (top left), iEM (top right), Online EM (bottom left) and h-FIEM (bottom right).

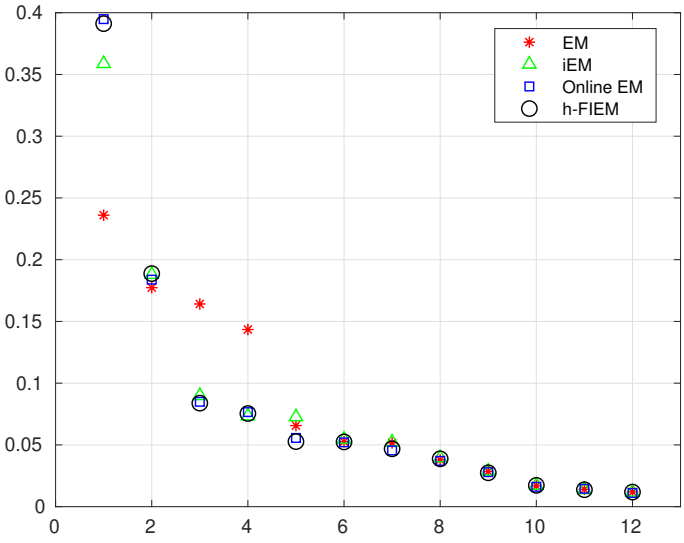

Figure 13: Estimation of the $L = 12$ weights of the mixture model. The estimator is the value of the parameter obtained at the end of a single path of length 100 epochs.

6 Proof

6.1 Proof of section 2

6.1.1 Proof of Proposition 1

(proof of item 1). From the Jensen's inequality, it holds

$$\mathcal{L}_i(\theta) - \mathcal{L}_i(\theta') \leq -\int_{\mathsf{Z}} \left\langle s_i(z), \phi(\theta) - \phi(\theta') \right\rangle \; p_i(z;\theta')\,\mu(\mathrm{d}z) = -\left\langle \bar{s}_i(\theta'), \phi(\theta) - \phi(\theta') \right\rangle \; ;$$

which concludes the proof. *(proof of item 2)* From (3) and item 1, it holds

$$F(\theta) \leq -\left\langle \bar{s}(\theta'), \phi(\theta) \right\rangle + \frac{1}{n}\sum_{i=1}^{n} \mathcal{C}_i(\theta') + \mathsf{R}(\theta) \; .$$

(proof of item 3) From item 2 and the definition of T, it holds

$$F(\mathsf{T} \circ \bar{s}(\theta^k)) \leq \overline{F}(\mathsf{T} \circ \bar{s}(\theta^k), \theta^k) \leq \overline{F}(\theta^k, \theta^k) = F(\theta^k) \; .$$

6.1.2 Proof of Proposition 2

(Proof of item 1). The statements are trivial and we only prove the first claim: if $s^\star = \bar{s} \circ \mathsf{T}(s^\star)$ then by applying T (under the uniqueness assumption H3), we have $\mathsf{T}(s^\star) = (\mathsf{T} \circ \bar{s}) \circ \mathsf{T}(s^\star)$ and the proof follows.

(Proof of item 2). For $\theta \in \Theta^v$, set $D\phi(\theta) \stackrel{\text{def}}{=} \left(\dot{\phi}(\theta)\right)^T$. By H4-item 2 and a chain rule,

$$\dot{V}(s) = \left(\dot{\mathsf{T}}(s)\right)^T \left\{\dot{\mathsf{R}}(\mathsf{T}(s)) - D\phi(\mathsf{T}(s))\ \bar{s} \circ \mathsf{T}(s)\right\} .$$

Moreover, using H3 and H4-item 1, the minimum $\mathsf{T}(s)$ is a critical point of $\theta \mapsto -\langle s, \phi(\theta)\rangle + \mathsf{R}(\theta)$: we have for any $s \in \mathbb{R}^q$, $\dot{\mathsf{R}}(\mathsf{T}(s)) - D\phi(\mathsf{T}(s))\, s = 0$. Hence,

$$\dot{V}(s) = -\left(\dot{\mathsf{T}}(s)\right)^T\ D\phi(\mathsf{T}(s))\ h(s) = -\left(B(s)\right)^T\ h(s) .$$

H4-item 3 implies that $B^T = B$ and the zeros of h are the zeros of $\dot{V}$.

6.1.3 Auxiliary results

Lemma 8. *Assume that Θ and $\phi(\Theta)$ are open; and ϕ is continuously differentiable on Θ. Then for all $i \in \{1, \ldots, n\}$, $\mathcal{L}_i$ is continuously differentiable on Θ.*

If in addition H1, H2, H3 and H4-item 1 hold, then F (resp. $V \stackrel{\text{def}}{=} F \circ \mathsf{T}$) is continuously differentiable on Θ (resp. on $\mathbb{R}^q$) and for any $\theta \in \Theta$,

$$\dot{F}(\theta) = -\left(\dot{\phi}(\theta)\right)^T\ \bar{s}(\theta) + \dot{\mathsf{R}}(\theta) .$$

Proof. H1 and (Sundberg, 2019, Proposition 3.8) (see also (Brown, 1986, Theorem 2.2.)) imply that $L_i : \tau \mapsto \int_Z h_i(z) \exp\left(\langle s_i(z), \tau\rangle\right) \mu(\mathrm{d}z)$ is continuously differentiable on the interior of the set $\{\tau \in \mathbb{R}^q, \int_Z h_i(z) \exp\left(\langle s_i(z), \tau\rangle\right) \mu(\mathrm{d}z) < \infty\}$ and its derivative is

$$\int_{\mathsf{Z}} s_i(z)\ h_i(z) \exp\left(\langle s_i(z), \tau\rangle\right) \mu(\mathrm{d}z) .$$

This set contains $\phi(\Theta)$ under H1. The equality $\mathcal{L}_i = -\log(L_i \circ \phi)$ and the differentiability of composition of functions conclude the proof of the first item. The second one easily follows. □

Lemma 9. *Assume H1 and H3. Assume also that for any $s \in \mathbb{R}^q$, $\tau \mapsto \mathsf{Q}(s, \tau) \stackrel{\text{def}}{=} -\langle s, \phi(\tau)\rangle + \mathsf{R}(\tau)$ is twice continuously differentiable on Θ^v where $\Theta^v \stackrel{\text{def}}{=} \Theta$ if Θ is open, or Θ^v is a neighborhood of Θ otherwise. Then for any $s \in \mathbb{R}^q$, $\dot{\phi \circ \mathsf{T}}$ is a symmetric $q \times q$ matrix satisfying*

$$\dot{\phi \circ \mathsf{T}}(s) = \left(\dot{\mathsf{T}}(s)\right)^T\ \partial_\tau^2 \mathsf{Q}(s, \mathsf{T}(s))\ \left(\dot{\mathsf{T}}(s)\right) .$$

Proof. The proof is adapted from the one of (Delyon et al., 1999, Lemma 2). H3 and the regularity conditions on Q imply that for any $s \in \mathbb{R}^q$:

$$\partial_\tau \mathsf{Q}(s, \mathsf{T}(s)) = -\left(\dot{\phi}(\mathsf{T}(s))\right)^T s + \dot{\mathsf{R}}(\mathsf{T}(s)) = 0$$

and (from the uniqueness assumption) $s \mapsto \partial_\tau^2 \mathsf{Q}(s, \mathsf{T}(s))\, \dot{\mathsf{T}}(s) - \left(\dot{\phi}(\mathsf{T}(s))\right)^T$ is positive-definite. This concludes the proof. □

Lemma 10. *Assume H1, H2 and H3. Assume in addition that (i) there exists $L_{i,p}$ such that for any $\theta, \theta' \in \Theta$*

$$\sup_{z \in \mathsf{Z}} \left| p_i(z;\theta) - p_i(z;\theta') \right| \leq L_{i,p}\, \|\theta - \theta'\|;$$

(ii) T is globally Lipschitz on $\mathbb{R}^q$, and (iii) $\int_{\mathsf{Z}} \|s_i\| \mathrm{d}\mu < \infty$. Then there exists a constant $0 < L_i < \infty$ such that for all $s, s' \in \mathbb{R}^q$, $\|\bar{s}_i \circ \mathsf{T}(s) - \bar{s}_i \circ \mathsf{T}(s')\| \leq L_i \|s - s'\|$.

