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► **To cite this version:**

Marc Dubois. Dominance Criteria on Grids for Measuring Seasonal Competitive Imbalance in Sports Leagues. 2020. <hal-02617635>

**HAL Id: hal-02617635**

**<https://hal.science/hal-02617635v1>**

Preprint submitted on 25 May 2020

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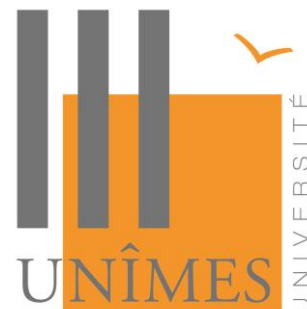
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Document de recherche #2020-02

Dominance Criteria on Grids for  
Measuring Seasonal  
Competitive Imbalance in Sports  
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Marc Dubois



# Dominance Criteria on Grids for Measuring Seasonal Competitive Imbalance in Sports Leagues.

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## Abstract

The paper proposes a dominance criterion that assesses whether a seasonal outcome of a sports league is more imbalanced than another. This criterion, known as downward seasonal balance (DSB), is proposed as a strategy to measure competitive imbalance when the Lorenz criterion is inconclusive. The DSB criterion places at least as much emphasis on competitive imbalance between leading competitors as on imbalance occurring among the non-leading competitors. DSB is a novel third-order stochastic dominance defined on finite sets of evenly spaced seasonal points (seasonal grids). An empirical application provides comparisons of seasonal outcomes of the five most competitive soccer leagues in Europe from season 2014-2015 to season 2018-2019.

*JEL Classification:* D63 ; L83 ; Z28. *Keywords:* Competitive imbalance ; Grids ; Lorenz criterion ; Stochastic dominance.

## 1 Introduction

“Though we consciously root for our favorite team to win, we find more unconscious satisfaction in the persistent struggles of the sports team that we root for than in its unqualified successes.” (McGowan 2016, p. 29).

Borlan and MacDonald (2003) report that demand for competitive sports is decreasing with respect to the degree of seasonal competitive imbalance among competitors. Hence, organizers of sports leagues desire that the current season be less imbalanced than the previous one. For this purpose, this paper aims at measuring competitive imbalance by means of pairwise comparisons of distributions of seasonal points.<sup>1</sup>

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<sup>1</sup>A distribution of seasonal points is how total rewards, here points, are distributed among competitors at the end of the contest. Seasonal competitive imbalance is one of several forms that competitive imbalance can take. For more information, see Cairns (1987).

As this work focuses on distributional aspects, the concepts and results of the measurement of income inequality can be translated into concepts and results of the measurement of competitive imbalance. One of the most-used methods for measuring inequality is the Lorenz, or second-order stochastic, dominance criterion.<sup>2</sup> Distribution  $\mathbf{y}$  is said to *Lorenz dominate* distribution  $\mathbf{x}$  when the Lorenz curve of distribution  $\mathbf{y}$  lies nowhere below that of distribution  $\mathbf{x}$ . In sports terms, this means that  $\mathbf{y}$  is at most as imbalanced as  $\mathbf{x}$ . Quirk and Fort (1992) and Szymanski and Kuypers (1999) employ the Lorenz criterion to measure long-term competitive imbalance. Also, Michie and Oughton (2004) employ the Lorenz criterion for analyzing seasonal competitive imbalance mainly because it allows for comparison of outcomes with different numbers of competitors.

The normative content of the Lorenz criterion is derived from the higher of two non-intersecting curves obtaining from the lower by a sequence of *progressive transfers*, that is, transfers of points from a stronger to a weaker competitor. For instance, assume a contest has three competitors:  $B$ ,  $C$ , and  $D$ . A first hypothetical distribution of points is  $\mathbf{x} = (B, 8; C, 2; D, 2)$ , where  $B$ , the strongest, is awarded 8 points, while the two weakest— $C$  and  $D$ —get 2 points. A second hypothetical distribution is  $\mathbf{y} = (B, 6; C, 4; D, 2)$ . Distribution  $\mathbf{y}$  can be obtained from  $\mathbf{x}$  through a progressive transfer of two points from  $B$  to  $C$ . According to the Lorenz criterion,  $\mathbf{y}$  is at most as imbalanced as  $\mathbf{x}$ . This judgment is consistent with the so-called *principle of transfers* (Pigou, 1912; Dalton, 1920). In sports terms, this principle requires competitive imbalance to decrease as the result of a progressive transfer. According to the principle of transfer,  $\mathbf{y}$  is less imbalanced than  $\mathbf{x}$ . Dasgupta, Sen, and Starrett (1973) show that a first-distribution Lorenz dominates a second distribution if and only if *all* inequality indices that satisfy the principle of transfers state that the first is at most as imbalanced as the second. This equivalence only stands for pairs of distributions with equal means.

Horowitz (1997), Schmidt and Berri (2001, 2002), Utt and Fort (2002), Borooah and Mangana (2012) and Gayant and Le Pape (2017) propose several inequality indices to measure competitive imbalance. Since these indices satisfy the principle of transfers, their judgment is unanimous when one distribution Lorenz dominates another. For instance, all indices state that  $\mathbf{y}$  is at most as imbalanced as  $\mathbf{x}$ . However, these indices render conflicting judgments when both progressive and *regressive* (weaker-to-stronger) transfers are needed to convert one distribution into another. Building on the previous example, consider a third hypothetical distribution  $\mathbf{z} = (B, 7; C, 4; D, 1)$ . The well-known Gini coefficient indicates that  $\mathbf{z}$  is as imbalanced as  $\mathbf{x}$ . Atkinson indices (Atkinson, 1970) state that distribution  $\mathbf{x}$  is less

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<sup>2</sup>Using results of Rothschild and Stiglitz (1970), Atkinson (1970) demonstrated equivalence between Lorenz dominance and second-order stochastic dominance employed to rank uncertain prospects.

imbalanced than  $\mathbf{z}$ . Alternatively, some generalized entropy indices (with parameters higher than 2) state that  $\mathbf{z}$  is less imbalanced than  $\mathbf{x}$ . When the indices that satisfy the principle of transfers are not unanimous, Lorenz curves intersect, and the criterion is inconclusive. With intersecting Lorenz curves, the criterion cannot be used to judge whether  $\mathbf{z}$  is at most as imbalanced as  $\mathbf{x}$ . Moyes (1999) and Aaberge (2009), among others, point out that Lorenz curves often intersect in practice.

To overcome this limitation, the paper proposes three equivalent aspects of the measurement of competitive imbalance. First, the notion of a reduction in imbalance is associated with a progressive transfer and an UNFAvorable Composite Transfer (UNFACT). A composition of two transfers, an UNFACT is a regressive transfer between weaker competitors and a progressive transfer between stronger competitors (Gayant and Le Pape, 2017). An UNFACT should decrease competitive imbalance because “a league is more attractive if it is more balanced among the leading teams, that is, at the top of the distribution of points” (Gayant and Le Pape, 2017, p. 775). Second, all indices that satisfy this requirement and the principle of transfers may be used to make a unanimous judgment on two distributions if and only if a distribution can be obtained from the other through a sequence of progressive transfers and/or UNFACTs. The set for which this unanimity is reached includes the generalized entropy indices (with parameters higher than 2) proposed by Gayant and Le Pape (2017). In practice, it seems impossible to check whether one of the two above equivalent conditions is fulfilled. For instance, an infinite number of comparisons should be performed to check whether these indices make a unanimous judgment.

