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The sterile insect technique used as a barrier control against reinfestation

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Abstract

The sterile insect technique consists in massive release of sterilized males in the aim to reduce the size of mosquitoes population or even eradicate it. In this work, we investigate the feasibility of using the sterile insect technique as a barrier against reinvasion. More precisely, we provide some numerical simulations and mathematical results showing that performing the sterile insect technique on a band large enough may stop reinvasion.

1 Introduction

Due to the number of diseases that they transmit, mosquitoes are considered as one of the most dangerous animal species for humans. According to the World Health Organization [1], vector-borne diseases account for more than 17% of all infectious diseases, causing more than 700 000 deaths annually. More than 3.9 billion people in over 128 countries are at risk of contracting dengue, with 96 millions cases estimated per year.

Several of the most prominent mosquito-transmitted diseases do not have a widely available vaccine; dengue, for instance, has only one licensed vaccine, Dengvaxia, with limited application [25]. Zika and chikungunya do not have licensed vaccines, although several candidates are on diverse stages of development [23] [31]. Hence, at this stage, a natural strategy to control them is to act directly on the mosquito population. For this purpose, several strategies have been developed and experimented. Some techniques aim at replacing the existing population of mosquitoes by a population unable to propagate the pathogens, using the Wolbachia bacteria [15]. Other techniques aim at reducing the size of the mosquito population : The Sterile Insect Technique (SIT) [10, 4], the release of insects carrying a dominant lethal (RIDL) [32, 14, 12], the driving of anti-pathogen genes into natural populations [13, 19, 33]. Finally, other strategies combine both reduce and replace strategies [26].

In this article, we focus on the SIT. This strategy has been introduced in the 50's by Raymond C. Bushland and Edward F. Knippling. It consists in using area-wide releases of sterile insects to

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reduce reproduction in a field population of the same species. Indeed wild female insects of the pest population do not reproduce when they are inseminated by released, radiation-sterilized males. For mosquito populations, this technique and the closely related incompatible insect technique (IIT) has been successfully used to drastically reduce their size in some isolated regions (see e.g. [29, 34]). In order to predict the dynamics of mosquito populations, mathematical modeling is an important tool. In particular, there is a growing interest in the study of control strategies (see e.g. [5, 18, 8, 2] and references therein). In order to conduct rigorous studies, these works usually neglect spatial dependency.

In fact, only few articles propose to incorporate the spatial variable in their study of SIT. In [16] the authors propose a simple scalar model to study the influence of the sterile insects density on the velocity of the spatial wave of spread of mosquitoes. The work [21] focuses on the influence of the release sites and the frequency of releases in the effectiveness of the sterile insect technique. In [28, 27], the authors conduct a numerical study on some mathematical models with spatial dependency to investigate the use of a barrier zone to prevent invasion of mosquitoes. However, up to our knowledge, there are no rigorous mathematical results on the existence of such barrier zone. In this paper, we conduct a similar study as in [28] for another mathematical model which has been recently introduced in [30]. Moreover, we propose a strategy to rigorously prove the existence of a barrier zone under some conditions on the parameters.

The outline of the paper is the following: In subsection 2.1 we introduce our dynamical system model and describe the variables and biological parameters, and also a simplified model that results from additional assumptions. We analyze the existence of positive equilibria and the stability of the mosquito-free equilibrium. In subsection 2.2 we introduce spatial models including diffusion. In section 3 we perform numerical simulations for the spatial models to observe the existence of wave-blocking for a large enough release of sterile males. In section 4 we prove the existence of a blocking phenomenon (some useful technical results are postponed to the appendix). We end this paper with a conclusion.

2 Mathematical model

2.1 Dynamical system

In the recent paper [30], the following mathematical model governing the dynamics of mosquitoes has been proposed:

$$\begin{aligned}
 \frac{dE}{dt} &= b\left(1 - \frac{E}{K}\right)F - (\nu_E + \mu_E)E, \\
 \frac{dM}{dt} &= (1 - r)\nu_E E - \mu_M M, \\
 \frac{dF}{dt} &= r\nu_E E(1 - e^{-\beta(M+\gamma M_s)})\frac{M}{M + \gamma M_s} - \mu_F F, \\
 \frac{dM_s}{dt} &= u - \mu_s M_s.
 \end{aligned}
 \tag{2.1}$$

In this system, the mosquito population is divided into several compartments. The number of mosquitoes in the aquatic phase is denoted E ; M and F denote respectively the number of adult wild males and adult females which have been fertilized; M_s is the number of sterile mosquitoes which are released, the release function being denoted u . The fraction $\frac{M}{M+\gamma M_s}$ corresponds to the probability that a female mates with a wild male. Moreover, the term $(1 - e^{-\beta(M+\gamma M_s)})$ has been introduced to model a strong Allee effect, that is, the fact that when the number of males is close to zero, it will be very difficult for a female find a wild male to mate with.

Finally, we have the following parameters :

- $b > 0$ is the oviposition rate;
- $\mu_E > 0$, $\mu_M > 0$, $\mu_F > 0$, and $\mu_s > 0$ denote the death rates for the mosquitoes in the aquatic phase, for wild adult males, for adult females, and for sterile males, respectively;
- K is an environmental capacity for the aquatic phase, taking also into account the intraspecific competition;
- $\nu_E > 0$ is the rate of emergence;
- $r \in (0, 1)$ is the probability that a female emerges and $(1 - r)$ is the probability that a male emerges;
- u is a control function, corresponding to the number of sterile males which are released into the field.

Before introducing the spatial dependance in this system, we will first investigate the equilibria of (2.1) and their stability. For future use, we introduce the notation

$$g(F, M_s) = \frac{r\nu_E bF}{\frac{b}{K}F + \mu_E + \nu_E} \left(1 - \exp \left(-\beta \left(\frac{(1-r)\nu_E bF}{\mu_M(\frac{b}{K}F + \mu_E + \nu_E)} + \gamma M_s \right) \right) \right) \quad (2.2)$$

$$\times \frac{(1-r)\nu_E bF}{(1-r)\nu_E bF + \gamma\mu_M M_s(\frac{b}{K}F + \mu_E + \nu_E)} - \mu_F F.$$

Then, we may use Lemma 3 in [30] to prove the following result:

Proposition 2.1 *Let $N = \frac{br\nu_E}{\mu_F(\nu_E + \mu_E)}$ and $\psi = \frac{\mu_M}{(1-r)\nu_E K \beta}$. Let us assume that $N > \max\{1, 4\psi\}$. Let $\theta_0 \in (0, 1)$ be the unique solution of $1 - \theta_0 = -\frac{4\psi}{N} \ln \theta_0$ and assume that*

$$\max_{\theta \in [\theta_0, 1]} \left(-\ln \theta - \frac{1}{2\psi} \left(1 - \sqrt{1 + \frac{4\psi \ln(\theta)}{N(1-\theta)}} \right) \right) > 0. \quad (2.3)$$

Then, we have :

1. When $u = 0$. There exist two positive equilibria for system (2.1). They are given by $(\bar{E}_1, \bar{M}_1, \bar{F}_1, 0)$ and $(\bar{E}_2, \bar{M}_2, \bar{F}_2, 0)$, with $\bar{F}_1 < \bar{F}_2$ and $\bar{E}_i = \frac{b\bar{F}_i}{\frac{b}{K}\bar{F}_i + \nu_E + \mu_E}$, $\bar{M}_i = \frac{(1-r)\nu_E \bar{E}_i}{\mu_M}$, for $i = 1, 2$.
2. There exists a positive constant \tilde{U} large enough, such that if $u = \bar{U}$ is a constant such that $\bar{U} > \tilde{U}$, then the unique equilibrium for system (2.1) is the mosquito-free equilibrium $(0, 0, 0, \bar{U}/\mu_s)$, which is globally stable.