Proof. Let $s, s' \in \mathbb{R}^q$. We have

$$\bar{s}_i \circ \mathsf{T}(s) - \bar{s}_i \circ \mathsf{T}(s') = \int_{\mathsf{Z}} s_i(z) \left[p_i(z; \mathsf{T}(s)) - p_i(z; \mathsf{T}(s')) \right] \mu(\mathrm{d}z)$$

so that

$$\begin{aligned} \|\bar{s}_i \circ \mathsf{T}(s) - \bar{s}_i \circ \mathsf{T}(s')\| &\leq \int_{\mathsf{Z}} \|s_i(z)\| \left| p_i(z; \mathsf{T}(s)) - p_i(z; \mathsf{T}(s')) \right| \mu(\mathrm{d}z) \\ &\leq L_p\, \|\mathsf{T}(s) - \mathsf{T}(s')\| \int_{\mathsf{Z}} \|s_i(z)\| \mu(\mathrm{d}z)\ . \end{aligned}$$

□

6.2 Proofs of section 3

For any $k \geq 0$ and $i \in \{1, \ldots, n\}$, we define $\widehat{S}^{<k,i}$ such that

$$\widetilde{S}^k = \frac{1}{n} \sum_{i=1}^{n} \bar{s}_i \circ \mathsf{T}(\widehat{S}^{<k,i})\ ;$$

it means $\widehat{S}^{<0,i} \stackrel{\text{def}}{=} \widehat{S}^0$ for all $i \in \{1, \ldots, n\}$ and for $k \geq 0$,

$$\widehat{S}^{<k+1,i} = \widehat{S}^{\ell}, \begin{cases} \ell = k & \text{if } I_{k+1} = i, \\ 1 \leq \ell \leq k-1 & \text{if } I_{k+1} \neq i, I_k \neq i, \ldots, I_{\ell+1} = i, \\ \ell = 0 & \text{otherwise.} \end{cases} \tag{26}$$

Define the filtrations, for $k \geq 0$,

$$\mathcal{F}_k \stackrel{\text{def}}{=} \sigma(\widehat{S}^0, I_1, J_1, \ldots, I_k, J_k), \qquad \mathcal{F}_{k+1/2} \stackrel{\text{def}}{=} \sigma(\widehat{S}^0, I_1, J_1, \ldots, I_k, J_k, I_{k+1}) \; ;$$

note that $\widehat{S}^k \in \mathcal{F}_k$ and $\mathsf{S}_{k+1,\cdot} \in \mathcal{F}_{k+1/2}$. Set

$$H_{k+1} \stackrel{\text{def}}{=} \bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \widehat{S}^k + \frac{1}{n} \sum_{i=1}^{n} \mathsf{S}_{k+1,i} - \mathsf{S}_{k+1,J_{k+1}} \; .$$

6.2.1 Proof of Theorem 4

By Proposition 2 and H5-item 3, $\dot{V}$ is $L_{\dot{V}}$-Lipschitz, and we have

$$\begin{aligned} V(\widehat{S}^{k+1}) &\leq V(\widehat{S}^k) + \left\langle \widehat{S}^{k+1} - \widehat{S}^k, \dot{V}(\widehat{S}^k) \right\rangle + \frac{L_{\dot{V}}}{2} \|\widehat{S}^{k+1} - \widehat{S}^k\|^2 \\ &\leq V(\widehat{S}^k) + \gamma_{k+1} \left\langle H_{k+1}, \dot{V}(\widehat{S}^k) \right\rangle + \gamma_{k+1}^2 \frac{L_{\dot{V}}}{2} \|H_{k+1}\|^2. \end{aligned}$$

Taking the expectation yields, upon noting that $\widehat{S}^k \in \mathcal{F}_k$

$$\begin{aligned} &\mathbb{E}\left[V(\widehat{S}^{k+1})\right] - \mathbb{E}\left[V(\widehat{S}^k)\right] \\ &\leq \gamma_{k+1} \mathbb{E}\left[\left\langle \mathbb{E}\left[H_{k+1}|\mathcal{F}_k\right], \dot{V}(\widehat{S}^k) \right\rangle\right] + \gamma_{k+1}^2 \frac{L_{\dot{V}}}{2} \mathbb{E}\left[\|H_{k+1}\|^2\right] \\ &\leq \gamma_{k+1} \mathbb{E}\left[\left\langle h(\widehat{S}^k), \dot{V}(\widehat{S}^k) \right\rangle\right] + \gamma_{k+1}^2 \frac{L_{\dot{V}}}{2} \mathbb{E}\left[\|H_{k+1}\|^2\right] \\ &\leq -\gamma_{k+1} v_{min} \mathbb{E}\left[\|h(\widehat{S}^k)\|^2\right] + \gamma_{k+1}^2 \frac{L_{\dot{V}}}{2} \mathbb{E}\left[\|H_{k+1}\|^2\right] \\ &\leq -\gamma_{k+1} \left(v_{min} - \gamma_{k+1} \frac{L_{\dot{V}}}{2}\right) \mathbb{E}\left[\|h(\widehat{S}^k)\|^2\right] + \gamma_{k+1}^2 \frac{L_{\dot{V}}}{2} \mathbb{E}\left[\|H_{k+1} - h(\widehat{S}^k)\|^2\right] \end{aligned}$$

where we used that $\mathbb{E}\left[H_{k+1}|\mathcal{F}_k\right] = h(\widehat{S}^k)$ and Proposition 3. Set

$$A_k \stackrel{\text{def}}{=} \mathbb{E}\left[\|h(\widehat{S}^k)\|^2\right] \; , \qquad B_{k+1} \stackrel{\text{def}}{=} \mathbb{E}\left[\|\widetilde{S}^{k+1} - \bar{s} \circ \mathsf{T}(\widehat{S}^k)\|^2\right] \; .$$

By 11 and 12, we have for any $k \geq 0$:

$$\begin{aligned} \mathbb{E}\left[V(\widehat{S}^{k+1})\right] - \mathbb{E}\left[V(\widehat{S}^k)\right] \leq -\gamma_{k+1} \left(v_{min} - \gamma_{k+1} \frac{L_{\dot{V}}}{2}\right) A_k - \gamma_{k+1}^2 \frac{L_{\dot{V}}}{2} B_{k+1} \\ + \gamma_{k+1}^2 \frac{L_{\dot{V}}}{2} \mathbb{E}\left[\|\bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \mathsf{S}_{k+1,J_{k+1}}\|^2\right] \leq T_{1,k} + T_{2,k+1} \end{aligned}$$

by setting

$$T_{1,k} \stackrel{\text{def}}{=} -\gamma_{k+1}\left(v_{min} - \gamma_{k+1}\frac{L_{\dot V}}{2}\right) A_k + \gamma_{k+1}^2 \frac{L_{\dot V}}{2} \sum_{j=0}^{k-1} \tilde\Lambda_{j+1,k} A_j$$

$$T_{2,k+1} \stackrel{\text{def}}{=} -\gamma_{k+1}^2 \frac{L_{\dot V}}{2}\left\{ B_{k+1} + \sum_{j=0}^{k-1} \tilde\Lambda_{j+1,k}(1+\beta_{j+1}^{-1})^{-1} B_{j+1}\right\} ;$$

