Second order local minimal-time Mean Field Games

Romain Ducasse, Guilherme Mazanti, Filippo Santambrogio

SECOND ORDER LOCAL MINIMAL-TIME MEAN FIELD GAMES

ROMAIN DUCASSE, GUILHERME MAZANTI, AND FILIPPO SANTAMBROGIO

Abstract. The paper considers a forward-backward system of parabolic PDEs arising in a Mean Field Game (MFG) model where every agent controls the drift of a trajectory subject to Brownian diffusion, trying to escape a given bounded domain Ω in minimal expected time. The important point is that agents are constrained by a bound on the drift depending on the density of the other agents at their location (the higher the density, the smaller the velocity). Existence for a finite time horizon T is proven via a fixed point argument but, because of the diffusion, the model should be studied in infinite horizon as the total mass inside the domain decreases in time, but never reaches zero in finite time. Hence, estimates are needed to pass the solution to the limit as $T \to \infty$, and the asymptotic behavior of the solution which is obtained in this way is also studied. This passes through a combination of classical parabolic arguments together with specific computations for MFGs. Both the Fokker–Planck equation ruling the evolution of the density of agents and the Hamilton–Jacobi–Bellman equation on the value function display Dirichlet boundary conditions as a consequence of the fact that agents stop as soon as they reach $\partial\Omega$. The initial datum for the density is given (and its regularity is discussed so as to have the sharpest result), and the long-time limit of the value function is characterized as the solution of a stationary problem.

Contents

1. Introduction

Introduced around 2006 by Jean-Michel Lasry and Pierre-Louis Lions [18–20] and at the same time by Peter Caines, Minyi Huang, and Roland Malhamé [12–14], the theory of Mean Field Games (MFGs, for short) describes the interaction of a continuum of players, assumed to be rational, indistinguishable, and negligible, when each one tries to solve a dynamical control problem influenced only by the average behavior of the other players (through a mean-field type interaction, using the physicists' terminology). The Nash

2020 *Mathematics Subject Classification.* 35Q89, 35K40, 35B40, 35A01, 35D30.

Key words and phrases. Mean Field Games, congestion games, parabolic PDEs, MFG system, existence of solutions, asymptotic behavior.

equilibrium in these continuous games is described by a system of PDEs: a Hamilton–Jacobi–Bellman equation for the value function of the control problem of each player, where the distribution (density) of the players appears, coupled with a continuity equation describing the evolution of such a density, where the velocity field is the optimal one in order to solve the control problem, and is therefore related to the gradient of the value function. This system is typically forward-backward in nature: the density evolves forward in time starting from a given initial datum, and the value function backward in time, according to Bellman's dynamical programming principle, and its final value at a given time horizon T is usually known.

The literature about MFG theory is quickly growing and many references are available. The 6-year course given by P.-L. Lions at Collège de France, for which video-recording is available in French [22], explains well the birth of the theory, but the reader can also refer to the lecture notes by P. Cardaliaguet [6], based on the same course.

In most of the MFG models studied so far the agents consider a fixed time interval $[0,T]$ and optimize a trajectory $x:[0,T]\to\Omega$ (where $\Omega\subset\mathbb{R}^d$ is the state space) trying to minimize a cost of the form $\int_0^T L(t,x(t),x'(t),\rho_t)\,\mathrm{d}t+\Psi(x(T),\rho_T)$, where ρ_t denotes the distribution of players at time t. The function L is typically increasing in $|x'|$ and, in some sense, in ρ. This means that high velocities are costly, and passing through areas where the population is strongly concentrated is also costly. Some MFGs, called *MFGs of congestion* (see, for instance, [1]), consider costs which include a product of the form $\rho_t(x(t))^\alpha|x'(t)|^\beta$ (for some exponents $\alpha,\beta>0$), which means that high velocities are costly, and that they are even more costly in the presence of high concentrations. These models present harder mathematical difficulties compared to those where the cost is decomposed into $L(t,x(t),x'(t))+g(t,x(t),\rho_t)$. Indeed, in many cases the latter MFG admits a variational formulation: equilibria can be found by minimizing a global energy among all possible evolutions $(\rho_t)_t$ (hence, they are *potential games*). This allows to prove the existence of the equilibrium via semicontinuity methods, and we refer to [4] and [25] for a detailed discussion of this branch of MFG theory.

When the MFG has no variational interpretation, then the existence of a solution is usually obtained via fixed-point theorems, but these theorems require much more regularity. Roughly speaking, given an evolution ρ one computes the corresponding value function φ as a solution to a Hamilton–Jacobi–Bellman equation and, given φ, one computes a new density evolution $\tilde\rho$ by following an evolution equation. We need existence, uniqueness, and stability results for these equations in order to find a fixed point $\tilde\rho=\rho$. This usually requires regularity of the velocity field $-\nabla\varphi$, which is difficult to prove, and can be essentially only obtained in two different frameworks: either the dependence of the cost functions on the distribution ρ is highly regularizing (which usually means that it is non-local, and passes through averaged quantities such as convolutions $\int\eta(x-y)\,\mathrm{d}\rho(y)$), or diffusion of the agents is taken into account, transforming the optimal control problem into a stochastic one. In this latter case, agents minimize $\mathbb{E}[\int_0^T L(t,X_t,\alpha(t),\rho_t)\,\mathrm{d}t+\Psi(X_T,\rho_T)]$ where the process X follows $\mathrm{d}X_t=\alpha_t\,\mathrm{d}t+\mathrm{d}B_t$ and $(B_t)_{t\geq0}$ represents a standard Brownian motion.

In [23], the second and third authors of the present paper introduced a different class of models, called *minimal-time MFGs*. The main difference is that instead of considering a cost for the players penalizing both the velocity and the density, and minimizing the integral of such a cost on a fixed time interval $[0,T]$, the dynamics is subject to a constraint where the maximal velocity of the agents cannot exceed a quantity depending on the density ρ_t, and the goal of each agent is to arrive to a given target as soon as possible. In the typical situation, the target of the agents is the boundary $\partial\Omega$ of the domain where the evolution occurs. This can model, for instance, an evacuation phenomenon in crowd

motion. The system that one obtains is the following

$$\begin{cases} \partial_t \rho - \nabla \cdot \left(\rho k[\rho] \dfrac{\nabla \varphi}{|\nabla \varphi|} \right) = 0, & \text{in } (0,T) \times \Omega, \\ -\partial_t \varphi + k[\rho] |\nabla \varphi| - 1 = 0, & \text{in } (0,T) \times \Omega, \\ \rho(0,x) = \rho_0(x), & \text{in } \Omega, \\ \varphi(t,x) = 0, & \text{on } (0,T) \times \partial\Omega, \end{cases} \tag{1.1}$$

where the function $k[\rho_t](x)$ denotes the maximal speed that agents can have at point x at time t, i.e., the dynamics is constrained to satisfy $|x'(t)| \leq k[\rho_t](x(t))$. Ideally, one would like to choose k to be a non-increasing function of the density itself, such as $k[\rho](x) = (1 - \rho(x))_+$. This choice is what is done in the well-known Hughes' model for crowd motion [15, 16]. Indeed, this model is very similar to Hughes', which also considers agents who aim at leaving in minimal time a bounded domain under a congestion-dependent constraint on their speeds.

The main difference between the model in [23] (from which the present paper stems) and Hughes' is that, in the latter, at each time, an agent moves in the optimal direction to the boundary assuming that the distribution of agents remains constant, whereas in [23] and here agents take into account the future evolution of the distribution of agents in the computation of their optimal trajectories. This accounts for the time derivative in the Hamilton–Jacobi–Bellman equation from (1.1), which is the main difference between (1.1) and the equations describing the motion of agents in Hughes' model and stands for the anticipation of future behavior of other agents.

Another crucial (and disappointing) similarity between the above MFG system and Hughes' model is the fact that general mathematical results do not exist in the case $k[\rho] = 1 - \rho$ and more generally in the local case (except few results in the Hughes case in 1D). Indeed, the lack of regularity makes the model too hard to study, and the MFG case is not variational. In some sense the closest MFG model to this one is the one with multiplicative costs in [1]. Indeed, an L^∞ constraint $|x'| \leq k[\rho]$ can be seen as a limit as $m \to \infty$ of an integral penalization

$$\int \left| \frac{|x'(t)|}{k[\rho_t](x(t))} \right|^m \mathrm{d}t$$

(note that the boundaries of the time interval have been omitted on purpose from the above integral, since the model in [1] is set on a fixed time horizon but this is not part of our setting).

Because of these difficulties, [23] studied the case of a non-local dependence of k w.r.t. ρ (say, $k[\rho](x) = \kappa(\int \eta(x-y)\,\mathrm{d}\rho(y))$, for a non-increasing function κ and a positive convolution kernel η), and proved existence of an equilibrium, characterized it as a solution of a non-local MFG system, and analyzed some examples, including numerical simulations. Instead, in the present paper we want to study the local case with diffusion.

This means that we will consider a local dependence $k[\rho](x) := \kappa(\rho(x))$, and each agent solves a stochastic control problem

$$\inf \left\{ \mathbb{E}[\tau] \,:\, X(\tau) \in \partial\Omega, X(0) = x_0,\, \mathrm{d}X_t = \alpha_t\,\mathrm{d}t + \sqrt{2\nu}\,\mathrm{d}B_t,\, |\alpha_t| \leq \kappa(\rho(t, X_t)) \right\},$$

where $(B_t)_{t \geq 0}$ denotes a standard Brownian motion and the Brownian motions for all players are assumed to be mutually independent. Defining the corresponding value function φ, from classical results on stochastic optimal control (see [10, Chapter IV]), under suitable assumptions, the optimal control is given in feedback form by

$$\alpha_t = -\kappa(\rho(t, X_t)) \frac{\nabla \varphi(t, X_t)}{|\nabla \varphi(t, X_t)|},$$

(a definition which has to be carefully adapted to the case $\nabla\varphi = 0$); moreover, the value function solves the Hamilton–Jacobi–Bellman equation

$$-\partial_t\varphi(t,x) - \nu\Delta\varphi(t,x) + K(t,x)|\nabla\varphi(t,x)| - 1 = 0, \quad (t,x) \in [0,T) \times \Omega,$$

for $K = \kappa(\rho)$. Hence, we know the drift of the optimal stochastic processes followed by each agent, and this allows to write the Fokker–Planck equation solved by the law of this process. Putting together all this information, we obtain the following MFG system

$$\begin{cases} \partial_t\rho - \nu\Delta\rho - \nabla\cdot\left(\rho\kappa(\rho)\dfrac{\nabla\varphi}{|\nabla\varphi|}\right) = 0, & \text{in } \mathbb{R}_+ \times \Omega, \\ -\partial_t\varphi - \nu\Delta\varphi + \kappa(\rho)|\nabla\varphi| - 1 = 0, & \text{in } \mathbb{R}_+ \times \Omega, \\ \rho(0,x) = \rho_0(x), & \text{in } \Omega, \\ \rho(t,x) = 0, \quad \varphi(t,x) = 0, & \text{on } \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^d$ is an open and bounded set, whose boundary will be supposed to be of class C^2 in this paper, $\nu > 0$ is a fixed constant, $\kappa : \mathbb{R} \to (0, +\infty)$, and $\rho_0 \geq 0$ is the initial density. The Dirichlet condition on φ comes as usual from the fact that, for agents who are already on the boundary, the remaining time to reach it is zero, and the Dirichlet condition on ρ comes from the fact that we stop the evolution of a particle as soon as it touches the boundary (absorbing boundary conditions).

A crucial difference with the previous paper [23] concerns the time horizon. If we suppose that κ is bounded from below in the model without diffusion, it is not difficult to see that all agents will have left the domain after a common finite time, so that the final value of φ is not really relevant, and the problem can be studied on a finite interval $[0,T]$. This is not the case when there is diffusion, as a density following a Fokker–Planck equation with a bounded drift cannot fully vanish in finite time. As a consequence, the model should be studied on the unbounded interval $[0,\infty)$. For every time $t < \infty$ there is still mass everywhere, but this mass decreases to 0 as $t \to +\infty$, which suggests that the value function φ should converge to a function, that we call Ψ, which is the value function for the corresponding control problem with no mass, i.e. when $\kappa = \kappa(0)$. Since in this control problem κ is independent of time, Ψ is a function of x only and solves a stationary Hamilton–Jacobi–Bellman equation which takes the form of an elliptic PDE

$$-\nu\Delta\Psi + \kappa(0)|\nabla\Psi| - 1 = 0$$

with Dirichlet boundary conditions on $\partial\Omega$. It is then reasonable to investigate whether solutions of the above system satisfy further $\rho_t \to 0$ and $\varphi_t \to \Psi$ as $t \to +\infty$.

In order to study the above system, we will first study an artificial finite-horizon setting, where we stop the game at time T, choose a penalization $\psi : \Omega \to \mathbb{R}_+$ with $\psi = 0$ on $\partial\Omega$, and look at the stochastic optimal control problem

$$\inf\Big\{\mathbb{E}[\min\{\tau,T\} + \psi(X_{\min\{\tau,T\}})] \,:\, X(\tau) \in \partial\Omega, X(0) = x_0, \, \mathrm{d}X_t = \alpha_t\,\mathrm{d}t + \sqrt{2\nu}\,\mathrm{d}B_t, \, |\alpha_t| \leq \kappa(\rho(t,X_t))\Big\}.$$

This gives rise to the MFG system

$$\begin{cases} \partial_t\rho - \nu\Delta\rho - \nabla\cdot\left(\rho\kappa(\rho)\dfrac{\nabla\varphi}{|\nabla\varphi|}\right) = 0, & \text{in } (0,T) \times \Omega, \\ -\partial_t\varphi - \nu\Delta\varphi + \kappa(\rho)|\nabla\varphi| - 1 = 0, & \text{in } (0,T) \times \Omega, \\ \rho(0,x) = \rho_0(x), \qquad \varphi(T,x) = \psi(x), & \text{in } \Omega, \\ \rho(t,x) = 0, \qquad\quad \varphi(t,x) = 0, & \text{on } (0,T) \times \partial\Omega, \end{cases} \tag{1.3}$$

which corresponds to (1.2) with the unbounded time interval $\mathbb{R}_+$ replaced by $(0,T)$ and the additional final condition $\varphi(T,x) = \psi(x)$. We will prove the existence of a solution

of the system for finite T, and then consider the limit as $T \to \infty$. In order to guarantee suitable bounds, we just need to choose a sequence of final data ψ_T, possibly depending on T, which is uniformly bounded. We will then get at the limit a solution of the limit system which automatically satisfies $\rho_t \to 0$ (in the sense of uniform convergence) and $\varphi_t \to \Psi$ (this convergence being both uniform and strong in H^1_0).

The paper is organized as follows. After this introduction, Section 2 presents the tools the we need to study the two separate equations appearing in System (1.3) on a finite horizon. These tools come from the classical theory of parabolic equations, but we developed in more details some precise statements which are difficult to find in the literature as far as the precise framework we need is concerned. Section 3 is devoted to the existence of solutions of (1.3). After providing a precise definition of solution of (1.3) taking care of the case $\nabla\varphi = 0$, we use the estimates of Section 2 to prove existence via a fixed-point argument based on Kakutani's theorem. Section 4 concerns the limit $T \to \infty$. In this section, some estimates of Section 2 need to be made more precise, in order to see how constants depend on the time horizon T. In this way we are able to prove existence of a limit of the solutions of (1.3) as the time horizon T tends to $+\infty$ and that this limit solves the limit system (1.2). Then we consider the asymptotic behavior of a solution (ρ, φ) of (1.2) as $t \to +\infty$, proving first $\rho_t \to 0$ in L^1 and, thanks to a parabolic regularization argument, also in L^∞. To prove convergence in L^1, which is true for general Fokker–Planck systems under very mild assumptions, we exploit the MFG nature of the system, i.e. the coupling between the two equations, which also provides exponential decrease. We then consider the limit in time of φ, and prove that any bounded solution of this equation, once we know $\kappa(t,x) \to \kappa(0)$, can only converge as $t \to +\infty$ to the stationary function Ψ. This convergence is a priori very weak, but we are able to improve it into $L^\infty \cap H^1_0$, and to prove that the uniform convergences of both ρ and φ occur exponentially fast. In all this analysis the initial datum ρ_0 is at first supposed to belong to $L^2(\Omega)$, because the general theory for the parabolic equation on ρ, including uniqueness results (which are crucial in order to prove the existence of a fixed point), usually requires this assumption. Yet, the sharp assumption on the initial datum which allows us to develop our analysis is weaker, as we only need that ρ_0 has finite entropy. At the end of Section 4 we explain how to adapt our study to such a more general case. The paper is then completed by two appendices. Appendix A details some global L^∞ estimates for a large class of parabolic equations, including the estimates that we use to prove uniform convergence in time of ρ_t and φ_t to 0 and Ψ, respectively. These estimates are not surprising and not difficult to prove, using standard Moser iterations, but are not easy to find in the literature under the only assumption of boundedness of the drift term in the divergence. The computations and the results are essentially the same as in the appendix of [7], but the boundary conditions are different. Appendix B is devoted, instead, to a full theory of the Fokker–Planck equation with bounded drift and initial datum with finite entropy. We give a notion of solution for which we prove both existence and uniqueness.

2. Preliminary results

This section presents some preliminary results on Fokker–Planck and Hamilton–Jacobi–Bellman equations which are useful for the analysis of the Mean Field Game systems (1.2) and (1.3). We recall that, in the whole paper, Ω denotes an open and bounded set whose boundary $\partial\Omega$ is assumed to be C^2. Even though some of the results presented in this preliminary section also hold without the smoothness assumption on $\partial\Omega$ (such as Propositions 2.2 and 2.5), this assumption is first used to obtain higher regularity of solutions of Hamilton–Jacobi–Bellman equations in Proposition 2.6 and is required for almost all of the subsequent results, including in particular our main results in Sections 3 and 4, as a consequence of the need of higher regularity of φ.