Proof. At equilibrium, we have

$$E = \frac{bF}{\frac{b}{K}F + \nu_E + \mu_E}, \quad M = \frac{(1-r)\nu_E}{\mu_M} E.$$

Injecting into the equilibrium equation for F , we deduce that any equilibrium should satisfy

$$g(F, M_s) = 0. \quad (2.4)$$

For $u = 0$, the only equilibrium of $\frac{dM_s}{dt} = u - \mu_s M_s$ is $M_s = 0$. Substituting $M_s = 0$ in (2.2), we obtain for 2.4

$$g(F, 0) = \frac{r\nu_E b F}{\frac{b}{K}F + \mu_E + \nu_E} \left(1 - \exp \left(-\beta \left(\frac{(1-r)\nu_E b F}{\mu_M (\frac{b}{K}F + \mu_E + \nu_E)} \right) \right) \right) - \mu_F F = 0.$$

$F = 0$ is always a solution. When $F > 0$ the equation $g(F, 0) = 0$ may be rewritten

$$g_1(F) := r\nu_E b \left(1 - \exp \left(-\beta \left(\frac{(1-r)\nu_E b F}{\mu_M (\frac{b}{K}F + \mu_E + \nu_E)} \right) \right) \right) = \mu_F \left(\frac{b}{K}F + \mu_E + \nu_E \right).$$

The right hand side is an affine function. It is straightforward to verify that g_1 is an increasing concave function on $(0, +\infty)$ such that $g_1(0) = 0$ and $\lim_{F \rightarrow +\infty} g_1(F) = r\nu_E b \left(1 - \exp \left(-\frac{\beta(1-r)\nu_E K}{\mu_M} \right) \right)$. Then, this concave function may intersect the affine function in 0, 1 or 2 points. Notice that a necessary (but not sufficient) condition for intersection is that $\lim_{F \rightarrow +\infty} g_1(F) > \mu_F(\mu_E + \nu_E)$. The fact that condition (2.3) guarantees existence of two positive roots is proved in [30, Lemma 3].

For the second part, for a given constant $u = \bar{U}$, the only equilibrium of $\frac{dM_s}{dt} = u - \mu_s M_s$ is $M_s = \bar{U}/\mu_s$. Then, injecting this relation into the expression for g in (2.2), we deduce that for any $F \geq 0$, we have $\lim_{\bar{U} \rightarrow +\infty} g(F, \frac{\bar{U}}{\mu_s}) = -\mu_F F$. For any $\alpha \in (0, \mu_F)$, we may find \bar{U} large enough such that for any $F \geq 0$, $g(F, \frac{\bar{U}}{\mu_s}) \leq -\alpha F$. Hence, the only solution of the equation $g(F, \frac{\bar{U}}{\mu_s}) = 0$ is $F = 0$, which implies that the only equilibrium for system (2.1) is the mosquito-free equilibrium $(0, 0, 0, \bar{U}/\mu_s)$.

Moreover, the Jacobian at the mosquito-free equilibrium is given by the following matrix

$$J(0, 0, 0, \bar{U}/\mu_s) = \begin{pmatrix} -(\nu_E + \mu_E) & 0 & b & 0 \\ (1-r)\nu_E & -\mu_M & 0 & 0 \\ 0 & 0 & -\mu_F & 0 \\ 0 & 0 & 0 & -\mu_s \end{pmatrix}.$$

We compute easily its eigenvalues which are given by $\{-\nu_E - \mu_E, -\mu_M, -\mu_F, -\mu_s\}$; they are all negative. Therefore the mosquito-free equilibrium is globally stable. \square

Remark 2.2 *Although the proof does not give us an analytic formula for the equilibria, remark 3.2 in [30] gives us a useful approximation of \bar{M}_2 when β is not too small, as $\bar{M}_2 \approx \frac{1}{\psi\beta} \left(1 - \frac{1}{N} \right)$.*

2.2 Spatial model

In order to model the spatial dynamics, we consider that adult mosquitoes diffuse according to a random walk. It is classical to model this active motion by adding a diffusion operator in the compartment of females and males. We denote by x the spatial variable. In order to simplify the approach, we only consider the one dimensional case, $x \in \mathbb{R}$. Then, all unknown functions depend

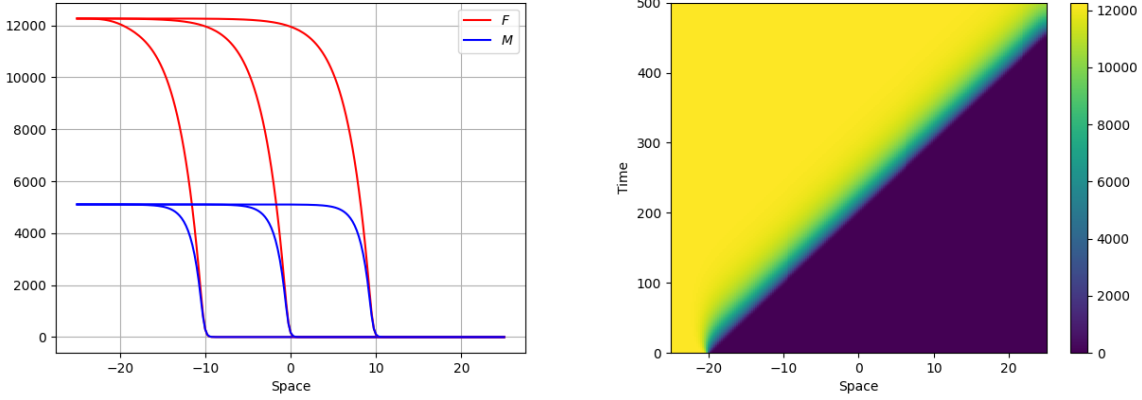


Figure 1: Numerical simulation of the time and space dynamics of (F, M) solving (2.5) when $M_s = 0$. Left : Profiles of solutions at three different times with same initial data; solutions are plotted at time $t = 100$, $t = 200$, $t = 300$. Right : Dynamics in time and space of the fertilized female density for system (2.5).

now on time $t > 0$ and position $x \in \mathbb{R}$. In this setting, model (2.1) becomes

$$\partial_t E = b\left(1 - \frac{E}{K}\right)F - (\nu_E + \mu_E)E, \quad (2.5a)$$

$$\partial_t M - D_u \partial_{xx} M = (1 - r)\nu_E E - \mu_M M, \quad (2.5b)$$

$$\partial_t F - D_u \partial_{xx} F = r\nu_E E(1 - e^{-\beta(M + \gamma M_s)}) \frac{M}{M + \gamma M_s} - \mu_F F, \quad (2.5c)$$

$$\partial_t M_s - D_u \partial_{xx} M_s = u - \mu_s M_s. \quad (2.5d)$$

In this model D_u is a given diffusion coefficient, which is assumed to be the same for both male and female mosquitoes. This system is complemented with some nonnegative initial data denoted E^0 , M^0 , F^0 , and $M_s^0 = 0$. Then, there exists a nonnegative solution of (2.5). From now on, we will always assume that the parameters of the model are such that there are two positive equilibria for system (2.1) (see Proposition 2.1).