by convention, $\sum_{j=0}^{1} a_j = 0$. By summing from $k=0$ to $k = K_{\max}-1$, we have

$$\sum_{k=0}^{K_{\max}-1} \gamma_{k+1}\left(v_{min} - \gamma_{k+1}\frac{L_{\dot V}}{2}\right) A_k - \frac{L_{\dot V} L^2}{2} \sum_{k=0}^{K_{\max}-2} \gamma_{k+1}^2 \Lambda_k \; A_k$$
$$\leq \Delta V - \frac{L_{\dot V}}{2} \sum_{k=0}^{K_{\max}-1} \gamma_{k+1}^2 \left(L^2 \Xi_k + 1\right) B_{k+1},$$

where for $0 \leq k \leq K_{\max}-2$ and with the convention $\Lambda_{K_{\max}-1} = \Xi_{K_{\max}-1} = 0$,

$$\Lambda_k \stackrel{\text{def}}{=} \left(1 + \frac{1}{\beta_{k+1}}\right) \sum_{j=k+1}^{K_{\max}-1} \gamma_{j+1}^2 \; \left(\frac{n-1}{n}\right)^{j-k} \prod_{\ell=k+2}^{j} \left(1 + \beta_\ell + \gamma_\ell^2 L^2\right)$$
$$\leq \left(1 + \frac{1}{\beta_{k+1}}\right) \sum_{j=k+1}^{K_{\max}-1} \gamma_{j+1}^2 \prod_{\ell=k+2}^{j} \left(1 - \frac{1}{n} + \beta_\ell + \gamma_\ell^2 L^2\right),$$
$$\Xi_k \stackrel{\text{def}}{=} \left(1 + \frac{1}{\beta_{k+1}}\right)^{-1} \Lambda_k = \frac{\Lambda_k \beta_{k+1}}{1+\beta_{k+1}} .$$

Hence,

$$\sum_{k=0}^{K_{\max}-1} \{\gamma_{k+1}\left(v_{min} - \gamma_{k+1}\frac{L_{\dot V}}{2}\right) - \gamma_{k+1}^2 \Lambda_k \frac{L_{\dot V} L^2}{2}\} \; A_k$$
$$+ \sum_{k=0}^{K_{\max}-1} \gamma_{k+1}^2 \{1 + \Xi_k L^2\} \frac{L_{\dot V}}{2} B_{k+1} \leq \Delta V .$$

6.2.2 Proof of Proposition 5

It is a follow-up of Theorem 4; the quantities $\alpha_k, \Lambda_k, \delta_k$ introduced in the statement of Theorem 4 are used below without being defined again. We consider the case when for $\ell = 1, \ldots, K_{\max}$,

$$\beta_\ell \stackrel{\text{def}}{=} \frac{1-\lambda}{n^{\mathsf{b}}} , \qquad \gamma_\ell^2 \stackrel{\text{def}}{=} \frac{C}{L^2 n^{2\mathsf{c}} K_{\max}^{2\mathsf{d}}} ,$$

for some $\lambda \in (0,1)$, $C > 0$ and $\mathsf{b}, \mathsf{c}, \mathsf{d}$ to be defined in the proof in such a way that *(i)* $\alpha_k \geq 0$, *(ii)* $\sum_{k=0}^{K_{\max}-1} \alpha_k$ is positive and as large as possible. Since there will be a discussion on (n, C, λ), we make more explicit the dependence of some constants upon these quantities: α_k will be denoted by $\alpha_k(n, C, \lambda)$.

With theses definitions, we have

$$1 - \frac{\rho_n}{n} \stackrel{\text{def}}{=} 1 - \frac{1}{n} + \beta_\ell + \gamma_\ell^2 L^2 = 1 - \frac{1}{n}\left(1 - \frac{1-\lambda}{n^{\mathsf{b}-1}} - \frac{C}{n^{2\mathsf{c}-1} K_{\max}^{2\mathsf{d}}}\right) ,$$

and choose $(\mathsf{b}, \mathsf{c}, \mathsf{d}, \lambda, C)$ such that

$$\frac{1-\lambda}{n^{\mathsf{b}-1}} + \frac{C}{n^{2\mathsf{c}-1} K_{\max}^{2\mathsf{d}}} < 1 , \tag{27}$$

which ensures that $\rho_n \in (0,1)$. Hence, for any $0 \leq k \leq K_{\max} - 2$,

$$\begin{aligned} \Lambda_k &\leq n^{\mathsf{b}} \left(\frac{1}{n^{\mathsf{b}}} + \frac{1}{1-\lambda}\right) \frac{C}{L^2 n^{2\mathsf{c}} K_{\max}^{2\mathsf{d}}} \sum_{j=k+1}^{K_{\max}-1} \left(1 - \frac{\rho_n}{n}\right)^{j-k-1} \\ &\leq \left(\frac{1}{n^{\mathsf{b}}} + \frac{1}{1-\lambda}\right) \frac{C}{L^2 \rho_n} \frac{1}{n^{2\mathsf{c}-\mathsf{b}-1} K_{\max}^{2\mathsf{d}}} . \end{aligned}$$

From this upper bound, we deduce for any $0 \leq k \leq K_{\max} - 1$: $\alpha_k(n, C, \lambda) \geq \underline{\alpha}_n(C, \lambda)$ where

$$\begin{aligned} \underline{\alpha}_n(C, \lambda) \stackrel{\text{def}}{=} \frac{\sqrt{C}}{L n^{\mathsf{c}} K_{\max}^{\mathsf{d}}} \Bigg(& v_{\min} - \frac{L_{\dot{V}}}{2L} \frac{\sqrt{C}}{n^{\mathsf{c}} K_{\max}^{\mathsf{d}}} \\ & - \frac{L_{\dot{V}}}{2L} \frac{C^{3/2}}{\rho_n \, n^{3\mathsf{c}-\mathsf{b}-1} K_{\max}^{3\mathsf{d}}} \left(\frac{1}{n^{\mathsf{b}}} + \frac{1}{1-\lambda}\right) \Bigg) . \end{aligned} \tag{28}$$

From (27) and (28), we choose $\mathsf{b} = 1$, $\mathsf{c} = 2/3$, $\mathsf{d} = 0$; which yields for $n \geq 1$, since $\rho_n = \lambda - C n^{-1/3}$

$$n^{2/3} \underline{\alpha}_n(C, \lambda) \geq \mathcal{L}_n(C, \lambda) ,$$

with

$$\begin{aligned} \mathcal{L}_n(C, \lambda) &\stackrel{\text{def}}{=} \frac{L_{\dot{V}} \sqrt{C}}{2L^2} \left(v_{\min} \frac{2L}{L_{\dot{V}}} - \sqrt{C} f_n(C, \lambda) \right) , \\ f_n(C, \lambda) &\stackrel{\text{def}}{=} \frac{1}{n^{2/3}} + \frac{C}{\lambda - C n^{-1/3}} \left(\frac{1}{n} + \frac{1}{1-\lambda}\right) . \end{aligned}$$

Let $\mu \in (0,1)$. Fix $\lambda \in (0,1)$ and $C > 0$ such that (see (27) for the second condition)

$$\sqrt{C} f_n(C, \lambda) = 2\mu v_{\min} \frac{L}{L_{\dot{V}}} , \qquad \frac{1}{n^{1/3}} < \frac{\lambda}{C} . \tag{29}$$

This implies that $n^{2/3}\alpha_k(n,C,\lambda) \geq n^{2/3}\underline{\alpha}_n(C,\lambda) \geq n^{2/3}\alpha_\star(C) \stackrel{\text{def}}{=} \sqrt{C}(1-\mu)v_{\min}/L$. We obtain an upper bound on E_1 by

$$\mathsf{E}_1 \leq \frac{1}{K_{\max}\,\alpha_\star(C)} \sum_{k=0}^{K_{\max}-1} \alpha_k(n,C,\lambda)\,\mathbb{E}\left[\|h(\widehat{S}^k)\|^2\right] .$$