2.1. **Fokker–Planck equation.** We recall some classical results on the Fokker–Planck equation on a bounded domain $\Omega \subset \mathbb{R}^d$ in finite time horizon $T \in (0, +\infty)$,

$$\begin{cases} \partial_t \rho - \nu \Delta \rho + \nabla \cdot (\rho V) = 0 & \text{in } (0,T) \times \Omega, \\ \rho(0,x) = \rho_0(x) & \text{in } \Omega, \\ \rho(t,x) = 0 & \text{in } [0,T] \times \partial\Omega, \end{cases} \tag{2.1}$$

where $V : (0,T) \times \Omega \to \mathbb{R}^d$ is a given velocity field.

Definition 2.1. Let $\nu > 0$, $V \in L^\infty((0,T) \times \Omega; \mathbb{R}^d)$, and $\rho_0 \in L^2(\Omega)$. We say that $\rho \in L^\infty((0,T); L^2(\Omega)) \cap L^2((0,T); H^1_0(\Omega))$ is a *weak solution* of (2.1) if, for every $\eta \in C^1([0,T] \times \Omega)$ such that $\eta\big|_{[0,T]\times\partial\Omega} = 0$ and $\eta\big|_{\{T\}\times\Omega} = 0$, one has

$$-\int_0^T \int_\Omega \rho \partial_t \eta \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \int_\Omega (\nu \nabla \rho - \rho V) \cdot \nabla \eta \,\mathrm{d}x \,\mathrm{d}t = \int_\Omega \rho_0(x) \eta(0,x) \,\mathrm{d}x. \tag{2.2}$$

We observe that if equality (2.2) holds for C^1 functions, then it also holds for less regular functions, by density. In particular, using the regularity of ρ, this becomes true at least for functions $\eta \in L^2((0,T); H^1_0(\Omega)) \cap C^0([0,T]; L^2(\Omega))$ such that $\partial_t \eta \in L^2((0,T); H^{-1}(\Omega))$ (which includes, for instance, functions $\eta \in H^1((0,T) \times \Omega)$) with suitable boundary conditions. In case ρ is more regular, other test functions can also be accepted.

The next proposition gathers existence and uniqueness results for weak solutions of (2.1) as well as an energy estimate, which can be found, for instance, in [9, Section 7.1, Theorems 3 and 4] or [17, Chapter III, Theorems 2.1 and 4.2]. It also states the non-negativity of solutions of (2.1) for non-negative initial conditions, in which case the total mass is non-increasing, a result whose proof, omitted here, can be done using classical arguments (see, e.g., [17, Chapter III, Theorem 7.1] for a similar proof).

Proposition 2.2. *Let $\nu > 0$, $V \in L^\infty((0,T) \times \Omega; \mathbb{R}^d)$, and $\rho_0 \in L^2(\Omega)$. Then (2.1) admits a unique weak solution $\rho \in L^\infty((0,T); L^2(\Omega)) \cap L^2((0,T); H^1_0(\Omega))$. In addition, we have $\rho \in C^0([0,T]; L^2(\Omega))$ and $\partial_t \rho \in L^2((0,T); H^{-1}(\Omega))$ and there exists $C > 0$ depending only on d, ν, T, Ω, and on an upper bound on $\|V\|_{L^\infty((0,T)\times\Omega)}$ such that*

$$\|\rho\|_{L^\infty((0,T);L^2(\Omega))} + \|\rho\|_{L^2((0,T);H^1_0(\Omega))} + \|\partial_t \rho\|_{L^2((0,T);H^{-1}(\Omega))} \le C \|\rho_0\|_{L^2(\Omega)}$$

holds. Moreover, if $\eta \in L^2((0,T); H^1_0(\Omega)) \cap C^0([0,T]; L^2(\Omega))$ is such that $\partial_t \eta \in L^2((0,T); H^{-1}(\Omega))$ and we take $0 \le t_0 < t_1 \le T$, then (2.2) becomes

$$\begin{aligned} -\int_{t_0}^{t_1} \int_\Omega \rho \partial_t \eta \,\mathrm{d}x \,\mathrm{d}t + \int_{t_0}^{t_1} \int_\Omega (\nu \nabla \rho - \rho V) \cdot \nabla \eta \,\mathrm{d}x \,\mathrm{d}t \\ = \int_\Omega \rho_{t_0}(x) \eta(t_0,x) \,\mathrm{d}x - \int_\Omega \rho_{t_1}(x) \eta(t_1,x) \,\mathrm{d}x, \end{aligned} \tag{2.3}$$

where the values of η and ρ at times t_0 and t_1 are well-defined thanks to their continuity as curves valued into L^2.

Lastly, if $\rho_0 \ge 0$ a.e. in Ω, then $\rho \ge 0$ a.e. in $(0,T) \times \Omega$ and the mass $\int_\Omega \rho(t,x) \,\mathrm{d}x$ is a non-increasing function of t.

2.2. **Hamilton–Jacobi–Bellman equation.** We consider the non-linear Hamilton–Jacobi–Bellman equation in a finite time horizon

$$\begin{cases} -\partial_t \varphi - \nu \Delta \varphi + K |\nabla \varphi| - 1 = 0 & \text{in } (0,T) \times \Omega, \\ \varphi(T,x) = \psi(x) & \text{in } \Omega, \\ \varphi(t,x) = 0 & \text{in } [0,T] \times \partial\Omega, \end{cases} \tag{2.4}$$

where $K : (0,T) \times \Omega \to \mathbb{R}$ is a given function.

Definition 2.3. Let $\nu > 0$, $K \in L^\infty((0,T)\times\Omega;\mathbb{R})$, and $\psi \in L^2(\Omega)$. We say that $\varphi \in L^\infty((0,T);L^2(\Omega)) \cap L^2((0,T);H^1_0(\Omega))$ is a *weak solution* of (2.4) if, for every $\eta \in C^1([0,T]\times\Omega)$ such that $\eta\big|_{[0,T]\times\partial\Omega} = 0$ and $\eta\big|_{\{0\}\times\Omega} = 0$, one has

$$\int_0^T \int_\Omega \varphi\partial_t\eta + \nu\int_0^T\int_\Omega \nabla\varphi\cdot\nabla\eta + \int_0^T\int_\Omega (K|\nabla\varphi|-1)\eta = \int_\Omega \psi(x)\eta(T,x)\,\mathrm{d}x. \tag{2.5}$$

As we did after Definition 2.1, we observe that the regularity of a solution φ allows to obtain the equality (2.5) for less regular functions η, and in particular for $\eta \in L^2((0,T);H^1_0(\Omega)) \cap C^0([0,T];L^2(\Omega))$ such that $\partial_t\eta \in L^2((0,T);H^{-1}(\Omega))$.

Remark 2.4. Note that (2.4), as a Hamilton–Jacobi–Bellman equation of an optimal control problem, is backward in time: the final condition $\varphi(T,x) = \psi(x)$ is given and one solves the equation in the time interval $[0,T]$. One can apply classical results on forward PDEs to (2.4) by using the standard time reversal $t \mapsto T - t$.

Proposition 2.5. *Let $\nu > 0$, $K \in L^\infty((0,T)\times\Omega)$, and $\psi \in L^2(\Omega)$. Then, the equation* (2.4) *admits a unique weak solution $\varphi \in L^\infty((0,T);L^2(\Omega)) \cap L^2((0,T);H^1_0(\Omega))$. In addition, we have $\varphi \in C^0([0,T];L^2(\Omega))$, and there exists $C > 0$ depending only on d, ν, T, Ω, and on an upper bound on $\|K\|_{L^\infty((0,T)\times\Omega)}$ such that*

$$\|\varphi\|_{L^\infty((0,T);L^2(\Omega))} + \|\varphi\|_{L^2((0,T);H^1_0(\Omega))} \le C\left(\|\psi\|_{L^2(\Omega)} + 1\right) \tag{2.6}$$

holds. Moreover, if $\eta \in L^2((0,T);H^1_0(\Omega)) \cap C^0([0,T];L^2(\Omega))$ is such that $\partial_t\eta \in L^2((0,T);H^{-1}(\Omega))$ and we take $0 \le t_0 < t_1 \le T$, then (2.5) *becomes*

$$\begin{aligned}\int_{t_0}^{t_1}\int_\Omega (\varphi\partial_t\eta + \nu\nabla\varphi\cdot\nabla\eta + (K|\nabla\varphi|-1)\eta) \\ = \int_\Omega \varphi(t_1,x)\eta(t_1,x)\,\mathrm{d}x - \int_\Omega \varphi(t_0,x)\eta(t_0,x)\,\mathrm{d}x,\end{aligned} \tag{2.7}$$

where the values of η and of φ at times t_0 and t_1 are well-defined thanks to their continuity as curves valued into L^2.

Lastly, if $\psi \ge 0$ a.e. in Ω, then the unique solution also satisfies $\varphi \ge 0$ a.e. in $(0,T)\times\Omega$.

Even if the above result is quite standard, we prefer to give a proof for completeness, since we did not find a proof adapted to the precise setting we need neither in the stochastic control literature nor in the references about parabolic equations.

Proof. Recall that the non-homogeneous backwards heat equation with Dirichlet boundary condition

$$\begin{cases} -\partial_t\phi - \nu\Delta\phi = f & \text{in } (t_1,t_2)\times\Omega, \\ \phi(t_2,x) \text{ given} & \text{in } \Omega, \\ \phi(t,x) = 0 & \text{in } [t_1,t_2]\times\partial\Omega \end{cases} \tag{2.8}$$

on an interval $[t_1,t_2] \subset [0,T]$ and with $f \in L^1((t_1,t_2);L^2(\Omega))$ is well-posed in $L^\infty((t_1,t_2);L^2(\Omega)) \cap L^2((t_1,t_2);H^1_0(\Omega))$ in the weak sense and satisfies the energy estimate

$$\|\phi\|_{L^\infty((t_1,t_2);L^2(\Omega))} + \|\phi\|_{L^2((t_1,t_2);H^1_0(\Omega))} \le C_0\left(\|\phi(t_2,\cdot)\|_{L^2(\Omega)} + \|f\|_{L^1((t_1,t_2);L^2(\Omega))}\right) \tag{2.9}$$

with a constant $C_0 > 0$ depending only on d, ν, T, and Ω (see, e.g., [17, Chapter III, Theorems 2.1 and 4.2]).

Let $s \in [0,T)$, $\mathsf{X}_s = L^\infty((s,T);L^2(\Omega)) \cap L^2((s,T);H^1_0(\Omega))$ and $S_s : \mathsf{X}_s \to \mathsf{X}_s$ be the map that associates with each $\varphi \in \mathsf{X}_s$ the unique weak solution $\widetilde{\varphi} = S_s(\varphi) \in \mathsf{X}$ of

$$\begin{cases} -\partial_t\widetilde{\varphi} - \nu\Delta\widetilde{\varphi} = 1 - K|\nabla\varphi| & \text{in } (s,T)\times\Omega, \\ \widetilde{\varphi}(T,x) = \psi(x) & \text{in } \Omega, \\ \widetilde{\varphi}(t,x) = 0 & \text{in } [s,T]\times\partial\Omega. \end{cases}$$

Notice that S_s is well-defined for every $s \in [0,T)$ and, thanks to [17, Theorem 4.2], one has moreover that $S_s(\varphi) \in C^0([s,T];L^2(\Omega))$ for every $\varphi \in \mathsf{X}_s$. Let $\varphi_1, \varphi_2 \in \mathsf{X}_s$ and denote $\widetilde{\varphi}_i = S_s(\varphi_i)$ for $i \in \{1,2\}$. Let $\widetilde{\varphi} = \widetilde{\varphi}_1 - \widetilde{\varphi}_2$. Then $\widetilde{\varphi}$ is a weak solution of

$$\begin{cases} -\partial_t \widetilde{\varphi} - \nu\Delta\widetilde{\varphi} = K\left(|\nabla\varphi_2| - |\nabla\varphi_1|\right) & \text{in } (s,T)\times\Omega, \\ \widetilde{\varphi}(T,x) = 0 & \text{in } \Omega, \\ \widetilde{\varphi}(t,x) = 0 & \text{in } [s,T]\times\partial\Omega \end{cases}$$

and, using the energy estimate (2.9), one obtains

$$\|\widetilde{\varphi}\|_{\mathsf{X}_s} \le C_0 \|K\left(|\nabla\varphi_2| - |\nabla\varphi_1|\right)\|_{L^1((s,T);L^2(\Omega))} \le C_0\sqrt{T-s}\|K\|_{L^\infty((0,T)\times\Omega)}\|\varphi_1 - \varphi_2\|_{\mathsf{X}_s}.$$

Hence, if $s > T - C_0^{-2}\|K\|_{L^\infty}^{-2}$, then S_s is a contraction, admitting thus a unique fixed point $\varphi \in \mathsf{X}_s$, which is then the unique weak solution of (2.4) in X_s and satisfies further $\varphi \in C^0([s,T];L^2(\Omega))$. If $T < C_0^{-2}\|K\|_{L^\infty}^{-2}$, this completes the proof of the first part of the theorem, since one can take $s = 0$. Otherwise, one may decompose $[0,T]$ in intervals of length $\ell < C_0^{-2}\|K\|_{L^\infty}^{-2}$ and apply the above argument inductively to obtain existence and uniqueness of weak solutions in X_0, which belong also to $C^0([0,T];L^2(\Omega))$.

To obtain (2.6), let $\ell = \frac{1}{4}C_0^{-2}\|K\|_{L^\infty((0,T)\times\Omega)}^{-2}$, $N = \left\lceil \frac{T}{\ell} \right\rceil$, and define $s_i = i\frac{T}{N}$ for $i \in \{0,1,\dots,N\}$. Thanks to (2.9), one has, for every $i \in \{1,\dots,N\}$,

$$\begin{aligned} &\|\varphi\|_{L^\infty((s_{i-1},s_i);L^2(\Omega))} + \|\varphi\|_{L^2((s_{i-1},s_i);H_0^1(\Omega))} \\ &\qquad\qquad \le C_0\left(\|\varphi(s_i,\cdot)\|_{L^2(\Omega)} + \|K|\nabla\varphi| - 1\|_{L^1((s_{i-1},s_i);L^2(\Omega))}\right). \end{aligned}$$

Since

$$\begin{aligned} \|K|\nabla\varphi| - 1\|_{L^1((s_{i-1},s_i);L^2(\Omega))} &\le \frac{T}{N}|\Omega|^{\frac{1}{2}} + \sqrt{\frac{T}{N}}\|K\|_{L^\infty((0,T)\times\Omega)}\|\varphi\|_{L^2((s_{i-1},s_i);H_0^1(\Omega))} \\ &\le \frac{T}{N}|\Omega|^{\frac{1}{2}} + \frac{1}{2C_0}\|\varphi\|_{L^2((s_{i-1},s_i);H_0^1(\Omega))}, \end{aligned}$$

we obtain

$$\|\varphi\|_{L^\infty((s_{i-1},s_i);L^2(\Omega))} + \frac{1}{2}\|\varphi\|_{L^2((s_{i-1},s_i);H_0^1(\Omega))} \le C_0\|\varphi(s_i,\cdot)\|_{L^2(\Omega)} + C_0\frac{T}{N}|\Omega|^{\frac{1}{2}}. \tag{2.10}$$

Hence, using

$$\begin{aligned} \|\varphi\|_{L^\infty((0,T);L^2(\Omega))} &= \max_{i\in\{1,\dots,N\}}\|\varphi\|_{L^\infty((s_{i-1},s_i);L^2(\Omega))}, \\ \|\varphi\|_{L^2((0,T);H_0^1(\Omega))} &\le \sqrt{N}\max_{i\in\{1,\dots,N\}}\|\varphi\|_{L^2((s_{i-1},s_i);H_0^1(\Omega))}, \end{aligned}$$

we obtain

$$\|\varphi\|_{L^\infty((0,T);L^2(\Omega))} + \|\varphi\|_{L^2((0,T);H_0^1(\Omega))} \le 2C_0\frac{T}{\sqrt{N}}|\Omega|^{\frac{1}{2}} + 2C_0\sqrt{N}\max_{i\in\{1,\dots,N\}}\|\varphi(s_i,\cdot)\|_{L^2(\Omega)}. \tag{2.11}$$

Since $\varphi \in C^0([0,T];L^2(\Omega))$, we have, using (2.10), that, for every $i \in \{1,\dots,N\}$,

$$\|\varphi(s_{i-1},\cdot)\|_{L^2(\Omega)} \le \|\varphi\|_{L^\infty((s_{i-1},s_i);L^2(\Omega))} \le C_0\|\varphi(s_i,\cdot)\|_{L^2(\Omega)} + C_0\frac{T}{N}|\Omega|^{\frac{1}{2}},$$

and one then verifies by induction, using $\varphi(s_N,\cdot) = \psi$, that, for every $i \in \{1,\dots,N\}$, we have

$$\|\varphi(s_i,\cdot)\|_{L^2(\Omega)} \le C_0^{N-i}\|\psi\|_{L^2(\Omega)} + C_0\frac{T}{N}|\Omega|^{\frac{1}{2}}\frac{C_0^{N-i}-1}{C_0-1}.$$

Together with (2.11) and noting that N can bounded from above in terms of T, C_0, and an upper bound on $\|K\|_{L^\infty((0,T)\times\Omega)}$, this yields (2.6).

Concerning (2.7), it is immediate to see that it follows from (2.5). First, the regularity of the test function can be weakened by standard density arguments. Second, we need to handle test function on $[t_0, t_1]$ instead of $[0, T]$ and accept that they do not vanish at $t = 0$. To do this, we start from the case $t_1 = T$, and we multiply the test function by a suitable cut-off in time, equal to 0 for $t < t_0 + 1/n$ and to 1 for $t > t_0 + 1/n$, whose derivative tends to a Dirac mass at $t = t_0$. This gives rise to the extra term at $t = t_0$ in the r.h.s. of (2.7). The general case with $t_1 < T$ can now be obtained by subtracting the expressions on $[t_0, T]$ and $[t_1, T]$.

Lastly, we need to address the positivity result which closes the statement of the proposition. This result is classical for smooth solutions, and can be obtained by easy applications of the maximum principle for parabolic equations. For solutions of HJB obtained as value functions of a stochastic control problem, the result is also straightforward, as the quantity which is minimized is positive. Yet, we want to obtain the same inequality for weak solutions. This can be obtained by applying, for instance, [2, Theorem 1] to $-\varphi$, after changing time orientation and paying attention to the observation at the end of the proof (page 98) that the inequality is enough (indeed, the source term 1 in the HJB equation has the good sign to preserve positivity). □

The next result states that, if the final condition ψ is more regular than $L^2(\Omega)$, then the corresponding solution φ is also more regular.