In the case when no control technique is implemented, i.e. $M_s = 0$, we expect that this model to predict the invasion of the whole domain by the wild mosquitoes. In order to illustrate this phenomenon, we perform some numerical simulations of system (2.5). This system is discretized thanks to a semi-implicit finite difference scheme on an uniform mesh. We use the numerical values in Table (1), which are taken from [30]. When $M_s = 0$, the numerical results are shown in Figure 1. We observe that there is a wave of mosquitoes that invades the whole domain. Hence, in the absence of control and in a homogeneous environment, it is expected that the solutions to (2.5) invade freely the spatial domain.

If we fix a constant value of the number of sterile males in the whole domain, then the invasion is expected to slow down. This observation has already been studied in [17]. Numerical illustrations of this phenomenon are provided in Figure 2. In this simulation, the number of sterile males is fixed to be $M_s = 5000$ on the whole domain. This figure should be compared to Figure 1 where $M_s = 0$: we observe that the invasion is slowed down.

In this paper, we want to investigate the possibility of blocking the propagation of the spreading of mosquitoes by releasing sterile mosquitoes on a band of width L . Can the sterile insect technique

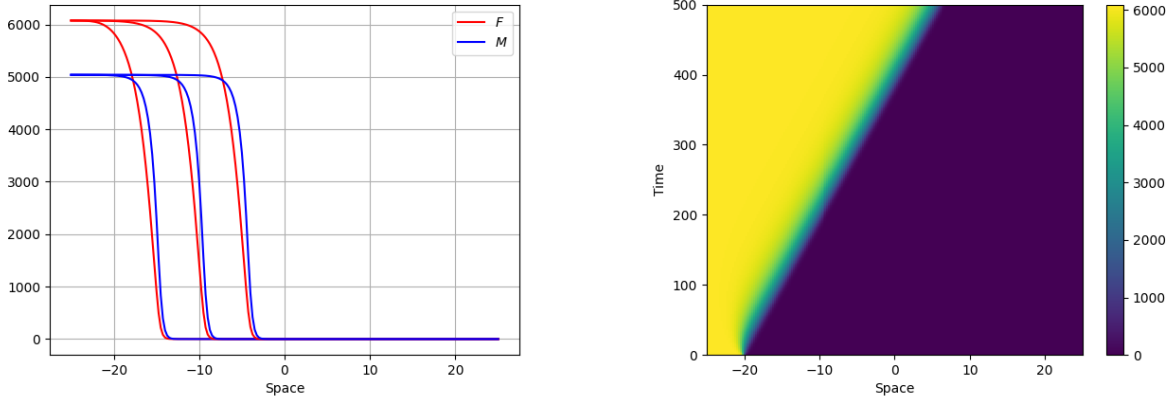


Figure 2: Numerical simulation of the time and space dynamics of (F, M) solving (2.5) when $M_s = 5000$. Left : Profiles of solutions at three different times with same initial data; solutions are plotted at time $t = 100$, $t = 200$, $t = 300$. Right : Dynamics in time and space of the females density for system (2.5). Comparing this figure with Fig. 1, we observe that the invasion is slowing down.

be used to act as a barrier to avoid re-invasion of wild mosquitoes in a mosquito-free region ? In order to answer this question, we first perform some numerical simulations in the next section.

3 Numerical simulations

We choose $u(t, x) = \bar{U}\mathbf{1}_{[0,L]}(x)$, where \bar{U} is a given positive constant. We propose some numerical simulations. As above, we implement a finite difference scheme on an uniform mesh. The values of the numerical parameters are taken from [30] and are given in Table 1.

Parameter	β	b	r	μ_E	ν_E	μ_F	μ_M	γ	μ_s	D_u
Value	10^{-2}	10	0.49	0.03	0.05	0.04	0.1	1	0.12	0.0125

Table 1: Table of the numerical values used for the numerical simulations. These values are taken from [30]

We show in Figures 3 the dynamics in time and space of the fertilized female population F for model (2.5). The domain where the release of sterile males is performed is of width $L = 5$ km. The release intensity is $\bar{U} = 10\,000 \text{ km}^{-1} \text{ day}^{-1}$ (left), $\bar{U} = 20\,000 \text{ km}^{-1} \text{ day}^{-1}$ (center), $\bar{U} = 30\,000 \text{ km}^{-1} \text{ day}^{-1}$ (right). These results illustrate the influence of the intensity of the release \bar{U} and of the width of the band on the efficiency of the blocking. We first notice that it seems that for \bar{U} or L large enough, the mosquito wave is not able to pass through the release zone. On the contrary, if \bar{U} and L are not large enough, the wave is only delayed by the release in the barrier zone.

It is also interesting to observe that when $\beta \rightarrow +\infty$, there is no blocking as it is illustrated in Figure 4 for model (2.5). In order to justify this observation, we observe that, when $\beta \rightarrow +\infty$ the mosquito steady state equilibrium is unstable for model (2.1). Indeed, when $M_s = 0$ in system (2.1), after passing to the limit when $\beta \rightarrow +\infty$, the Jacobian matrix of the resulting system at the point

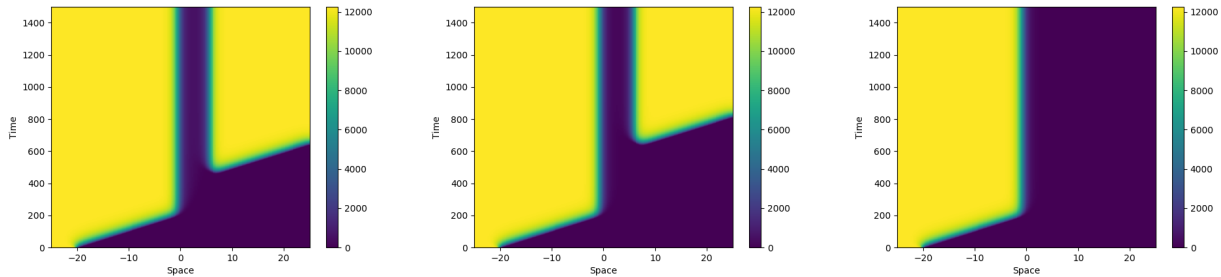


Figure 3: Numerical simulations of system (2.5) with $L = 5$ km and $\bar{U} = 10\,000$ km $^{-1}$ day $^{-1}$ (left), $\bar{U} = 20\,000$ km $^{-1}$ day $^{-1}$ (center), and $\bar{U} = 30\,000$ km $^{-1}$ day $^{-1}$ (right).

