

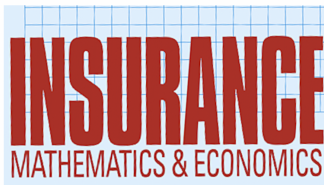
Bounding Basis-Risk Using s-convex Orders on Beta-unimodal Distributions

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Basis risk

"Risk that the recoveries obtained by the entity which has faced a loss are too different from what it would have received with an indemnity-based risk transfer."

Motivations

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Parametric Insurance

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Measurement of basis risk



Definition of some risk limit for basis risk management

Model-free approach ?

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Link between the Loss and the Index:

$$L = Z - c + X,$$

where $X = SY$, $X \in [0, a + c]$.

Beta-unimodal Distributions

Definition

An \mathbb{R}_+ -valued rv X has a continuous Beta-unimodal distribution if it has the product representation

$$X =_d SY, \quad (1)$$

where Y is a positive continuous rv, and $S \sim \mathcal{B}(\alpha, \beta)$ is a random contracting factor independent of Y .

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From Y to X : if $Y \sim F_Y$ and if X is Beta-unimodal

$$\bar{F}_X(x) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)} x^\alpha (I_\beta \phi)(x), \quad (2)$$

where:

- $(I_\beta \phi)(x) = \int_x^\infty \frac{1}{\Gamma(\beta)} (t-x)^{\beta-1} \phi(t) dt$ is called the Weyl fractional-order integral operator,
- $\phi(t) = \bar{F}_Y(t) t^{-\alpha-\beta}$.

Identification: s-convex orders

Definition: s-convexity

A function ϕ defined on \mathcal{S} is said to be **s-convex** if the inequality

$$[x_0, \dots, x_s; \phi] \geq 0,$$

holds for any choice of distinct points x_0, \dots, x_s in \mathcal{S} .

$[x_0, \dots, x_s; \phi]$ denotes a divided difference of the function ϕ at the different points x_0, \dots, x_s .

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s differentiability condition: if $\phi^{(s)}$ exists in \mathcal{S} , then ϕ is s-convex if and only if $\phi^{(s)} \geq 0$.

Definition: s-convexity

Definition: s-increasing convexity

A function ϕ is said to be **s-increasing convex** on its domain \mathcal{S} if and only if for all choices of $k + 1$ distinct points $x_0 < x_1 < \dots < x_k$ in \mathcal{S} , we have

$$[x_0, x_1, \dots, x_k; \phi] \geq 0, \quad k = 2, \dots, s.$$

We denote by $\mathcal{U}_{s-icx}^{\mathcal{S}}$ the class of the s-increasing convex functions on \mathcal{S} .

Identification: s-convex orders

Definition: s-convexity

Definition: s-increasing convexity

Definition: s-convex order

Let X_1 and X_2 be two random variables that take on values in \mathcal{S} . Then X_1 is said to be smaller than X_2 in the s -convex order, denoted by $X_1 \leq_{s-cx}^{\mathcal{S}} X_2$ if

$$\mathbb{E}[\phi(X_1)] \leq \mathbb{E}[\phi(X_2)] \text{ for all } \phi \in \mathcal{U}_{s-cx}^{\mathcal{S}}, \quad (3)$$

where $\mathcal{U}_{s-cx}^{\mathcal{S}}$ is the class of all the s -convex functions $\phi : \mathcal{S} \rightarrow \mathbb{R}$.

Identification: s-convex extrema

Definition: moment space

We denote by $\mathcal{B}_s([a, b], \mu_1, \mu_2, \dots, \mu_{s-1})$ the moment space of all the random variables valued in $[a, b]$ and with known $s - 1$ moments μ_1, \dots, μ_{s-1} .

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Theorem: s-convex extrema

Let $Y \in \mathcal{B}_s([a, b], \mu_1, \mu_2, \dots, \mu_{s-1})$.

Within $\mathcal{B}_s([a, b], \mu_1, \mu_2, \dots, \mu_{s-1})$, there exist two unique random variables $Y_{\min}^{(s)}$ and $Y_{\max}^{(s)}$ such that


$$Y_{\min}^{(s)} \leq_{s\text{-cx}} Y \leq_{s\text{-cx}} Y_{\max}^{(s)}.$$

Proof. See Denuit et al. (1999).


s-convex extrema are actually extremal distributions built from the s-1 first moments of Y .

3-convex extremal distributions

Let $Y \in \mathcal{B}_3([a, b], \mu_1, \mu_2)$.


$$Y_{\min}^{(3)} = \begin{cases} a & \text{with proba } \frac{\mu_2 - \mu_1^2}{(a - \mu_1)^2 + \mu_2 - \mu_1^2}, \\ b & \text{with proba } \frac{(a - \mu_1)^2}{(a - \mu_1)^2 + \mu_2 - \mu_1^2}, \end{cases}$$

and


$$Y_{\max}^{(3)} = \begin{cases} \mu_1 - \frac{\mu_2 - \mu_1^2}{b - \mu_1} & \text{with proba } \frac{(b - \mu_1)^2}{(b - \mu_1)^2 + \mu_2 - \mu_1^2}, \\ b & \text{with proba } \frac{\mu_2 - \mu_1^2}{(b - \mu_1)^2 + \mu_2 - \mu_1^2}. \end{cases}$$

Generalization to Beta-unimodal distributions

Proposition (Beta-unimodal s-convex extrema)