Proposition 2.6. *Let $\nu > 0$, $K \in L^\infty((0,T)\times\Omega)$, and $\psi \in H^1_0(\Omega)$. Then the unique weak solution φ of (2.4) satisfies $\varphi \in C^0([0,T]; H^1_0(\Omega)) \cap L^2((0,T); H^2(\Omega))$ and its distributional time derivative $\partial_t\varphi$ satisfies $\partial_t\varphi \in L^2((0,T)\times\Omega)$. In addition, there exists $C > 0$ depending only on d, ν, T, Ω, and on an upper bound on $\|K\|_{L^\infty((0,T)\times\Omega)}$ such that*

$$\|\varphi\|_{L^\infty((0,T);H^1_0(\Omega))} + \|\varphi\|_{L^2((0,T);H^2(\Omega))} + \|\partial_t\varphi\|_{L^2((0,T)\times\Omega)} \le C\left(\|\psi\|_{H^1_0(\Omega)} + 1\right). \tag{2.12}$$

Proof. As in the proof of Proposition 2.5, let us consider the non-homogeneous backwards heat equation with Dirichlet boundary condition (2.8) on an interval $[t_1, t_2] \subset [0, T]$. Classical improved regularity results on this equation (see, e.g., [9, Section 7.1, Theorem 5] and [17, Chapter III, § 6, Equation (6.10) and Theorem 6.1]) show that, if $\phi(t_2, \cdot) \in H^1_0(\Omega)$ and $f \in L^2((t_1, t_2) \times \Omega)$, then the unique solution ϕ of (2.8) satisfies $\phi \in L^\infty((t_1, t_2); H^1_0(\Omega)) \cap L^2((t_1, t_2); H^2(\Omega))$ and $\partial_t\phi \in L^2((t_1, t_2) \times \Omega)$ and one has the energy estimate

$$\begin{aligned}\|\phi\|_{L^\infty((t_1,t_2);H^1_0(\Omega))} + \|\phi\|_{L^2((t_1,t_2);H^2(\Omega))} + \|\partial_t\phi\|_{L^2((t_1,t_2)\times\Omega)} \\ \le C_0\left(\|\phi(t_2,\cdot)\|_{H^1_0(\Omega)} + \|f\|_{L^2((t_1,t_2)\times\Omega)}\right),\end{aligned} \tag{2.13}$$

where C_0 depends only on d, ν, T, and Ω.

Note that (2.4) can be written as $-\partial_t\varphi - \nu\Delta\varphi = f$ with $f = 1 - K|\nabla\varphi| \in L^2((0,T) \times \Omega)$, and thus one obtains at once the improved regularity $\varphi \in L^\infty((0,T); H^1_0(\Omega)) \cap L^2((0,T); H^2(\Omega))$ and $\partial_t\varphi \in L^2((0,T)\times\Omega)$. Since $\varphi \in L^2((0,T); H^2(\Omega))$, $\partial_t\varphi \in L^2((0,T)\times\Omega)$, and $\partial\Omega$ is C^2, then, by classical results in Sobolev spaces (see, e.g., [9, Section 5.9, Theorem 3]), one obtains that φ also satisfies $\varphi \in C^0([0,T]; H^1_0(\Omega))$. The energy estimate (2.12) can be obtained from (2.13) in the same way as (2.6) was obtained from (2.9) in the proof of Proposition 2.5. □

The next proposition gives bounds for solutions of (2.4) with bounded final condition.

Proposition 2.7. *In addition to the hypotheses of Proposition 2.6, assume that $K \ge 0$, $\psi \in H^1_0(\Omega) \cap L^\infty(\Omega)$, and $\psi \ge 0$. Let φ be the unique solution of (2.4). Then there is $C > 0$ depending on ν, Ω, and $\|\psi\|_{L^\infty}$ such that*

$$0 \le \varphi(t,x) \le C, \quad \text{for } t \in (0,T),\ x \in \Omega.$$

Proof. The non-negativity of φ comes directly from Proposition 2.5. To get the upper bound, the idea is to compare the solution φ of (2.4) with a stationary supersolution. To do so, we will apply a parabolic comparison theorem. Even if this is completely standard we give a full proof of this fact, by regularization, as most of the results which can be found in the literature deal either with linear equations or with smooth solutions.

We start with defining $\Omega^+ := \Omega + B_1$, where B_1 is the open unit ball in $\mathbb{R}^d$. We define φ^+ to be equal to φ on $(0,T)\times\Omega$, and zero elsewhere. We let $(\eta_\varepsilon)_{\varepsilon>0}$ denote an approximation of the identity, that is, for every $\varepsilon > 0$, $\eta_\varepsilon \in C^\infty(\mathbb{R}^d)$, $\eta_\varepsilon \geq 0$, $\int_{\mathbb{R}^d} \eta_\varepsilon = 1$, and η_ε converges in the sense of measures to a Dirac measure centered on 0 as ε goes to zero. We assume moreover that the support of η_ε is contained in B_1.

We define

$$\varphi_\varepsilon := \varphi^+ \star \eta_\varepsilon,$$

where the convolution is only performed in space. The function φ_ε is hence C^∞ in x, but not in t. Yet, since $\varphi \in C^0([0,T]; L^2(\Omega))$, this function is at least continuous in t. It satisfies $\|\varphi_\varepsilon(T)\|_{L^\infty} \leq \|\psi\|_{L^\infty}$. We then take an approximation of the identity $(\chi_\delta)_{\delta>0}$ which acts on time and is supported on the non-positive half-line, that is, for every $\delta > 0$, $\chi_\delta \in C^\infty(\mathbb{R})$, $\chi_\delta \geq 0$, $\int_{\mathbb{R}} \chi_\delta = 1$, χ_δ converges in the sense of measures to a Dirac measure centered on 0 as δ goes to zero, and the support of χ_δ is contained in $\mathbb{R}_-$.

We now define $\varphi_{\varepsilon,\delta} := \varphi_\varepsilon \star \chi_\delta$. This function is $C^\infty((0,T)\times\Omega^+)$ and satisfies

$$-\partial_t \varphi_{\varepsilon,\delta} - \nu\Delta\varphi_{\varepsilon,\delta} \leq -\chi_\delta \star \eta_\varepsilon \star (K|\nabla\varphi^+|) + 1 \leq 1,$$

with $\varphi_{\varepsilon,\delta} = 0$ on $(0,T)\times\partial\Omega^+$ and $\varphi_{\varepsilon,\delta}(T,\cdot) \leq \|\psi\|_{L^\infty} + \omega(\delta)C(\varepsilon)$, where we have $\omega(\delta) \to 0$ as $\delta \to 0$ and $C(\varepsilon)$ is a constant depending on ε (to state it differently, for fixed ε the function $(t,x) \mapsto \varphi_\varepsilon(t,x)$ is continuous, with a modulus of continuity $\omega(\cdot)C(\varepsilon)$ depending on ε).

Now, let Φ be the solution of the torsion equation on Ω^+, that is,

$$-\nu\Delta\Phi = 1, \quad x \in \Omega^+,$$

with Dirichlet boundary conditions. Owing to standard elliptic regularity results, Φ is $C^2(\Omega^+)$. Observe that $v := \Phi + \|\psi\|_{L^\infty} + \omega(\delta)C(\varepsilon)$ also satisfies $-\nu\Delta v = 1$ and the trace of v is non-negative on $\partial\Omega^+$. Owing to the parabolic comparison principle for nonlinear equations, see [21, Theorem 9.1] for instance, we find that

$$\varphi_{\varepsilon,\delta} \leq v \leq \|\Phi\|_{L^\infty} + \|\psi\|_{L^\infty} + \omega(\delta)C(\varepsilon), \quad \text{ on } (0,T)\times\Omega^+.$$

Taking first the limit $\delta \to 0$ and then $\varepsilon \to 0$ we get the result. □

We now state a parabolic comparison principle for the Hamilton–Jacobi–Bellman equation (2.4).

Proposition 2.8. *Let φ_1, φ_2 be two solutions of (2.4) with $T < +\infty$, with final data such that $\varphi_1(T,\cdot) \geq \varphi_2(T,\cdot)$. Then*

$$\varphi_1 \geq \varphi_2, \quad \text{ on } \ (0,T)\times\Omega.$$

Proof. Let $w(t,x) := \varphi_1(t,x) - \varphi_2(t,x)$. It solves

$$-\partial_t w - \nu\Delta w + V\cdot\nabla w = 0, \quad \text{ on } \ (0,T)\times\Omega,$$

where $V := K\frac{\nabla\varphi_1 + \nabla\varphi_2}{|\nabla\varphi_1| + |\nabla\varphi_2|} \in L^\infty((0,T)\times\Omega)$ and with final datum $w(T,\cdot) \geq 0$. Then, owing to the positivity principle, see [17, Theorem 7.2, Chapter III], we find $w \geq 0$ on $(0,T)\times\Omega$, hence the result. □

Note that the above comparison principle actually holds true if φ_1 is supersolution and if φ_2 is subsolution of (2.4) as it is stated, for instance, in [21, Theorem 9.1], but only for smooth super- and sub- solutions, which explains why we decided to give a more precise reference for it, in the case of interest for us.

3. The MFG system with a finite time horizon

We now consider the MFG system with a finite time horizon (1.3). One of the difficulties in the analysis of (1.3) is that the velocity field in the continuity equation depends on $\frac{\nabla\varphi}{|\nabla\varphi|}$, which is defined only when $\nabla\varphi \neq 0$. In order to handle this difficulty, we make use of the following definition of weak solution.

Definition 3.1. Let $\nu > 0$, $T \in (0,+\infty)$, $\kappa : \mathbb{R} \to (0,+\infty)$ be continuous and bounded, $\rho_0 \in L^2(\Omega)$, and $\psi \in L^2(\Omega)$. We say that $(\rho,\varphi) \in \left[L^\infty((0,T);L^2(\Omega)) \cap L^2((0,T);H^1_0(\Omega))\right]^2$ is a *weak solution* of (1.3) with initial condition ρ_0 and final condition ψ if there exists $V \in L^\infty((0,T)\times\Omega;\mathbb{R}^d)$ such that $|V(t,x)| \leq \kappa(\rho(t,x))$ and $V(t,x)\cdot\nabla\varphi(t,x) = -\kappa(\rho(t,x))|\nabla\varphi(t,x)|$ a.e. on $(0,T)\times\Omega$ and such that ρ is a solution of the Fokker–Planck equation (2.1) with initial datum ρ_0 and vector field V on $[0,T]\times\Omega$ in the sense of Definition 2.1, and φ is a solution of the Hamilton–Jacobi–Bellman equation (2.4) with final datum ψ and $K = \kappa(\rho)$ in the sense of Definition 2.3 on the same domain.

Remark 3.2. If (ρ,φ) is a weak solution of (1.3) and V is any function satisfying the properties stated in Definition 3.1, then we have $V(t,x) = \kappa(\rho(t,x))\frac{\nabla\varphi(t,x)}{|\nabla\varphi(t,x)|}$ wherever $\nabla\varphi(t,x) \neq 0$. The introduction of the function V in Definition 3.1 has the advantages of providing a meaning to the first equation of (1.3) and handling its velocity field even when $\nabla\varphi(t,x) = 0$, which might a priori happen in a set of positive measure.

The main result of this section is the following.

Theorem 3.3. *Let $\nu > 0$, $T \in (0,+\infty)$, $\kappa : \mathbb{R} \to (0,+\infty)$ be continuous and bounded, $\rho_0 \in L^2(\Omega)$, and $\psi \in H^1_0(\Omega)$. Then there exists a weak solution (ρ,φ) of (1.3) with initial condition ρ_0 and final condition ψ.*

The proof of Theorem 3.3 relies on a fixed-point argument on the velocity field V of the Fokker–Planck equation in (1.3). Before turning to the proof, we need some continuity results on solutions of (2.1) with respect to the velocity field V and on solutions of (2.4) with respect to the function K, which we state and prove now.

Proposition 3.4. *Let $\nu > 0$ and $\rho_0 \in L^2(\Omega)$. Given $V \in L^\infty((0,T)\times\Omega;\mathbb{R}^d)$, let $(V_n)_{n\in\mathbb{N}}$ be a sequence in $L^\infty((0,T)\times\Omega;\mathbb{R}^d)$ such that $V_n \overset{*}{\rightharpoonup} V$ as $n\to\infty$. For $n\in\mathbb{N}$, let ρ_n (resp. ρ) be the unique weak solution of (2.1) in $L^\infty((0,T);L^2(\Omega)) \cap L^2((0,T);H^1_0(\Omega))$ with velocity field V_n (resp. V). Then $\rho_n \to \rho$ in $L^2((0,T)\times\Omega)$ as $n\to\infty$.*

Proof. Since $(V_n)_{n\in\mathbb{N}}$ converges weakly-$*$ to V in L^∞, there exists a constant $M > 0$ such that $\|V_n\|_{L^\infty((0,T)\times\Omega)} \leq M$ for every $n\in\mathbb{N}$ and thus, by Proposition 2.2, there exists $C > 0$ depending only on d, ν, and M such that, for every $n\in\mathbb{N}$,

$$\|\rho_n\|_{L^\infty((0,T);L^2(\Omega))} + \|\rho_n\|_{L^2((0,T);H^1_0(\Omega))} + \|\partial_t\rho_n\|_{L^2((0,T);H^{-1}(\Omega))} \leq C\|\rho_0\|_{L^2(\Omega)}. \tag{3.1}$$

It follows from (3.1) and Aubin–Lions Lemma (see, e.g., [26, Corollary 4]) that $(\rho_n)_{n\in\mathbb{N}}$ is relatively compact in $L^2((0,T)\times\Omega)$. Let $\rho^* \in L^2((0,T)\times\Omega)$ be a limit point of $(\rho_n)_{n\in\mathbb{N}}$ and $(\rho_{n_k})_{k\in\mathbb{N}}$ a subsequence of $(\rho_n)_{n\in\mathbb{N}}$ converging to ρ^* in $L^2((0,T)\times\Omega)$. Since, by (3.1), the sequence $(\rho_{n_k})_{k\in\mathbb{N}}$ is bounded in $L^\infty((0,T);L^2(\Omega))$ and in $L^2((0,T);H^1_0(\Omega))$, one also has the weak-$*$ convergence $\rho_{n_k} \overset{*}{\rightharpoonup} \rho^*$ in $L^\infty((0,T);L^2(\Omega))$ and the weak convergence $\rho_{n_k} \rightharpoonup \rho^*$ in $L^2((0,T);H^1_0(\Omega))$ as $k\to\infty$, and $\rho^* \in L^\infty((0,T);L^2(\Omega)) \cap L^2((0,T);H^1_0(\Omega))$.

We easily obtain from the above convergences that ρ^* is a weak solution of (2.1) and, by the uniqueness of such solution from Proposition 2.2, one concludes $\rho^* = \rho$. In particular, ρ is the unique limit point of the relatively compact sequence $(\rho_n)_{n\in\mathbb{N}}$ in $L^2((0,T)\times\Omega)$, which yields the result. □

Proposition 3.5. *Let $\nu > 0$ and $\psi \in H^1_0(\Omega)$. Given $K \in L^\infty((0,T)\times\Omega)$, let $(K_n)_{n\in\mathbb{N}}$ be a sequence in $L^\infty((0,T)\times\Omega)$ such that $K_n \overset{*}{\rightharpoonup} K$ as $n\to\infty$. For $n\in\mathbb{N}$, let φ_n (resp. φ) be the unique weak solution of (2.4) in $L^\infty((0,T);L^2(\Omega))\cap L^2((0,T);H^1_0(\Omega))$ with K_n (resp. K). Then $\varphi_n\to\varphi$ in $L^2((0,T);H^1_0(\Omega))$ as $n\to\infty$.*

Proof. Again, there exists a constant $M>0$ such that $\|K_n\|_{L^\infty((0,T)\times\Omega)}\le M$ for every $n\in\mathbb{N}$ and thus, by Proposition 2.6, there exists $C>0$ depending only on d, ν, T, Ω, and M such that

$$\|\varphi_n\|_{L^\infty((0,T);H^1_0(\Omega))} + \|\varphi_n\|_{L^2((0,T);H^2(\Omega))} + \|\partial_t\varphi_n\|_{L^2((0,T)\times\Omega)} \le C\left(\|\psi\|_{H^1_0(\Omega)}+1\right). \tag{3.2}$$

Hence, by Aubin–Lions Lemma (see, e.g., [26, Corollary 4]), $(\varphi_n)_{n\in\mathbb{N}}$ is relatively compact in $L^2((0,T);H^1_0(\Omega))$. Let φ^* be a limit point of $(\varphi_n)_{n\in\mathbb{N}}$ and $(\varphi_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(\varphi_n)_{n\in\mathbb{N}}$ converging to φ^* in $L^2((0,T);H^1_0(\Omega))$. By (3.2), we also have $\varphi^*\in L^\infty((0,T);H^1_0(\Omega)))$.

Now, because of the non-linearity in the equation, we prefer to provide details on how to pass it to the limit. For every k and every $\eta\in H^1((0,T)\times\Omega)$ such that $\eta\big|_{[0,T]\times\partial\Omega}=0$ and $\eta\big|_{\{0\}\times\Omega}=0$, one has

$$\int_0^T\int_\Omega \varphi_{n_k}\partial_t\eta + \nu\int_0^T\int_\Omega \nabla\varphi_{n_k}\cdot\nabla\eta + \int_0^T\int_\Omega (K_{n_k}|\nabla\varphi_{n_k}|-1)\eta = \int_\Omega \psi(x)\eta(T,x)\,\mathrm{d}x.$$

Since $K_{n_k}\overset{*}{\rightharpoonup}K$ in $L^\infty((0,T)\times\Omega)$ and $\varphi_{n_k}\to\varphi^*$ in $L^2((0,T);H^1_0(\Omega))$, one obtains, letting $k\to\infty$, that

$$\int_0^T\int_\Omega \varphi^*\partial_t\eta + \nu\int_0^T\int_\Omega \nabla\varphi^*\cdot\nabla\eta + \int_0^T\int_\Omega (K|\nabla\varphi^*|-1)\eta = \int_\Omega \psi(x)\eta(T,x)\,\mathrm{d}x.$$

Hence $\varphi^*\in L^\infty((0,T);L^2(\Omega))\cap L^2((0,T);H^1_0(\Omega))$ is a weak solution of (2.4) and, by the uniqueness of solutions of (2.4) from Proposition 2.5, one deduces that $\varphi^*=\varphi$. Thus φ is the unique limit point in $L^2((0,T);H^1_0(\Omega))$ of the relatively compact sequence $(\varphi_n)_{n\in\mathbb{N}}$, yielding the conclusion. □

We can now turn to the proof of Theorem 3.3.