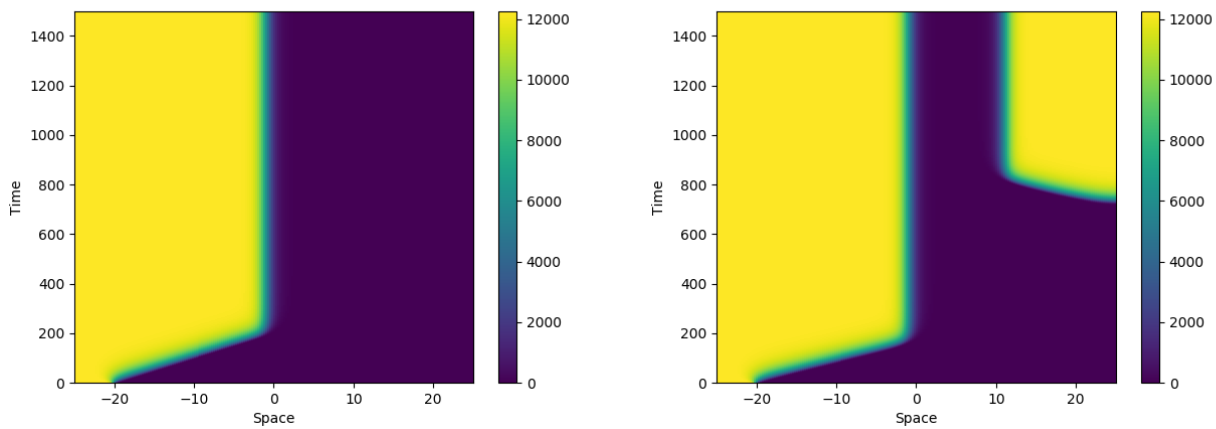


Figure 4: Numerical simulations for the model (2.5) with $\beta \rightarrow +\infty$, $L = 10$ km and $\bar{U} = 40\,000$ km $^{-1}$ day $^{-1}$. In the figure on the left $\beta = 10^{-2}$ as in Table 1, while on the right we have the solution of the limit equation when $\beta \rightarrow +\infty$.

$(0, 0, 0)$ is given by

$$\begin{pmatrix} -\nu_E - \mu_E & 0 & b \\ (1-r)\nu_E & -\mu_M & 0 \\ r\nu_E & 0 & -\mu_F \end{pmatrix}$$

Clearly $-\mu_M$ is one of the eigenvalues. For $br\nu_E > \mu_F(\nu_E + \mu_E)$, there is a positive real eigenvalue and therefore, the steady state would be unstable.

4 Mathematical approach

These numerical simulations seem to show that it is possible to block the spreading by releasing enough sterile males on a sufficiently wide domain. However, to be sure that it is not a numerical artifact and that the propagation is really blocked, we have to prove rigorous mathematical results. The study of blocking by local action has been done by several authors with applications for instance in biology or criminal studies [22, 16, 9, 7, 20, 11]. In this section, we will always consider that

$$u(t, x) = \bar{U}\mathbf{1}_{[0, L]}(x),$$

and we assume that (2.3) is satisfied such that when $u = 0$ there are two stable steady states. We justify rigorously that for $L > 0$ and \bar{U} large enough, we may block the propagation.

We first provide some useful bounds and comparisons on the variables of our system. Then, we construct a stationary supersolution for (2.5) that vanishes at $+\infty$. When it exists, we shall call this stationary supersolution *a barrier* for (2.5), since such a solution acts as a barrier to block the propagation.

4.1 Estimates

We first recall that, since initial data is nonnegative, the solution (E, F, M, M_s) of (2.5) is also nonnegative. Moreover, we have the following estimates :

Lemma 4.1 *Let us assume that $E^0 \leq K$. Then, for any $t \geq 0$, we have $E \leq K$. Moreover, M and F are uniformly bounded.*

Proof. The inequality is assumed to be true for $t = 0$. If there exists some $t_0 > 0$ and $x \in \mathbb{R}$ such that $E(t_0, x) \geq K$, then we deduce from the first equation in (2.5) that $\frac{d}{dt}E(t_0, x) < 0$, hence $E(\cdot, x)$ is decreasing in the vicinity of t_0 . Therefore, if this inequality is true initially, it should be true for any larger time. It follows then from a standard application of the maximum principle that M and F are also uniformly bounded. \square

Lemma 4.2 *Let $u(t, x) = \bar{U}\mathbf{1}_{[0, L]}(x)$. Then, the solution $M_s(t, x)$ to (2.5d) with initial data $M_s(t = 0) = 0$ converges uniformly on x , when $t \rightarrow +\infty$, to the solution \bar{M}_s of the stationary equation*

$$-D_u \partial_{xx} \bar{M}_s = \bar{U}\mathbf{1}_{[0, L]} - \mu_s \bar{M}_s, \quad \text{on } \mathbb{R}. \quad (4.6)$$

Proof. We have

$$\partial_t(M_s - \bar{M}_s) - \partial_{xx}(M_s - \bar{M}_s) = -\mu_s(M_s - \bar{M}_s), \quad M_s(t = 0) = 0.$$

Then, using the heat kernel, we deduce

$$M_s(t, x) = \bar{M}_s(x) - \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{y^2}{4t} - \mu_s t} \bar{M}_s(x - y) dy.$$

The stationary solution \bar{M}_s has the expression

$$\bar{M}_s(x) = \frac{\bar{U}}{2\sqrt{D_u \mu_s}} \int_0^L e^{-\sqrt{\frac{\mu_s}{D_u}}|x-y|} dy.$$

We can see from (4.1) that \bar{M}_s is symmetric with respect to $x = \frac{L}{2}$ and decreasing on $(\frac{L}{2}, +\infty)$. Therefore \bar{M}_s has a global maximum at $x = \frac{L}{2}$, which we compute to be $\bar{M}_s(\frac{L}{2}) = \frac{\bar{U}}{\mu_s} \left(1 - e^{-\sqrt{\frac{\mu_s}{D_u}} \frac{L}{2}}\right)$, and we have

$$0 \leq \bar{M}_s(x) - M_s(t, x) = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{y^2}{4t} - \mu_s t} \bar{M}_s(x - y) dy \leq e^{-\mu_s t} \bar{M}_s\left(\frac{L}{2}\right)$$

where we use the fact that the integral of the heat kernel equals 1. Therefore, $M_s(t, x)$ converges to \bar{M}_s exponentially in t and uniformly in x . \square

Lemma 4.3 *Let $L > 0$. Then, for any $L^* > 0$, the solution \overline{M}_s of (4.6) verifies, for any $x \in [0, L^*]$*

$$\overline{M}_s(x) \geq \frac{\overline{U}}{2\mu_s} \min \left\{ 1 - e^{-\sqrt{\frac{\mu_s}{D_u}}L}, e^{-\sqrt{\frac{\mu_s}{D_u}}L^*} (e^{\sqrt{\frac{\mu_s}{D_u}}L} - 1) \right\}.$$

Proof.

Using the symmetry of \overline{M}_s , we have that for any $x \in [0, L^*]$ $\overline{M}_s(x) \geq \overline{M}_s(0) = \overline{M}_s(L)$ if $L > L^*$, and $\overline{M}_s(x) \geq \overline{M}_s(L^*)$ if $L \leq L^*$. If $L > L^*$, we see that

$$\overline{M}_s(0) = \frac{\overline{U}}{2\mu_s} (1 - e^{-\sqrt{\frac{\mu_s}{D_u}}L}),$$

and, if $L \leq L^*$, we have that

$$\overline{M}_s(L^*) = \frac{\overline{U}}{2\mu_s} e^{-\sqrt{\frac{\mu_s}{D_u}}L^*} (e^{\sqrt{\frac{\mu_s}{D_u}}L} - 1).$$

□

An immediate consequence of this result is the following :

Corollary 4.4 *Let $\varepsilon > 0$ and $L^* > 0$. Then, there exists \overline{U} large enough such that, for any bounded nonnegative function M and for any $x \in [0, L^*]$, we have*

$$(1 - e^{-\beta(M(x) + \gamma\overline{M}_s(x))}) \frac{M(x)}{M(x) + \gamma\overline{M}_s(x)} \leq \varepsilon,$$

where \overline{M}_s is the solution of (4.6).