The third aspect of the measurement of competitive imbalance regards an implementable method to verify the validity of these conditions. It is shown that a first-distribution downward seasonal balance (DSB) dominates a second distribution if and only if a sequence of progressive transfers and/or UNFACTs is needed to convert the second into the first distribution.<sup>3</sup> The DSB criterion is a novel third-order stochastic dominance quasiorder. It is weaker, and so less incomplete, than the Lorenz criterion. When the Lorenz criterion can be employed to make a judgment, the DSB criterion allows the same judgment, whereas the converse is not true. According to the previous example,  $\mathbf{y}$  DSB dominates  $\mathbf{x}$  because  $\mathbf{y}$  Lorenz dominates  $\mathbf{x}$ . Moreover,  $\mathbf{z}$  DSB dominates  $\mathbf{x}$  because  $\mathbf{z}$  can be obtained from  $\mathbf{x}$  through an UNFACT of one point among  $B$ ,  $C$ , and  $D$ .

In contrast to measuring income inequality, measuring seasonal competitive imbalance generally includes populations with fewer than 33 competitors, each with an integer-valued number of points. Thus, both Lorenz and DSB criteria are designed to compare distributions

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<sup>3</sup>The name of the DSB criterion is inspired by Michie and Oughton (2004), who called the Lorenz criterion the “Lorenz Seasonal Balance” criterion.

of finite numbers of integers (grids). They are easily implementable as the required data are lists of wins-ties-losses of each competitor. Such data are freely available and do not suffer from bias or measurement errors.

This work performs intra-national judgments of seasonal distributions from 2014-2015 to 2018-2019 of the five most competitive soccer leagues in Europe. The Lorenz criterion was conclusive in only 34% of the 50 pairwise comparisons that were possible to be made, whereas the DSB criterion was conclusive for 66% of these cases. According to these criteria, the competitive imbalance of the English *Premier League*, the German *Bundesliga*, and the Italian *Serie A* globally increased during the period. Only the competitive imbalance of the Spanish *LaLiga* globally decreased from 2014-2015 to 2018-2019. More precisely, the competitive imbalance of the French *Ligue 1*, the Bundesliga, and Serie A first increased generally from 2014-2015 to 2016-2017, and later generally decreased from season 2017-2018 to 2018-2019.

Section 2 introduces the framework. Section 3 presents the second-order stochastic dominance criterion on grids (equivalently, the Lorenz criterion). Section 4 examines the DSB criterion. Section 5 discusses an empirical application, whose results are summarized in Table 5.1. Section 6 concludes the article.

## 2 Setup

A population of  $n \geq 2$  competitors (teams) is investigated. Teams compete in a “one home-one away” championship without playoffs so that each team plays  $2(n - 1)$  games, and the total number of games played is  $n(n - 1)$ . The degree of competitive imbalance of a league is measured according to the distribution of points at the end of the contest. Assume a *point award system* where, for every game it plays, a team is awarded  $q_v$  points for a victory,  $q_t$  for a tie, and  $q_d$  for a defeat such that  $q_v, q_t, q_d \in \mathbb{Z}_+$  and  $q_v > q_t > q_d$ .<sup>4</sup> Denote  $p_0 \in \mathbb{Z}_+$  as the number of points of every team at the outset of the contest. A team has the smallest possible number of points,  $p_0 + 2(n - 1)q_d$ , if it loses the  $2(n - 1)$  games. The second smallest,  $p_0 + (2n - 3)q_d + q_t$ , is the number of points of a team if it loses all the games it plays but one for which it makes a tie. This calculation continues up to the greatest possible number of points,  $p_0 + 2(n - 1)q_v$ , if a team wins the  $2(n - 1)$  games it plays. The seasonal number of points of a team lies in the following finite set:

$$\mathcal{P} = \{p_0 + j : j = 2(n - 1)q_d, (2n - 3)q_d + q_t, \dots, (2n - 3)q_v + q_t, 2(n - 1)q_v\}.$$

---

<sup>4</sup>The set  $\mathbb{Z}_+$  is the set of non-negative integers.

The *distribution* (function) of points  $f : \mathcal{P} \rightarrow [0, 1] \cap \mathbb{Q}$  lies in the following set:<sup>5</sup>

$$\Omega = \left\{ f : f(p_0 + j) \in [0, 1] \cap \mathbb{Q}, \forall p_0 + j \in \mathcal{P} \text{ and } \sum_{j=2(n-1)q_d}^{2(n-1)q_v} f(p_0 + j) = 1 \right\}$$

where  $f(p_0 + j) \times 100$  is the percentage of teams that hold  $p_0 + j$  points at the end of the championship. The cumulative distribution function  $F : \mathcal{P} \rightarrow [0, 1] \cap \mathbb{Q}$  is such that

$$F(p_0 + j) = \sum_{\substack{i \in \mathcal{P}: \\ i \leq p_0 + j}} f(i)$$

where  $F(p_0 + j) \times 100$  is the percentage of teams awarded by at most  $p_0 + j$  points at the end of the contest.

Assume that the degree of competitive imbalance of a distribution can be measured through the following *aggregate value of points*:

$$A_f = \sum_{j=2(n-1)q_d}^{2(n-1)q_v} v(p_0 + j) f(p_0 + j)$$

where  $v(p_0 + j)$  is the *value* of points  $p_0 + j$  and  $v : \mathcal{P} \rightarrow \mathbb{R}$  is assumed to be strictly increasing. The aggregate value of points is formally analogous to a standard additively separable social welfare function. Following Kolm (1969), Atkinson (1970), and Sen (1973), additively separable competitive imbalance (inequality) indices may be derived from the aggregate value of points.<sup>6</sup> Therefore, each competitive imbalance index  $I$  can be considered a function of  $A_f$  with a particular value function  $v$  so that for two distributions  $f, g \in \Omega$ ,  $A_f \geq A_g$  if and only if  $I_f \leq I_g$ . An aggregate value of points ranks  $f$  not below  $g$  if and only if  $f$  is at most as imbalanced as  $g$  according to  $I$ .

Inequality indices may suitably describe a totally equal distribution, say  $\mathbf{b}$ , as the least imbalanced one. In the study of competitive imbalance measurement, a further postulate involves the *perfect competitive imbalance distribution*  $\mathbf{i}$ . Under a regular schedule, distribution  $\mathbf{i}$  can be objectified as the weakest team loses its  $2(n - 1)$  games, the second weakest team wins 2 games and loses  $2(n - 2)$  games, ..., the strongest team wins its  $2(n - 1)$  games. Gayant and Le Pape (2015, 2017) argue that every index should state  $\mathbf{i}$  as the most imbalanced distribution. The authors demonstrated that this requirement is respected only if the point award system is such that  $2q_t = q_v + q_d$ . Although some realistic point award

<sup>5</sup>The set  $\mathbb{Q}$  is the set of rational numbers.

<sup>6</sup> The aggregate value of points introduced here forms the basis for determining the generalized entropy indices proposed by Gayant and Le Pape (2017), but it is inconsistent with the rank-dependent family of inequality indices, such as the Gini index. For more information on this point, see Chateauneuf, Gajdos, and Wilthien (2002).

systems are not consistent with  $2q_t = q_v + q_d$ , this condition is imposed throughout the paper. Consequently, the mean number of points only depends on the size of the population.

To simplify the notation, it is henceforth assumed that  $p_0 = 0$ ,  $q_v = 2$ ,  $q_t = 1$ , and  $q_d = 0$ . Each team is awarded a number of points, which lies in the following grid:

$$\mathcal{P}_* = \{p \in \mathbb{Z}_+ : p \leq 4n - 4\}.$$

### 3 Second-order stochastic dominance for points

As seen above, an essential definition shared in the measurement of both income inequality and competitive imbalance is progressive transfer. According to the following definition, a progressive transfer is a “stronger-to-weaker” transfer. The stronger team is awarded  $p + \epsilon + \delta$  points before the transfer and  $p + \epsilon$  after the transfer. The weaker team is awarded with  $p$  points before the transfer and  $p + \delta$  afterward.