Proof of Theorem 3.3. Let κ_0 be an upper bound on κ. We endow the space $L^\infty((0,T)\times\Omega;\mathbb{R}^d)$ with its weak-$*$ topology and consider the ball of radius κ_0 given by

$$\mathcal{B} = \left\{ V\in L^\infty((0,T)\times\Omega;\mathbb{R}^d) \;\middle|\; \|V\|_{L^\infty((0,T)\times\Omega;\mathbb{R}^d)}\le\kappa_0 \right\}.$$

Note that $\mathcal{B}$ is clearly convex and, by the Banach–Alaoglu theorem, $\mathcal{B}$ is a compact subset of $L^\infty((0,T)\times\Omega;\mathbb{R}^d)$.

Let $\mathcal{S}_{\mathrm{FP}}: L^\infty((0,T)\times\Omega;\mathbb{R}^d)\to L^2((0,T)\times\Omega)$ be the function that associates, with each $V\in L^\infty((0,T)\times\Omega;\mathbb{R}^d)$, the unique weak solution $\rho=\mathcal{S}_{\mathrm{FP}}(V)\in L^2((0,T)\times\Omega)$ of (2.1) with initial condition ρ_0. Note that, by Proposition 3.4, $\mathcal{S}_{\mathrm{FP}}$ is continuous with respect to the weak-$*$ topology of $L^\infty((0,T)\times\Omega;\mathbb{R}^d)$ and the strong topology of $L^2((0,T)\times\Omega)$. Similarly, we define $\mathcal{S}_{\mathrm{HJB}}: L^\infty((0,T)\times\Omega)\to L^2((0,T);H^1_0(\Omega))$ as the function that associates, with each $K\in L^\infty((0,T)\times\Omega)$, the unique weak solution $\varphi=\mathcal{S}_{\mathrm{HJB}}(K)\in L^2((0,T);H^1_0(\Omega))$ of (2.4) with terminal condition ψ. Proposition 3.5 ensures that $\mathcal{S}_{\mathrm{HJB}}$ is continuous with respect to the weak-$*$ topology of $L^\infty((0,T)\times\Omega)$ and the strong topology of $L^2((0,T);H^1_0(\Omega))$.

We define the set-valued map $\mathcal{V}$ that, with each $V \in \mathcal{B}$, associates the set $\mathcal{V}(V) \subset \mathcal{B}$ given by

$$\mathcal{V}(V) = \Big\{ \widetilde{V} \in \mathcal{B} \;\Big|\; |\widetilde{V}(t,x)| \le \kappa(\rho(t,x)) \text{ for a.e. } (t,x) \in (0,T)\times\Omega, \\ \widetilde{V}(t,x)\cdot\nabla\varphi(t,x) = -\kappa(\rho(t,x))|\nabla\varphi(t,x)| \text{ for a.e. } (t,x)\in(0,T)\times\Omega, \\ \text{where } \rho = \mathcal{S}_{\mathrm{FP}}(V) \text{ and } \varphi = \mathcal{S}_{\mathrm{HJB}}(\kappa\circ\rho)\Big\}.$$

In order to prove the existence of a weak solution (ρ,φ) of (1.3), we first prove the existence of a fixed point of the set-valued map $\mathcal{V}$, i.e., of a $V \in L^\infty((0,T)\times\Omega;\mathbb{R}^d)$ such that $V \in \mathcal{V}(V)$. This is done by applying Kakutani's fixed point theorem (see, e.g., [11, §7, Theorem 8.6]) to the set-valued map $\mathcal{V}$. To do so, we first need to verify some properties of $\mathcal{V}$ and its graph $\mathcal{G}$ defined by

$$\mathcal{G} = \Big\{ (V,\widetilde{V}) \in \mathcal{B}\times\mathcal{B} \;\Big|\; \widetilde{V} \in \mathcal{V}(V) \Big\}.$$

Claim 1. *For every $V \in L^\infty((0,T)\times\Omega;\mathbb{R}^d)$, the set $\mathcal{V}(V)$ is non-empty and convex.*

Proof. It is immediate to verify that $\mathcal{V}(V)$ is convex. To prove that it is non-empty, let $V \in L^\infty((0,T)\times\Omega;\mathbb{R}^d)$, $\rho = \mathcal{S}_{\mathrm{FP}}(V)$, and $\varphi = \mathcal{S}_{\mathrm{HJB}}(\kappa\circ\rho)$. Then, the function $\widetilde{V} \in L^\infty((0,T)\times\Omega;\mathbb{R}^d)$ defined for a.e. $(t,x)\in(0,T)\times\Omega$ by

$$\widetilde{V}(t,x) = \begin{cases} -\kappa(\rho(t,x))\dfrac{\nabla\varphi(t,x)}{|\nabla\varphi(t,x)|} & \text{if } \nabla\varphi(t,x)\neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

clearly satisfies $\widetilde{V} \in \mathcal{V}(V)$. □

Claim 2. *The graph $\mathcal{G}$ is a closed subset of $\mathcal{B}\times\mathcal{B}$.*

Proof. Let $(V_n,\widetilde{V}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{G}$ converging weakly-$*$ in $\mathcal{B}\times\mathcal{B}$ to a point $(V,\widetilde{V})$. We want to prove $(V,\widetilde{V})\in\mathcal{G}$, i.e., $\widetilde{V}\in\mathcal{V}(V)$.

Define, for $n\in\mathbb{N}$, the functions $\rho_n \in L^2((0,T)\times\Omega)$ and $\varphi_n \in L^2((0,T);H^1_0(\Omega))$ by $\rho_n = \mathcal{S}_{\mathrm{FP}}(V_n)$ and $\varphi_n = \mathcal{S}_{\mathrm{HJB}}(\kappa\circ\rho_n)$ and, similarly, let $\rho = \mathcal{S}_{\mathrm{FP}}(V)$ and $\varphi = \mathcal{S}_{\mathrm{HJB}}(\kappa\circ\rho)$. Since $\mathcal{S}_{\mathrm{FP}} : L^\infty((0,T)\times\Omega;\mathbb{R}^d) \to L^2((0,T)\times\Omega)$ is continuous with respect to the weak-$*$ topology of $L^\infty((0,T)\times\Omega;\mathbb{R}^d)$ and the strong topology of $L^2((0,T)\times\Omega)$, one deduces $\rho_n\to\rho$ in $L^2((0,T)\times\Omega)$ as $n\to\infty$. Hence, up to extracting subsequences (which we still denote using the same notation for simplicity), one has $\rho_n\to\rho$ a.e. in $(0,T)\times\Omega$. Since κ is continuous, we deduce $\kappa\circ\rho_n\to\kappa\circ\rho$ a.e. in $(0,T)\times\Omega$, and it follows $\kappa\circ\rho_n \overset{*}{\rightharpoonup} \kappa\circ\rho$ in $L^\infty((0,T)\times\Omega)$. The continuity of $\mathcal{S}_{\mathrm{HJB}} : L^\infty((0,T)\times\Omega;\mathbb{R}^d) \to L^2((0,T);H^1_0(\Omega))$ with respect to the weak-$*$ topology of $L^\infty((0,T)\times\Omega;\mathbb{R}^d)$ and the strong topology of $L^2((0,T);H^1_0(\Omega))$ implies $\varphi_n\to\varphi$ in $L^2((0,T);H^1_0(\Omega))$ as $n\to\infty$.

From the weak convergence of $\widetilde{V}_n$ to $\widetilde{V}$, the convexity of the function $|\cdot|$, and the (strong) convergence of $\kappa(\rho_n)$ to $\kappa(\rho)$, the inequality $\left|\widetilde{V}_n\right| \le \kappa(\rho_n)$ gives at the limit

$$\left|\widetilde{V}(t,x)\right| \le \kappa(\rho(t,x)) \qquad \text{for a.e. } (t,x)\in(0,T)\times\Omega. \tag{3.3}$$

Since $\widetilde{V}_n\in\mathcal{V}(V_n)$ for every $n\in\mathbb{N}$, we have $\widetilde{V}_n(t,x)\cdot\nabla\varphi_n(t,x) = -\kappa(\rho_n(t,x))|\nabla\varphi_n(t,x)|$ for a.e. $(t,x)\in(0,T)\times\Omega$. Then, for every $v\in L^2((0,T)\times\Omega)$, one has

$$\int_0^T\int_\Omega \widetilde{V}_n(t,x)\cdot\nabla\varphi_n(t,x)v(t,x)\,\mathrm{d}x\,\mathrm{d}t = -\int_0^T\int_\Omega \kappa(\rho_n(t,x))|\nabla\varphi_n(t,x)|v(t,x)\,\mathrm{d}x\,\mathrm{d}t.$$

Recalling that, as $n\to\infty$, one has $V_n \overset{*}{\rightharpoonup} V$ in $L^\infty((0,T)\times\Omega)$, $\nabla\varphi_n\to\nabla\varphi$ in $L^2((0,T)\times\Omega)$, and $\kappa\circ\rho_n \overset{*}{\rightharpoonup} \kappa\circ\rho$ in $L^\infty((0,T)\times\Omega)$, we obtain, letting $n\to\infty$, that

$$\int_0^T\int_\Omega \widetilde{V}(t,x)\cdot\nabla\varphi(t,x)v(t,x)\,\mathrm{d}x\,\mathrm{d}t = -\int_0^T\int_\Omega \kappa(\rho(t,x))|\nabla\varphi(t,x)|v(t,x)\,\mathrm{d}x\,\mathrm{d}t$$

for every $v \in L^2((0,T)\times\Omega)$, which implies that

$$\widetilde{V}(t,x)\cdot\nabla\varphi(t,x) = -\kappa(\rho(t,x))|\nabla\varphi(t,x)| \qquad \text{for a.e. } (t,x)\in(0,T)\times\Omega. \tag{3.4}$$

Combining (3.3) and (3.4), we conclude that $\widetilde{V}\in\mathcal{V}(V)$, as required. □

Claim 3. *For every $V\in L^\infty((0,T)\times\Omega;\mathbb{R}^d)$, the set $\mathcal{V}(V)$ is compact.*

Proof. This is a consequence of the fact that $\mathcal{G}$ is a closed subset of the compact set $\mathcal{B}\times\mathcal{B}$. □

Thanks to Claims 2 and 3, it follows from [3, Proposition 1.4.8] that the set-valued map $\mathcal{V}$ is upper semi-continuous. Using this fact and Claims 1 and 3, it follows from Kakutani's fixed point theorem that $\mathcal{V}$ admits a fixed point $V\in\mathcal{B}$. Let $\rho = \mathcal{S}_{\mathrm{FP}}(V)$ and $\varphi = \mathcal{S}_{\mathrm{HJB}}(\kappa\circ\rho)$. Using the facts that ρ and φ are solutions of (2.1) and (2.4), respectively, and that $V\in\mathcal{V}(V)$, it is immediate to verify, using Definitions 2.1, 2.3, and 3.1, that (ρ,φ) is a weak solution of (1.3) with initial condition ρ_0 and final condition ψ, as required. □

4. The MFG system with an infinite time horizon

Now that we have established in Section 3 the existence of solutions to the Mean Field Game system (1.3) in a finite time horizon T, we consider in this section the Mean Field Game system (1.2) with an infinite time horizon. Let us first provide the definition of a weak solution in this setting.

Definition 4.1. Let $\nu>0$, $\kappa:\mathbb{R}\to(0,+\infty)$ be continuous and bounded, and $\rho_0\in L^2(\Omega)$. We say that $(\rho,\varphi)\in\left[L^\infty_{\mathrm{loc}}(\mathbb{R}_+;L^2(\Omega))\cap L^2_{\mathrm{loc}}(\mathbb{R}_+;H^1_0(\Omega))\right]^2$ is a *weak solution* of (1.2) with initial condition ρ_0 if $\varphi\in L^\infty(\mathbb{R}_+\times\Omega)$ and if there exists $V\in L^\infty(\mathbb{R}_+\times\Omega;\mathbb{R}^d)$ such that $|V(t,x)|\le\kappa(\rho(t,x))$ and $V(t,x)\cdot\nabla\varphi(t,x) = -\kappa(\rho(t,x))|\nabla\varphi(t,x)|$ a.e. on $\mathbb{R}_+\times\Omega$ and such that, for every $T>0$, ρ is a solution of the Fokker–Planck equation (2.1) with initial datum ρ_0 and vector field V on $[0,T]\times\Omega$ in the sense of Definition 2.1 and φ is a solution of the Hamilton–Jacobi–Bellman equation (2.4) with $K=\kappa(\rho)$ in the sense of Definition 2.3 on the same domain[1].

Notice that, with respect to Definition 3.1, we make the additional requirement that $\varphi\in L^\infty(\mathbb{R}_+\times\Omega)$. This is done mainly for three reasons. Firstly, boundedness of the solution of a Hamilton–Jacobi–Bellman equation is a condition usually required in order to ensure that this solution is the value function of an optimal control problem (see, e.g., [5, Theorem 8.1.10] and [10, Chapter II, Corollary 9.1]). Secondly, the strategy we use in this section to prove existence of a solution of (1.2), based on a limit argument from solutions of (1.3) in finite time horizon T as $T\to+\infty$, allows us to ensure that the function $\varphi:\mathbb{R}_+\times\Omega\to\mathbb{R}$ we construct is indeed bounded. Finally, boundedness of φ is an important property in order to establish the results on the the asymptotic behavior of solutions to (1.2) provided in Theorem 4.2 and Propositions 4.5 and 4.6.

4.1. Existence of solutions and their asymptotic behavior.

From now on, we let Ψ denote the solution of the (stationary) Hamilton–Jacobi–Bellman equation

$$-\nu\Delta\Psi + \kappa(0)|\nabla\Psi| = 1, \quad x\in\Omega,$$

with Dirichlet boundary conditions $\Psi=0$ on $\partial\Omega$. Owing to standard results on elliptic equations, Ψ is unique, and it is C^2 and positive in Ω.

The main result of this section is the following.

[1]Note that Definition 2.3 requires to fix a final value, and we did not define the notion of solution independently of the final value ψ. This could be formalized as "there exists $\psi\in L^2(\Omega)$ such that φ is a solution of (2.4)". Yet, since the function φ will be finally continuous as a function valued into $L^2(\Omega)$, the final datum on $[0,T]$ will be necessarily given by its own value $\varphi(T,\cdot)$.

Theorem 4.2. *Let $\rho_0 \in L^2$. Then, there exists at least one solution (ρ, φ) to the Mean Field Game system with infinite time horizon* (1.2).

In addition, any such solution satisfies

$$\rho_t \underset{t\to+\infty}{\longrightarrow} 0, \quad \varphi_t \underset{t\to+\infty}{\longrightarrow} \Psi,$$

and the above convergences hold uniformly.

The sequel of this section is devoted to the proof of Theorem 4.2. Let us start by giving an idea of the proof. First, we will construct solutions to the problem with infinite time horizon as limits of solutions of the problem with finite time horizon T by letting T go to $+\infty$. Then, to prove the long-time uniform convergence of the solutions, we shall make a crucial use of some regularity results for parabolic equations. More precisely, we will use local maximum principles for Fokker–Planck and for (forward) Hamilton–Jacobi–Bellman equations; roughly speaking, these results state that the $L^\infty(\Omega)$ norm of solutions of such equations at some time t_2 is controlled by some L^p norms of the same solution at some previous time $t_1 < t_2$. The results we use are proved in Appendix A, see Proposition A.1 and Corollaries A.2 and A.3.

We start with a lemma that gathers some useful estimates. These estimates have already been discussed in Section 2, but we need now to track possible dependencies of the constant on the time horizon T.

Lemma 4.3. *Let (ρ, φ) be solution of the finite horizon MFG system* (1.3) *on $[0,T]\times\Omega$ in the sense of Definition 3.1, with final datum $\psi \in H^1_0(\Omega)\cap L^\infty(\Omega)$ with $\psi \geq 0$. Then, there are $C_1, C_2 > 0$, depending on $\|\psi\|_{L^\infty} + \|\psi\|_{H^1_0}$, $\sup\kappa$, ν, Ψ and Ω such that*

$$\|\nabla\varphi(t,\cdot)\|_{L^2} \leq C_1, \quad \text{for all } \ t\in[0,T], \tag{4.1}$$

and

$$\|\varphi\|_{L^2((T_1,T_2);H^2)} \leq C_2(1+|T_2-T_1|). \tag{4.2}$$

Proof. Step 1. Two preliminary estimates.

Let us start with proving some bounds on the Laplacian and on the gradient of φ. First, multiplying by φ the equation satisfied by φ and integrating on Ω for a fixed $t \in (0,T)$, we find

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_\Omega \varphi^2\right) = -\nu\int_\Omega |\nabla\varphi|^2 - \int_\Omega \kappa(\rho)|\nabla\varphi|\varphi + \int_\Omega \varphi.$$

Therefore, since φ is bounded, we have

$$\int_{T_1}^{T_2}\int_\Omega |\nabla\varphi|^2 \leq C(1+|T_2-T_1|), \tag{4.3}$$

for some $C > 0$ depending on $\sup\varphi$, $|\Omega|$, $\sup\kappa$, ν and for every $T_1, T_2 \in [0,T]$ with $0 \leq T_1 \leq T_2 \leq T$. Note that $\sup\varphi$ is bounded in terms of ν, Ω, and $\|\psi\|_{L^\infty}$, as shown in Proposition 2.7.

Step 2. Bound on $\|\nabla\varphi(t,\cdot)\|_{L^2}$.