4.2 Comparison principle

Let $\overline{M}_s \geq 0$ be a given nonnegative function. Let us consider the stationary problem associated to (2.5) :

$$0 = b(1 - \frac{\overline{E}}{K})\overline{F} - (\nu_E + \mu_E)\overline{E} \tag{4.7a}$$

$$- D_u \overline{M}'' = (1 - r)\nu_E \overline{E} - \mu_M \overline{M} \tag{4.7b}$$

$$- D_u \overline{F}'' = r\nu_E \overline{E} (1 - e^{-\beta(\overline{M} + \gamma\overline{M}_s)}) \frac{\overline{M}}{\overline{M} + \gamma\overline{M}_s} - \mu_F \overline{F}. \tag{4.7c}$$

We are particularly interested by stationary solutions which link the two stable equilibria $(0, 0, 0)$ and $(\overline{E}_2, \overline{M}_2, \overline{F}_2)$ defined in Proposition 2.1. Therefore, we complement this system with conditions at infinity : $(\overline{E}, \overline{M}, \overline{F})(-\infty) = (\overline{E}_2, \overline{M}_2, \overline{F}_2)$ and $(\overline{E}, \overline{M}, \overline{F})(+\infty) = (0, 0, 0)$.

Definition 4.5 *We call **barrier** any super-solution of (4.7) such that $(\overline{E}, \overline{M}, \overline{F})(-\infty) = (\overline{E}_2, \overline{M}_2, \overline{F}_2)$ and $(\overline{E}, \overline{M}, \overline{F})(+\infty) = (0, 0, 0)$.*

We recall that super-solutions are obtained by replacing the equality signs in (4.7) by the inequalities \geq .

The following result explains why we call barrier such a solution : If the initial data is below this barrier, the solution stays below the barrier and hence the propagation (invasion of the whole domain by the wild population) will be blocked.

Lemma 4.6 *Let $M_s \geq \overline{M}_s$. Let us assume that the initial data are nonnegative and such that*

$$E^0 \leq K. \quad (4.8)$$

Let us assume that there exists a barrier $(\overline{E}, \overline{M}, \overline{F})$ in the sense of definition 4.5.

If we assume that $E^0 \leq \overline{E}$, $M^0 \leq \overline{M}$, $F^0 \leq \overline{F}$. Then, the solution (E, M, F) of (2.5a)–(2.5c) is such that for any $t \geq 0$ and any $x \in \mathbb{R}$, we have $E(t, x) \leq \overline{E}$, $M(t, x) \leq \overline{M}$ and $F(t, x) \leq \overline{F}$.

Proof. From Lemma 4.1, we may consider the system (2.5a)–(2.5c) on the set $[0, K] \times \mathbb{R}^+ \times \mathbb{R}^+$. Let us denote

$$g(M, M_s) := \frac{M}{M + \gamma M_s} (1 - e^{-\beta(M + \gamma M_s)}).$$

We verify easily that, on this set, the map g is nonincreasing with respect to M_s and nondecreasing with respect to M . Therefore, the system is monotonous and we deduce directly the result from standard arguments.

For the sake of completeness, we explain briefly these arguments. Let us denote $\delta E = E - \overline{E}$, $\delta M = M - \overline{M}$, and $\delta F = F - \overline{F}$. Subtracting and (4.7) from (2.5), we deduce

$$\begin{aligned} \partial_t \delta E &\leq b \left(1 - \frac{E}{K}\right) \delta F - \left(\frac{b\overline{F}}{K} + \nu_E + \mu_E\right) \delta E \\ \partial_t \delta M - D_u \partial_{xx} \delta M &\leq (1 - r) \nu_E \delta E - \mu_M \delta M \\ \partial_t \delta F - D_u \partial_{xx} \delta F &\leq r \nu_E \delta E g(M, M_s) + r \nu_E \overline{E} (g(M, M_s) - g(\overline{M}, M_s)) - \mu_F \delta F, \end{aligned}$$

where we use the fact that $g(\overline{M}, M_s) \leq g(\overline{M}, \overline{M}_s)$. Denoting by $v_+ := \max\{v, 0\}$ the positive part of any real v , we multiply the first equation by $(\delta E)_+$, the second by $(\delta M)_+$, and the third by $(\delta F)_+$. Integrating over \mathbb{R} , we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (\delta E)_+^2 dx &\leq \int_{\mathbb{R}} b \left(1 - \frac{E}{K}\right) (\delta F)_+ (\delta E)_+ dx - \int_{\mathbb{R}} \left(\frac{b\overline{F}}{K} + \nu_E + \mu_E\right) (\delta E)_+^2 dx \\ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (\delta M)_+^2 dx + D_u \int_{\mathbb{R}} |\partial_x (\delta M)_+|^2 dx &\leq \int_{\mathbb{R}} (1 - r) \nu_E (\delta E)_+ (\delta M)_+ dx - \int_{\mathbb{R}} \mu_M (\delta M)_+^2 dx \\ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (\delta F)_+^2 dx + D_u \int_{\mathbb{R}} |\partial_x (\delta F)_+|^2 dx &\leq \int_{\mathbb{R}} r \nu_E (\delta E)_+ (\delta F)_+ dx - \int_{\mathbb{R}} \mu_F (\delta F)_+^2 dx \\ &\quad + \int_{\mathbb{R}} r \nu_E \overline{E} (g(M, M_s) - g(\overline{M}, M_s)) (\delta F)_+ dx, \end{aligned}$$

where we use also the fact that $g(M, M_s) \leq 1$. Since $M \mapsto g(M, M_s)$ is increasing and uniformly Lipschitz-continuous, we have for some nonnegative constant C

$$g(M, M_s) - g(\overline{M}, M_s) \leq C (\delta M)_+.$$

Hence, adding these latter inequalities, we deduce that for some nonnegative constant C , we have

$$\frac{d}{dt} \int_{\mathbb{R}} ((\delta E)_+^2 + (\delta F)_+^2 + (\delta M)_+^2) dx \leq C \int_{\mathbb{R}} ((\delta E)_+^2 + (\delta F)_+^2 + (\delta M)_+^2) dx.$$

We conclude by using a Gronwall lemma, noticing that from our assumption on the initial data we have that at time $t = 0$, $\delta E(t = 0) = 0$, $\delta M(t = 0) = 0$, $\delta F(t = 0) = 0$. \square

4.3 Existence of a barrier

Let $L^* > 0$ and $\alpha \in (0, \mu_F)$, we deduce from Corollary 4.4 that for any $L > 0$, there exists \bar{U} large enough such that for all $x \in [0, L^*]$ the solution \bar{M}_s of (4.6) verifies, for any nonnegative and bounded functions \bar{M} and \bar{F} ,

$$\frac{r\nu_E\bar{F}(x)}{\frac{b}{K}\bar{F}(x) + \mu_E + \nu_E}(1 - e^{-\beta(\bar{M}(x) + \gamma\bar{M}_s(x))})\frac{\bar{M}(x)}{\bar{M}(x) + \gamma\bar{M}_s(x)} \leq (\mu_F - \alpha)\bar{F}(x).$$