**Definition 3.1.** *Let  $f$  and  $g$  belong to  $\Omega$ . Distribution  $f$  is obtained from  $g$  by means of a progressive transfer if four point levels exist:  $p, p + \delta, p + \epsilon, p + \epsilon + \delta \in \mathcal{P}_*$  with  $\delta > 0$ , and  $\epsilon \geq \delta$ , such that:*

$$\begin{aligned} f(p) &= g(p) - \frac{1}{n} ; & f(p + \delta) &= g(p + \delta) + \frac{1}{n} ; \\ f(p + \epsilon) &= g(p + \epsilon) + \frac{1}{n} ; & f(p + \epsilon + \delta) &= g(p + \epsilon + \delta) - \frac{1}{n} ; \end{aligned}$$

*$f$  and  $g$  are identical everywhere else. Equivalently,  $g$  is obtained from  $f$  by means of a regressive transfer.*

“ $f(p) = g(p) - \frac{1}{n}$ ” means that at point level  $p$ , the mass of teams in distribution  $f$  is lower by  $\frac{1}{n} \times 100$  percent than in distribution  $g$ . Moreover, “ $f(p + \delta) = g(p + \delta) + \frac{1}{n}$ ” means that at point level  $p + \delta$ , the mass of teams in distribution  $f$  is higher by  $\frac{1}{n} \times 100$  percent than in distribution  $g$ . Aggregating both interpretations yields that  $\frac{1}{n} \times 100$  percent of the teams—in fact, one team—with points  $p$  in distribution  $g$  receives  $\delta$  additional points in distribution  $f$ . Definition 3.1 means that distribution  $f$  is obtained from distribution  $g$  through a progressive transfer of points  $\delta$ . The restriction  $\epsilon \geq \delta$  implies that the stronger team before a transfer is still stronger than the weaker team after a transfer.

**Axiom 3.1.** *Principle of transfers: A principle requiring that competitive imbalance does not increase as the result of a progressive transfer.*

The principle of transfers holds if and only if the value of points increases at a (non-strictly) decreasing rate over  $\mathcal{P}_*$ . That is, the marginal gain in competitiveness due to an

increase in points received by a weaker team at least compensates for the marginal loss due to a decrease in points suffered by a stronger team. Formally, for

$$\Delta_1 v(p) = v(p+1) - v(p) \text{ and } \Delta_2 v(p) = \Delta_1 v(p+1) - \Delta_1 v(p),$$

the value function should lie in  $V_2$  defined with respect to  $V_1$  as follows:

$$\begin{aligned} V_1 &= \{v : \mathcal{P}_* \rightarrow \mathbb{R} : \Delta_1 v(p) > 0, \forall p = 0, \dots, 4n-5\}; \\ V_2 &= \{v \in V_1 : \Delta_2 v(p) \leq 0, \forall p = 0, \dots, 4n-6\}. \end{aligned}$$

Functions  $v$  in  $V_2$  are usually referred as to Jensen concave of order 1.

To align with the principle of transfers, it is possible to perform comparisons of distributions without specifying the form of  $v$  insofar as the function belongs to  $V_2$ . However, this way of proceeding raises the practical question of its implementation, as without specifying  $v \in V_2$ , it is impossible to directly verify whether a distribution is less imbalanced than another. Consider two alternative distributions,  $f$  and  $g$  in  $\Omega$ , and denote the difference by  $\Delta f = f - g$ , then the difference in the aggregate value of points is:

$$\Delta A_f = \sum_{p=0}^{4n-4} v(p) \Delta f(p).$$

Applying Abel's lemma, we then have:<sup>7</sup>

$$\Delta A_f = \sum_{p=0}^{4n-5} \Delta F(p) [v(p) - v(p+1)] + \Delta F(4n-4) v(4n-4)$$

where  $\Delta F \equiv F - G$ . Making use of the fact that  $\Delta F(4n-4) = 0$ , we obtain:

$$\Delta A_f = \sum_{p=0}^{4n-5} \Delta F(p) [v(p) - v(p+1)] = \sum_{p=0}^{4n-5} \Delta D^1(p) \Delta_1 v(p) \quad (1)$$

where  $\Delta D^1(p) = -\Delta F(p)$ . Once again applying Abel's lemma:

$$\Delta A_f = - \sum_{p=0}^{4n-6} \Delta D^2(p) \Delta_2 v(p) + \Delta D^2(4n-5) \Delta_1 v(4n-5) \quad (2)$$

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<sup>7</sup>Abel's lemma is a summation formula which shows the following: Let  $b_0, \dots, b_n, c_0, \dots, c_n$  be real numbers. Set  $B_j = \sum_{k=0}^j b_k$ . Then:

$$\sum_{j=0}^n b_j c_j = \sum_{j=0}^{n-1} B_j (c_j - c_{j+1}) + B_n c_n.$$

where  $\Delta D^2(p) = \sum_{q=0}^p \Delta D^1(q) = -\sum_{q=0}^p \Delta F(q)$  for  $p = 0, \dots, 4n - 5$ .

The following remark recalls the well-known equivalence between the criterion expressed by  $\Delta D^2(p)$ , which is formally the second-order stochastic dominance relation (Rothschild and Stiglitz, 1970) and the Lorenz criterion.

**Remark 3.1.** *For  $f$  and  $g$  in  $\Omega$ , the two following assertions are equivalent:*

- (a)  $f$  Lorenz dominates  $g$ .
- (b)  $\Delta D^2(p) \geq 0$  for all  $p = 0, \dots, 4n - 5$ .

*Proof.* The proof follows from two equivalences. First, statement (b) is equivalent to *traditional* second-order stochastic dominance, that is, dominance of distributions defined on a continuum (Fishburn and Lavalley 1995). Second, traditional second-order stochastic dominance is equivalent to Lorenz dominance (Atkinson 1970).  $\square$

The following theorem proposes a simple procedure in order to decide whether  $f$  is at most as imbalanced as  $g$  when attention is restricted to unspecified value functions being first-order Jensen concave.

**Theorem 3.1.** *For  $f$  and  $g$  in  $\Omega$ , the three following assertions are equivalent:*

- (i) For all  $v \in V_2$ ,  $\Delta A_f \geq 0$ .
- (ii)  $f$  can be reached from  $g$  through a finite sequence of progressive transfers.
- (iii)  $\Delta D^2(p) \geq 0$  for all  $p = 0, \dots, 4n - 5$ .

*Proof.* Fishburn and Lavalley (1995) show that statement (iii) is equivalent to traditional second-order stochastic dominance. Therefore, the theorem is a direct adaptation of the result of Hardy, Littlewood, and Polya (1934). A proof of this theorem is also provided by Marshall, Olkin, and Arnold (2011). Moreover, (iii)  $\Rightarrow$  (i) is straightforward from (2).  $\square$

Statement (i) makes a judgment: Distribution  $f$  is at most as imbalanced as  $g$ . Moreover, the value function  $v$  should be Jensen concave of order 1 for the functions  $A$  to be unanimous on this judgment. This means that all functions  $A$ , equivalently all indices, that satisfy the principle of transfers unanimously state  $f$  at most is as imbalanced as  $g$ . Statement (ii) stresses that progressive transfers are the required transformations to convert a distribution into another for this unanimity be possible. Statement (iii) exhibits how to perform a pairwise comparison between  $f$  and  $g$  in practice. Distribution  $f$  Lorenz dominates  $g$  if and only if the verdict implied in (i) and stated by (ii) holds.<sup>8</sup> On the contrary, if the Lorenz

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<sup>8</sup>The statement is valid for all indices that are additively separable. This point is discussed in the conclusion.

curves of two distributions cross, then two indices exist that make opposite conclusions on these two distributions.

## 4 Downward third-order stochastic dominance for points

Gayant and Le Pape (2017) base their work on the notion of UNFACT to provide a formal characterization of what a decrease in competitive imbalance of the league could be.<sup>9</sup> As mentioned above, the UNFACT is a combination of a regressive transfer between weaker teams (e.g., two arrows on the left in Figure 4.1) and a progressive transfer between stronger teams (e.g., two arrows on the right in Figure 4.1), involving the same number of points and the same amplitude.

