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Tables of Quantiles of the Distribution of the Empirical Chiral Index in the Case of the Uniform Law and in the Case of the Normal Law

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Abstract: The empirical distribution of the chiral index is simulated for various sample sizes for the uniform law and for the normal law. The estimated quantiles $K_{0.90}$, $K_{0.95}$, $K_{0.98}$, and $K_{0.99}$, are tabulated for use in symmetry testing in the uniform case and in the normal case.

Keywords: symmetry test; skewness; asymmetry coefficient; chiral index; location-free; scale-free; order statistics; midranges; range lengths; Monte-Carlo quantiles estimates; uniform; normal

2020 MSC codes: primary 62Q05; secondary: 62F03, 62G30

1 Introduction

The chiral index χ is a quantitative measure of chirality which can be used as an asymmetry coefficient of multivariate distributions [1, 2]. It takes values in the interval $[0; 1]$. The chiral index is null if and only if there is a mirror symmetry. It is insensitive to isometries and scaling. In the case of a d -variate distribution, its upper bound χ^* depends on the dimensionality d . $\chi^* = 1/2$ when $d = 1$ [1], $\chi^* \in [1 - 1/\pi; 1 - 1/2\pi]$ when $d = 2$ [3], and $\chi^* \in [1/2; 1]$ when $d \geq 3$ [4]. As a consequence of a convergence theorem [1], the empirical chiral index of a mirror symmetric distribution converges to 0, provided that the parent distribution has a finite and non null inertia. So, the chiral index can be used for testing the symmetry of the parent law. The purpose of this paper is to provide tables of the estimated quantiles $K_{0.90}$, $K_{0.95}$, $K_{0.98}$, and $K_{0.99}$, in the univariate uniform case and in the normal case.

2 Theory

The chiral index χ of an univariate distribution is given in equation 1 [1, 2]:

$$\chi = (1 + r_m)/2 \quad (1)$$

where r_m is the lower bound of the correlation coefficient between two identical distributions, taken over all their joint distributions. r_m cannot be positive, $\chi = 0$ if and only if the distribution is symmetric, and its upper bound is $\chi^* = 1/2$. This latter is asymptotically reached for the Bernoulli distribution with parameter tending to 0 or 1 [1].

We consider a sample of size n from a parent distribution with finite and non null inertia. $X_{i:n} \leq \dots \leq X_{n:n}$ are the order statistics, \bar{X} is their mean, and σ is their standard deviation. $M_i = (X_{n+1-i:n} + X_{i:n})/2$ is the midrange of the i^{th} interval, and $L_i = (X_{n+1-i:n} - X_{i:n})/2$ is the

half range length of the i^{th} interval, $i = 1 \dots n$. The i^{th} interval and the $(n-i)^{\text{th}}$ interval are identical except in their orientation. Thus L_i is negative for $i > (n/2)$. The chiral index can be expressed from the studentized variance of the midranges or of the half ranges lengths: see equations 2.9.4 and 2.9.5 in [2].

$$\chi = \left[\sum_{i=1}^{i=n} M_i^2 - n \cdot \bar{X}^2 \right] / (n\sigma^2) \quad (2)$$

$$\chi = 1 - \left[\sum_{i=1}^{i=n} L_i^2 \right] / (n\sigma^2) \quad (3)$$

For practical computations, it is much simpler to use the expression of r_m in equation 1, for which the minimal correlation exists and is such that the sequence of observations sorted in increasing order is correlated with the sequence of observations sorted in decreasing order [2, 5]. Thus, the algorithm to compute χ is very simple:

- (a) Sort the sequence of n observations in increasing order
- (b) Correlate it with the sequence sorted in decreasing order
- (c) Add 1 and then divide the result by 2.

Step (a) is operating in $O(n \cdot \text{Log}(n))$ and step (b) in $O(n)$, and the global algorithm is optimal worst-case in $O(n \cdot \text{Log}(n))$. The whole computation is easy on a pocket calculator.