For E_2, since $\delta_k \geq L_{\dot{V}}\gamma_{k+1}^2/2$,

$$\begin{aligned}\frac{L_{\dot{V}}\sqrt{C}}{2L(1-\mu)n^{2/3}}\frac{1}{v_{\min}}\mathsf{E}_2 &\leq \frac{L_{\dot{V}}\,C}{2L^2n^{4/3}}\frac{1}{K_{\max}\,\alpha_\star(C)}\sum_{k=0}^{K_{\max}-1}\mathbb{E}\left[\|\widetilde{S}^{k+1}-\bar{s}\circ\mathsf{T}(\widehat{S}^k)\|^2\right] \\ &\leq \frac{1}{K_{\max}\,\alpha_\star(C)}\sum_{k=0}^{K_{\max}-1}\delta_k\mathbb{E}\left[\|\widetilde{S}^{k+1}-\bar{s}\circ\mathsf{T}(\widehat{S}^k)\|^2\right] .\end{aligned}$$

We then conclude by

$$\frac{1}{K_{\max}\,\alpha_\star(C)} = \frac{n^{2/3}}{K_{\max}}\frac{L}{\sqrt{C}(1-\mu)v_{\min}} , \tag{30}$$

and use $\sqrt{C}f_n(C,\lambda) = 2\mu v_{\min}L/L_{\dot{V}}$.

• **The choice $C=\lambda$.** Since $n \geq 2$, the second condition in (29) is satisfied with $\lambda = C$. (30) is a decreasing function of C so that by the first condition in (29), C solves

$$\sqrt{C}\left(\frac{1}{n^{2/3}} + \frac{1}{1-n^{-1/3}}\left(\frac{1}{n}+\frac{1}{1-C}\right)\right) = 2\mu v_{\min}\frac{L}{L_{\dot{V}}}$$

A solution exists in $(0,1)$ and is unique (see Lemma 14); it is denoted by $C^\star$. Since the LHS is lower bounded by $C \mapsto (1-C)^{-1}$ on $(0,1)$, $C^\star$ is upper bounded by $C^+ \in (0,1)$ solving

$$\sqrt{C} = 2\mu v_{\min}\frac{L}{L_{\dot{V}}}(1-C) .$$

This yields $C^+ = (\sqrt{1+4A^2}-1)/(2A)$ with $A \stackrel{\text{def}}{=} 2\mu v_{\min}L/L_{\dot{V}}$. Note that $f_n(C,C) \leq f_2(C,C) \leq f_2(C^+,C^+)$ for any $C \in (0,1)$.

• **Another choice, for any n large enough.** We have when $n \to \infty$

$$\mathcal{L}_n(C,\lambda) \uparrow \mathcal{L}_\infty(C,\lambda) \stackrel{\text{def}}{=} \frac{L_{\dot{V}}\sqrt{C}}{2L^2}\left(v_{\min}\frac{2L}{L_{\dot{V}}} - \frac{C^{3/2}}{\lambda}\frac{1}{1-\lambda}\right) .$$

By Lemma 15 applied with $A \leftarrow v_{\min}/L$ and $B \leftarrow 2L_{\dot V}/L^2$, we have $\mathcal{L}_\infty(C,\lambda) \le \mathcal{L}_\infty(C_\star,\lambda_\star)$ where

$$\lambda_\star \stackrel{\text{def}}{=} \frac{1}{2}\,, \qquad C_\star \stackrel{\text{def}}{=} \frac{1}{4}\left(\frac{v_{\min}L}{L_{\dot V}}\right)^{2/3}\,, \qquad \mathcal{L}_\infty(C_\star,\lambda_\star) = \frac{3}{8}\frac{v_{\min}}{L}\left(\frac{v_{\min}L}{L_{\dot V}}\right)^{1/3}\,.$$

Set $N_\star \stackrel{\text{def}}{=} (v_{\min}L/L_{\dot V})^2/8$; for any $n \ge N_\star$, the second condition in (27) is satisfied and we have

$$\lim_n n^{2/3}\alpha_k(n,C_\star,\lambda_\star) \ge \lim_n n^{2/3}\underline{\alpha}_n(C_\star,\lambda_\star) \ge \mathcal{L}_\infty(C_\star,\lambda_\star) > 0\,,$$

thus showing that for any n large enough (with a bound which only depends upon $L, L_{\dot V}, v_{\min}$), we have

$$\frac{1}{K_{\max}\sum_{k=0}^{K_{\max}-1}\alpha_k(n,C_\star,\lambda_\star)} \le \frac{n^{2/3}}{K_{\max}\mathcal{L}_\infty(C_\star,\lambda_\star)} = \frac{n^{2/3}}{K_{\max}}\frac{8}{3}\frac{L}{v_{\min}}\left(\frac{L_{\dot V}}{v_{\min}L}\right)^{1/3}\,.$$

6.2.3 Proof of Proposition 6

It is a follow-up of Theorem 4; the quantities $\alpha_k, \Lambda_k, \delta_k$ introduced in the statement of Theorem 4 are used below without being defined again.

We consider the case when, for $\ell = 1, \dots, K_{\max}$,

$$\beta_\ell \stackrel{\text{def}}{=} \frac{1-\lambda}{n^{\mathsf{b}}}, \qquad \gamma_\ell^2 \stackrel{\text{def}}{=} \frac{C}{L^2 n^{2\mathsf{c}} K_{\max}^{2\mathsf{d}}}$$

for some $\lambda \in (0,1)$, $C > 0$ and $\mathsf{b},\mathsf{c},\mathsf{d}$ to be defined in the proof in such a way that *(i)* $\alpha_k \ge 0$, *(ii)* $\sum_{k=0}^{K_{\max}-1}\alpha_k$ is positive and as large as possible. Since there will be a discussion on (n,C,λ), we make more explicit the dependence of some constants upon these quantities: α_k will be denoted by $\alpha_k(n,C,\lambda)$.

With theses definitions, we have

$$\rho \stackrel{\text{def}}{=} 1 - \frac{1}{n} + \beta_\ell + L^2\gamma_\ell^2 = 1 - \frac{1}{n}\left(1 - \frac{1-\lambda}{n^{\mathsf{b}-1}} - \frac{C}{n^{2\mathsf{c}-1}K_{\max}^{2\mathsf{d}}}\right)\,,$$

and choose $(\mathsf{b},\mathsf{c},\mathsf{d},\lambda,C)$ such that

$$\frac{1-\lambda}{n^{\mathsf{b}-1}} + \frac{C}{n^{2\mathsf{c}-1}K_{\max}^{2\mathsf{d}}} \le 1\,, \tag{31}$$

which ensures that $\rho \in (0,1]$. Hence, for any $0 \le k \le K_{\max}-2$,

$$\begin{aligned}\Lambda_k &\le n^{\mathsf{b}}\left(\frac{1}{n^{\mathsf{b}}} + \frac{1}{1-\lambda}\right)\frac{C}{L^2 n^{2\mathsf{c}}K_{\max}^{2\mathsf{d}}}\sum_{j=k+1}^{K_{\max}-1}\rho^{j-k-1} \\ &\le \left(\frac{1}{n^{\mathsf{b}}} + \frac{1}{1-\lambda}\right)\frac{C}{L^2 n^{2\mathsf{c}-\mathsf{b}}K_{\max}^{2\mathsf{d}-1}}\,.\end{aligned}$$