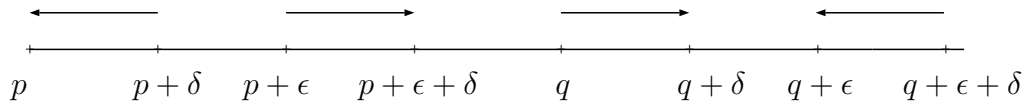


Figure 4.1. An example of UNFACT

The number of points  $\delta$  is transferred from a team with  $p + \delta$  to a stronger team with  $p + \epsilon$  and also transferred from a team with  $q + \epsilon + \delta$  to a weaker team with  $q$ . The integer  $\epsilon$  ensures that the pre-transfer difference in points between teams involved in the regressive transfer is the same as the post-transfer difference in points between teams involved in the progressive transfer.

The following definition sets a general presentation of the UNFACT.

**Definition 4.1.** *Let  $f$  and  $g$  belong to  $\Omega$ . Distribution  $f$  is obtained from  $g$  by means of an UNFACT if eight point levels exist,  $p, p + \epsilon, p + \epsilon + \delta, q, q + \delta, q + \epsilon, q + \epsilon + \delta \in \mathcal{P}_*$  with  $p < q$ ,  $0 < \delta \leq q - p$ , and  $\epsilon \geq \delta$  such that:*

$$\begin{aligned} \Delta f(p) &= \frac{1}{n} ; \Delta f(p + \delta) = -\frac{1}{n} ; \Delta f(p + \epsilon) = -\frac{1}{n} ; \Delta f(p + \epsilon + \delta) = \frac{1}{n} ; \\ \Delta f(q) &= -\frac{1}{n} ; \Delta f(q + \delta) = \frac{1}{n} ; \Delta f(q + \epsilon) = \frac{1}{n} ; \Delta f(q + \epsilon + \delta) = -\frac{1}{n} ; \end{aligned}$$

*$f$  and  $g$  are identical everywhere else. Equivalently,  $g$  is obtained from  $f$  by means of a FACT.*

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<sup>9</sup>The favorable composite transfer (FACT) allows for characterizing aversion to downside inequality (Shorrocks and Foster 1987). Although this attitude generally is desirable when dealing with comparisons of income distributions, Gayant and Le Pape (2017) claim that the competitive imbalance of a league increases as the result of a FACT.

In Definition 4.1, both  $\Delta f(p) = \frac{1}{n}$  and  $\Delta f(p + \delta) = -\frac{1}{n}$  formally correspond to the left arrow on the left side of Figure 4.1. That is,  $\frac{1}{n} \times 100$  percent of the teams move from point level  $p + \delta$  to point level  $p$ . Hence, the weakest team affected by the UNFACT “loses” points at the expense of a stronger team. The other pairs of equations in the definition correspond to the other arrows of Figure 4.1.

As an illustration, consider  $\mathbf{x} = (B, 8; C, 2; D, 2)$  and  $\mathbf{y} = (B, 6; C, 6; D, 0)$ . Distribution  $\mathbf{y}$  may be obtained from  $\mathbf{x}$  through an UNFACT. In sports terms,  $\mathbf{x}$  may be viewed as a distribution, where  $B$  wins all its games,  $C$  loses its games against  $B$  and makes two ties against  $D$  (so  $D$  loses its games against  $B$  and makes two ties against  $C$ ). Moreover,  $\mathbf{y}$  may be viewed as a distribution where  $B$  wins its games against  $D$  and makes two ties against  $C$ ;  $C$  wins its games against  $D$  and makes two ties against  $B$ , and  $D$  loses all its games. While  $\mathbf{x}$  is locally less imbalanced at the bottom,  $\mathbf{y}$  is less imbalanced at the top of the distribution. The following principle states that  $\mathbf{y}$  is at most as imbalanced as  $\mathbf{x}$ .

**Axiom 4.1.** *Upside transfer sensitivity (Gayant and Le Pape, 2017). The principle requires that competitive imbalance does not increase as the result of an UNFACT.*

Upside transfer sensitivity requires at least as much emphasis be placed on the imbalance among stronger teams as among weaker teams. The aggregate function of points  $A$  satisfies this principle and the principle of transfers if and only if its value function  $v$  has a first-order forward difference that is decreasing and concave for  $\mathcal{P}_*$ . In brief, the marginal valuation of points decreases at a non-decreasing rate over  $\mathcal{P}_*$ . That is,  $v$  belongs to the following class:

$$V_3 = \{v \in V_2 : \Delta_3 v(p) = \Delta_1(\Delta_2 v(p)) \leq 0, \forall p = 0, \dots, 4n - 7\}.$$

Functions in  $V_3$  are Jensen concave of order 2.

Applying the lemma of downward summation formula to (2), it follows:<sup>10</sup>

$$\Delta A_f = - \sum_{p=1}^{4n-6} \Delta D^3(p) \Delta_3 v(p-1) - \Delta D^3(0) \Delta_2 v(0) + \Delta D^2(4n-5) \Delta_1 v(4n-5) \quad (3)$$

where

$$\Delta D^3(p) = \sum_{q=p}^{4n-6} \Delta D^2(q) \text{ for } p = 0, \dots, 4n - 6.$$

From (3), distribution  $f$  is at most as imbalanced as  $g$ , that is,  $\Delta A_f \geq 0$  if

$$\Delta D^3(p) \geq 0 \text{ for all } p = 0, \dots, 4n - 6 \text{ and } \Delta D^2(4n - 5) \geq 0.$$

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<sup>10</sup>The lemma of downward summation formula is presented and demonstrated in the appendix.

This sufficient condition is the formal means of stating that  $f$  DSB dominates  $g$ . The DSB dominance criterion greatly depends on the function  $D^3$ , which aggregates the  $D^2$  values from the third-largest point level,  $4n - 6$ , to the lowest one in  $\mathcal{P}_*$ . Aaberge (2009) names this kind of reversal condition as downward dominance. It is part of the following theorem.

**Theorem 4.1.** *For  $f$  and  $g$  in  $\Omega$ , the three following assertions are equivalent:*

- (i) *For all  $v \in V_3$ ,  $\Delta A_f \geq 0$ .*
- (ii)  *$f$  can be reached from  $g$  through a finite sequence of progressive transfers and/or UNFACTs.*
- (iii)  *$\Delta D^3(p) \geq 0$  for all  $p = 0, \dots, 4n - 6$  and  $\Delta D^2(4n - 5) \geq 0$ .*

*Proof.* See the appendix. □

Statement (i) displays a judgment: Distribution  $f$  is at most as imbalanced as  $g$ . Moreover, the value function  $v$  should be Jensen concave of order 2 for the functions  $A$  to be unanimous on this judgment. This means that all functions  $A$  (equivalently, all indices), which satisfy the principle of transfers and upside transfer sensitivity, unanimously state  $f$  at most is as imbalanced as  $g$ . Statement (ii) stresses that progressive transfers and/or UNFACTs are the necessary transformations to convert one distribution into another for this unanimity to be possible. Statement (iii) exhibits how to perform the pairwise comparison between  $f$  and  $g$  in practice. Distribution  $f$  DSB dominates  $g$  if and only if the verdict implied in (i) and stated by (ii) holds. Assertions (i–iii) in Theorem 3.1 imply assertions (i–iii) in Theorem 4.1, demonstrating that the DSB criterion is weaker than the Lorenz criterion.