Let $Y \in \mathcal{B}_s(\mathcal{S}, \mu_1, \mu_2, \dots, \mu_{s-1})$, and let $\phi \in \mathcal{U}_{s-cx}^{\mathcal{S}}$. If X is Beta-unimodal, then $X \in \mathcal{B}_s(\mathcal{S}, v_1, v_2, \dots, v_{s-1})$ and the s-convex extrema in this set are

$$X_{min}^{(s)} = SY_{min}^{(s)}, \text{ and } X_{max}^{(s)} = SY_{max}^{(s)}.$$

Besides, if $\phi \in \mathcal{U}_{s-icx}^{\mathcal{S}}$, then

$$\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(SY_{max}^{(s)})] \leq \mathbb{E}[\phi(SY_{max}^{(s-1)})] \leq \dots \leq \mathbb{E}[\phi(SY_{max}^{(2)})],$$

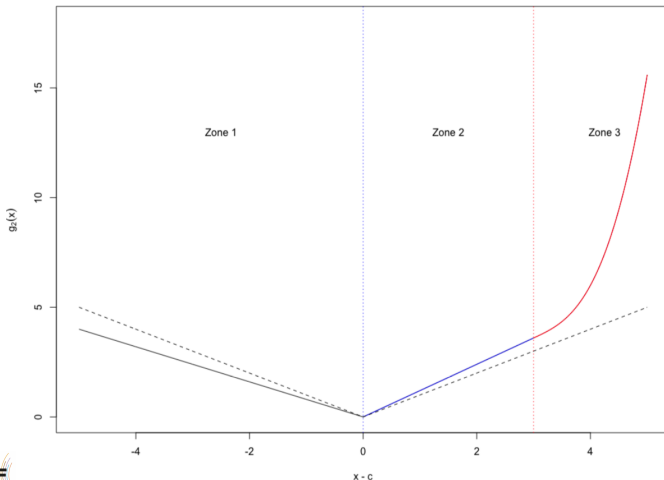
which can be written as,

$$\forall k \in \llbracket 2, s \rrbracket, X \leq_{k-cx} X_{max}^{(s)} \leq_{k-cx} X_{max}^{(s-1)} \leq_{k-cx} \dots \leq_{k-cx} X_{max}^{(2)}.$$

When $\phi \in \mathcal{U}_{s-icx}^{\mathcal{S}}$, the more moments, the sharper the bounds !

Penalty functions

A penalty function g allows us to represent the consequences of a positive or negative difference between the Loss and the Index.



Two examples

$$g(L-Z) = \eta(c-X)_x + \gamma(X-c)_+ + \begin{cases} \psi_n(x), \\ -\gamma(x-(c+d))_+ + \varphi_\kappa(x), \end{cases}$$

where

$$\psi_n(x) = \gamma(x-(c+d))_+^n,$$

and

$$\varphi_\kappa(x) = \gamma\kappa \left(\exp\left(\frac{(x-(c+d))_+}{\kappa}\right) - 1 \right)$$

Note: ψ_n is n increasing convex, φ_κ is ∞ increasing convex.

From basis risk budget to risk limits

Framework

First, we consider a symmetric range of possible values for uncertainty: $a = c$, i.e. $L = Z - a + X$, with $X \in [0, 2a]$.

Let BR^{Sym} be the two-sided uncertainty budget. The way we limit the consequences of a difference is finding the window of uncertainty (the value of a) such that the budget is not overpassed.

We look for $\hat{a}_f^{(s)}$ defined as,

$$\hat{a}_f^{(s)} = \operatorname{argmin} \left| BR^{Sym} - \mathbb{E}[g(X_{max}^{(s)} - c)] \right|, \text{ with } f = \psi_n, \varphi_\kappa.$$

Numerical illustrations

Parameters

$$\begin{aligned} a = c = 5, \quad d = 2.5, \quad \eta = \gamma = 1 \\ Y \sim \mathcal{U}[0, 10], \quad \alpha = 4, \beta = 2, \quad BR^{Sym} = 3.5417. \end{aligned}$$

Numerical illustrations

Parameters

$$a = c = 5, \quad d = 2.5, \quad \eta = \gamma = 1$$
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Convex order used	$\hat{a}_{\psi_2}^{(s)}$	$\hat{a}_{\psi_3}^{(s)}$	$\hat{a}_{\psi_4}^{(s)}$	$\hat{a}_{\psi_6}^{(s)}$
2	4.5268	4.4686	4.4015	4.2746
3	-	4.6292	4.5521	4.4002
4	-	-	4.6335	4.4730

Table: Values of $\hat{a}_{\psi_n}^{(s)}$ for $s = 2, 3$ and 4 , for $n = 2, 3, 4$ and 6

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Convex order used	$\hat{a}_{\varphi_{\kappa_2}}^{(s)}$	$\hat{a}_{\varphi_{\kappa_3}}^{(s)}$	$\hat{a}_{\varphi_{\kappa_4}}^{(s)}$	$\hat{a}_{\varphi_{\kappa_6}}^{(s)}$
2	4.6324	4.4783	4.3351	4.1020
3	5.0058	4.8033	4.6172	4.3206
4	5.2321	5.0000	4.7870	4.4510

Table: Values of $\hat{a}_{\varphi_{\kappa}}^{(s)}$ for $s = 2, 3$ and 4 , for $\kappa_2, \kappa_3, \kappa_4$ and κ_6

Conclusion

Promise kept ?



Identification of s-convex extrema for Beta-Unimodal random variables,

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Measurement of basis risk in "model-free" context, and show the impact of information (in terms of moments) on basis risk assessment,

Propose a way to set up **risk limit** for basis risk, relative to extremal s-convex bounds.

Is basis risk always bounded ?