From this upper bound, we obtain the following lower bound for any $0 \leq k \leq K_{\max}-1$: $\alpha_k(n,C,\lambda) \geq \underline{\alpha}_n(C,\lambda)$ where

$$(n^{\mathsf{c}} K_{\max}^{\mathsf{d}})\ \underline{\alpha}_n(C,\lambda) \stackrel{\text{def}}{=} \frac{\sqrt{C}}{L}\left(v_{\min} - \sqrt{C}\frac{L_{\dot{V}}}{2L}\left\{\frac{1}{n^{\mathsf{c}} K_{\max}^{\mathsf{d}}} + \frac{C}{n^{3\mathsf{c}-\mathsf{b}} K_{\max}^{3\mathsf{d}-1}}\left(\frac{1}{n^{\mathsf{b}}} + \frac{1}{1-\lambda}\right)\right\}\right) .$$

Based on this inequality and on (31), we choose $\mathsf{b} = 1$ and $\mathsf{c} = \mathsf{d} = 1/3$; which yields for $n \geq 1$,

$$(nK_{\max})^{1/3}\ \underline{\alpha}_n(C,\lambda) = \mathcal{L}_n(C,\lambda) \stackrel{\text{def}}{=} \frac{\sqrt{C}L_{\dot{V}}}{2L^2}\left(v_{\min}\frac{2L}{L_{\dot{V}}} - \sqrt{C}\tilde{f}_n(C,\lambda)\right) ,$$
$$\tilde{f}_n(C,\lambda) \stackrel{\text{def}}{=} \frac{1}{(nK_{\max})^{1/3}} + C\left(\frac{1}{n} + \frac{1}{1-\lambda}\right) .$$

Let $\mu \in (0,1)$. Fix $\lambda \in (0,1)$ and $C > 0$ such that (see (31) for the second condition)

$$\sqrt{C}\tilde{f}_n(C,\lambda) = 2\mu v_{\min}\frac{L}{L_{\dot{V}}} , \qquad \frac{n^{1/3}}{K_{\max}^{2/3}} \leq \frac{\lambda}{C} . \tag{32}$$

This implies that

$$(nK_{\max})^{1/3}\alpha_k(n,C,\lambda) \geq (nK_{\max})^{1/3}\underline{\alpha}_n(C,\lambda)$$
$$\geq (nK_{\max})^{1/3}\alpha_\star(C) \stackrel{\text{def}}{=} \sqrt{C}(1-\mu)v_{\min}/L .$$

We obtain the upper bound on E_1 by

$$\mathsf{E}_1 \leq \frac{1}{K_{\max}\ \alpha_\star(C)} \sum_{k=0}^{K_{\max}-1} \alpha_k(n,C,\lambda)\ \mathbb{E}\left[\|h(\widehat{S}^k)\|^2\right] .$$

For E_2 and since $\delta_k \geq L_{\dot{V}}\gamma_{k+1}^2/2$

$$\frac{L_{\dot{V}}\sqrt{C}}{2(1-\mu)Ln^{1/3}}\frac{1}{K_{\max}^{1/3}v_{\min}}\mathsf{E}_2$$
$$\leq \frac{L_{\dot{V}}\,C}{2L^2n^{2/3}K_{\max}^{2/3}}\frac{1}{K_{\max}\ \alpha_\star(C)} \sum_{k=0}^{K_{\max}-1} \mathbb{E}\left[\|\widetilde{S}^{k+1} - \bar{s}\circ\mathsf{T}(\widehat{S}^k)\|^2\right]$$
$$\leq \frac{1}{K_{\max}\ \alpha_\star(C)} \sum_{k=0}^{K_{\max}-1} \delta_k\mathbb{E}\left[\|\widetilde{S}^{k+1} - \bar{s}\circ\mathsf{T}(\widehat{S}^k)\|^2\right] .$$

We then conclude by

$$\frac{1}{K_{\max}\alpha_\star(C)} = \frac{n^{1/3}}{K_{\max}^{2/3}} \frac{L}{\sqrt{C}(1-\mu)v_{\min}} \ , \tag{33}$$

and use $\sqrt{C}\tilde{f}_n(C,\lambda) = 2\mu v_{\min} L/L_{\dot{V}}$.

Complexity. For $\tau > 0$, set $C = \lambda\tau$. Then for any $\lambda \in (0,1)$,

$$\sqrt{\lambda\tau}\tilde{f}_n(\lambda\tau,\lambda) = \frac{\sqrt{\lambda}\sqrt{\tau}}{(nK_{\max})^{1/3}} + \lambda^{3/2}\tau^{3/2}\left(\frac{1}{n} + \frac{1}{1-\lambda}\right) \ ,$$

which is a continuous increasing function of λ, which tends to zero when $\lambda \to 0$ and to $+\infty$ when $\lambda \to 1$. Hence, there exists an unique $\lambda_\star \in (0,1)$, depending upon $L, L_{\dot{V}}, v_{\min}, \tau, \mu$ and $n, K_{\max}$ such that $\sqrt{\lambda_\star\tau}\tilde{f}_n(\lambda_\star\tau,\lambda_\star) = 2\mu v_{\min}L/L_{\dot{V}}$. Note however that since $\sqrt{\lambda\tau}\tilde{f}_n(\lambda\tau,\lambda) \geq \lambda^{3/2}\tau^{3/2}/(1-\lambda)$ for any $\lambda \in (0,1)$, then $\lambda_\star$ is upper bounded by the unique solution $\lambda^+ \in (0,1)$ satisfying $L_{\dot{V}}\lambda^{3/2}\tau^{3/2}/(2L(1-\lambda)) = \mu v_{\min}$ (see Lemma 16). Such a solution λ^+ only depends upon $L, L_{\dot{V}}, v_{\min}, \tau, \mu$. Hence, for any $\tau > 0$,

$$\tilde{f}_n(\lambda\tau,\lambda) \leq \sup_{n,K_{\max}} \tilde{f}_n(\lambda^+(\tau)\tau, \lambda^+(\tau))$$

and the RHS does not depend on $n, K_{\max}$. There exists $M > 0$ depending upon $L, L_{\dot{V}}, v_{\min}, \tau, \mu$ such that for any $\varepsilon > 0$,

$$K_{\max} \geq \left(\tau^{3/2}\sqrt{n}\right) \vee \left(M\sqrt{n}\varepsilon^{-3/2}\right) \Longrightarrow \frac{n^{1/3}}{K_{\max}^{2/3}} \frac{L\,\tilde{f}_n(\lambda\tau,\lambda)}{\mu(1-\mu)v_{\min}^2} \leq \varepsilon \ .$$

Another choice of (λ, C), for any n large enough. In this section, we consider that there exists $\tau > 0$ such that $\sup_{n,K_{\max}} n^{1/3}K_{\max}^{-2/3} \leq \tau$, and $n \to \infty$, $nK_{\max} \to \infty$. In this asymptotic, we have $\mathcal{L}_n(C,\lambda) \uparrow \mathcal{L}_\infty(C,\lambda)$ where

$$\mathcal{L}_\infty(C,\lambda) \stackrel{\text{def}}{=} \frac{\sqrt{C}}{L}\left(v_{\min} - \frac{L_{\dot{V}}}{2L}\frac{C^{3/2}}{1-\lambda}\right) \ .$$

For any $(C,\lambda) \in \mathbb{R}^+ \times (0,1)$ s.t. $\tau \leq \lambda/C$, we have $\mathcal{L}_\infty(C,\lambda) \leq \mathcal{L}_\infty(C_\star(\lambda),\lambda)$ where

$$C_\star(\lambda) \stackrel{\text{def}}{=} \left(\frac{v_{\min}L}{2L_{\dot{V}}}\right)^{2/3}(1-\lambda)^{2/3} \ ;$$

see Lemma 15. The condition $C\tau \leq \lambda$ implies that this inequality holds for any $\lambda \in [\lambda_\star, 1)$ where $\lambda_\star$ is the unique solution of (see Lemma 16)

$$\left(\frac{v_{\min} L}{2L_{\dot{V}}}\right)^2 (1-\lambda_\star)^2 = \lambda_\star^3/\tau^3 \; .$$