We define, for $t\in[0,T]$,

$$u(t) := \frac{1}{2}\int_\Omega |\nabla\varphi(t,x)|^2\,\mathrm{d}x.$$

We differentiate u to obtain

$$\dot{u}(t) = \nu\int_\Omega (\Delta\varphi)^2 - \int_\Omega \kappa(\rho)|\nabla\varphi|\Delta\varphi + \int_\Omega \Delta\varphi. \tag{4.4}$$

Using Young's inequality, we find that there are $K_1, K_2 > 0$ depending only on $|\Omega|$, $\sup\kappa$, ν such that

$$\dot{u}(t) + K_1 u(t) + K_2 \geq 0.$$

This implies, for any $0 \leq t < s \leq T$,

$$u(s) + \frac{K_2}{K_1}\left(1 - e^{-K_1(s-t)}\right) \geq u(t)e^{-K_1(s-t)}. \tag{4.5}$$

We integrate (4.5) for $s \in (t, t+1)$ to get

$$\frac{1}{2}\int_t^{t+1}\int_\Omega |\nabla\varphi|^2(s,x)\,\mathrm{d}x\,\mathrm{d}s + \frac{K_2}{K_1}\int_0^1 (1 - e^{-K_1 r})\,\mathrm{d}r \geq \frac{1}{2}\left(\int_0^1 e^{-K_1 r}\,\mathrm{d}r\right)\int_\Omega |\nabla\varphi|^2(t,x)\,\mathrm{d}x.$$

Using (4.3) yields the $L^\infty(H^1)$ bound (4.1) for $t \in [0, T-1)$. To get the $L^\infty(H^1)$ bound (4.1) for $t \in [T-1, T]$, we use (4.5) with $s = T$. The result follows, with a constant also depending on $u(T) = \frac{1}{2}\int_\Omega |\nabla\psi|^2 < +\infty$.

Step 3. Bound in $L^2((T_1, T_2); H^2)$.

Let us integrate (4.4) on (T_1, T_2). We find

$$\nu\int_{T_1}^{T_2}\int_\Omega (\Delta\varphi)^2 = u(T_2) - u(T_1) + \int_{T_1}^{T_2}\int_\Omega \kappa(\rho)|\nabla\varphi|\Delta\varphi - \int_{T_1}^{T_2}\int_\Omega \Delta\varphi.$$

Using Young's inequality on $\int_{T_1}^{T_2}\int_\Omega \kappa(\rho)|\nabla\varphi|\Delta\varphi$ and $\int_{T_1}^{T_2}\int_\Omega \Delta\varphi$ and the estimate (4.3), we get the desired bound (4.2) on $L^2((T_1, T_2); H^2)$. □

The next lemma shows that the time derivative of $\int_\Omega \rho\varphi$ is equal to $-\int_\Omega \rho$. Differentiating the average value of the value function is a classical computation in Mean Field Game theory. Since here the value function is an exit time, it is expected that it should decrease with rate 1, and one can guess the result from the fact that the total mass of the agents in this model is not fixed but decreases in time and is equal to $\int_\Omega \rho$.

Lemma 4.4. *Let (ρ, φ) be a solution of the finite-horizon MFG (1.3) on $[0,T] \times \Omega$ in the sense of Definition 3.1. Then, for a.e. t, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \rho(t,x)\varphi(t,x)\,\mathrm{d}x\right) = -\int_\Omega \rho(t,x)\,\mathrm{d}x.$$

Proof. We can use φ as a test function in (2.3) and ρ in (2.7) since both φ and ρ are continuous as curves valued in L^2, belong to $L^2((0,T); H_0^1(\Omega))$, and their time-derivatives belong to $L^2((0,T); H^{-1}(\Omega))$. We subtract the two equalities that we obtain, which provides

$$\begin{aligned}\int_{t_0}^{t_1}\int_\Omega \rho\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_{t_0}^{t_1}\int_\Omega (\nu\nabla\rho - \rho V)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t \\ + \int_{t_0}^{t_1}\int_\Omega (\varphi\partial_t\rho + \nu\nabla\varphi\cdot\nabla\rho + (\kappa(\rho)|\nabla\varphi| - 1)\rho) \\ = 2\int_\Omega \varphi(t_1,x)\rho(t_1,x)\,\mathrm{d}x - 2\int_\Omega \varphi(t_0,x)\rho(t_0,x)\,\mathrm{d}x.\end{aligned}$$

After canceling the terms with $\nabla\rho\cdot\nabla\varphi$ and using $V\cdot\nabla\varphi + \kappa(\rho)|\nabla\varphi| = 0$ we are left with

$$\begin{aligned}\int_{t_0}^{t_1}\int_\Omega (\rho\partial_t\varphi + \varphi\partial_t\rho)\,\mathrm{d}x\,\mathrm{d}t - \int_{t_0}^{t_1}\int_\Omega \rho\,\mathrm{d}x\,\mathrm{d}t \\ = 2\int_\Omega \varphi(t_1,x)\rho(t_1,x)\,\mathrm{d}x - 2\int_\Omega \varphi(t_0,x)\rho(t_0,x)\,\mathrm{d}x.\end{aligned}$$

It is then easy to see, by approximation via smooth functions, that for every pair (ρ, φ) such that $\rho, \varphi \in L^2((0,T); H^1(\Omega))$ and $\partial_t\rho, \partial_t\varphi \in L^2((0,T); H^{-1}(\Omega))$, we have

$$\int_{t_0}^{t_1}\int_\Omega (\rho\partial_t\varphi + \varphi\partial_t\rho)\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \varphi(t_1,x)\rho(t_1,x)\,\mathrm{d}x - \int_\Omega \varphi(t_0,x)\rho(t_0,x)\,\mathrm{d}x.$$

We are then left with

$$\int_\Omega \varphi(t_1,x)\rho(t_1,x)\,\mathrm{d}x - \int_\Omega \varphi(t_0,x)\rho(t_0,x)\,\mathrm{d}x = -\int_{t_0}^{t_1}\int_\Omega \rho\,\mathrm{d}x\,\mathrm{d}t,$$

which is equivalent to the claim. □

We are now in position to prove Theorem 4.2.

Proof of Theorem 4.2. Let $\rho_0 \in L^2(\Omega)$ be fixed.

Step 1. Existence.

For $T > 0$, we let (ρ^T, φ^T) denote a solution of (1.3) with $T > 0$, with initial datum ρ_0 for ρ and with final datum ψ^T for φ, where $(\psi^T)_{T>0}$ is any family of non-negative functions, bounded in $L^\infty(\Omega) \cap H^1_0(\Omega)$.

Recall that, by Proposition 2.7, $\|\varphi^T\|_{L^\infty((0,T)\times\Omega)}$ is bounded independently of T. Let $0 < T_1 < T_2$ be fixed. Lemma 4.3 implies that, as soon as $T > T_2$, φ^T is bounded in $L^2((T_1, T_2); H^2(\Omega))$ independently of $T > 0$. Moreover, because $\partial_t \varphi^T \in L^2((T_1, T_2) \times \Omega)$ owing to Proposition 2.6, we can apply Aubin–Lions Lemma to the sequence $(\varphi^T)_{T>0}$ to get that, up to extraction, it converges strongly in $L^2_{loc}((0, +\infty); H^1(\Omega))$ to some limit φ_∞. Up to another extraction, we ensure that the convergence of $\varphi^T, \nabla\varphi^T$ also holds pointwise.

Using Aubin–Lions Lemma for the sequence $(\rho^T)_{T>0}$ as in the proof of Proposition 3.4, we find that, up to another extraction, it converges strongly to a limit ρ_∞ in $L^2((T_1, T_2)\times\Omega)$ and weakly in $L^2((T_1, T_2); H^1_0(\Omega))$. The solutions (ρ^T, φ^T) are associated with a bounded vector field V_T, which will converge weakly-$*$ in L^∞ to a vector field V_∞. Using the same arguments as in the proof of Theorem 3.3, Claim 2, we can pass to the limit $T \to +\infty$ in the equation to find that the pair $(\rho_\infty, \varphi_\infty)$ solves (1.2).

Step 2. Long-time behavior of ρ.

Let (ρ, φ) be a solution of (1.2), as built in the previous step. The integral version of Lemma 4.4, which is valid for (ρ^T, φ^T), also applies to (ρ, φ) at the limit, and we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho(t,x)\varphi(t,x)\,\mathrm{d}x \le \frac{-1}{\sup\varphi} \left(\int_\Omega \rho(t,x)\varphi(t,x)\,\mathrm{d}x \right),$$

hence, for all $t \ge 0$, we have

$$\int_\Omega \rho(t,x)\varphi(t,x)\,\mathrm{d}x \le \left(\int_\Omega \rho_0(x)\varphi(0,x)\,\mathrm{d}x \right) e^{-\frac{1}{\sup\varphi}t}.$$

Moreover, using the fact that $t \mapsto \int_\Omega \rho(t,x)\,\mathrm{d}x$ is non-increasing, we get, integrating the relation from Lemma 4.4,

$$\int_\Omega \rho(t,x)\,\mathrm{d}x \le \int_{t-1}^t \int_\Omega \rho(\tau,x)\,\mathrm{d}x d\tau \le \int_\Omega \rho(t-1,x)\varphi(t-1,x)\,\mathrm{d}x,$$

from which we get that there are $\alpha, \beta > 0$ such that

$$\int_\Omega \rho(t,x)\,\mathrm{d}x \le \beta e^{-\alpha t}.$$

Now, denote $u(t) := \int_\Omega \rho^2(t,x)\,\mathrm{d}x$. We have

$$\dot{u}(t) = -2\nu \int_\Omega |\nabla\rho|^2 - 2\int_\Omega \rho V \cdot \nabla\rho,$$

and, using Young's inequality, we get that there is $\delta > 0$ (depending on $\sup\kappa$ and ν) such that

$$\dot{u} - 2\delta u \le 0.$$

Hence

$$\int_\Omega \rho^2(t,x)\,\mathrm{d}x \le \left(\int_\Omega \rho_0^2\,\mathrm{d}x \right) e^{2\delta t}.$$

Now, let $\theta \in (0,1)$ be close enough to 1 so that $\alpha\theta > \delta(1-\theta)$. Let $p_\theta := \theta + 2(1-\theta) > 1$. By classical interpolation arguments on L^p spaces, one has

$$\|\rho(t,\cdot)\|_{L^{p_\theta}} \le \|\rho(t,\cdot)\|^\theta_{L^1} \|\rho(t,\cdot)\|^{1-\theta}_{L^2} \le A e^{-(\alpha\theta - \delta(1-\theta))t},$$

where $A = \beta^\theta \|\rho_0\|^{1-\theta}_{L^2}$. Now that we have that the L^{p_θ} norm of $\rho(t,\cdot)$ goes to zero as t goes to $+\infty$, Corollary A.2 gives us that the L^∞ norm of $\rho(t,\cdot)$ also goes to zero when t goes to $+\infty$.

Step 3. Long-time behavior of φ.

We now turn to the convergence of φ as $t \to +\infty$. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers diverging to $+\infty$. Define

$$\varphi_n(t,x) := \varphi(t+t_n,x).$$

Then, φ_n solves

$$-\partial_t\varphi_n - \nu\Delta\varphi_n + \kappa(\rho(t+t_n,x))|\nabla\varphi_n| - 1 = 0, \quad t > -t_n,\ x \in \Omega.$$

Using the same estimates as in the first step, we find that, up to a subsequence, φ_n converges to some $\overline{\varphi}(t,x)$ in the $L^2_{loc}(H^1)$ sense, that satisfies

$$-\partial_t\overline{\varphi} - \nu\Delta\overline{\varphi} + \kappa(0)|\nabla\overline{\varphi}| - 1 = 0, \quad t \in \mathbb{R},\ x \in \Omega,$$

where we have used the uniform convergence $\rho(t,\cdot) \to 0$ as $t \to +\infty$ from the previous step in order to get the convergence of $\kappa(\rho(t+t_n,x))$ to $\kappa(0)$ as $n \to +\infty$. We now want to prove $\overline{\varphi} = \Psi$. From the boundedness of φ, the function $\overline{\varphi}$ is also bounded.

Let $T > 0$ be fixed. Let u_T, v_T be the solutions of

$$-\partial_t u - \nu\Delta u + \kappa(0)|\nabla u| - 1 = 0, \quad t \in (0,T),\ x \in \Omega, \tag{4.6}$$

with Dirichlet boundary conditions and final data $u_T(T,\cdot) = 0$ and $v_T(T,\cdot) = \Phi|_\Omega + M$, where $M \geq \overline{\varphi}$ and $\Phi|_\Omega \geq 0$ is the restriction to Ω of the solution of the torsion equation $-\nu\Delta\Phi = 1$ in Ω^+ (with Ω^+ a domain that contains Ω, say $\Omega^+ := \Omega + B_1$) with Dirichlet boundary conditions. We recall that the existence of u_T, v_T is guaranteed by Proposition 2.5.

The parabolic comparison principle, Proposition 2.8, implies that, for every $T > 0$,

$$u_T(t,\cdot) \leq \overline{\varphi}(t,\cdot) \leq v_T(t,\cdot), \quad \text{for } \ t \in (0,T).$$

Let us prove that u_T, v_T converge to Ψ, the stationary solution of (4.6), as T goes to $+\infty$. To get this, let us show that the sequences of functions $(u_T)_{T>0}$ and $(v_T)_{T>0}$ are non-decreasing and non-increasing respectively, in the sense that $u_T \leq u_{T+h}$ and $v_T \geq v_{T+h}$ on $(0,T)\times\Omega$ for every $h \in (0,T)$.

Let $T > 0$ be fixed and let $h \in (0,T)$. Because (4.6) is autonomous, u_{T+h} and u_T are both solutions of (4.6) on $(0,T)\times\Omega$, with final data $u_{T+h}(T,\cdot)$ and $u_T(T,\cdot) = 0$ respectively. However, because $u_{T+h}(t,\cdot) \geq 0$ for $t \in (0,T+h)$ (as recalled in Proposition 2.5), we have $u_{T+h}(T,\cdot) \geq u_T(T,\cdot)$. To phrase it differently, u_{T+h} and u_T are solutions of the same equation with ordered final data, hence, we can apply the comparison principle Proposition 2.8 to find that $u_{T+h} \geq u_T$ on $(0,T)\times\Omega$.

Similarly , we have that v_{T+h} and v_T solve (4.6) on $(0,T)\times\Omega$, with final data $v_{T+h}(T,\cdot)$ and $v_T(T,\cdot) = \Phi|_\Omega + M$. However, arguing as in the proof of Proposition 2.7, we have that $v_{T+h} \leq \Phi|_\Omega + M$. Therefore, we can apply the parabolic comparison principle Proposition 2.8 to get that $v_{T+h} \leq v_T$ on $(0,T)\times\Omega$.

Therefore, owing to theses monotonicities, the sequences $(u_T)_{T>0}$ and $(v_T)_{T>0}$ converge a.e. as T goes to $+\infty$ to functions that do not depend on the t variable (this last fact comes from the equality $u_T(\cdot,\cdot) = u_{T+h}(\cdot+h,\cdot)$, which is true because (4.6) is autonomous and because the solutions are unique). Moreover, arguing as in the first step, we have that these limiting functions are solutions of (4.6). The only stationary solution of (4.6) being Ψ, we get that $\overline{\varphi}(t,\cdot) = \Psi$ for every t. We have thus proven that

$$w_n(t,x) := \varphi(t+t_n,x) - \Psi(x) \underset{n\to+\infty}{\longrightarrow} 0,$$

in the $L^2_{loc}(H^1)$ sense.

Let us prove that this convergence is actually uniform. To this aim, observe that w_n is a weak solution of

$$-\partial_t w_n = \nu\Delta w_n - \kappa(\rho(\cdot+t_n,\cdot))z_n\cdot\nabla w_n + (\kappa(0) - \kappa(\rho(\cdot+t_n,\cdot)))|\nabla\Psi|,$$

where $z_n := \frac{\nabla\varphi_n+\nabla\Psi}{|\nabla\varphi_n|+|\nabla\Psi|}$ is bounded. Then, for every t_1, t_2 such that $t_2+1 < t_1 < t_2+2$, using Corollary A.3, we find that

$$\|w_n(t_2,\cdot)\|_{L^\infty} \le C\left(\|w_n(t_1,\cdot)\|_{L^2} + \|(\kappa(0)-\kappa(\rho(\cdot+t_n,\cdot)))|\nabla\Psi|\|_{L^\infty((t_1,t_2)\times\Omega))}\right).$$

Integrating this for $t_1 \in (t_2+1, t_2+2)$, we find

$$\|w_n(t_2,\cdot)\|_{L^\infty} \le C\left(\|w_n\|_{L^2((t_2+1,t_2+2)\times\Omega)} + \|(\kappa(0)-\kappa(\rho(\cdot+t_n,\cdot)))|\nabla\Psi|\|_{L^\infty((t_1,t_2)\times\Omega))}\right).$$

Because w_n goes to zero in the $L^2_{loc}(H^1)$ sense and $|\nabla\Psi|$ is bounded, observing that $|\kappa(\rho(\cdot+t_n,\cdot))-\kappa(0)|$ converges uniformly to zero (this comes from the uniform convergence to zero of ρ from Step 2), we obtain that w_n goes to zero uniformly, whence

$$\varphi(t,x) \underset{t\to+\infty}{\longrightarrow} \Psi(x)$$

in the L^∞ sense. □

4.2. **Improved convergence results.** In the previous section, Theorem 4.2 proved the existence of solutions (ρ,φ) to the MFG system with infinite time horizon (1.2) and characterized the asymptotic behavior of any such solution by providing uniform convergence $\rho_t \to 0$ and $\varphi_t \to \Psi$. We want here to improve this result in two ways: first, we will prove that this convergence is actually exponential (in what concerns φ this requires a very small extra assumption on the function κ); second, we will prove that the convergence of $\varphi(t,\cdot)$ to Ψ as $t \to +\infty$, in addition to being uniform, is also a strong convergence in $H^1_0(\Omega)$. This last result is natural to evoke, because of the role played by $\nabla\varphi$ in the dynamics.