Graphically, a distribution  $f$  is at most as imbalanced as a distribution  $g$  if two conditions are fulfilled. First, the DSB point of  $f$  does not lie below the DSB point of  $g$  at every abscissa value from the third largest point level up to the smallest in  $\mathcal{P}_*$ . Second,  $\Delta D^2(4n - 5) \geq 0$ .<sup>11</sup> Consider a toy example. For a population of 3 teams,  $\mathcal{P}_* = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ , let  $(0, 6, 6)$

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<sup>11</sup>In that variables are defined on grids, Fishburn and Lavallo (1995, Corollary 3) remark that a third-order stochastic dominance condition is stronger than its counterpart defined on continuous variables. It turns out that, at least for this sub-case, assertion (3) in Theorem 4.1 is stronger than its counterpart defined on continuous variables.

be represented by  $f \in \Omega$ , and  $(2, 2, 8)$  be represented by  $g \in \Omega$ :

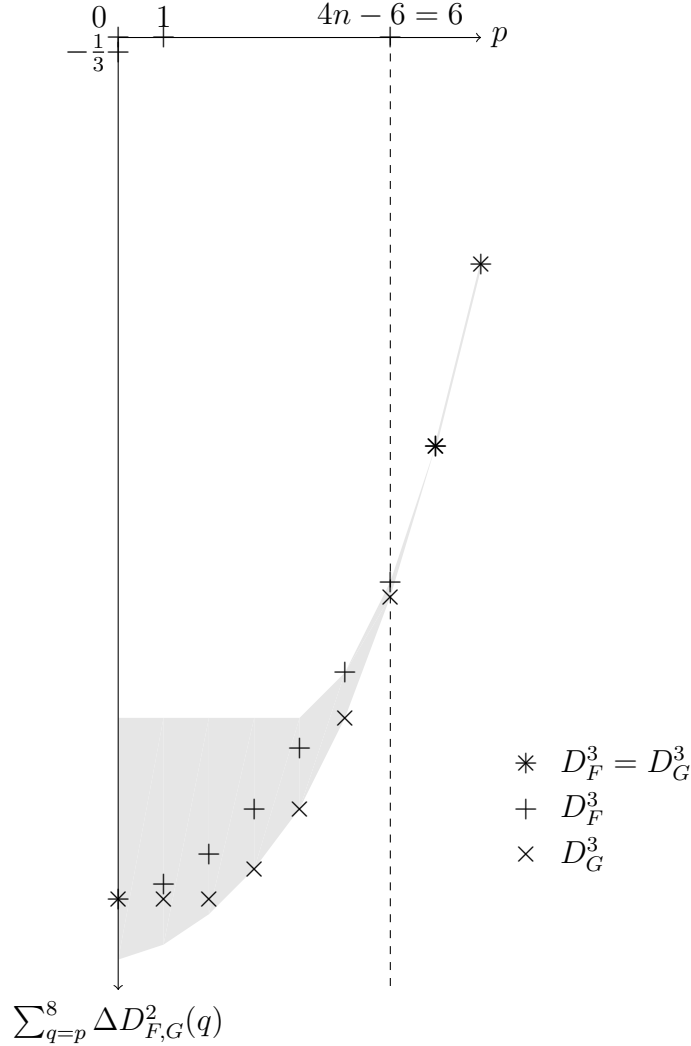
$$f = \begin{bmatrix} 0 & \frac{1}{3} \\ 1 & 0 \\ 2 & 0 \\ 3 & 0 \\ 4 & 0 \\ 5 & 0 \\ 6 & \frac{2}{3} \\ 7 & 0 \\ 8 & 0 \end{bmatrix}, \quad g = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 2 & \frac{2}{3} \\ 3 & 0 \\ 4 & 0 \\ 5 & 0 \\ 6 & 0 \\ 7 & 0 \\ 8 & \frac{1}{3} \end{bmatrix}.$$

Denote  $D_F^2(p) := \sum_{q=0}^p (-F(q))$ , hence  $D_F^3(p) = \sum_{q=p}^8 D_F^2(q)$ . Moreover,  $D_F^2(4n - 5) = D_G^2(4n - 5) = -4$ .

$p_i$	0	1	2	3	4	5	6	7	8
$D_F^3(p)$	-19	-18.67	-18	-17	-15.67	-14	-12	-9	-5
$D_G^3(p)$	-19	-19	-19	-18.33	-17	-15	-12.33	-9	-5

**Table 4.1.** Coordinates of the DSB points of the 3-team example.

The following figure illustrates that distribution  $f$  is at most as imbalanced as  $g$  according to the DSB criterion.



**Figure 4.2.** An example of DSB dominance.

The totally equal distribution, **b**, is (roughly) represented by the upper contour of the gray area. Moreover, the lower contour stands for the perfect competitive imbalance distribution, **i**.<sup>12</sup>

## 5 Competitive imbalance in European soccer leagues

This section briefly illustrates the results from the data covering features of the five best-ranked national soccer leagues of the Union of European Football Associations (UEFA).<sup>13</sup>

<sup>12</sup>Rigorously, distributions **b** and **i** should be represented by 18 points. However, these additional points would worsen the clarity of the figure.

<sup>13</sup>The statement relies on the UEFA clubs' coefficient rankings from the 2018-2019 season, which itself depends on the four seasons considered in the application. More information is available at: <https://www.uefa.com/memberassociations/uefarankings/country/#/yr/2019>

The five soccer leagues comprise the German Bundesliga, the English Premier League, the Spanish LaLiga, the French Ligue 1, and the Italian Serie A. The period analyzed covers from the 2014-2015 season to the 2018-2019 season.

The relevant data include numbers of victories, ties, and defeats made by each team at the end of each season. The data are freely available on *skysports.com*, *google.com* and *espn.deportes.espn.com*, among others. Distributions of points can be generated from these data in accordance with the point award system proposed in the setup. For instance, at the end of Ligue 1 season 2018-2019, the weakest team won 5 games, tied 12 times, and lost 21 games, so the team was awarded 22 points. The second-weakest team won 7 games, had 12 ties, and lost 19 games, so it was awarded 26 points; and so on up to the champion, who won 29 games, had 4 ties, and lost 5 games, so that it was awarded 62 points.

Only intra-national comparisons were examined to check whether competitive imbalance decreased during the covered period. Obviously, international comparisons could be performed to check which league was the least imbalanced and which was the most imbalanced.<sup>14</sup> The Lorenz criterion was able to be used to assert whether a season is more or less imbalanced than another season in the same league in only 17 cases out of 50 pairwise comparisons. In such cases, all the inequality indices unanimously made the same conclusion as the Lorenz criterion if they satisfy the principle of transfers. The DSB criterion was conclusive in 33 cases out of the 50 pairwise comparisons. All the inequality indices that satisfy the principle of transfers and upside transfer sensitivity were unanimously in line with the DSB criterion in these cases.

All comparisons are presented in Table 5.1. The table should be read as follows: Below the mention of “Premier League,” the row named 2015-2016 “ $\succ_D$ ” (DSB dominates) the column named 2014-2015. This means that the Premier League season 2015-2016 was at most as imbalanced as the Premier League season of 2014-2015, according to the DSB criterion. Still, below the mention of “Premier League,” the row named 2016-2017 “ $\prec_L$ ” (is Lorenz dominated by) the column named 2014-2015. This means that the Premier League season 2016-2017 was at least as imbalanced as the 2014-2015 Premier League season, according to both the Lorenz and DSB criteria. The symbols “ $\prec_D$ ” and “ $\succ_D$ ” allow for making a conclusion according to the DSB criterion while the Lorenz criterion was inconclusive. In Table 5.1, “NC” indicates that both the Lorenz and the DSB criteria were unable to be used to draw a conclusive comparison between the two distributions under consideration.

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<sup>14</sup>One technical difficulty would arise from the fact that the Bundesliga is an 18 team-league, whereas the others are composed of 20 teams. A standard method for providing comparisons in such a case is to consider relative points  $s$  rather than seasonal points  $p \in \mathcal{P}$  in the analysis. The relative points of a team should be such that:  $s = \frac{p}{n \sum_{q \in \mathcal{P}} qf(q)}$ , that is, the (absolute) point level of the team divided by the sum of points all teams are awarded in the league.

Competitive imbalance of the Premier League tended to increase from season 2014-2015. Indeed, season 2018-2019 was at least as imbalanced as seasons 2016-2017, 2015-2016, and 2014-2015. Also, season 2017-2018 was at least as imbalanced as seasons 2015-2016 and 2014-2015. The least imbalanced season of the period was 2015-2016.