3 Simulations and results

Table 1 was generated for the continuous uniform law $U(0, 1)$. Table 2 was generated for the normal law $N(0, 1)$. For each sample size n from $U(0, 1)$ or from $N(0, 1)$, we compute one chiral index, i.e., one observation from the distribution of the chiral index. So, to approach this distribution, 10000 observations of the chiral index are generated. For each set of 10000 observations of the chiral index, four quantiles are computed as follow ($\chi_{i:10000}$ denotes the i^{th} order statistic):

$$K_{0.90} = (\chi_{9000:10000} + \chi_{9001:10000})/2 \quad (4)$$

$$K_{0.95} = (\chi_{9500:10000} + \chi_{9501:10000})/2 \quad (5)$$

$$K_{0.98} = (\chi_{9800:10000} + \chi_{9801:10000})/2 \quad (6)$$

$$K_{0.99} = (\chi_{9900:10000} + \chi_{9901:10000})/2 \quad (7)$$

The whole simulation is repeated 100 times, and both the mean values $\bar{K}_{0.90}$, $\bar{K}_{0.95}$, $\bar{K}_{0.98}$, and $\bar{K}_{0.99}$, and the estimated standard deviations $S_{K_{0.90}}$, $S_{K_{0.95}}$, $S_{K_{0.98}}$, and $S_{K_{0.99}}$ of the four distributions of the quantiles are calculated and reported in tables 1 and 2.

The random generator is a long period pseudo-random congruential multiplicative one, used in the routine *g05saf* of the NAG library [6]. All data of both tables were generated sequentially within a single executable program, without reinitializing the random generator.

Tables 1 or 2 can be used as follows. Compute the chiral index χ of your sample of size n .

- If you assume that your sample is issued from the continuous uniform law $U(a, b)$, $a < b$, look in Table 1 if χ exceeds the quantile value at the desired confidence level (e.g. 0.95). If yes, you may reject the symmetry assumption, and thus you may reject the uniform law assumption.
- If you assume that your sample is issued from the normal law $N(m, s)$, look in Table 2 if χ exceeds the quantile value at the desired confidence level (e.g. 0.95). If yes, you may reject the symmetry assumption, and thus you may reject the normal law assumption.

Table 1. Simulations for the uniform law $U(0, 1)$. n : sample size. $\bar{K}_{0.90}, \bar{K}_{0.95}, \bar{K}_{0.98}, \bar{K}_{0.99}$: mean values of estimated quantiles over 100 experiments. $S_{K_{0.90}}, S_{K_{0.95}}, S_{K_{0.98}}, S_{K_{0.99}}$: estimated standard deviations of the quantiles distributions.