Proposition 4.5. *Suppose that the function $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ is Hölder continuous. Then, there exist constants $C, \alpha > 0$ (depending on κ, ν, and Ω), such that we have, for any $(t,x) \in [0,+\infty)\times\Omega$,*

$$|\rho(t,x)| + |\varphi(t,x)-\Psi(x)| \le Ce^{-\alpha t}.$$

Proof. The exponential convergence of ρ to 0 is indeed part of the proof of Theorem 4.2, since we proved that, for p close to 1, the L^p norm of ρ_t tends exponentially to 0, and we then used the parabolic regularization estimate $\|\rho_t\|_{L^\infty} \le C\|\rho_{t-1}\|_{L^p}$.

Thanks to the assumption that κ is Hölder continuous, up to modifying the coefficient in the exponent, we obtain $|K(t,x)-\kappa(0)| \le Ce^{-\alpha t}$, where $K(t,x) = \kappa(\rho(t,x))$.

We need now to discuss the exponential convergence of φ. Let us fix a time t_1 and define

$$a_\pm := 1 \pm 3C\|\nabla\psi\|_{L^\infty}e^{-\alpha t_1}, \quad \Psi_\pm := a_\pm\Psi \pm e^{-\alpha t_1}.$$

We will use a comparison principle between φ and $\Psi_\pm$. The functions $\Psi_\pm$ solve

$$-\partial_t\Psi_\pm - \nu\Delta\Psi_\pm + \kappa(0)|\nabla\Psi_\pm| - a_\pm = 0,$$

where the time-derivative term is actually 0 since they are functions of the x variable only. If we set $v_\pm := \varphi - \Psi_\pm$, the functions $v_\pm$ solve a linear PDE of the form

$$-\partial_t v_\pm - \nu\Delta v_\pm + w_\pm\cdot\nabla v_\pm \pm 3C\|\nabla\psi\|_{L^\infty}e^{-\alpha t_1} + (K(t,x)-\kappa(0))a_\pm|\nabla\Psi| = 0,$$

where the vector fields $w_\pm$ are such that $|w_\pm(t,x)| \le K(t,x)$. In particular, if we note that, for t_1 large enough, we have $0 \le a_\pm \le 2$, we have $(K(t,x)-\kappa(0))a_\pm|\nabla\Psi| \le 2C\|\nabla\psi\|_{L^\infty}e^{-\alpha t_1}$. Hence, for v_+ we have

$$-\partial_t v_+ - \nu\Delta v_+ + w_+\cdot\nabla v_+ < 0$$

and for v_-

$$-\partial_t v_- - \nu\Delta v_- + w_-\cdot\nabla v_- > 0.$$

Let us look now at the boundary conditions of $v_\pm$ relative to the parabolic domain $[t_1,t_2]\times\Omega$. If t_2 is large enough, using the uniform convergence $\varphi_t \to \Psi$, we can infer $v_+(t_2,x) < 0$ for every $x \in \Omega$. Moreover, we also have $v_+(t,x) < 0$ for every t and every $x \in \partial\Omega$. The

inequalities are opposite for v_-, i.e. we have $v_-(t,x) > 0$ for $t = t_2$ or $x \in \partial\Omega$. This implies, by the maximum principle in [2] (see [2, Theorem 1], adapted to this backward equation, and using again the version with the inequality presented at the end of the proof, page 98), the inequalities $v_+(t_1,x) \le 0 \le v_-(t_1,x)$, i.e.

$$(1 - 3C\|\nabla\psi\|_{L^\infty}e^{-\alpha t_1})\Psi - e^{-\alpha t_1} \le \varphi \le (1 + 3C\|\nabla\psi\|_{L^\infty}e^{-\alpha t_1})\Psi + e^{-\alpha t_1}.$$

This shows $\|\varphi_t - \Psi\|_{L^\infty} \le Ce^{-\alpha t_1}$, for a new constant C. □

We can now pass to the following statement, which proves the convergence of the gradient of φ.

Proposition 4.6. *Let (ρ,φ) be a solution to the Mean Field Game system with infinite time horizon* (1.2)*. Then*

$$\varphi(t,\cdot) \underset{t\to+\infty}{\longrightarrow} \Psi$$

in $H^1_0(\Omega)$.

Proof. We first observe that, by Lemma 4.3, the family of functions $(\varphi(t,\cdot))_{t\ge0}$ is bounded in $H^1_0(\Omega)$. This, together with the uniform convergence to Ψ, implies that one has the weak convergence $\varphi(t,\cdot) \rightharpoonup \Psi$ in $H^1_0(\Omega)$ as $t \to +\infty$.

In order to conclude the proof, it suffices to show $\|\varphi(t,\cdot)\|_{H^1_0(\Omega)} \to \|\Psi\|_{H^1_0(\Omega)}$ as $t \to +\infty$. Since $\varphi(t,\cdot) \rightharpoonup \Psi$ in $H^1_0(\Omega)$ as $t \to \infty$, one has

$$\|\Psi\|^2_{H^1_0(\Omega)} \le \liminf_{t\to+\infty} \|\varphi(t,\cdot)\|^2_{H^1_0(\Omega)}.$$

Let us argue by contradiction and assume that there exists $\varepsilon > 0$ and an increasing sequence $(s_n)_{n\in\mathbb{N}}$ with $s_n \to +\infty$ as $n \to +\infty$ such that

$$\|\varphi(s_n)\|^2_{H^1_0(\Omega)} \ge \|\Psi\|^2_{H^1_0(\Omega)} + 2\varepsilon \tag{4.7}$$

for every $n \in \mathbb{N}$. Recall that, from (4.5) in the proof of Lemma 4.3, there exists $C > 0$ depending only on $\sup\varphi$, $\sup\kappa$, ν, and $|\Omega|$ such that

$$\|\varphi(t,\cdot)\|^2_{H^1_0(\Omega)} \ge \|\varphi(s_n)\|^2_{H^1_0(\Omega)}e^{-C(t-s_n)} - C(1 - e^{-C(t-s_n)})$$

for every $n \in \mathbb{N}$ and $t \in (s_n, s_n+1)$. Combining this with (4.7), one obtains that there exists $\delta \in (0,1)$ depending only on $\|\Psi\|_{H^1_0(\Omega)}$, ε, and C such that

$$\|\varphi(t,\cdot)\|^2_{H^1_0(\Omega)} \ge \|\Psi\|^2_{H^1_0(\Omega)} + \varepsilon \tag{4.8}$$

for every $n \in \mathbb{N}$ and $t \in (s_n, s_n+\delta)$. By Lemma 4.3, there exists a constant $C' > 0$ depending only on $\sup\varphi$, $\sup\kappa$, ν, $|\Omega|$, and Ψ such that

$$\|\varphi\|^2_{L^2((s_n,s_n+\delta);H^2(\Omega))} \le C' \qquad \text{for every } n \in \mathbb{N}.$$

In particular, for every $n \in \mathbb{N}$, there exists $t_n \subset (s_n, s_n+\delta)$ such that $\|\varphi(t_n)\|^2_{H^2(\Omega)} \le C'/\delta$. Hence $(\varphi(t_n))_{n\in\mathbb{N}}$ is bounded in $H^2(\Omega)$ and thus, up to extracting subsequences (which we still denote by $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ for simplicity), $(\varphi(t_n))_{n\in\mathbb{N}}$ converges strongly in $H^1_0(\Omega)$ as $n \to \infty$. Since $\varphi(t,\cdot) \rightharpoonup \Psi$ as $t \to +\infty$, the strong limit of $(\varphi(t_n))_{n\in\mathbb{N}}$ in $H^1_0(\Omega)$ is necessarily Ψ, and thus, in particular, $\|\varphi(t_n)\|^2_{H^1_0(\Omega)} \to \|\Psi\|^2_{H^1_0(\Omega)}$ as $n \to +\infty$. This, however, contradicts (4.8), and establishes the desired result. □

4.3. **Initial conditions with finite entropy.** In all the analysis in this section, we assumed $\rho_0 \in L^2(\Omega)$. As we said in the introduction, this relies on the fact that the general theory for parabolic equations is usually stated in an L^2 framework. Note that, because of the fixed-point argument that we use, even if we do not prove uniqueness for the solution of the MFG system, we do need uniqueness and stability for the solution of each of the two equations separately, so we needed to choose a setting where the equation is well-posed, at least in our very first result (the one in finite-time horizon, in Section 3). This is only an issue for the Fokker–Planck part of the system, since for the value function φ we have, as we saw, uniform estimates in L^∞.

Now, Appendix B provides a notion of solution for the Fokker–Planck equation with bounded drift when the initial condition only satisfies a finite entropy condition

$$e(\rho_0) := \int_\Omega (\rho_0(x)+1)\log(\rho_0(x)+1)\,\mathrm{d}x < \infty.$$

With the use of these tools, it is possible to generalize the results of our paper to this more general setting. Appendix B aims at detailing a general theory to deal with this class of equations, and hence provides both existence and uniqueness in a common setting; yet, for our scopes, the existence part, and more precisely just the uniform estimates depending on the initial entropy, would be enough. Indeed, we can first prove existence of a solution of the MFG system supposing $\rho_0 \in L^2$, and then act by approximation.

We therefore obtain the following theorem.

Theorem 4.7. *Let $\rho_0 \geq 0$ be an initial density with $e(\rho_0) < \infty$. Then, there exists at least one solution (ρ, φ) to the Mean Field Game system with infinite time horizon* (1.2)*, where the Fokker–Planck equation is solved in the sense of Appendix B: the solution ρ belongs to $L^\infty_{loc}((0,\infty)\times\Omega)$, solves the equation in the sense of Definition 2.1 on $(0,\infty)\times\Omega$, and $t \mapsto \rho_t$ is a continuous curve in the sense of distributions, till $t = 0$ included. Moreover, this solution satisfies*

$$\rho(t,x) \underset{t\to+\infty}{\longrightarrow} 0, \quad \varphi(t,x) \underset{t\to+\infty}{\longrightarrow} \Psi,$$

and the above convergences hold uniformly, with $\varphi_t \to \Psi$ in H^1_0 as well.

Proof. It is enough to use the results of this section in order to build a solution (ρ^n, φ^n) to the same system corresponding to initial conditions $\rho^n_0 \in L^2$. These initial data can be chosen so as to approximate ρ_0 in the X' sense (see Appendix B, where X' is defined as the dual of Lipschitz functions vanishing on the boundary) and to have bounded entropy. Using the estimates presented in the appendix, the solutions ρ^n are bounded, for every time $T > 0$, in $L^2((0,T); BV(\Omega))$. This also implies a bound for $\partial_t \rho^n$ in $L^2((0,T); X')$ and allows to apply Aubin–Lions theory to get strong compactness in $L^1((0,T)\times\Omega)$. Since the usual $L^2_t H^2_x \cap L^\infty_t H^1_x$ bounds are true for φ^n, we can pass to the limit the equations after extracting a converging subsequence. We thus get a solution, in $(0,T)\times\Omega$, of the system. The initial datum is preserved since Appendix B provides a uniform $C^{0,\frac{1}{2}}([0,T]; X')$ bound. This means that we can assume that the subsequence we extract also converges uniformly in time as curves valued into X', and provides continuity of the limit curve $t \mapsto \rho_t$ in X' (and hence in the distributional sense), with the initial datum ρ_0 which is also the pointwise limit in X' of the initial data.

The limit solution that we obtain still satisfies the same bounds in $L^2_t BV_x$, which means that for a.e. $t > 0$ we have $\rho_t \in BV(\Omega) \subset L^p(\Omega)$ for a certain $p > 1$ (depending on the space dimension). We can now apply our analysis starting from a time $t_0 > 0$ with $\rho_{t_0} \in L^p$. The L^∞ regularization arguments of Appendix A implies that ρ is locally (in time) L^∞ and all the arguments of the rest of this section apply. □

Appendix A. Regularizing effects of parabolic equations

This appendix is concerned with the regularizing properties of a class of parabolic equations including both the Fokker–Planck and the Hamilton–Jacobi–Bellman equations we consider in this paper. More precisely, we consider the increase of the exponent p of the L^p integrability in space of the solution of the system. As recalled in the introduction, the computations and results presented here are very similar to those from the appendix of [7], the main difference lying in the boundary condition. The main result of this appendix is the following.

Proposition A.1. *Let $T \in (0,+\infty]$. Let $V, F, f, g, u \in C^\infty((0,T)\times\Omega)$ with $V, g \in L^\infty((0,T)\times\Omega)$, $u \geq 0$, $u = 0$ on $\partial\Omega$, such that*

$$\partial_t u - \nu\Delta u + \nabla\cdot(uV) + \nabla\cdot F + f + g\cdot\nabla u \leq 0, \quad \text{on} \quad (0,T)\times\Omega. \tag{A.1}$$

Then, for every $p > 1$, every number $a \in (0,1)$ and t_1, t_2 such that $0 < t_1 < t_2 < T$ and $a < |t_1 - t_2| < a^{-1}$, there is $C > 0$, depending only on $p, a, \|V\|_{L^\infty}, \|g\|_{L^\infty}$ such that

$$\|u(t_2,\cdot)\|_{L^\infty} \leq C\left(\|u(t_1,\cdot)\|_{L^p} + \|F\|_{L^\infty((t_1,t_2)\times\Omega)} + \|f\|_{L^\infty((t_1,t_2)\times\Omega)}\right).$$

The same result is true omitting the assumption $u \geq 0$ if the PDE (A.1) *is satisfied as an equality instead of an inequality.*

The proof follows a standard method based on Moser's iterations that will be detailed here. This appendix is included for completeness: the experienced reader will recognize well-known computations, which are simplified in this setting thanks in particular to the Dirichlet boundary conditions we use.

Proof. Let u be as in the proposition. For $k > 1$, we define

$$m_k(t) := \int_\Omega u^k(t,x)\,\mathrm{d}x.$$

We also define $\alpha := \frac{2^\star}{2} = \frac{n}{n-2}$ if $n > 2$ (here $2^\star$ is the Sobolev exponent in dimension n). When $n = 1, 2$ we set $\alpha := 2$ (but any number larger than 1 and smaller than $+\infty$ could be used here). Moreover, we set

$$M := \|F\|_{L^\infty((t_1,t_2)\times\Omega)} + \|f\|_{L^\infty((t_1,t_2)\times\Omega)}$$

Step 1. L^p estimates.

Let us start with proving that, for $k_0 > 1$, there is $C > 0$ depending on k_0 and on the L^∞ norms of V, g, such that, for every $k > k_0 > 1$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_k e^{-Ck^2t}) + \frac{1}{C}m_{\alpha k}^{\frac{1}{\alpha}} e^{-Ck^2t} \leq Ce^{-Ck^2t}k^2M^k. \tag{A.2}$$

In order to do so, we differentiate m_k with respect to t, to get

$$\begin{aligned} m_k'(t) &\leq k\int_\Omega(\nu\Delta u - \nabla\cdot(uV) - g\cdot\nabla u - \nabla\cdot F - f)u^{k-1} \\ &\leq -k(k-1)\nu\int_\Omega|\nabla u|^2u^{k-2} + k(k-1)\int_\Omega(V\cdot\nabla u)u^{k-1} - k\int_\Omega(g\cdot\nabla u)u^{k-1} \\ &\qquad + k(k-1)\int_\Omega(F\cdot\nabla u)u^{k-2} - k\int_\Omega fu^{k-1}. \end{aligned}$$

Now, owing to Young's inequality, we can find $C_1, C_2, C_3 > 0$ depending only on $\|V\|_{L^\infty}$, $\|g\|_{L^\infty}$, k_0, ν such that

$$m_k'(t) \leq -C_1k^2\int_\Omega|\nabla u|^2u^{k-2} + C_2k^2\int_\Omega u^k + C_3k^2\int_\Omega|F|^2u^{k-2} + k\int_\Omega|f|u^{k-1}$$

(note that we replaced the coefficient $k(k-1)$ with k^2, as these two numbers are equivalent up to multiplicative constants as far as $k > k_0 > 1$). Moreover, thanks again to a Young inequality, we have

$$|F|^2 u^{k-2} \le \frac{2}{k}|F|^k + \frac{k-2}{k}u^k \quad \text{and} \quad |f|u^{k-1} \le \frac{1}{k}|f|^k + \frac{k-1}{k}u^k.$$

Therefore, up to increasing C_2, C_3, we get

$$\begin{aligned} m_k'(t) &\le -C_1k^2\int_\Omega |\nabla u|^2 u^{k-2} + C_2k^2\int_\Omega u^k + C_3k^2\int_\Omega |F|^k + k\int_\Omega |f|^k \\ &\le -C_1k^2\int_\Omega |\nabla u|^2 u^{k-2} + C_2k^2\int_\Omega u^k + C_3k^2M^k. \end{aligned}$$

Now, owing to the Gagliardo–Nirenberg–Sobolev inequality, we have, for some $C_4 > 0$,

$$k^2\int_\Omega |\nabla u|^2 u^{k-2} = 4\int_\Omega |\nabla(u^{\frac{k}{2}})|^2 \ge C_4\left(\int_\Omega u^{k\alpha}\right)^{\frac{1}{\alpha}} = C_4 m_{k\alpha}^{\frac{1}{\alpha}}.$$

Hence

$$m_k'(t) + C_1C_4 m_{\alpha k}^{\frac{1}{\alpha}} \le C_2k^2 m_k + C_3k^2M^k.$$

Let us denote $C := \max\{\frac{1}{C_1C_4}, C_2, C_3\}$. Then, the above equation gives us

$$m_k'(t) - Ck^2 m_k + \frac{1}{C}m_{\alpha k}^{\frac{1}{\alpha}} \le Ck^2M^k,$$

which we can rewrite in order to get (A.2).

Step 2. Estimates on $m_{\alpha k}$.

We show in this step that, for $k > k_0 > 1$ and for $0 < t_1 < t_2 < T$, we have

$$m_{\alpha k}^{\frac{1}{\alpha}}(t_2) \le e^{C\alpha k^2(t_2-t_1)}\frac{1}{t_2-t_1}e^{Ck^2t_2}\int_{t_1}^{t_2} m_{\alpha k}^{\frac{1}{\alpha}}(s)e^{-Ck^2s}\,\mathrm{d}s + e^{C\alpha k^2(t_2-t_1)}M^k, \tag{A.3}$$

for some C depending on k_0 and on the L^∞ norms of V, g.