Since $\mathcal{L}_\infty(C_\star(\lambda), \lambda) = \frac{3}{4}\left(\frac{v_{\min}^4}{2L^2 L_{\dot{V}}}\right)^{1/3}(1-\lambda)^{1/3}$, this quantity is maximal by choosing $\lambda = \lambda_\star$. Therefore, we have for any $(C, \lambda) \in \mathbb{R}^+ \times (0,1)$, s.t. $\tau \leq \lambda/C$, we have

$$\lim_n n^{1/3} K_{\max}^{1/3} \underline{\alpha}_n(C_\star(\lambda_\star), \lambda_\star) = \mathcal{L}_\infty(C_\star(\lambda_\star), \lambda_\star) > 0 \; .$$

For any n large enough (with a bound which only depends upon $L, L_{\dot{V}}, v_{\min}, \tau$), we have

$$\frac{1}{K_{\max}\alpha_\star(C_\star, \lambda_\star)} = \frac{n^{1/3}}{K_{\max}^{2/3}} \frac{4}{3}\left(\frac{2L^2 L_{\dot{V}}}{v_{\min}^4}\right)^{1/3} (1-\lambda_\star)^{-1/3} \; .$$

6.2.4 Proof of Proposition 7

It is a follow-up of Theorem 4; the quantities $\alpha_k, \Lambda_k, \delta_k$ introduced in the statement of Theorem 4 are used below without being defined again.

Let $p_0, \ldots, p_{K_{\max}-1}$ be positive real numbers such that $\sum_{k=0}^{K_{\max}-1} p_k = 1$. We consider the case when

$$\beta_\ell \stackrel{\text{def}}{=} \frac{1-\lambda}{n^{\mathsf{b}}} \; , \qquad \gamma_\ell^2 \stackrel{\text{def}}{=} \frac{C_\ell}{L^2 n^{2\mathsf{c}} K_{\max}^{2\mathsf{d}}} \; ,$$

for $\lambda \in (0,1)$, $C_\ell > 0$, and $\mathsf{b}, \mathsf{c}, \mathsf{d}$ to be defined in the proof.

The first step consists in the definition of a function F and of a family $\mathcal{C}$ of vectors $\underline{C} = (C_1, \ldots, C_{K_{\max}}) \in \mathbb{R}_+^{K_{\max}}$ such that $\alpha_k \geq F(C_{k+1}) \geq 0$, and $\sum_{\ell=0}^{K_{\max}-1} F(C_{\ell+1}) > 0$. The second step proves that we can find $\underline{C} \in \mathcal{C}$ such that $p_k = F(C_{k+1}) / \sum_{\ell=0}^{K_{\max}-1} F(C_{\ell+1})$ for any $k = 0, \ldots, K_{\max}-1$.

Such a pair $(F, \underline{C})$ is not unique, and among the possible ones, we indicate two strategies, all motivated by making the sum $\sum_{\ell=0}^{K_{\max}-1} F(C_{\ell+1})$ as large as possible.

Step 1- Definition of the function F. With the definition of the sequences γ_ℓ and β_ℓ, we have

$$1 - \frac{\rho_{n,\ell}}{n} \stackrel{\text{def}}{=} 1 - \frac{1}{n} + \beta_\ell + \gamma_\ell^2 L^2 = 1 - \frac{1}{n}\left(1 - \frac{1-\lambda}{n^{\mathsf{b}-1}} - \frac{C_\ell}{n^{2\mathsf{c}-1} K_{\max}^{2\mathsf{d}}}\right)$$

and choose $(\mathsf{b}, \mathsf{c}, \mathsf{d}, \lambda, C_\ell)$ such that

$$\frac{1-\lambda}{n^{\mathsf{b}-1}} + \frac{C_{\max}}{n^{2\mathsf{c}-1} K_{\max}^{2\mathsf{d}}} < 1 \; , \text{ where } C_{\max} \stackrel{\text{def}}{=} \max_\ell C_\ell \; , \tag{34}$$

which ensures that $\rho_{n,\ell} \in (0,1)$. Define

$$\rho_n \stackrel{\text{def}}{=} \min_\ell \rho_{n,\ell} = 1 - \frac{1-\lambda}{n^{\mathsf{b}-1}} - \frac{C_{\max}}{n^{2\mathsf{c}-1} K_{\max}^{2\mathsf{d}}} \, .$$

Hence, for any $0 \le k \le K_{\max} - 2$,

$$\begin{aligned}\Lambda_k &\le n^{\mathsf{b}} \left(\frac{1}{n^{\mathsf{b}}} + \frac{1}{1-\lambda} \right) \frac{1}{L^2 n^{2\mathsf{c}} K_{\max}^{2\mathsf{d}}} \sum_{j=k+1}^{K_{\max}-1} C_{j+1} \left(1 - \frac{\rho_n}{n}\right)^{j-k-1} \\ &\le \left(\frac{1}{n^{\mathsf{b}}} + \frac{1}{1-\lambda} \right) \frac{C_{\max}}{L^2 \rho_n} \frac{1}{n^{2\mathsf{c}-\mathsf{b}-1} K_{\max}^{2\mathsf{d}}} \, .\end{aligned}$$

From this upper bound, we obtain the following lower bound on α_k, for any $0 \le k \le K_{\max} - 1$,

$$\alpha_k \;\ge\; \frac{\sqrt{C_{k+1}}}{L n^{\mathsf{c}} K_{\max}^{\mathsf{d}}} \left(v_{\min} - \frac{L_{\dot V}}{2L} \frac{\sqrt{C_{k+1}}}{n^{\mathsf{c}} K_{\max}^{\mathsf{d}}} - \frac{L_{\dot V}}{2L} \frac{C_{\max} \sqrt{C_{k+1}}}{\rho_n \, n^{3\mathsf{c}-\mathsf{b}-1} K_{\max}^{3\mathsf{d}}} \left(\frac{1}{n^{\mathsf{b}}} + \frac{1}{1-\lambda} \right) \right) \quad .$$

Based on this inequality and on (34), we choose $\mathsf{b} = 1$, $\mathsf{c} = 2/3$, $\mathsf{d} = 0$: this yields $\rho_n = \lambda - C_{\max} n^{-1/3}$ and $\alpha_k \ge \underline{\alpha}_k$ with (see (16) for the definition of f_n)

$$\underline{\alpha}_k \stackrel{\text{def}}{=} \frac{\sqrt{C_{k+1}} L_{\dot V}}{2L^2 n^{2/3}} \left(v_{\min} \frac{2L}{L_{\dot V}} - \sqrt{C_{k+1}} f_n(C_{\max}, \lambda) \right) \; ; \tag{35}$$

the condition (34) gets into $n^{-1/3} < \lambda / C_{\max}$.

Define the quadratic function $x \mapsto F(x) \stackrel{\text{def}}{=} Ax(v_{\min} - Bx)$ where

$$A \stackrel{\text{def}}{=} \frac{1}{L n^{2/3}} \, , \quad B \stackrel{\text{def}}{=} f_n(C, \lambda) \frac{L_{\dot V}}{2L} \; ; \tag{36}$$

we have $\underline{\alpha}_k = F(\sqrt{C_{k+1}})$. By Lemma 13, F is increasing on $(0, v_{\min}/(2B)]$, reaches its maximum at $x_\star \stackrel{\text{def}}{=} v_{\min}/(2B)$ and its maximal value is $F_\star \stackrel{\text{def}}{=} A v_{\min}^2/(4B)$. In addition, its inverse F^{-1} exists on $(0, F_\star]$.