n	$\bar{K}_{0.90}$	$\bar{K}_{0.95}$	$\bar{K}_{0.98}$	$\bar{K}_{0.99}$	$S_{K_{0.90}}$	$S_{K_{0.95}}$	$S_{K_{0.98}}$	$S_{K_{0.99}}$
3	0.212764	0.231432	0.242661	0.246342	0.001020	0.000774	0.000583	0.000320
4	0.185211	0.229221	0.268552	0.287850	0.002169	0.002272	0.002310	0.002153
5	0.154894	0.195716	0.242379	0.270442	0.001717	0.002531	0.003358	0.003770
6	0.138459	0.174426	0.214915	0.242201	0.001695	0.001989	0.002476	0.003498
7	0.122632	0.156176	0.195287	0.221293	0.001495	0.001935	0.002647	0.003295
8	0.109606	0.139940	0.176472	0.201666	0.001339	0.001867	0.002653	0.003400
9	0.099866	0.127905	0.161753	0.185403	0.001367	0.001822	0.002665	0.003770
10	0.090742	0.116454	0.148675	0.171121	0.001184	0.001687	0.002358	0.003258
11	0.083883	0.107905	0.137897	0.158657	0.001161	0.001501	0.002188	0.003212
12	0.077375	0.099899	0.128247	0.148157	0.000985	0.001459	0.002142	0.002723
13	0.072242	0.093095	0.119908	0.138950	0.000913	0.001188	0.002112	0.002306
14	0.067347	0.087017	0.112577	0.130622	0.000790	0.001316	0.001573	0.002496
15	0.063401	0.081884	0.105740	0.123299	0.000871	0.001163	0.001712	0.002299
16	0.059669	0.077378	0.100094	0.116943	0.000788	0.001031	0.001639	0.002331
17	0.056655	0.073521	0.095419	0.111137	0.000743	0.001169	0.001649	0.002057
18	0.053513	0.069685	0.090491	0.105711	0.000691	0.001029	0.001668	0.002266
19	0.050912	0.066281	0.086226	0.100894	0.000624	0.000967	0.001355	0.002063
20	0.048617	0.063213	0.082350	0.096221	0.000623	0.000875	0.001360	0.001722
21	0.046461	0.060632	0.078984	0.092450	0.000607	0.000980	0.001557	0.001941
22	0.044460	0.058067	0.075878	0.088865	0.000586	0.000829	0.001383	0.001741
23	0.042783	0.055897	0.073019	0.085646	0.000673	0.000865	0.001218	0.001951
24	0.040955	0.053483	0.070111	0.082124	0.000591	0.000734	0.001247	0.001733
25	0.039345	0.051406	0.067428	0.079209	0.000463	0.000868	0.001282	0.001677
26	0.037959	0.049700	0.065121	0.076492	0.000457	0.000749	0.001253	0.001640
27	0.036764	0.048079	0.063067	0.074118	0.000524	0.000689	0.001010	0.001453
28	0.035401	0.046299	0.060849	0.071566	0.000507	0.000681	0.001274	0.001703
29	0.034266	0.044886	0.058923	0.069457	0.000519	0.000673	0.001147	0.001431
30	0.033099	0.043416	0.057067	0.067353	0.000409	0.000569	0.000986	0.001319
31	0.032121	0.042051	0.055308	0.065082	0.000429	0.000658	0.000945	0.001362
32	0.031084	0.040791	0.053830	0.063512	0.000405	0.000621	0.001171	0.001500
33	0.030233	0.039744	0.052259	0.061845	0.000389	0.000501	0.000990	0.001308
34	0.029371	0.038641	0.050966	0.060222	0.000427	0.000659	0.001019	0.001513
35	0.028665	0.037600	0.049672	0.058667	0.000424	0.000611	0.000893	0.001281
36	0.027763	0.036530	0.048173	0.057204	0.000354	0.000584	0.000949	0.001177
37	0.027185	0.035736	0.047072	0.055737	0.000371	0.000599	0.000971	0.001313
38	0.026346	0.034660	0.045839	0.054346	0.000359	0.000478	0.000817	0.001306
39	0.025722	0.033817	0.044709	0.053001	0.000307	0.000518	0.000888	0.001334
40	0.025175	0.033091	0.043765	0.051885	0.000352	0.000512	0.000833	0.001107
41	0.024560	0.032330	0.042778	0.050509	0.000328	0.000461	0.000848	0.001111
42	0.024013	0.031578	0.041900	0.049560	0.000275	0.000440	0.000796	0.001129
43	0.023479	0.030914	0.040873	0.048470	0.000314	0.000473	0.000764	0.001153
44	0.022904	0.030150	0.039905	0.047332	0.000296	0.000401	0.000707	0.000884
45	0.022436	0.029536	0.039130	0.046380	0.000308	0.000474	0.000796	0.001086

Table 1. Continued.