It invites us to introduce the stationary problem

$$\bar{E} = \frac{b\bar{F}}{\frac{b}{K}\bar{F} + \mu_E + \nu_E} \quad (4.9a)$$

$$-D_u\bar{M}'' = (1-r)\nu_E\bar{E} - \mu_M\bar{M} \quad (4.9b)$$

$$-D_u\bar{F}'' = -\alpha\bar{F}\mathbf{1}_{[0, L^*]} + \left(r\nu_E\bar{E}(1 - e^{-\beta\bar{M}}) - \mu_F\bar{F}\right)\mathbf{1}_{[0, L^*]^c}. \quad (4.9c)$$

Any super-solution of (4.9) which satisfies the boundary conditions $(\bar{E}, \bar{M}, \bar{F})(-\infty) = (\bar{E}_2, \bar{M}_2, \bar{F}_2)$ and $(\bar{E}, \bar{M}, \bar{F})(+\infty) = (0, 0, 0)$ is a barrier, as defined in Definition 4.5. Then, in order to prove existence of a barrier, it suffices to prove that there exist super-solutions of (4.9), i.e. (\bar{M}, \bar{F}) such that

$$-D_u\bar{M}'' \geq \frac{(1-r)\nu_E b\bar{F}}{\frac{b}{K}\bar{F} + \mu_E + \nu_E} - \mu_M\bar{M}, \quad \text{on } \mathbb{R} \setminus \{0, L\}, \quad (4.10a)$$

$$-D_u\bar{F}'' \geq -\alpha\bar{F}\mathbf{1}_{[0, L]} + \left(\frac{r\nu_E b\bar{F}}{\frac{b}{K}\bar{F} + \mu_E + \nu_E}(1 - e^{-\beta\bar{M}}) - \mu_F\bar{F}\right)\mathbf{1}_{[0, L]^c}, \quad \text{on } \mathbb{R} \setminus \{0, L\}, \quad (4.10b)$$

$$\bar{M}'(\xi^-) \geq \bar{M}'(\xi^+), \quad \bar{F}'(\xi^-) \geq \bar{F}'(\xi^+), \quad \text{for } \xi \in \{0, L\}, \quad (4.10c)$$

$$(\bar{M}, \bar{F})(-\infty) = (\bar{M}_2, \bar{F}_2), \quad (\bar{M}, \bar{F})(+\infty) = (0, 0). \quad (4.10d)$$

We recall that (\bar{M}_2, \bar{F}_2) is the larger mosquito equilibrium defined in (2.1). The main result of this work is the following :

Theorem 4.7 *Under the same assumptions as in Proposition 2.1, for any $L^* > 0$, there exists \bar{U} large enough, such that there exists a solution (\bar{M}, \bar{F}) of (4.10).*

Hence, there exists a barrier for system (2.5).

Proof. Let $\alpha \in (0, \mu_F)$ be fixed. We take \bar{U} and L large enough such that Lemma 4.6 holds. The last point of the Theorem is a consequence of the comparison principle. Therefore, we just have to construct a super-solution of (4.9) which verifies the boundary conditions. We construct this solution piecewise.

First, on $(-\infty, 0)$, we notice that system (4.9) is the stationary system for (2.5) when $M_s = 0$. From Proposition 2.1, we deduce that we may take \bar{M}, \bar{F} on $(-\infty, 0)$ as the larger values of the equilibria in 2.1, that is, $\bar{M} = \bar{M}_2$ and $\bar{F} = \bar{F}_2$.

On $(0, L)$, we introduce the notation $\tau := \frac{\nu_E b}{\mu_E + \nu_E}$ and take $(\bar{M}(L), \bar{F}(L))$ such that

$$0 < \bar{M}(L) < -\frac{1}{\beta} \ln\left(1 - \frac{\mu_F}{2r\tau}\right), \quad 0 < \bar{F}(L) < \frac{\mu_M}{(1-r)\tau}\bar{M}(L). \quad (4.11)$$

Then, we solve (4.9) on $(0, L)$ with boundary conditions $(\overline{M}(0), \overline{F}(0)) = (\overline{M}_2, \overline{F}_2)$ and $(\overline{M}(L), \overline{F}(L))$. From (4.9c), we deduce that

$$\overline{F}(x) = \frac{1}{2 \sinh(\sqrt{\frac{\alpha}{D_u}} L)} \left((\overline{F}(L) - \overline{F}(0) e^{-\sqrt{\frac{\alpha}{D_u}} L}) e^{\sqrt{\frac{\alpha}{D_u}} x} - (\overline{F}(L) - \overline{F}(0) e^{\sqrt{\frac{\alpha}{D_u}} L}) e^{-\sqrt{\frac{\alpha}{D_u}} x} \right),$$

which is positive on $(0, L)$. From (4.9b), we deduce using the comparison principle that $\overline{M} \geq \underline{M} > 0$, where \underline{M} is the solution of $-D_u \underline{M}'' + \mu_M \underline{M} = 0$ with $\underline{M}(0) = \overline{M}_2$ and $\underline{M}(L) = \overline{M}(L)$.

On $(L, +\infty)$, we first observe that for $F \geq 0$, we have $\frac{\nu b F}{\frac{b}{K} F + \nu_E + \mu_E} \leq \tau F$. Thus, it suffices to prove that there exists a nonnegative solution on $(L, +\infty)$, that tends to 0 at $+\infty$, of the system

$$\begin{aligned} -D_u \overline{M}'' &= (1-r)\tau \overline{F} - \mu_M \overline{M}, \\ -D_u \overline{F}'' &= (r\tau(1 - e^{-\beta \overline{M}}) - \mu_F) \overline{F}, \end{aligned}$$

with $(\overline{M}(L), \overline{F}(L))$ as in (4.11). Notice that from (4.11), we have that (A.3) holds with $\phi(u) = r\tau(1 - e^{-\beta u})$ (which is clearly increasing and Lipschitz continuous). Then, we may apply the result in Proposition A.1 in the Appendix : For $\overline{F}(L)$ small enough, there exists $(\overline{M}'(L^+), \overline{F}'(L^+))$ such that the solution of the above Cauchy problem is nonincreasing, nonnegative and goes exponentially fast to 0 as x tends to $+\infty$. Thus, our solution so far satisfies (4.10a) and (4.10b), and it only remains to prove that (4.10c) holds in order to conclude our proof.

From the expression of \overline{F} on $(0, L)$, we compute

$$\begin{aligned} \overline{F}'(0^+) &= \frac{\sqrt{\frac{\alpha}{D_u}}}{\sinh\left(\sqrt{\frac{\alpha}{D_u}} L\right)} \left(\overline{F}(L) - \overline{F}(0) \cosh\left(\sqrt{\frac{\alpha}{D_u}} L\right) \right) \\ \overline{F}'(L^-) &= \frac{\sqrt{\frac{\alpha}{D_u}}}{\sinh\left(\sqrt{\frac{\alpha}{D_u}} L\right)} \left(\overline{F}(L) \cosh\left(\sqrt{\frac{\alpha}{D_u}} L\right) - \overline{F}(0) \right). \end{aligned}$$

We already know that $\overline{F}'(0^-) = 0$, and from Lemma A.2, we have $\overline{F}'(L^+) \leq -\sqrt{\frac{\mu_F}{2D_u}} \overline{F}(L)$. In the same manner $\overline{M}'(0^-) = 0$, and we have from Lemma A.3 that $\overline{M}'(L^+) \leq \frac{1}{\sqrt{\mu_M D_u}} ((1-r)\tau \overline{F}(L) - \mu_M \overline{M}(L))$. By the comparison principle, \overline{M} is bounded from above by the solution on $(0, L)$ of

$$-D_u y'' + \mu_M y = \frac{(1-r)\nu_E b \overline{F}(0)}{\frac{b}{K} \overline{F}(0) + \nu_E + \mu_E} = \mu_M \overline{M}(0), \quad y(0) = \overline{M}(0), \quad y(L) = \overline{M}(L).$$