The relation (A.2) provides

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_k e^{-Ck^2t}) \le Ce^{-Ck^2t}k^2M^k.$$

Let us take $s \in (t_1, t_2)$. We integrate the above inequality for $t \in (s, t_2)$ to get:

$$m_k(t_2)e^{-Ck^2t_2} \le m_k(s)e^{-Ck^2s} + CM^k\int_s^{t_2} e^{-Ck^2t}\,\mathrm{d}t \le m_k(s)e^{-Ck^2s} + M^ke^{-Ck^2s}.$$

Taking the power $\frac{1}{\alpha} < 1$ and using its subadditivity yields

$$m_k^{\frac{1}{\alpha}}(t_2)e^{-\frac{C}{\alpha}k^2t_2} \le m_k^{\frac{1}{\alpha}}(s)e^{-\frac{C}{\alpha}k^2s} + M^{\frac{k}{\alpha}}e^{-\frac{C}{\alpha}k^2s}.$$

We replace k by αk so as to re-write the above inequality as

$$m_{\alpha k}^{\frac{1}{\alpha}}(t_2)e^{-C\alpha k^2t_2} \le m_{\alpha k}^{\frac{1}{\alpha}}(s)e^{-C\alpha k^2s} + M^ke^{-C\alpha k^2s}.$$

We multiply by $e^{Ck^2s(\alpha-1)}$ and integrate this for $s \in (t_1, t_2)$ in order to obtain

$$m_{\alpha k}^{\frac{1}{\alpha}}(t_2)\int_{t_1}^{t_2} e^{C\alpha k^2(s-t_2)}e^{-Ck^2s}\,\mathrm{d}s \le \int_{t_1}^{t_2} m_{\alpha k}^{\frac{1}{\alpha}}(s)e^{-Ck^2s}\,\mathrm{d}s + M^k\int_{t_1}^{t_2} e^{-Ck^2s}\,\mathrm{d}s.$$

We then use $e^{C\alpha k^2(s-t_2)} \ge e^{C\alpha k^2(t_1-t_2)}$ in order to obtain

$$m_{\alpha k}^{\frac{1}{\alpha}}(t_2) \le e^{C\alpha k^2(t_2-t_1)}\left(\int_{t_1}^{t_2} e^{-Ck^2s}\,\mathrm{d}s\right)^{-1}\int_{t_1}^{t_2} m_{\alpha k}^{\frac{1}{\alpha}}(s)e^{-Ck^2s}\,\mathrm{d}s + e^{C\alpha k^2(t_2-t_1)}M^k$$

and finally we use $\int_{t_1}^{t_2} e^{-Ck^2s}\,\mathrm{d}s \ge (t_2-t_1)e^{-Ck^2t_2}$, which provides the desired inequality.

Step 3. Higher integrability estimates.

Let us now show that, for $k > k_0$, there is $C > 0$ (depending on the same quantities as in the previous steps), such that

$$m_{\alpha k}^{\frac{1}{\alpha k}}(t_2) \leq \frac{e^{Ck(t_2-t_1)}}{|t_2-t_1|^{1/k}} \left(m_k(t_1) + M^k\right)^{\frac{1}{k}}, \tag{A.4}$$

First of all, integrating (A.2) for $t \in (t_1, t_2)$, and discharging the final value $m_k(t_2)e^{-Ct_2}$, we obtain

$$\frac{1}{C}\int_{t_1}^{t_2} m_{\alpha k}^{\frac{1}{\alpha}}(t)e^{-Ck^2t}\,\mathrm{d}t \leq m_k(t_1)e^{-Ck^2t_1} + M^k\int_{t_1}^{t_2} Ck^2e^{-Ck^2t}\,\mathrm{d}t \leq e^{-Ck^2t_1}\left(M^k + m_k(t_1)\right).$$

Combining this with (A.3), we get

$$m_{\alpha k}^{\frac{1}{\alpha}}(t_2) \leq e^{C\alpha k^2(t_2-t_1)}\left(\frac{e^{Ck^2(t_2-t_1)}}{t_2-t_1}(m_k(t_1)+M^k) + M^k\right).$$

Up to enlarging the constant C and using $0 < t_2 - t_1 < a^{-1}$, we can write the above inequality in a simpler form, i.e.

$$m_{\alpha k}^{\frac{1}{\alpha}}(t_2) \leq \frac{e^{C(\alpha+1)k^2(t_2-t_1)}}{t_2-t_1}\left(m_k(t_1)+M^k\right),$$

hence, (A.4) holds true, after raising to the power $1/k$ and including $\alpha+1$ in the constant C.

Step 4. Iterations.

We conclude the proof in this step by proving that, for p, t_1, t_2 as in the statement of the proposition, there is $C > 0$ such that

$$\|u(t_2,\cdot)\|_{L^\infty} \leq C(\|u(t_1,\cdot)\|_{L^p} + \|F\|_{L^\infty} + \|f\|_{L^\infty}). \tag{A.5}$$

We denote

$$s_n := t_2 - \frac{t_2-t_1}{(2\alpha)^n},\ k_n := \alpha^n p,\ \beta_n := \frac{e^{Ck_n(s_{n+1}-s_n)}}{(s_{n+1}-s_n)^{\frac{1}{k_n}}},$$

and

$$a_n := m_{k_n}^{\frac{1}{k_n}}(s_n), \quad \tilde{a}_n := \max\{a_n, M\}.$$

Then, (A.4) gives us that

$$a_{n+1} \leq \beta_n(a_n^{k_n} + M^{k_n})^{\frac{1}{k_n}} \leq \beta_n 2^{\frac{1}{k_n}}\tilde{a}_n.$$

Hence, up to replacing the constant C with a larger one so as to suppose $\beta_n 2^{\frac{1}{k_n}} \geq 1$, we find

$$\tilde{a}_{n+1} \leq \beta_n 2^{\frac{1}{k_n}}\tilde{a}_n.$$

We observe that we have $\prod_{n=0}^{+\infty}\beta_n 2^{\frac{1}{k_n}} < +\infty$ as a consequence of the logarithmic estimate

$$\sum_{n=0}^{+\infty}\log(\beta_n 2^{\frac{1}{k_n}}) \leq \sum_{n=0}^{+\infty} Ck_n\frac{t_2-t_1}{(2\alpha)^n} + \frac{1}{k_n}(\log 2 + n\log(2\alpha) - \log(t_2-t_1)) < +\infty.$$

Therefore, we obtain

$$\max\{\lim_{n\to+\infty} a_n, M\} \leq \left(\prod_{n=0}^{+\infty}\beta_n 2^{\frac{1}{k_n}}\right)\max\{a_0, M\} \leq C(a_0+M).$$

Therefore, thanks to $\lim_{n\to+\infty} a_n = \|u(t_2,\cdot)\|_{L^\infty}$, we obtain (A.5). This concludes the proof. □

Corollary A.2. *Let $T \in (0,+\infty]$. Let $V \in L^\infty((0,T)\times\Omega)$. Let $u \in L^1((0,T)\times\Omega)$ be a positive distributional solution of*

$$\partial_t u - \nu\Delta u - \nabla\cdot(uV) \le 0, \qquad \text{on } \ (0,T)\times\Omega,$$

satisfying the following mild regularity assumption: u is obtained as a measurable curve $(u_t)_t$ of functions of the x variable, which is such that $t \mapsto \int_\Omega \eta(x)u_t(x)\,\mathrm{d}x$ is continuous in time for every $\eta \in C^\infty(\Omega)$ (note that we do not restrict to $\eta \in C^\infty_c(\Omega)$). Then, for every $p > 1$ and $a \in (0,1)$, there is $C > 0$, depending only on p, a, $\|V\|_{L^\infty}$, such that we have

$$\|u(t_2,\cdot)\|_{L^\infty} \le C\|u(t_1,\cdot)\|_{L^p}$$

for every $0 < t_1 < t_2 < T$ with $a < |t_2 - t_1| < a^{-1}$.

Proof. To prove this estimate the only important point is to regularize the equation so as to apply Proposition A.1. We will do a two-step approximation exactly as we did in Proposition 2.7.

We convolve the equation by an approximation of the identity and to apply Proposition A.1. However, convolving will not preserve the Dirichlet boundary conditions, so we first have to extend u by zero on a bigger set.

We define Ω^+ to be a open bounded regular set such that $\Omega + B_1 \subset \Omega^+$, where B_1 is unit ball in $\mathbb{R}^N$.

We define $u^+(t,x) := u(t,x)$ if $x \in \Omega$, and $u^+(t,x) = 0$ elsewhere. Let $\eta_\varepsilon(x)$ be an approximation of the identity whose support is included in B_1. We define $u_\varepsilon := u^+ \star \eta_\varepsilon$ (here, $\star$ is the convolution in space only). It is a function which is smooth in x and continuous in t. We then convolve in time as well, taking $\chi_\delta(t)$ an approximation of the identity whose support is included in $\mathbb{R}_+$. Defining $u_{\varepsilon,\delta} := \chi_\delta \star u_\varepsilon$ we have now a function which is smooth in time and space. It satisfies, in the classical sense,

$$\partial_t u_{\varepsilon,\delta} - \nu\Delta u_{\varepsilon,\delta} - \nabla\cdot(u_{\varepsilon,\delta}V_{\varepsilon,\delta}) \le 0, \qquad \text{for } \ t \in (0,T),\ x \in \Omega^+,$$

with $V_{\varepsilon,\delta} := \frac{\chi_\delta\star\eta_\varepsilon\star(uV)}{u_{\varepsilon,\delta}} \in C^\infty$. Moreover, the L^∞ norm of $V_{\varepsilon,\delta}$ is bounded independently of ε and δ. Then, $u_{\varepsilon,\delta}$ is positive, regular and is a (classical) subsolution of a Fokker–Planck equation with regular coefficients, hence we can apply Proposition A.1. We then take the limit $\delta \to 0$, and we observe that we have

$$\|u_\varepsilon(t,\cdot)\|_{L^p} = \lim_{\delta\to 0}\|u_{\varepsilon,\delta}(t,\cdot)\|_{L^p}$$

for every t, since u_ε is continuous. Then, we have

$$\|u(t,\cdot)\|_{L^p} = \lim_{\varepsilon\to 0}\|u_\varepsilon(t,\cdot)\|_{L^p}$$

from standard properties of convolutions (with the possibility, of course, that this limit and this norm take the value $+\infty$). □

Corollary A.3. *Let $T \in (0,+\infty]$. Let $f, g \in L^\infty$ and let $u \in L^\infty((0,T);L^2(\Omega)) \cap L^2((0,T);H^1_0(\Omega))$ be solution (in the weak sense) of*

$$\partial_t u - \nu\Delta u + f + g\cdot\nabla u = 0, \quad \text{on} \quad (0,T)\times\Omega,$$

with Dirichlet boundary conditions and initial datum $u(0,\cdot) = u_0 \in L^2$.

Then, for every $p > 1$, and $a \in (0,1)$ there is $C > 0$, depending only on $p, a, \|g\|_{L^\infty}$ such that

$$\|u(t_2,\cdot)\|_{L^\infty} \le C\left(\|u(t_1,\cdot)\|_{L^p} + \|f\|_{L^\infty}\right)$$

for every $t_1 < t_2$ with $a < |t_2 - t_1| < a^{-1}$.

Proof. Let f_n, g_n be C^∞ and such that $f_n \to f$ and $g_n \to g$ in the L^2 norm. Assume moreover that we have $\|f_n\|_{L^\infty} \to \|f\|_{L^\infty}$ and $\|g_n\|_{L^\infty} \to \|g\|_{L^\infty}$. Let u_n be the solution of

$$\partial_t u_n - \nu\Delta u_n + f_n + g_n \cdot \nabla u_n = 0, \quad \text{on} \quad (0,+\infty)\times\Omega,$$

with Dirichlet boundary condition and with initial datum u_0^n, where u_0^n is a smooth L^2 approximation of u_0.

Then, u_n is smooth enough to apply Proposition A.1 to u_n, to get, for p, t_1, t_2 as in the statement of the corollary,

$$\|u_n(t_2,\cdot)\|_{L^\infty} \le C\left(\|u_n(t_1,\cdot)\|_{L^p} + \|f_n\|_{L^\infty}\right). \tag{A.6}$$

Then, as n goes to $+\infty$, u_n converges (the arguments to prove this are standard and based on the weak L^2 convergence of ∇u_n) to a solution (in the weak sense) of

$$\partial_t u - \nu\Delta u + f + g \cdot \nabla u = 0, \quad \text{on} \quad (0,+\infty)\times\Omega,$$

with Dirichlet boundary conditions and with initial datum u_0. The convergence is also strong in the L^2 sense. Because this solution is unique, it necessarily coincides with the original solution u of the statement. In order to obtain the result, we need to pass to the limit the inequality (A.6). The left-hand side can easily be dealt with by semicontinuity, while for the right-hand side, we suppose $p \le 2$ and we use strong L^2 convergence. Since this convergence is L^2 in space-time, we have convergence of the right-hand side only for a.e. t_1. Yet, using the fact that the solution u is continuous in time as a function valued into $L^2(\Omega)$, the result extends to all t_1. The inequality for $p = 2$ implies that with $p > 2$, up to modifying the constant in a way depending on $|\Omega|$. □

The reader may observe that the regularization strategies to prove the two above corollaries are very much different from each other. Indeed, regularizing the coefficients only work in a setting where we already know uniqueness. For parabolic equations, this works in an L^2 setting. It is not restrictive for us to apply it to the equation in Corollary A.3, as this corollary is meant to deal with differences of solutions of the Hamilton–Jacobi–Bellman equation. In this case, we always have uniform L^∞ bounds and it is not restrictive for us to stick to L^2 functions. On the other hand, as we will see in the next section of the appendix and as we did in Subsection 4.2, we are interested in studying the Fokker–Planck equation outside the L^2 framework and we do not know yet uniqueness in this case. Yet, for the Fokker–Planck case we can exploit the simpler structure of the equation: we convolve the solution and define a new drift vector field which preserves the same L^∞ bound, which was not possible in the setting of the equation in Corollary A.3.

Appendix B. Fokker–Planck equation with finite-entropy initial condition

The goal of this section is to prove the existence and uniqueness of solution to the Fokker–Planck equation

$$\partial_t \rho - \nu\Delta\rho + \nabla\cdot(\rho V) = 0, \quad t > 0,\ x \in \Omega, \tag{B.1}$$

with Dirichlet boundary conditions, $V \in L^\infty$, and initial datum ρ_0 with finite entropy.

In this whole section, if $u \in L^1_{loc}(\Omega)$, $u \ge 0$, we denote

$$e(u) := \int_\Omega (u+1)\log(u+1) \ge 0$$

the entropy of u. We define $X := \{\phi \in \mathrm{Lip},\ \phi = 0 \ \text{ on } \partial\Omega\}$ and we denote by X' the topological dual of X. The notion of solution has then to be understood in the following sense:

Definition B.1. Let $V \in L^\infty((0,T)\times\Omega)$ and $\rho_0 \ge 0$ be such that $e(\rho_0) < +\infty$. We say that ρ solves (B.1) with Dirichlet boundary conditions and with initial datum ρ_0 if

- $t \mapsto \rho_t$ is $C^{0,1/2}$ as a curve valued in X', with initial datum equal to ρ_0;
- ρ_t belongs to $L^\infty(\Omega)$ for every $t > 0$, and $\|\rho_t\|_{L^\infty}$ is locally bounded on $(0, T]$;
- ρ solves (B.1) in the sense of Definition 2.1 on $[t_0, T] \times \Omega$ for every $t_0 > 0$.

Theorem B.2. *Let $V \in L^\infty$ and let $\rho_0 \in L^1_{loc}$ be such that $e(\rho_0) < +\infty$. Then, there exists a unique solution to the Fokker–Planck equation* (B.1) *with initial datum ρ_0, in the sense of Definition B.1.*

To prove the existence part of the theorem, we will proceed by approximation: we shall build the solution as the limit of a sequence of solutions to the equation with regularized data.

Lemma B.3. *There is a constant $C > 0$ (depending on $\|V\|_{L^\infty}$, $\|\rho_0\|_{L^1}$, and ν) such that any smooth solution $\rho \geq 0$ of* (B.1) *with Dirichlet boundary condition satisfies, for every t_1, t_2, $0 \leq t_1 \leq t_2$:*

$$e(\rho(t_2, \cdot)) + \frac{\nu}{2} \int_{t_1}^{t_2} \int_\Omega \frac{|\nabla \rho|^2}{\rho + 1} \, \mathrm{d}x \, \mathrm{d}t \leq e(\rho(t_1, \cdot)) + C|t_2 - t_1|.$$

Proof of Lemma B.3. We define, for notational simplicity,

$$e(t) := e(\rho(t, \cdot)).$$

Then, we compute

$$\begin{aligned} \dot{e}(t) &= \int_\Omega \partial_t \rho (\log(\rho + 1) + 1) \\ &= -\int_\Omega (\nu \nabla \rho - \rho V) \cdot \frac{\nabla \rho}{\rho + 1} + \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho \\ &= -\nu \int_\Omega \frac{|\nabla \rho|^2}{\rho + 1} + \int_\Omega \frac{\rho}{\rho + 1} V \cdot \nabla \rho + \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho. \end{aligned}$$

Using Young's inequality and the fact that $V \in L^\infty$, we obtain

$$\dot{e}(t) + \frac{\nu}{2} \int_\Omega \frac{|\nabla \rho|^2}{\rho + 1} \leq \frac{1}{2\nu} \|V\|^2_{L^\infty} \int_\Omega \frac{\rho^2}{\rho + 1} + \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho.$$

The mass of a positive solution of this Fokker–Planck equation being non-increasing in time, we have $\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho \leq 0$ and $\|\rho(t, \cdot)\|_{L^1} \leq \|\rho_0\|_{L^1}$. Hence

$$\dot{e}(t) + \frac{\nu}{2} \int_\Omega \frac{|\nabla \rho|^2}{\rho + 1} \leq C \int_\Omega \frac{\rho^2}{\rho + 1} \leq C, \tag{B.2}$$

which implies the claim. □

We deduce in particular the following estimates.