46	0.021912	0.028844	0.038205	0.045312	0.000271	0.000395	0.000677	0.001032
47	0.021513	0.028343	0.037557	0.044496	0.000293	0.000427	0.000736	0.000993
48	0.021034	0.027734	0.036730	0.043550	0.000261	0.000407	0.000606	0.000907
49	0.020630	0.027197	0.035990	0.042803	0.000279	0.000415	0.000791	0.001129
50	0.020233	0.026654	0.035312	0.041995	0.000260	0.000389	0.000583	0.000858
51	0.019919	0.026211	0.034795	0.041392	0.000283	0.000397	0.000710	0.000930
52	0.019437	0.025630	0.034018	0.040360	0.000289	0.000382	0.000645	0.000909
53	0.019145	0.025207	0.033497	0.039760	0.000257	0.000393	0.000566	0.000738
54	0.018779	0.024782	0.032827	0.038877	0.000271	0.000368	0.000610	0.000911
55	0.018481	0.024346	0.032247	0.038246	0.000234	0.000399	0.000669	0.000914
56	0.018195	0.023973	0.031819	0.037913	0.000223	0.000348	0.000622	0.000879
57	0.017848	0.023510	0.031156	0.036994	0.000235	0.000317	0.000510	0.000796
58	0.017540	0.023125	0.030707	0.036498	0.000227	0.000330	0.000580	0.000791
59	0.017256	0.022719	0.030166	0.035924	0.000244	0.000360	0.000608	0.000911
60	0.016933	0.022354	0.029726	0.035475	0.000199	0.000327	0.000583	0.000872
61	0.016717	0.022031	0.029217	0.034741	0.000206	0.000330	0.000589	0.000812
62	0.016420	0.021642	0.028772	0.034233	0.000215	0.000315	0.000526	0.000739
63	0.016188	0.021349	0.028339	0.033746	0.000210	0.000340	0.000574	0.000838
64	0.015933	0.021102	0.028073	0.033321	0.000208	0.000322	0.000554	0.000765
65	0.015717	0.020733	0.027559	0.032849	0.000210	0.000341	0.000517	0.000701
66	0.015451	0.020362	0.027101	0.032306	0.000208	0.000292	0.000490	0.000679
67	0.015220	0.020104	0.026754	0.031902	0.000212	0.000319	0.000473	0.000708
68	0.014980	0.019793	0.026426	0.031379	0.000220	0.000301	0.000510	0.000687
69	0.014789	0.019509	0.026005	0.030905	0.000200	0.000311	0.000505	0.000675
70	0.014596	0.019222	0.025610	0.030422	0.000180	0.000279	0.000464	0.000701
71	0.014384	0.018999	0.025335	0.030181	0.000197	0.000306	0.000495	0.000731
72	0.014147	0.018706	0.024913	0.029688	0.000171	0.000258	0.000490	0.000679
73	0.013969	0.018462	0.024527	0.029260	0.000163	0.000254	0.000424	0.000756
74	0.013788	0.018167	0.024190	0.028830	0.000201	0.000281	0.000457	0.000682
75	0.013624	0.018002	0.023958	0.028546	0.000193	0.000268	0.000456	0.000623
76	0.013447	0.017747	0.023662	0.028168	0.000190	0.000273	0.000414	0.000566
77	0.013302	0.017557	0.023380	0.027778	0.000177	0.000261	0.000425	0.000659
78	0.013119	0.017334	0.023157	0.027641	0.000177	0.000285	0.000470	0.000709
79	0.012959	0.017119	0.022789	0.027187	0.000178	0.000272	0.000400	0.000622
80	0.012777	0.016868	0.022473	0.026816	0.000175	0.000272	0.000441	0.000686
81	0.012627	0.016715	0.022264	0.026520	0.000164	0.000264	0.000417	0.000662
82	0.012499	0.016512	0.022052	0.026281	0.000178	0.000289	0.000403	0.000529
83	0.012352	0.016319	0.021718	0.025897	0.000158	0.000248	0.000408	0.000606
84	0.012192	0.016114	0.021558	0.025664	0.000182	0.000264	0.000398	0.000553
85	0.012077	0.015989	0.021302	0.025331	0.000169	0.000262	0.000411	0.000573
86	0.011906	0.015734	0.020964	0.024974	0.000153	0.000249	0.000426	0.000618
87	0.011756	0.015554	0.020740	0.024612	0.000149	0.000214	0.000363	0.000521
88	0.011650	0.015398	0.020566	0.024557	0.000172	0.000220	0.000373	0.000492
89	0.011506	0.015238	0.020295	0.024188	0.000179	0.000242	0.000417	0.000572
90	0.011421	0.015099	0.020147	0.024046	0.000166	0.000230	0.000428	0.000545

Table 1. Continued.