Step 2- Choice of $C_1, \ldots, C_{K_{\max}}$. We are now looking for $C_1, \ldots, C_{K_{\max}}$ such that $p_k = F(\sqrt{C_{k+1}}) / \sum_{\ell=0}^{K_{\max}-1} F(\sqrt{C_{\ell+1}})$ or equivalently

$$\frac{p_k}{p_I} = \frac{F(\sqrt{C_{k+1}})}{F(\sqrt{C_I})}, \qquad I \in \operatorname{argmax}_k p_k \, . \tag{37}$$

It remains to fix $F(\sqrt{C_I})$ in such a way that F is invertible on $\left(0, \sqrt{C_I}\right]$. Since we also want $\sum_\ell F(\sqrt{C_{\ell+1}}) = F(\sqrt{C_I})/p_I$ as large as possible, and F is increasing on $(0, x_\star]$, we choose

$$\sqrt{C_I} = \sqrt{C_{\max}} = x_\star = \frac{v_{\min}}{2B} \, . \tag{38}$$

Therefore, $C_{\max}$ solves the equation $\sqrt{C_{\max}} = v_{\min}/(2B)$ or equivalently

$$\frac{v_{\min} L}{L_{\dot{V}}} = \sqrt{C_{\max}} f_n(C_{\max}, \lambda) \ , \tag{39}$$

under the constraint that $\lambda \in (0,1)$ and $n^{-1/3} < \lambda / C_{\max}$. When $C_{\max}$ is fixed, we set

$$\sqrt{C_{k+1}} \stackrel{\text{def}}{=} F^{-1}\left(\frac{p_k}{\max_\ell p_l} F(\sqrt{C_{\max}}) \right).$$

With these definitions, we have (see (37))

$$\frac{1}{\sum_{k=0}^{K_{\max}-1} F(\sqrt{C_{k+1}})} = \frac{\max_\ell p_\ell}{F(\sqrt{C_{\max}})}.$$

Remember that $F(\sqrt{C_{\max}}) = F(x_\star) = v_{\min}\sqrt{C_{\max}}/(2Ln^{2/3})$.

Step 3. Lower bound on δ_k We write

$$\delta_k \geq \frac{L_{\dot{V}}}{2} \gamma_{k+1}^2 \ ,$$

so that

$$\frac{\delta_k}{\sum_{k=0}^{K_{\max}-1} F(\sqrt{C_{k+1}})} \geq \frac{L_{\dot{V}} L}{v_{\min}} n^{2/3} \frac{\max_\ell p_\ell}{\sqrt{C_{\max}}} \gamma_{k+1}^2 \ .$$

Case $\lambda = C$. A simple strategy is to choose $n \geq 2$ and $C_{\max} = \lambda$ solution of $v_{\min}/2 = \sqrt{C} f_n(C, C)$. This solution exists and is unique, and it is upper bounded by a quantity C^+ which depends only on $L, L_{\dot{V}}, v_{\min}$ - the same discussion is proved in subsubsection 6.2.2.

Case $\lambda = 1/2$. $f_n(C, \lambda)$ controls the errors E_i and we can choose $\lambda \in (0,1)$ and then $C > 0$ such that this quantity is minimal; to make the computations easier, we minimize w.r.t. λ the function $\lim_n f_n(C, \lambda)$: it behaves like $\lambda^{-1}(1-\lambda)^{-1}$ so that we set $\lambda = 1/2$. The equation $\sqrt{C} f_n(C, 1/2) = v_{\min} L / L_{\dot{V}}$ possesses an unique solution in $\left(0, \lambda n^{1/3}\right)$.

Upon noting that $x \mapsto \sqrt{x} f_n(x, 1/2)$ is lower bounded by $x \mapsto 4x^{3/2}$, the solution to the equation $\sqrt{C} f_n(C, 1/2) = v_{\min} L / L_{\dot{V}}$ satisfies

$$C \leq \left(\frac{v_{\min} L}{4 L_{\dot{V}}} \right)^{2/3} \ ,$$

thus showing that the constraint $n^{-1/3} < \lambda / C = 1/(2C)$ is satisfied for any n such that $8n > \left(v_{\min} L / L_{\dot{V}}\right)^2$.

6.2.5 Auxiliary results

Lemma 11. *Assume H1, H2 and H3. For any $k \geq 0$,*

$$\mathbb{E}\left[\|H_{k+1}\|^2\right] = \mathbb{E}\left[\|H_{k+1} - h(\widehat{S}^k)\|^2\right] + \mathbb{E}\left[\|h(\widehat{S}^k)\|^2\right],$$

and

$$\mathbb{E}\left[\|H_{k+1} - h(\widehat{S}^k)\|^2\right] + \mathbb{E}\left[\|\widetilde{S}^{k+1} - \bar{s} \circ \mathsf{T}(\widehat{S}^k)\|^2\right] = \mathbb{E}\left[\|\bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \mathsf{S}_{k+1,J_{k+1}}\|^2\right] .$$

Proof. Since $\mathbb{E}\left[H_{k+1}\middle|\mathcal{F}_{k+1/2}\right] = h(\widehat{S}^k)$, we have

$$\mathbb{E}\left[\|H_{k+1}\|^2\right] = \mathbb{E}\left[\|H_{k+1} - h(\widehat{S}^k)\|^2\right] + \mathbb{E}\left[\|h(\widehat{S}^k)\|^2\right] .$$

In addition, upon noting that $\mathsf{S}_{k+1,i} \in \mathcal{F}_{k+1/2}$ for any i,

$$\begin{aligned} H_{k+1} - h(\widehat{S}^k) &= \bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \mathsf{S}_{k+1,J_{k+1}} - \bar{s} \circ \mathsf{T}(\widehat{S}^k) + \widetilde{S}^{k+1} \\ &= \bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \mathsf{S}_{k+1,J_{k+1}} - \mathbb{E}\left[\bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \mathsf{S}_{k+1,J_{k+1}}\middle|\mathcal{F}_{k+1/2}\right] , \end{aligned}$$

we have

$$\mathbb{E}\left[\|H_{k+1} - h(\widehat{S}^k)\|^2\right] + \mathbb{E}\left[\|\widetilde{S}^{k+1} - \bar{s} \circ \mathsf{T}(\widehat{S}^k)\|^2\right] = \mathbb{E}\left[\|\bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \mathsf{S}_{k+1,J_{k+1}}\|^2\right] .$$

□

Proposition 12. *Assume H1, H2, H3 and H5-item 2. Set $L^2 \stackrel{\text{def}}{=} n^{-1}\sum_{i=1}^n L_i^2$. Then*

$$\mathbb{E}\left[\|\bar{s}_{J_1} \circ \mathsf{T}(\widehat{S}^0) - \mathsf{S}_{1,J_1}\|^2\right] = 0 ,$$

and for any $k \geq 1$ and $\beta_1, \ldots, \beta_k > 0$,

$$\mathbb{E}\left[\|\bar{s}_{J_{k+1}} \circ \mathsf{T}(\widehat{S}^k) - \mathsf{S}_{k+1,J_{k+1}}\|^2\right] \leq \sum_{j=1}^k \widetilde{\Lambda}_{j,k} \left\{ \mathbb{E}\left[\|h(\widehat{S}^{j-1})\|^2\right] - \left(1 + \frac{1}{\beta_j}\right)^{-1} \mathbb{E}\left[\|\widetilde{S}^j - \bar{s} \circ \mathsf{T}(\widehat{S}^{j-1})\|^2\right] \right\} ,$$

where

$$\widetilde{\Lambda}_{j,k} \stackrel{\text{def}}{=} L^2 \left(\frac{n-1}{n}\right)^{k-j+1} \gamma_j^2 \left(1+\frac{1}{\beta_j}\right) \prod_{\ell=j+1}^{k} \left(1+\beta_\ell+\gamma_\ell^2 L^2\right).$$

By convention, $\prod_{\ell=k+1}^{k} a_\ell = 1$.