Corollary B.4. *Any smooth solution $\rho(t, x)$ of* (B.1) *arising from a smooth initial datum $\rho_0 \geq 0$ belongs to $C^{0,1/2}([0, T]; X') \cap L^2((0, T); L^p(\Omega))$, for a certain exponent $p > 1$ only depending on the dimension d, and the norm in these spaces is bounded by a constant only depending on $e(\rho_0)$, $\|\rho_0\|_{L^1}$, $\|V\|_{L^\infty}$, and ν.*

Proof. If ρ is smooth and $\phi \in X$ we can compute

$$\begin{aligned} \int_\Omega \phi(\rho_{t_1} - \rho_{t_2}) \, \mathrm{d}x = -\int_{t_1}^{t_2} \mathrm{d}t \int_\Omega \phi \, \partial_t \rho_t \, \mathrm{d}x &= \int_{t_1}^{t_2} \mathrm{d}t \int_\Omega \nabla \phi \cdot (\nu \nabla \rho_t - \rho_t V) \, \mathrm{d}x \\ &\leq \|\phi\|_X \int_{t_1}^{t_2} \mathrm{d}t (\nu \|\nabla \rho_t\|_{L^1} + C). \end{aligned}$$

We then use the following estimate

$$\int_\Omega |\nabla \rho| \leq \left(\int_\Omega \frac{|\nabla \rho|^2}{\rho + 1} \right)^{1/2} \left(\int_\Omega \rho + 1 \right)^{1/2},$$

which shows that $t \mapsto \|\nabla \rho_t\|_{L^1}$ belongs to $L^2((0,T))$, with norm bounded by a constant depending on $e(\rho_0)$ and $\|\rho_0\|_{L^1}$, thanks to the previous lemma. This allows to obtain

$$\int_\Omega \phi(\rho_{t_1} - \rho_{t_2})\,\mathrm{d}x \le C|t_2 - t_1|^{1/2}.$$

Moreover, using the injection $BV \subset L^p$, for $p = d/(d-1) > 1$, we also obtain the $L^2_t L^p_x$ behavior of the claim. □

In the above corollary, the summability exponent $p = d/(d-1)$ could also be raised to $p = d/(d-2)$ if we use, instead, the injection $H^1 \subset L^{2^\star}$, for $2^\star = 2d/(d-2)$, and apply it to $\sqrt{\rho+1} - 1 \approx \sqrt{\rho}$. Indeed, the bound on $\int\int |\nabla \rho|^2/(\rho+1)$ exactly means $\sqrt{\rho+1} - 1 \in L^2_t H^1_x$.

Lemma B.5. *Given $t_1 > 0$, there is a constant $C > 0$, depending on t_1, $e(\rho_0)$, $\|\rho_0\|_{L^1}$, $\|V\|_{L^\infty}$, and ν such that, for any smooth $\rho_0 \ge 0$, the solution ρ of the Fokker–Planck equation* (B.1) *arising from the initial datum ρ_0 satisfies*

$$\|\rho(t,\cdot)\|_{L^\infty(\Omega)} \le C \quad \text{for every } t \ge t_1.$$

Proof. From Corollary A.2, if we fix an exponent $p > 1$, we know that, for every $s \in (\frac{t_1}{2}, t_1)$, we have

$$\|\rho(t,\cdot)\|_{L^\infty(\Omega)} \le C\|\rho(s,\cdot)\|_{L^p(\Omega)}.$$

Integrating this for $s \in (\frac{t_1}{2}, t_1)$, we find

$$\|\rho(t,\cdot)\|_{L^\infty(\Omega)} \le C \int_{t_1/2}^{t_1} \|\rho(s,\cdot)\|_{L^p(\Omega)}\,\mathrm{d}s,$$

and the result follows by combining this with Corollary B.4. □

We are now in a position to prove Theorem B.2.

Proof of Theorem B.2. Step 1. Existence.

We take a sequence of smooth initial data $(\rho_0^n)_{n\in\mathbb{N}}$ and a sequence of smooth vector fields V_n, where $\rho_n \to \rho$ in the L^1 sense (which implies the convergence in X') and such that $e(\rho_0^n) \to e(\rho_0)$, and $V_n \to V$ in L^1 together with $\|V_n\|_{L^\infty} \to \|V\|_{L^\infty}$.

We let ρ^n be the solution of (B.1) arising from the initial datum ρ_0^n. From Corollary B.4 the curves $t \mapsto \rho_t^n \in X'$ are equicontinuous, and we can extract a uniformly converging subsequence. We call ρ this uniform limit, which is also a $C^{0,1/2}$ curve valued into X', with initial datum ρ_0. From Lemma B.5 we have a uniform bound on $\|\rho_t^n\|_{L^\infty}$ for $t \ge t_1 > 0$, and this bound propagates to ρ_t. Hence, we also have $\rho_t^n \overset{*}{\rightharpoonup} \rho_t$ in the sense of weak-$*$ convergence in L^∞ which allows, together with the strong L^1 convergence of the vector fields, to pass to the limit the Fokker–Planck equation in the weak sense.

Step 2. Uniqueness.

Consider two possible solutions ρ^1 and ρ^2 of the Fokker–Planck equation, and set $u := \rho^1 - \rho^2$. The curve $t \mapsto u_t$ is also a $C^{0,1/2}$ curve valued into X', and the function u is locally L^∞ far from $t = 0$ and solves the Fokker–Planck equation in the sense of Definition 2.1 on $(0,T]$ (i.e. on every compact subinterval of $(0,T]$).

Consider $\Theta \in C_c^\infty(\Omega)$, $|\Theta| \le 1$, and let ψ_n be the solution of the approximate adjoint equation

$$-\partial_t \psi = \nu\Delta\psi + V_n \cdot \nabla\psi, \quad t \in (0,T),\ x \in \Omega,$$

with Dirichlet boundary conditions and with final datum $\psi(T,\cdot) = \Theta$, for a smooth vector field V_n satisfying $V_n \to V$ in L^2 and $\|V_n\|_{L^\infty} \to \|V\|_{L^\infty}$. The solution ψ_n is smooth and, by the maximum principle, we have $|\psi_n| \le 1$. Moreover, differentiating in time $t \mapsto \int |\psi_n|^2(t,x)\,\mathrm{d}x$, we obtain a uniform bound, independent of n, on $\|\nabla\psi_n\|_{L^2}$ (in the sense of the time-space L^2 norm). Up to extracting a subsequence and passing to the

limit, we have $\psi_n \rightharpoonup \psi$ in $L^2(H^1_0)$, where ψ is a weak solution of $-\partial_t\psi = \nu\Delta\psi + V\cdot\nabla\psi$ with Dirichlet boundary conditions. This solution also satisfies $|\psi| \le 1$. Because of the $L^2_t H^1_x$ bound on ψ_n, which also implies a bound in $L^2_t H^{-1}_x$ for $\partial_t\psi_n$, Aubin–Lions Lemma improves the convergence of ψ_n to ψ, and proves that it is actually strong in L^2 in space-time. In particular, for a.e. t, we have $\psi_n(t,\cdot)\to\psi(t,\cdot)$ in $L^2(\Omega)$.

Using ψ_n as a test function in the equation (2.3) satisfied by u far from $t=0$, choosing times $0<t<T$, we obtain

$$\int \Theta u_T - \int \psi_n(t,x)u_t(x) = \int_t^T \mathrm{d}s \int_\Omega u(V_n - V)\cdot\nabla\psi_n \,\mathrm{d}x.$$

We then pass to the limit as $n\to\infty$ on a time $t>0$ where we have L^2 convergence of $\psi_n(t,\cdot)$ and use the L^2 bound on $\nabla\psi_n$, the L^2 convergence $V_n\to V$, and the L^∞ bound on u on $(t,T)\times\Omega$. We obtain

$$\int_\Omega \Theta u_T = \int_\Omega \psi(t,x)u_t(x)\,\mathrm{d}x.$$

This equality is valid for a.e. $t>0$, so that we can write

$$\begin{aligned}\int_\Omega \psi(T,x)u(T,x) &= \frac{1}{t}\int_0^t\int_\Omega \psi(s,x)u(s,x)\,\mathrm{d}x\,\mathrm{d}s \\ &= \frac{1}{t}\int_0^t\int_\Omega \psi(s,x)(u(s,x)-u(0,x))\,\mathrm{d}x\,\mathrm{d}s + \int_\Omega\left(\frac{1}{t}\int_0^t\psi(s,x)\,\mathrm{d}s\right)u(0,x)\,\mathrm{d}x.\end{aligned} \tag{B.3}$$

We now observe that the function ψ satisfies $\nabla\psi\in L^2_t L^\infty_x$. This can be obtained in the following way: we already know, passing to the limit from ψ_n, that we have $\nabla\psi\in L^2_tL^2_x$; hence ψ satisfies a backward parabolic equation of the form $-\partial_t\psi-\nu\Delta\psi = f$ with $f\in L^2_tL^2_x$, and maximal regularity for parabolic PDEs[2] implies $D^2\psi\in L^2_tL^2_x$, hence $\nabla\psi\in L^2_tL^{2^\star}_x$. Iterating the argument we can improve the space summability exponent of $\nabla\psi$ and in finitely many steps arrive at $\nabla\psi\in L^2_tL^p_x$ for $p>d$, hence $f\in L^2_tL^p_x$ for the same p, and $D^2\psi\in L^2_tL^p_x$, i.e. $\nabla\psi\in L^2_tW^{1,p}_x\subset L^2_tC^{0,\alpha}_x\subset L^2_tL^\infty_x$.

We now come back to (B.3) and write

$$\begin{aligned}\frac{1}{t}\int_0^t\int_\Omega \psi(s,x)(u(s,x)-u(0,x))\,\mathrm{d}x\,\mathrm{d}s &\le \frac{1}{t}\int_0^t \|\nabla\psi(s,\cdot)\|_{L^\infty}\sqrt{s}\,\mathrm{d}s \\ &\le \frac{\sqrt{t}}{t}\left(\int_0^t\|\nabla\psi(s,\cdot)\|^2_{L^\infty}\,\mathrm{d}s\right)^{\frac12}\sqrt{t}.\end{aligned}$$

Because $\nabla\psi\in L^2_tL^\infty_x$, this goes to zero as t goes to zero.

On the other hand, we have, using $|\psi|\le 1$,

$$\int_\Omega\left(\frac{1}{t}\int_0^t\psi(s,x)\,\mathrm{d}s\right)u(0,x)\,\mathrm{d}x \le \|u_0\|_{L^1}.$$

Hence, we obtain

$$\int_\Omega \Theta(x)u(T,x)\,\mathrm{d}x = \int_\Omega\psi(T,x)u(T,x)\,\mathrm{d}x \le \|u_0\|_{L^1}.$$

Taking the sup among $\Theta\in C^\infty_c$ with $|\Theta|\le 1$ we obtain

$$\|\rho^1_T-\rho^2_T\|_{L^1}\le\|\rho^1_0-\rho^2_0\|_{L^1},$$

which shows that the solution with given initial datum is unique and that the flow of this Fokker–Planck equation is a contraction in L^1. □

[2]We refer to [24] for a survey on the subject. The very same estimate that we need is also presented, with a reference to the previous survey, in [8], formula (3), in the setting of Dirichlet conditions on a bounded domain. Adding a non-zero initial or final datum, as far as it is smooth, does not provide extra difficulties as one can consider the difference between the solution and the datum, provided the Laplacian of the datum is in L^∞.

The reader could be disappointed about the choice to define solutions in this setting under the assumption that ρ is a $C^{0,1/2}$ curve valued in X' and could have preferred to see a more elegant assumption such as just continuous into X'. Yet, in order to adapt the proof to the case where only continuity is assumed, one would need to prove $\nabla\psi \in L^\infty_{t,x}$ which is not easy to get. To this concern, we observe that, if we differentiate in time $\int_\Omega |\nabla\psi|^2$ instead of $\int_\Omega |\psi|^2$, using a suitable Gronwall lemma, one obtains $\psi \in L^\infty_t H^1_x \cap L^2_t H^2_x$ (an estimate that we already used many times in this paper). Unfortunately, maximal regularity for parabolic equations, exactly as for elliptic regularity, cannot be performed for $p = \infty$. The best that we can easily obtain with these methods is $\nabla\psi \in L^p_t L^\infty_x$ for every $p < \infty$. Looking attentively at the proof the reader can see that that this improved estimate could allow to handle the case $\rho \in C^{0,\alpha}([0,T]; X')$ for arbitrary $\alpha > 0$, but still we are not able to prove uniqueness under the only assumption of continuity.

Acknowledgments. The authors wish to thank Alessio Porretta for useful suggestions after a talk the second author gave on the topic of the present paper. Without his guidance, the strategy to achieve convergence to a solution in the limit $T \to +\infty$ could have been completely different, the result less general, and the time needed to achieve it much longer.

References

[1] Y. Achdou and A. Porretta. Mean field games with congestion. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 35(2):443–480, 2018.

[2] D. G. Aronson and J. Serrin. Local behavior of solutions of quasilinear parabolic equations. *Arch. Rational Mech. Anal.*, 25:81–122, 1967.

[3] J.-P. Aubin and H. Frankowska. *Set-valued analysis*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2009. Reprint of the 1990 edition.

[4] J.-D. Benamou, G. Carlier, and F. Santambrogio. Variational mean field games. In *Active particles. Vol. 1. Advances in theory, models, and applications*, Model. Simul. Sci. Eng. Technol., pages 141–171. Birkhäuser/Springer, Cham, 2017.

[5] P. Cannarsa and C. Sinestrari. *Semiconcave functions, Hamilton–Jacobi equations, and optimal control*. Progress in Nonlinear Differential Equations and their Applications, 58. Birkhäuser Boston, Inc., Boston, MA, 2004.

[6] P. Cardaliaguet. Notes on mean field games (from P.-L. Lions' lectures at Collège de France). Available at `https://www.ceremade.dauphine.fr/~cardaliaguet/MFG20130420.pdf`, 2013.

[7] P. Cardaliaguet, J.-M. Lasry, P.-L. Lions, and A. Porretta. Long time average of mean field games with a nonlocal coupling. *SIAM J. Control Optim.*, 51(5):3558–3591, 2013.

[8] R. Danchin. On the solvability of the compressible Navier–Stokes system in bounded domains. *Nonlinearity*, 23(2):383–407, jan 2010.

[9] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

[10] W. H. Fleming and H. M. Soner. *Controlled Markov processes and viscosity solutions*, volume 25 of *Stochastic Modelling and Applied Probability*. Springer, New York, second edition, 2006.

[11] A. Granas and J. Dugundji. *Fixed point theory*. Springer Monographs in Mathematics. Springer-Verlag, New York, 2003.

[12] M. Huang, P. E. Caines, and R. P. Malhamé. Individual and mass behaviour in large population stochastic wireless power control problems: centralized and Nash equilibrium solutions. In *42nd IEEE Conference on Decision and Control, 2003. Proceedings*, volume 1, pages 98–103. IEEE, 2003.

[13] M. Huang, P. E. Caines, and R. P. Malhamé. Large-population cost-coupled LQG problems with nonuniform agents: individual-mass behavior and decentralized ϵ-Nash equilibria. *IEEE Trans. Automat. Control*, 52(9):1560–1571, 2007.

[14] M. Huang, R. P. Malhamé, and P. E. Caines. Large population stochastic dynamic games: closed-loop McKean–Vlasov systems and the Nash certainty equivalence principle. *Commun. Inf. Syst.*, 6(3):221–251, 2006.

[15] R. L. Hughes. A continuum theory for the flow of pedestrians. *Transportation Research Part B: Methodological*, 36(6):507–535, jul 2002.

[16] R. L. Hughes. The flow of human crowds. In *Annual review of fluid mechanics, Vol. 35*, volume 35 of *Annu. Rev. Fluid Mech.*, pages 169–182. Annual Reviews, Palo Alto, CA, 2003.

[17] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural′ceva. *Linear and quasilinear equations of parabolic type*. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968. Translated from the Russian by S. Smith.
[18] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen. I. Le cas stationnaire. *C. R. Math. Acad. Sci. Paris*, 343(9):619–625, 2006.
[19] J.-M. Lasry and P.-L. Lions. Jeux à champ moyen. II. Horizon fini et contrôle optimal. *C. R. Math. Acad. Sci. Paris*, 343(10):679–684, 2006.
[20] J.-M. Lasry and P.-L. Lions. Mean field games. *Jpn. J. Math.*, 2(1):229–260, 2007.
[21] G. M. Lieberman. *Second order parabolic differential equations*. World Scientific Publishing Co., Inc., River Edge, NJ, 1996.
[22] P.-L. Lions. Courses at Collège de France, 2006–2012. `http://www.college-de-france.fr/site/pierre-louis-lions/_co`
[23] G. Mazanti and F. Santambrogio. Minimal-time mean field games. *Math. Models Methods Appl. Sci.*, 29(8):1413–1464, 2019.
[24] S. Monniaux. Maximal regularity and application to PDEs. In *Analytical and Numerical Aspects of Partial Differential Equations*, De Gruyter Proceedings in Mathematics, pages 247–288. De Gruyter, 2009.
[25] F. Santambrogio. Lecture notes on variational mean field games. Preprint cvgmt. `http://cvgmt.sns.it/paper/4646/`.
[26] J. Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, 146:65–96, 1987.

Institut Camille Jordan, Université Claude Bernard - Lyon 1; 43 boulevard du 11 novembre 1918, 69622 Villeurbanne cedex, France.

Email address: `ducasse@math.univ-lyon1.fr`

Université Paris-Saclay, CNRS, CentraleSupélec, Inria, Laboratoire des signaux et systèmes, 91190, Gif-sur-Yvette, France & Inria Saclay–Île-de-France, DISCO Team & Institut Polytechnique des Sciences Avancées, 63 boulevard de Brandebourg, 94200 Ivry-sur-Seine, France.

Email address: `guilherme.mazanti@centralesupelec.fr`

Institut Camille Jordan, Université Claude Bernard - Lyon 1; 43 boulevard du 11 novembre 1918, 69622 Villeurbanne cedex, France & Institut Universitaire de France.

Email address: `santambrogio@math.univ-lyon1.fr`