91	0.011254	0.014856	0.019800	0.023658	0.000148	0.000235	0.000372	0.000564
92	0.011142	0.014747	0.019618	0.023369	0.000137	0.000223	0.000367	0.000542
93	0.011026	0.014585	0.019455	0.023258	0.000137	0.000218	0.000377	0.000570
94	0.010927	0.014433	0.019267	0.023014	0.000173	0.000265	0.000363	0.000512
95	0.010822	0.014315	0.019090	0.022762	0.000153	0.000213	0.000364	0.000586
96	0.010670	0.014148	0.018876	0.022528	0.000146	0.000221	0.000355	0.000482
97	0.010588	0.014029	0.018659	0.022190	0.000136	0.000200	0.000335	0.000470
98	0.010484	0.013846	0.018503	0.022106	0.000146	0.000224	0.000319	0.000460
99	0.010364	0.013718	0.018361	0.021858	0.000140	0.000224	0.000321	0.000513
100	0.010268	0.013574	0.018043	0.021550	0.000143	0.000230	0.000368	0.000502
110	0.009351	0.012355	0.016481	0.019638	0.000132	0.000204	0.000356	0.000485
120	0.008593	0.011360	0.015157	0.018115	0.000132	0.000185	0.000281	0.000418
130	0.007919	0.010494	0.014075	0.016789	0.000117	0.000193	0.000276	0.000412
140	0.007373	0.009770	0.013062	0.015599	0.000105	0.000143	0.000224	0.000348
150	0.006883	0.009105	0.012201	0.014561	0.000102	0.000154	0.000235	0.000368
160	0.006453	0.008527	0.011409	0.013678	0.000111	0.000142	0.000247	0.000392
170	0.006071	0.008048	0.010753	0.012889	0.000077	0.000126	0.000203	0.000263
180	0.005741	0.007606	0.010169	0.012198	0.000078	0.000124	0.000215	0.000298
190	0.005426	0.007185	0.009627	0.011511	0.000084	0.000122	0.000207	0.000280
200	0.005169	0.006854	0.009178	0.010955	0.000071	0.000102	0.000167	0.000262
210	0.004942	0.006542	0.008748	0.010466	0.000067	0.000112	0.000178	0.000253
220	0.004710	0.006233	0.008337	0.010005	0.000063	0.000091	0.000161	0.000242
230	0.004500	0.005972	0.008006	0.009560	0.000064	0.000099	0.000162	0.000220
240	0.004309	0.005715	0.007667	0.009196	0.000063	0.000096	0.000151	0.000233
250	0.004148	0.005500	0.007370	0.008818	0.000059	0.000086	0.000150	0.000186
260	0.003986	0.005283	0.007073	0.008479	0.000049	0.000076	0.000137	0.000199
270	0.003834	0.005083	0.006809	0.008153	0.000048	0.000074	0.000130	0.000185
280	0.003702	0.004906	0.006573	0.007869	0.000049	0.000074	0.000118	0.000194
290	0.003577	0.004748	0.006358	0.007602	0.000053	0.000081	0.000127	0.000171
300	0.003457	0.004583	0.006140	0.007360	0.000049	0.000072	0.000126	0.000180
310	0.003345	0.004435	0.005942	0.007107	0.000044	0.000064	0.000117	0.000174
320	0.003241	0.004302	0.005772	0.006915	0.000051	0.000067	0.000116	0.000164
330	0.003145	0.004173	0.005590	0.006682	0.000039	0.000067	0.000119	0.000156
340	0.003051	0.004047	0.005413	0.006486	0.000044	0.000065	0.000109	0.000153
350	0.002964	0.003929	0.005261	0.006327	0.000038	0.000064	0.000106	0.000153
360	0.002887	0.003826	0.005119	0.006132	0.000040	0.000065	0.000097	0.000126
370	0.002808	0.003725	0.005005	0.005994	0.000037	0.000054	0.000096	0.000139
380	0.002736	0.003627	0.004863	0.005830	0.000037	0.000060	0.000097	0.000144
390	0.002663	0.003537	0.004752	0.005690	0.000037	0.000056	0.000090	0.000135
400	0.002601	0.003447	0.004639	0.005532	0.000037	0.000062	0.000094	0.000124
410	0.002527	0.003357	0.004514	0.005406	0.000033	0.000051	0.000095	0.000123
420	0.002467	0.003278	0.004406	0.005275	0.000038	0.000055	0.000078	0.000128
430	0.002411	0.003199	0.004288	0.005136	0.000037	0.000051	0.000095	0.000112
440	0.002358	0.003129	0.004184	0.005017	0.000033	0.000052	0.000080	0.000115
450	0.002308	0.003067	0.004123	0.004934	0.000036	0.000051	0.000090	0.000115

Table 1. Continued.