Proof. For $k=0$,

$$\mathbb{E}\left[\|\bar{s}_{J_1} \circ \mathsf{T}(\widehat{S}^0) - \mathsf{S}_{1,J_1}\|^2\right] = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left[\|\bar{s}_i \circ \mathsf{T}(\widehat{S}^0) - \mathsf{S}_{1,i}\|^2\right] = 0\,.$$

Let $k \geq 1$. We write (see (11))

$$\mathsf{S}_{k+1,i} = \mathsf{S}_{k,i}\mathbb{1}_{I_{k+1}\neq i} + \bar{s}_i \circ \mathsf{T}(\widehat{S}^k)\mathbb{1}_{I_{k+1}=i} = \bar{s}_i \circ \mathsf{T}(\widehat{S}^{<k,i})\mathbb{1}_{I_{k+1}\neq i} + \bar{s}_i \circ \mathsf{T}(\widehat{S}^k)\mathbb{1}_{I_{k+1}=i},$$

where $\widehat{S}^{<\ell,i}$ is defined by (26). This yields, by H5-item 2

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left[\|\bar{s}_i \circ \mathsf{T}(\widehat{S}^k) - \mathsf{S}_{k+1,i}\|^2\right] &= \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left[\|\bar{s}_i \circ \mathsf{T}(\widehat{S}^k) - \bar{s}_i \circ \mathsf{T}(\widehat{S}^{<k,i})\|^2 \mathbb{1}_{I_{k+1}\neq i}\right] \\ &\leq \Delta_k \stackrel{\text{def}}{=} \frac{n-1}{n^2}\sum_{i=1}^{n} L_i^2\, \mathbb{E}\left[\|\widehat{S}^k - \widehat{S}^{<k,i}\|^2\right]. \end{aligned} \tag{40}$$

We have

$$\Delta_k = \frac{n-1}{n^2}\sum_{i=1}^{n} L_i^2\, \mathbb{E}\left[\|\widehat{S}^k - \widehat{S}^{k-1} + \left(\widehat{S}^{k-1} - \widehat{S}^{<k-1,i}\right)\mathbb{1}_{I_k\neq i}\|^2\right]$$

where we used in the last inequality that $\widehat{S}^{<k,i} = \widehat{S}^{k-1}\mathbb{1}_{I_k=i} + \widehat{S}^{<k-1,i}\mathbb{1}_{I_k\neq i}$. Upon noting that $2\left\langle \tilde{U}, V\right\rangle \leq \beta^{-1}\|\tilde{U}\|^2 + \beta\|V\|^2$ for any $\beta > 0$, we have for any $\mathcal{G}$-measurable r.v. V

$$\mathbb{E}\left[\|U+V\|^2\right] \leq \mathbb{E}\left[\|U\|^2\right] + \beta^{-1}\mathbb{E}\left[\|\mathbb{E}\left[U|\mathcal{G}\right]\|^2\right] + (1+\beta)\mathbb{E}\left[\|V\|^2\right].$$

Applying this inequality with $\beta \leftarrow \beta_k$, $U \leftarrow \widehat{S}^k - \widehat{S}^{k-1} = \gamma_k H_k$ and $\mathcal{G} \leftarrow \mathcal{F}_{k-1/2}$ yields

$$\begin{aligned}\Delta_k \leq \gamma_k^2 \frac{n-1}{n} L^2\, \mathbb{E}\left[\|H_k\|^2\right] + \frac{\gamma_k^2}{\beta_k}\frac{n-1}{n} L^2\, \mathbb{E}\left[\|\mathbb{E}\left[H_k|\mathcal{F}_{k-1/2}\right]\|^2\right] \\ + (1+\beta_k)\frac{n-1}{n^2}\sum_{i=1}^{n} L_i^2\, \mathbb{E}\left[\|\widehat{S}^{k-1} - \widehat{S}^{<k-1,i}\|^2 \mathbb{1}_{I_k\neq i}\right].\end{aligned}$$

By Lemma 11 and (40), we have

$$\mathbb{E}\left[\|H_k\|^2\right] \leq \mathbb{E}\left[\|h(\widehat{S}^{k-1})\|^2\right] + \Delta_{k-1} - \mathbb{E}\left[\|\widetilde{S}^k - \bar{s}\circ\mathsf{T}(\widehat{S}^{k-1})\|^2\right];$$

for the second term, we use again $\mathbb{E}\left[H_k|\mathcal{F}_{k-1/2}\right] = h(\widehat{S}^{k-1})$; for the third term, since $I_k \in \mathcal{F}_{k-1/2}$, $\widehat{S}^{k-1} \in \mathcal{F}_{k-1}$, $\widehat{S}^{<k-1,i} \in \mathcal{F}_{k-1}$, then

$$\sum_{i=1}^{n} L_i^2\, \mathbb{E}\left[\|\widehat{S}^{k-1} - \widehat{S}^{<k-1,i}\|^2 \mathbb{1}_{I_k\neq i}\right] = n\Delta_{k-1}.$$

Therefore, we established

$$\Delta_k \leq \left(1+\beta_k+\gamma_k^2L^2\right)\frac{n-1}{n}\Delta_{k-1} + \gamma_k^2(1+\frac{1}{\beta_k})L^2\ \frac{n-1}{n}\,\mathbb{E}\left[\|h(\widehat{S}^{k-1})\|^2\right] - \gamma_k^2L^2\frac{n-1}{n}\mathbb{E}\left[\|\widetilde{S}^k - \bar{s}\circ\mathsf{T}(\widehat{S}^{k-1})\|^2\right].$$

The proof is then concluded by standard algebra upon noting that $\Delta_0 = 0$. □

6.2.6 Technical lemmas

Lemma 13. *Let $A, B, v > 0$ and define $F(x) \stackrel{\text{def}}{=} Ax(v - Bx)$ on $\mathbb{R}$. Then the roots of F are $\{0, v/B\}$; F is positive on $(0, v/B)$; the maximal value of F is $Av^2/(4B)$ and it is reached at $x_\star \stackrel{\text{def}}{=} v/2B$.*

Lemma 14. *Let $a, b > 0$ and define F on $(0,1)$ by $F(x) = \sqrt{x}(a + b/(1-x))$. For any $v > 0$, there exists an unique $x \in (0,1)$ such that $F(x) = v$.*

Proof. $x \mapsto F(x)$ is continuous and increasing on $(0,1)$, tends to zero when $x \to 0$ and to $+\infty$ when $x \to 1$; therefore for any $v > 0$, there exists an unique $x \in (0,1)$ such that $F(x) = v$. □

Lemma 15. *Let $A, B > 0$. The function $F : x \mapsto Ax - Bx^4$ defined on $(0,\infty)$ reaches its unique maximum at $x_\star \stackrel{\text{def}}{=} A^{1/3}B^{-1/3}4^{-1/3}$ and $F(x_\star) = 3A^{4/3}/(B4^4)^{1/3}$.*

Proof. $F'(x) = A - 4Bx^3$ and $F''(x) = -12Bx^2 < 0$; hence, F' is decreasing. $F'(x) = 0$ iff $x^3 = A/(4B)$, showing $F' > 0$ on $(0, x_\star)$ with $x_\star \stackrel{\text{def}}{=} A^{1/3}/(4B)^{1/3}$. Hence, F is increasing on $[0, x_\star]$ and then decreasing. □

Lemma 16. *For any $v > 0$, the function $x \mapsto (1-x)^2/x^3$ is decreasing on $(0,1)$ and there exists an unique $x \in (0,1)$ solving $(1-x)^2/x^3 = v$.*

Proof. The derivative of $x \mapsto (1-x)^2/x^3$ is $-x^{-4}(x-3)(x-1)$ thus showing that the function is decreasing on $(0,1)$; it tends to $+\infty$ when $x \to 0$ and to 0 when $x \to 1$. This concludes the proof. □

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