The DSB criterion provided a small number of judgments considering Ligue 1. In the first part of the period, competitive imbalance did not decrease insofar as seasons 2017-2018 and 2016-2017 were at least as imbalanced as season 2014-2015. Ultimately, competitive imbalance did not increase: Season 2018-2019 was at most as imbalanced as seasons 2016-2017 and 2017-2018.

Competitive imbalance of the Bundesliga tended to increase during the observed period: Season 2018-2019 was at least as imbalanced as season 2014-2015. Beyond this trend, the evolution of the competitive imbalance of the Bundesliga is similar to that of Ligue 1. In the first part of the period, competitive imbalance did not decrease insofar as seasons 2016-2017 and 2015-2016 were at least as imbalanced as season 2014-2015. Later, competitive imbalance did not increase: Season 2017-2018 was at most as imbalanced as 2016-2017, which was at most as imbalanced as 2015-2016.

The competitive imbalance of Serie A tended to increase during the period under consideration. Indeed, season 2018-2019 was at least as imbalanced as season 2014-2015. In the first part of the period, competitive imbalance did not decrease, reaching a maximum level in 2017-2018. Season 2017-2018 was at least as imbalanced as season 2016-2017, which was at least as imbalanced as season 2015-2016, which, in turn, was at least as imbalanced as season 2014-2015. At the end of the period, competitive imbalance did not increase since season 2018-2019 was at most as imbalanced as seasons 2017-2018 and 2016-2017.

The DSB criterion provided a small number of judgments for LaLiga. Competitive imbalance tended to decrease from season 2014-2015 to season 2018-2019. Moreover, season 2018-2019 was the least imbalanced season of the period.

	2014-2015	2015-2016	2016-2017	2017-2018
<b>Premier League</b>				
2015-2016	$\succ_D$			
2016-2017	$\prec_L$	$\prec_D$		
2017-2018	$\prec_D$	$\prec_D$	NC	
2018-2019	$\prec_L$	$\prec_L$	$\prec_D$	NC
<b>Ligue 1</b>				
2015-2016	NC			
2016-2017	$\prec_D$	NC		
2017-2018	$\prec_D$	NC	NC	
2018-2019	NC	NC	$\succ_D$	$\succ_L$
<b>Bundesliga</b>				
2015-2016	$\prec_L$			
2016-2017	$\prec_L$	$\succ_D$		
2017-2018	NC	$\succ_D$	$\succ_D$	
2018-2019	$\prec_L$	NC	NC	NC
<b>Serie A</b>				
2015-2016	$\prec_D$			
2016-2017	$\prec_L$	$\prec_L$		
2017-2018	$\prec_L$	$\prec_L$	$\prec_D$	
2018-2019	$\prec_L$	NC	$\succ_L$	$\succ_L$
<b>LaLiga</b>				
2015-2016	NC			
2016-2017	NC	NC		
2017-2018	NC	NC	NC	
2018-2019	$\succ_L$	$\succ_D$	$\succ_L$	$\succ_L$

**Table 5.1.** Competitive imbalance comparisons in five soccer leagues in Europe.

## 6 Concluding remarks

The DSB criterion is proposed as a strategy to measure competitive imbalance when the Lorenz criterion is inconclusive. It makes a conclusive comparison of any pair of distributions of points when a sequence of progressive transfers and/or UNFACTs is necessary to convert one distribution into another. The normative view of competitive imbalance supported by this criterion is characterized by two axioms: The principle of transfers and upside transfer sensitivity. Moreover, the DSB criterion draws a conclusion if and only if all competitive imbalance indices that satisfy these axioms unanimously make the same conclusion as this criterion. The set for which unanimity is reached includes the indices proposed by Gayant and Le Pape (2017).

Chateauneuf, Gajdos, and Wilthien (2002) show that a principle that involves a FACT

and/or an UNFACT tacitly rules out all rank-dependent methods to measure income inequality. Consequently, upside transfer sensitivity implies excluding all these methods to measure competitive imbalance. The Gini coefficient is one of the rank-dependent methods. Upside sensitive rank-dependent measures satisfy a principle that requires progressive transfers be at least as valuable when taking place between two stronger competitors as between two weaker competitors. Moreover this principle requires that the difference in ranks between the two stronger competitors should be the same as the difference in ranks between the two weaker.<sup>15</sup> For example, consider three hypothetical distributions:  $\mathbf{x} = (10, 4, 2)$ ,  $\mathbf{y} = (10, 3, 3)$ , and  $\mathbf{z} = (9, 5, 2)$ . According to rank-dependent indices that are upside sensitive,  $\mathbf{z}$  is at most as imbalanced as  $\mathbf{y}$ . Indeed,  $\mathbf{z}$  is obtained from  $\mathbf{x}$  by a progressive transfer of one point between the first and second-ranked, whereas  $\mathbf{y}$  is obtained from  $\mathbf{x}$  by a progressive transfer of one point between the second and third-ranked. Both progressive transfers involve the same difference in ranks (equal to one). In my view,  $\mathbf{z}$  is not at most as imbalanced as  $\mathbf{y}$ . Distribution  $\mathbf{y}$  is obtained from  $\mathbf{x}$  by a progressive transfer that equalizes the numbers of points of the second and third-ranked competitors, whereas  $\mathbf{z}$  is obtained from  $\mathbf{x}$  by a transfer that marginally reduces a big gap between the first and the second-ranked.

The DSB criterion is a quasiorder; hence there are cases where it is inconclusive. To overcome this limitation, two strategies might be used: (A) An additional criterion which places even more emphasis on transfers that occur at the top of the distribution than the DSB criterion does. (B) An additional criterion that satisfies the same principles as the DSB criterion and places more emphasis on the bottom of the distribution of points than the middle. The choice of strategy (B) rather than (A) may be motivated by the possibility of relegation and promotion in some leagues, such as European soccer leagues. This point is stressed by Gayant and Le Pape (2017).

## Appendix

### Downward summation formula

**Lemma. *Downward summation formula.*** *Let  $b_0, \dots, b_n, c_0, \dots, c_n$  be real numbers. Set  $C_j = \sum_{k=j}^n c_k$ . Then:*

$$\sum_{j=0}^n b_j c_j = \sum_{j=1}^n C_j (b_j - b_{j-1}) + C_0 b_0.$$

---

<sup>15</sup>To measure income inequality, Aaberge (2009) provides a downward dominance criterion, which is consistent with rank-dependent inequality indices that are upside sensitive.

*Proof.* Let  $b_0, \dots, b_n, c_0, \dots, c_n$  be real numbers. Set  $C_j = \sum_{k=j}^n c_k$ . Then for every  $j > 0$ ,

$$c_j = \sum_{k=j}^n c_k - \sum_{k=j+1}^n c_k = C_j - C_{j+1}.$$

It turns out that:

$$\begin{aligned} \sum_{j=0}^n b_j c_j &= \sum_{j=0}^{n-1} b_j (C_j - C_{j+1}) + b_n c_n \\ &= \sum_{j=0}^{n-1} b_j C_j - \sum_{j=0}^{n-1} b_j C_{j+1} + b_n c_n = \sum_{j=1}^{n-1} b_j C_j + b_0 C_0 - \sum_{j=0}^{n-2} b_j C_{j+1} - b_{n-1} C_n + b_n c_n \\ &= \sum_{j=1}^{n-1} (b_j - b_{j-1}) C_j + b_0 C_0 - b_{n-1} C_n + b_n c_n \end{aligned}$$

By definition  $C_n = c_n$ , thus:

$$\sum_{j=0}^n b_j c_j = \sum_{j=1}^n (b_j - b_{j-1}) C_j + b_0 C_0$$

which concludes the proof. □

## Proof of Theorem 4.1

*Proof.* [(i)  $\Rightarrow$  (ii)] Building on Theorem 3.1 For  $v$  lying in  $V_2$

$$\sum_{p=0}^{4n-4} v(p) f(p) \geq \sum_{p=0}^{4n-4} v(p) g(p)$$

implies that  $f$  can be reached from  $f$  by a finite sequence of progressive transfers. Since  $V_3 \subseteq V_2$ , then for  $v$  lying in  $V_3$  the same implication remains valid.