Then, we have $\overline{M}'(0^+) \leq y'(0)$ and $\overline{M}'(L^-) \geq y'(L)$ where, after some straightforward computations,

$$y'(0) = \frac{\sqrt{\frac{\mu_M}{D_u}}}{\sinh\left(\sqrt{\frac{\mu_M}{D_u}} L\right)} (\overline{M}(L) - \overline{M}(0)), \quad y'(L) = \frac{\sqrt{\frac{\mu_M}{D_u}} \cosh\left(\sqrt{\frac{\mu_M}{D_u}} L\right)}{\sinh\left(\sqrt{\frac{\mu_M}{D_u}} L\right)} (\overline{M}(L) - \overline{M}(0)).$$

We can always find L large enough and $(\overline{M}(L), \overline{F}(L))$, with $\overline{F}(L)$ small, such that (4.11) holds true, $\overline{F}'(0^+) \leq 0$, $-\sqrt{\frac{\mu_F}{2D_u}} \overline{F}(L) \leq \overline{F}'(L^-)$, $y'(0) \leq 0$, and $y'(L) \geq \frac{1}{\sqrt{\mu_M D_u}} ((1-r)\tau \overline{F}(L) - \mu_M \overline{M}(L))$. It concludes the construction of $(\overline{M}, \overline{F})$. \square

Theorem 4.7 establishes that it is possible to block a propagating front of invading mosquitoes by performing a sterile insect technique on a domain wide enough with releases of a sufficiently large amount of sterile males. However, the mathematical proof presented above provides only a sufficient condition to block the front and it is difficult to quantify the width of the domain and the number of sterile males. This is due to the fact that here we have to deal with a system of differential equations, whereas for a scalar equation it is possible to obtain a better characterization of the barrier.

5 Conclusion

In this work we have proved that mosquito invasion may be blocked by releasing a sufficient amount of sterile males over a sufficiently wide band. The main motivation of this result is to use the sterile insect technique to build a sanitary cordon to protect a certain region (e.g. an urban area) from wild mosquitoes living in an exterior region (a reservoir area where they are abundant like, for instance, a forest) and which are vectors of a disease (like dengue, zika or any other mosquito borne disease). We remark that the mathematical result is not exclusive to mosquitoes. This technique of building a barrier to avoid invasion may be applied to other invading insect species (for instance, other disease vectors or agricultural pests). It is however important that the population dynamics be characterized by an Allee effect such that the extinction steady state is stable.

Appendix

Existence of nonincreasing solution vanishing at infinity

In this appendix, we consider the second order differential system on $(0, +\infty)$

$$-u'' = \alpha v - \beta u, \tag{A.1a}$$

$$-v'' = (\phi(u) - \mu)v, \tag{A.1b}$$

$$u(0) = u_0, \quad v(0) = v_0, \quad u'(0) = u'_0, \quad v'(0) = v'_0. \tag{A.1c}$$

We assume

$$\phi \text{ nonnegative, nondecreasing, and Lipschitz continuous,} \tag{A.2}$$

$$\phi(u_0) \leq \frac{\mu}{2}, \quad \alpha v_0 - \beta u_0 < 0. \tag{A.3}$$

Our main result is the following

Proposition A.1 *Under assumptions (A.2) and (A.3), for v_0 small enough, there exist (u'_0, v'_0) such that the solution (u, v) to (A.1) is such that u and v are nonincreasing and go exponentially fast to 0 as $x \rightarrow +\infty$.*

Before proving this result, we will prove two technical lemmas which concern (A.1a) and (A.1b) independently.

Lemma A.2 *Let $\mu > 0$ and ψ be a nonincreasing and nonnegative function such that $\psi(0) \leq \frac{\mu}{2}$, and let $v_0 > 0$. Let us consider the Cauchy problem on $(0, +\infty)$*

$$-v'' = (\psi(x) - \mu)v, \quad v(0) = v_0, \quad v'(0) = v'_0. \tag{A.4}$$

Then, there exists $v'_0 \in [-\sqrt{\mu}v_0, -\sqrt{\frac{\mu}{2}}v_0]$ such that the solution of (A.4) is decreasing and tends to 0 at $+\infty$. Moreover, $v(x) \leq Ce^{-\sqrt{\mu}x}$ for some nonnegative constant C .

Proof. From the Duhamel formula, we have that

$$\begin{aligned} v(x) &= v_0 \cosh(\sqrt{\mu}x) + \frac{v'_0}{\sqrt{\mu}} \sinh(\sqrt{\mu}x) - \frac{1}{\sqrt{\mu}} \int_0^x \psi(z)v(z) \sinh(\sqrt{\mu}(x-z)) dz \\ &\leq v_0 \cosh(\sqrt{\mu}x) + \frac{v'_0}{\sqrt{\mu}} \sinh(\sqrt{\mu}x), \end{aligned} \tag{A.5}$$

as long as $v \geq 0$. Hence, we deduce that if $v'_0 < -\sqrt{\mu}v_0$, the right hand side vanishes which implies that v vanishes. From the assumptions on ψ , we also have that

$$\begin{aligned} v(x) &= v_0 \cosh(\sqrt{\frac{\mu}{2}}x) + v'_0 \sqrt{\frac{2}{\mu}} \sinh(\sqrt{\frac{\mu}{2}}x) - \sqrt{\frac{2}{\mu}} \int_0^x (\psi(z) - \frac{\mu}{2})v(z) \sinh(\sqrt{\frac{\mu}{2}}(x-z)) dz \\ &\geq v_0 \cosh(\sqrt{\frac{\mu}{2}}x) + v'_0 \sqrt{\frac{2}{\mu}} \sinh(\sqrt{\frac{\mu}{2}}x). \end{aligned}$$

Then, if $v'_0 > -\sqrt{\frac{\mu}{2}}v_0$, we have $v > 0$ and $v(x)$ goes to $+\infty$ as $x \rightarrow +\infty$.

Since we are looking for a solution v which is decreasing, we may invert it into a function $x(v)$. Let us denote $w(v) = -v'(x(v))$. Then, we have

$$w'(v) = \frac{(\mu - \psi(x(v)))v}{w(v)}, \quad w(v_0) = -v'_0.$$

The question is to know whether there exists $v'_0 < 0$ such that the solution of this equation is defined on $(0, v_0)$ and such that $w(0) = 0$.

There are two possibilities: either w vanishes on $v_c \in (0, v_0)$ and then the solution is defined only on (v_c, v_0) and we say that this solution is type I, or w does not vanish on $(0, v_0)$ and then we set $v_c = 0$ and we say that this solution is type II. We are looking for a solution which is between type I and type II.