Moreover, assume that  $v$  is such that

$$\begin{aligned} \Delta_3 v(p) &\leq 0 \\ \Leftrightarrow v(p+3) - v(p+2) - v(p+2) + v(p+1) \\ &\quad - v(p+2) + v(p+1) + v(p+1) - v(p) \geq 0 \end{aligned}$$

Consider

$$\begin{aligned} p+3 &= q + \epsilon + \delta; \quad p+2 = q + \epsilon = p + \epsilon + \delta; \\ p+1 &= q = p + \delta; \quad p = p \end{aligned}$$

It comes:

$$\begin{aligned} & v(q + \epsilon + \delta) - v(q + \epsilon) - v(q + \delta) + v(q) \\ & -v(p + \epsilon + \delta) + v(p + \epsilon) + v(p + \delta) - v(p) \leq 0 \end{aligned}$$

which is equivalent to

$$\frac{1}{n} \begin{bmatrix} v(q + \epsilon + \delta) - v(q + \epsilon) - v(q + \delta) + v(q) \\ -v(p + \epsilon + \delta) + v(p + \epsilon) + v(p + \delta) - v(p) \end{bmatrix} \leq 0$$

Or equivalently,

$$\begin{aligned} & [g(q + \epsilon + \delta) - f(q + \epsilon + \delta)]v(q + \epsilon + \delta) + [g(q + \epsilon) - f(q + \epsilon)]v(q + \epsilon) \\ & + [g(q + \delta) - f(q + \delta)]v(q + \delta) + [g(q) - f(q)]v(q) + [g(p + \epsilon + \delta) - f(p + \epsilon + \delta)]v(p + \epsilon + \delta) \\ & + [g(p + \epsilon) - f(p + \epsilon)]v(p + \epsilon) + [g(p + \delta) - f(p + \delta)]v(p + \delta) + [g(p) - f(p)]v(p) \leq 0 \end{aligned}$$

which amounts to

$$\sum_{p=0}^{4n-4} v(p)f(p) \geq \sum_{p=0}^{4n-4} v(p)g(p)$$

in which  $f$  can be reached from  $g$  by a UNFACT.

[[*ii*)  $\Rightarrow$  (*iii*)] Building on Theorem 3.1, if  $f$  is reached from  $g$  by a finite sequence of progressive transfers, then

$$\Delta D^2(k) \geq 0 \quad \forall k = 0, \dots, 4n - 5$$

Since

$$\Delta D^3(k) = \sum_{j=k}^{4n-6} \Delta D^2(j)$$

then

$$\Delta D^3(k) \geq 0 \quad \forall k = 0, \dots, 4n - 6$$

Moreover, assume that  $f$  is reached from  $g$  by a UNFACT as in Definition 4.1 such that  $\delta < \epsilon$

and  $q > p + \delta + \epsilon$ : We have

$$\Delta D^2(k) = 0 \text{ for } k = 0, \dots, p-1 \quad (\text{A1})$$

$$\Delta D^2(k) = \frac{-1}{n}(k-p+1) \text{ for } k = p, \dots, p+\delta-1 \quad (\text{A2})$$

$$\Delta D^2(k) = \frac{-\delta}{n} \text{ for } k = p+\delta, \dots, p+\epsilon-1 \quad (\text{A3})$$

$$\Delta D^2(k) = \frac{1}{n}(k-p-\epsilon+1) - \frac{\delta}{n} \text{ for } k = p+\epsilon, \dots, p+\delta+\epsilon-1 \quad (\text{A4})$$

$$\Delta D^2(k) = 0 \text{ for } k = p+\delta+\epsilon, \dots, q-1 \quad (\text{A5})$$

$$\Delta D^2(k) = \frac{1}{n}(k-q+1) \text{ for } k = q, \dots, q+\delta-1 \quad (\text{A6})$$

$$\Delta D^2(k) = \frac{\delta}{n} \text{ for } k = q+\delta, \dots, q+\epsilon-1 \quad (\text{A7})$$

$$\Delta D^2(k) = \frac{\delta}{n} - \frac{1}{n}(k-q-\epsilon+1) \text{ for } k = q+\epsilon, \dots, q+\delta+\epsilon-1 \quad (\text{A8})$$

$$\Delta D^2(k) = 0 \text{ for all } k = q+\delta+\epsilon, \dots, 4n-4 \quad (\text{A9})$$

To check whether  $\Delta D^2(4n-5) \geq 0$ , three cases should be studied: For the first two cases, *i.e.*  $4n-5 > q+\delta+\epsilon$  and  $4n-5 = q+\delta+\epsilon$ , it is straightforward to verify that  $\Delta D^2(4n-5) = 0$ , so  $\Delta D^2(4n-5) \geq 0$ . For the third case, *i.e.*  $4n-5 = q+\delta+\epsilon-1$  we have

$$\Delta D^2(4n-5) = \frac{\delta}{n} - \frac{1}{n}(4n-5-q-\epsilon+1) = \frac{\delta}{n} - \frac{1}{n}(q+\delta+\epsilon-1-q-\epsilon+1) = 0$$

It follows that  $\Delta D^2(4n-5) \geq 0$  is verified.

Now in order to check whether  $\Delta D^3(k) \geq 0$  for all  $k \in \{0, \dots, 4n-6\}$ , we proceed by a step-by-step analysis. The first step consists of checking for the sign of  $\Delta D^3(k)$  for all  $k \in \{q+\delta+\epsilon, \dots, 4n-6\}$ , which is a subset of  $\{0, \dots, 4n-6\}$ . The second step consists of checking for the sign of  $\Delta D^3(k)$  for all  $k \in \{q+\delta, \dots, q+\delta+\epsilon-1\}$  and so down until the last step, which consists of checking for the sign of  $\Delta D^3(k)$  for all  $k \in \{0, \dots, p\}$ . Let us begin by the *first step*: By (A9), it is straightforward that  $\Delta D^3(k) \geq 0$  for  $k = q+\delta+\epsilon, \dots, 4n-6$ .

*Second step*: By (A8),

$$\Delta D^3(k) = \frac{1}{n} \sum_{j=k}^{q+\delta+\epsilon-1} (q+\delta+\epsilon-1-j) \text{ for } k = q+\epsilon, \dots, q+\delta+\epsilon-1$$

Then, it is clear that  $\Delta D^3(k) \geq 0$  for  $k = q+\epsilon, \dots, q+\delta+\epsilon-1$ .

*Third step*: By (A8) and (A7),

$$\Delta D^3(k) = \frac{1}{n} \sum_{j=q+\epsilon}^{q+\delta+\epsilon-1} (q+\delta+\epsilon-1-j) + \frac{\delta}{n}(q+\epsilon-k) \text{ for } k = q+\delta, \dots, q+\epsilon-1$$

As already stated, the first term is nonnegative. Moreover since  $\epsilon \geq \delta$  it is clear that the second term is also nonnegative for  $k = q + \delta, \dots, q + \epsilon - 1$ . Then,  $\Delta D^3(k) \geq 0$  for  $k = q + \delta, \dots, q + \epsilon - 1$ .

*Fourth step:* By (A8), (A7) and (A6),

$$\Delta D^3(k) = \frac{1}{n} \sum_{j=q+\epsilon}^{q+\delta+\epsilon-1} (q + \delta + \epsilon - 1 - j) + \frac{(\epsilon - \delta)\delta}{n} + \frac{1}{n} \sum_{j=k}^{q+\delta-1} (j - q + 1) \text{ for } k = q, \dots, q + \delta - 1$$

Since the last term of the above equation is nonnegative for  $k = q, \dots, q + \delta - 1$ , it is clear that  $\Delta D^3(k) \geq 0$  for  $k = q, \dots, q + \delta - 1$ .

*Fifth step:* By (A8), (A7), (A6) and (A5),

$$\Delta D^3(k) = \frac{1}{n} \sum_{j=q+\epsilon}^{q+\delta+\epsilon-1} (q + \delta + \epsilon - 1 - j) + \frac{(\epsilon - \delta)\delta}{n} + \frac{1}{n} \sum_{j=k}^{q+\delta-1} (j - q + 1) \text{ for } k = p + \delta + \epsilon, \dots, q - 1$$

It is clear that  $\Delta D^3(k) \geq 0$  for  $k = p + \delta + \epsilon, \dots, q - 1$ .

*Sixth step:* By (A8), (A7), (A6), (A5) and (A4),

$$\begin{aligned} \Delta D^3(k) &= \frac{1}{n} \sum_{j=q+\epsilon}^{q+\delta+\epsilon-1} (q + \delta + \epsilon - 1 - j) + \frac{(\epsilon - \delta)\delta}{n} + \frac{1}{n} \sum_{j=k}^{q+\delta-1} (j - q + 1) \\ &\quad - \frac{1}{n} \sum_{j=k}^{p+\delta+\epsilon-1} (p + \delta + \epsilon - 1 - j) \text{ for } k = p + \epsilon, \dots, p + \delta + \epsilon - 1 \end{aligned}$$

Making a focus on the first sum and the last one in the right-hand side of the above equation,

$$\frac{1}{n} \sum_{j=q+\epsilon}^{q+\delta+\epsilon-1} (q + \delta + \epsilon - 1 - j) - \frac{1}{n} \sum_{j=k}^{p+\delta+\epsilon-1} (p + \delta + \epsilon - 1 - j)$$

For  $k = p + \delta + \epsilon - 1$ , the expression becomes:

$$0 + \sum_{j=q+\epsilon}^{q+\delta+\epsilon-2} (q + \delta + \epsilon - 1 - j) - 0$$

which is nonnegative. One sub-step further, for  $k = p + \delta + \epsilon - 2$ , it becomes:

$$\frac{1}{n} + \sum_{j=q+\epsilon}^{q+\delta+\epsilon-3} (q + \delta + \epsilon - 1 - j) - \frac{1}{n}$$

which is nonnegative. This sub-step can be repeated up to  $k = p + \epsilon$ , and the expression becomes:

$$\frac{1}{n} \sum_{j=q+\epsilon}^{q+\delta+\epsilon-1} (q + \delta + \epsilon - 1 - j) - \frac{1}{n} \sum_{j=p+\epsilon}^{p+\delta+\epsilon-1} (p + \delta + \epsilon - 1 - j) = 0$$

Then  $\Delta D^3(k) \geq 0$  for  $k = p + \epsilon, \dots, p + \delta + \epsilon - 1$ .

*Seventh step:* By (A8), (A7), (A6), (A5), (A4) and (A3),

$$\Delta D^3(k) = \frac{(\epsilon - \delta)\delta}{n} + \frac{1}{n} \sum_{j=q}^{q+\delta-1} (j - q + 1) - \frac{\delta(p + \epsilon - k)}{n} \quad \text{for } k = p + \delta, \dots, p + \epsilon - 1$$

Making a focus on the first and the last terms in the right-hand side of the above equation:

$$\frac{(\epsilon - \delta)\delta}{n} - \frac{\delta(p + \epsilon - k)}{n}$$

It is straightforward that  $(\epsilon - \delta)\delta - (p + \epsilon - k)\delta \geq 0$  for  $k = p + \delta, \dots, p + \epsilon - 1$  and so,  $\Delta D^3(k) \geq 0$  for  $k = p + \delta, \dots, p + \epsilon - 1$ .

*Eighth step:* By (A8), (A7), (A6), (A5), (A4), (A3) and (A2),

$$\Delta D^3(k) = \frac{1}{n} \sum_{j=q}^{q+\delta-1} (j - q + 1) - \frac{1}{n} \sum_{j=k}^{p+\delta-1} (j - p + 1) \quad \text{for } k = p, \dots, p + \delta - 1$$

For  $k = p + \delta - 1$ :

$$\Delta D^3(k) = \frac{\delta}{n} + \frac{1}{n} \sum_{j=q}^{q+\delta-2} (j - q + 1) - \frac{\delta}{n}$$

which is nonnegative. If  $\delta \leq 2$ , then there is nothing more to show for this step. Otherwise, one sub-step further, for  $k = p + \delta - 2$ , the expression becomes

$$\Delta D^3(k) = \frac{\delta}{n} + \frac{\delta - 1}{n} + \frac{1}{n} \sum_{j=q}^{q+\delta-3} (j - q + 1) - \frac{\delta}{n} - \frac{\delta - 1}{n}$$

which is nonnegative. This sub-step can be repeated up to  $k = p$ , and the expression becomes:

$$\Delta D^3(k) = \frac{1}{n} \sum_{j=q}^{q+\delta-1} (j - q + 1) - \frac{1}{n} \sum_{j=p}^{p+\delta-1} (j - p + 1) = 0$$

Then  $\Delta D^3(k) \geq 0$  for  $k = p, \dots, p + \delta - 1$ .

*Last step:* By (A1), it follows that  $\Delta D^3(k) = 0$  for  $k = 0, \dots, p$ .

Since we have set  $\delta < \epsilon$  and  $q > p + \delta + \epsilon$ , the UNFACT taken in this proof is not compelling with all the cases of Definition 4.1. All other cases fulfilling  $q > p$ ,  $0 < \delta \leq q - p$ , and  $\delta \leq \epsilon$  follow directly from the proof provided above. To illustrate it, consider  $\delta = \epsilon = 1 = q - p$ . It

comes that:

$$\begin{aligned}
\Delta f(k) &= 0 \quad \text{for } k = 0, \dots, p-1 \\
\Delta f(k) &= \frac{1}{n} \quad \text{for } k = p \\
\Delta f(k) &= -\frac{3}{n} \quad \text{for } k = p + \delta = p + \epsilon = q \\
\Delta f(k) &= \frac{3}{n} \quad \text{for } k = p + \delta + \epsilon = q + \epsilon = q + \delta \\
\Delta f(k) &= -\frac{1}{n} \quad \text{for } k = q + \delta + \epsilon \\
\Delta f(k) &= 0 \quad \text{for } k = q + \delta + \epsilon + 1, \dots, 4n - 4
\end{aligned}$$

Whence:

$$\begin{aligned}
\Delta D^2(k) &= 0 \quad \text{for } k = 0, \dots, p-1 \\
\Delta D^2(k) &= -\frac{1}{n} \quad \text{for } k = p \\
\Delta D^2(k) &= \frac{1}{n} \quad \text{for } k = p + \delta = p + \epsilon = q \\
\Delta D^2(k) &= 0 \quad \text{for } k = p + \delta + \epsilon = q + \epsilon = q + \delta \\
\Delta D^2(k) &= 0 \quad \text{for } k = q + \delta + \epsilon \\
\Delta D^2(k) &= 0 \quad \text{for } k = q + \delta + \epsilon + 1, \dots, 4n - 4
\end{aligned}$$

Thus  $\Delta D^2(4n - 5) \geq 0$ . Moreover:

$$\begin{aligned}
\Delta D^3(k) &= 0 \quad \text{for } k = p + \delta + \epsilon, \dots, 4n - 4 \\
\Delta D^3(k) &= \frac{1}{n} \quad \text{for } k = p + \delta \\
\Delta D^3(k) &= 0 \quad \text{for } k = p \\
\Delta D^3(k) &= 0 \quad \text{for } k = 0, \dots, p-1
\end{aligned}$$

It follows that  $\Delta D^3(k) \geq 0$  for  $k = 0, \dots, 4n - 6$ .

[(iii)  $\Rightarrow$  (i)] This implication follows directly from (3). □

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