Clearly, if $v'_0 < \bar{v}'_0$, then, from the Cauchy-Lipschitz theorem the corresponding solutions verify $w < \bar{w}$. Therefore, the map $v'_0 \mapsto v_c$ is nonincreasing and continuous. Moreover, we have seen at the beginning of this proof that if $v'_0 > -\sqrt{\frac{\mu}{2}}v_0$ the solution is type I and if $v'_0 < -\sqrt{\mu}v_0$ the solution is type II. Thus, by continuity there exists v'_0 such that $v_c = 0$ and $w(0) = 0$. Moreover, we have also $v'_0 \in [-\sqrt{\mu}v_0, -\sqrt{\frac{\mu}{2}}v_0]$ and with (A.5) we deduce the estimate $v(x) \leq \frac{1}{2}(v_0 + \frac{v'_0}{\sqrt{\mu}})e^{-\sqrt{\mu}x}$. \square

In the same spirit we have the following result:

Lemma A.3 *Let v be a nonincreasing and nonnegative function on $[0, \infty)$. Let $u_0 > 0$. We assume that $\alpha v(0) < \beta u_0$. We consider the Cauchy problem on $(0, +\infty)$*

$$-u'' = \alpha v - \beta u, \quad u(0) = u_0, \quad u'(0) = u'_0. \quad (\text{A.6})$$

Then, there exists $u'_0 \in [-\sqrt{\beta}u_0, \frac{1}{\sqrt{\beta}}(\alpha v(0) - \beta u_0)]$ such that the solution of (A.6) is decreasing and tends to 0 at $+\infty$. Moreover, $u(x) \leq Ce^{-\sqrt{\beta}x}$ for some nonnegative constant C .

Proof. Following the idea of the proof of Lemma A.2, we are left to prove that there exist type I and type II solutions. From the Duhamel formula, we have

$$\begin{aligned} u(x) &= u_0 \cosh(\sqrt{\beta}x) + \frac{u'_0}{\sqrt{\beta}} \sinh(\sqrt{\beta}x) - \frac{\alpha}{\sqrt{\beta}} \int_0^x v(z) \sinh(\sqrt{\beta}(x-z)) dz \\ &\leq u_0 \cosh(\sqrt{\beta}x) + \frac{u'_0}{\sqrt{\beta}} \sinh(\sqrt{\beta}x), \end{aligned}$$

where we use the fact that v is nonnegative. Hence the solution is type I if $u'_0 < -\sqrt{\beta}u_0$. Moreover, using the fact that $v(x) \leq v(0)$, we have

$$\begin{aligned} u(x) &\geq u_0 \cosh(\sqrt{\beta}x) + \frac{u'_0}{\sqrt{\beta}} \sinh(\sqrt{\beta}x) + \frac{\alpha}{\beta} v(0)(1 - \cosh(\sqrt{\beta}x)) \\ &= \frac{\alpha}{\beta} v(0) + (u_0 - \frac{\alpha}{\beta} v(0) + \frac{u'_0}{\sqrt{\beta}}) e^{\sqrt{\beta}x} + (u_0 - \frac{\alpha}{\beta} v(0) - \frac{u'_0}{\sqrt{\beta}}) e^{-\sqrt{\beta}x}. \end{aligned}$$

Therefore, the solution is type II if $u'_0 > \frac{1}{\sqrt{\beta}}(\alpha v(0) - \beta u_0)$.

By a continuity and monotonicity argument, as in the proof of Lemma A.2, there exists $u'_0 \in [-\sqrt{\beta}u_0, \frac{1}{\sqrt{\beta}}(\alpha v(0) - \beta u_0)]$ such that the solution of (A.6) is such that $\lim_{x \rightarrow +\infty} u(x) = 0$. Moreover, we have from the first estimate in this proof that $u(x) \leq \frac{1}{2}(u_0 + \frac{u'_0}{\sqrt{\beta}})e^{\sqrt{\beta}x}$. \square

Proof of Proposition A.1 Let us consider a nonincreasing and nonnegative function u on $[0, +\infty)$ such that $u(0) = u_0$. Then, from Lemma A.2, there exists v'_0 such that the solution of

$$-v'' = (\phi(u) - \mu)v, \quad v(0) = v_0, \quad v'(0) = v'_0,$$

is nonincreasing, nonnegative, and decays exponentially fast to 0 at $+\infty$. With such a function v , from Lemma A.3, there exists u'_0 such that the solution of (A.6) is nonincreasing, nonnegative, and decays exponentially fast to 0 at $+\infty$. We denote by $\mathcal{F}(u)$ this solution, which allows to define the map \mathcal{F} . Let us denote \mathcal{A} the closed subset of functions in $H^1(0, +\infty)$ which are nonnegative and nonincreasing and take the value u_0 at 0. Clearly \mathcal{F} maps \mathcal{A} into itself. Let us prove that \mathcal{F} is a contraction on \mathcal{A} for v_0 small enough.

Let u_1 and u_2 be in \mathcal{A} . Then, by definition of \mathcal{F} , we have

$$-(\mathcal{F}(u_1) - \mathcal{F}(u_2))'' + \beta(\mathcal{F}(u_1) - \mathcal{F}(u_2)) = \alpha(v_1 - v_2), \quad (\text{A.7})$$

$$-(v_1 - v_2)'' = (\phi(u_1) - \mu)(v_1 - v_2) + v_2(\phi(u_1) - \phi(u_2)). \quad (\text{A.8})$$

On the one hand, multiplying (A.7) by $(\mathcal{F}(u_1) - \mathcal{F}(u_2))$ and integrating, we obtain, after an integration by parts

$$\int_0^\infty |(\mathcal{F}(u_1) - \mathcal{F}(u_2))'|^2 dx + \beta \int_0^\infty (\mathcal{F}(u_1) - \mathcal{F}(u_2))^2 dx = \int_0^\infty \alpha(v_1 - v_2)(\mathcal{F}(u_1) - \mathcal{F}(u_2)) dx.$$

We deduce from a Cauchy-Schwarz inequality

$$\|\mathcal{F}(u_1) - \mathcal{F}(u_2)\|_{H^1(0, +\infty)} \leq \frac{\alpha}{\min(1, \beta)} \|v_1 - v_2\|_{L^2(0, +\infty)}. \quad (\text{A.9})$$

On the other hand, multiplying (A.8) by $(v_1 - v_2)$ and integrating we get, in a similar way,

$$\begin{aligned} \int_0^\infty |(v_1 - v_2)'|^2 dx &= \int_0^\infty (\phi(u_1) - \mu)|v_1 - v_2|^2 dx + \int_0^\infty v_2(\phi(u_1) - \phi(u_2))(v_1 - v_2) dx \\ &\leq -\frac{\mu}{2} \int_0^\infty |v_1 - v_2|^2 dx + v_0 \|\phi'\|_\infty \int_0^\infty |u_1 - u_2| |v_1 - v_2| dx, \end{aligned}$$

where we use (A.3), the Lipschitz continuity of ϕ (see (A.2)), and the fact that v_2 is decreasing. Applying again a Cauchy-Schwarz inequality, we obtain

$$\|v_1 - v_2\|_{L^2(0, +\infty)} \leq \frac{2v_0 \|\phi'\|_\infty}{\mu} \|u_1 - u_2\|_{L^2(0, +\infty)}.$$

With (A.9), we conclude that

$$\|\mathcal{F}(u_1) - \mathcal{F}(u_2)\|_{H^1(0, +\infty)} \leq \frac{2v_0 \alpha \|\phi'\|_\infty}{\mu \min(1, \beta)} \|v_1 - v_2\|_{L^2(0, +\infty)}.$$

Hence, \mathcal{F} is a contraction for v_0 small enough. Thus, there exists a unique fixed point, which is a solution of our problem. \square

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