460	0.002258	0.002997	0.004027	0.004833	0.000028	0.000049	0.000081	0.000111
470	0.002210	0.002933	0.003937	0.004725	0.000031	0.000050	0.000079	0.000126
480	0.002166	0.002875	0.003854	0.004624	0.000030	0.000050	0.000075	0.000115
490	0.002122	0.002810	0.003771	0.004528	0.000030	0.000041	0.000070	0.000097
500	0.002078	0.002763	0.003699	0.004432	0.000032	0.000043	0.000076	0.000122
510	0.002038	0.002710	0.003645	0.004374	0.000029	0.000043	0.000074	0.000107
520	0.001999	0.002652	0.003554	0.004268	0.000028	0.000045	0.000072	0.000112
530	0.001966	0.002605	0.003487	0.004178	0.000026	0.000039	0.000061	0.000093
540	0.001927	0.002561	0.003430	0.004108	0.000029	0.000043	0.000071	0.000104
550	0.001890	0.002506	0.003367	0.004041	0.000027	0.000041	0.000070	0.000099
560	0.001857	0.002466	0.003310	0.003980	0.000028	0.000046	0.000067	0.000098
570	0.001819	0.002414	0.003229	0.003875	0.000027	0.000039	0.000064	0.000093
580	0.001791	0.002382	0.003199	0.003829	0.000027	0.000034	0.000067	0.000092
590	0.001766	0.002344	0.003142	0.003776	0.000025	0.000039	0.000055	0.000096
600	0.001733	0.002297	0.003087	0.003701	0.000023	0.000040	0.000064	0.000093
610	0.001705	0.002264	0.003035	0.003642	0.000023	0.000036	0.000058	0.000089
620	0.001677	0.002227	0.002978	0.003571	0.000022	0.000035	0.000057	0.000086
630	0.001649	0.002193	0.002936	0.003520	0.000022	0.000034	0.000064	0.000084
640	0.001620	0.002151	0.002892	0.003471	0.000024	0.000033	0.000058	0.000093
650	0.001601	0.002121	0.002851	0.003415	0.000022	0.000038	0.000059	0.000094
660	0.001571	0.002089	0.002805	0.003366	0.000022	0.000034	0.000053	0.000083
670	0.001553	0.002063	0.002765	0.003318	0.000021	0.000034	0.000051	0.000081
680	0.001529	0.002032	0.002726	0.003262	0.000019	0.000032	0.000056	0.000084
690	0.001506	0.002003	0.002676	0.003205	0.000021	0.000034	0.000052	0.000072
700	0.001487	0.001976	0.002653	0.003180	0.000020	0.000029	0.000054	0.000080
710	0.001465	0.001944	0.002612	0.003137	0.000021	0.000030	0.000050	0.000079
720	0.001446	0.001919	0.002581	0.003088	0.000021	0.000030	0.000048	0.000068
730	0.001427	0.001890	0.002536	0.003044	0.000020	0.000027	0.000044	0.000069
740	0.001407	0.001870	0.002509	0.003005	0.000018	0.000030	0.000046	0.000070
750	0.001384	0.001838	0.002461	0.002958	0.000020	0.000028	0.000049	0.000072
760	0.001365	0.001815	0.002440	0.002926	0.000018	0.000028	0.000050	0.000069
770	0.001353	0.001795	0.002408	0.002886	0.000020	0.000029	0.000049	0.000068
780	0.001330	0.001769	0.002376	0.002847	0.000018	0.000028	0.000046	0.000075
790	0.001317	0.001748	0.002349	0.002815	0.000018	0.000029	0.000045	0.000065
800	0.001301	0.001727	0.002316	0.002770	0.000017	0.000028	0.000049	0.000072
810	0.001286	0.001709	0.002297	0.002752	0.000016	0.000023	0.000039	0.000055
820	0.001270	0.001685	0.002264	0.002718	0.000016	0.000024	0.000043	0.000071
830	0.001252	0.001661	0.002229	0.002674	0.000017	0.000028	0.000050	0.000064
840	0.001241	0.001649	0.002206	0.002651	0.000017	0.000026	0.000041	0.000064
850	0.001225	0.001627	0.002184	0.002621	0.000017	0.000026	0.000043	0.000062
860	0.001210	0.001606	0.002152	0.002581	0.000019	0.000028	0.000043	0.000064
870	0.001196	0.001583	0.002128	0.002547	0.000016	0.000022	0.000041	0.000062
880	0.001182	0.001568	0.002105	0.002521	0.000015	0.000024	0.000040	0.000058
890	0.001170	0.001552	0.002087	0.002500	0.000016	0.000027	0.000049	0.000064
900	0.001155	0.001536	0.002067	0.002481	0.000015	0.000024	0.000039	0.000060

Table 1. Continued.

910	0.001143	0.001519	0.002041	0.002441	0.000014	0.000026	0.000043	0.000064
920	0.001132	0.001504	0.002018	0.002417	0.000017	0.000025	0.000043	0.000056
930	0.001119	0.001488	0.001996	0.002393	0.000014	0.000022	0.000041	0.000068
940	0.001109	0.001473	0.001972	0.002361	0.000015	0.000021	0.000038	0.000054
950	0.001093	0.001453	0.001952	0.002342	0.000015	0.000023	0.000042	0.000061
960	0.001080	0.001434	0.001928	0.002314	0.000015	0.000024	0.000040	0.000052
970	0.001071	0.001420	0.001907	0.002288	0.000017	0.000024	0.000041	0.000057
980	0.001059	0.001406	0.001895	0.002277	0.000014	0.000022	0.000040	0.000053
990	0.001050	0.001395	0.001870	0.002244	0.000014	0.000022	0.000033	0.000052
1000	0.001041	0.001384	0.001861	0.002228	0.000015	0.000025	0.000042	0.000055
10000	0.000104	0.000138	0.000186	0.000223	0.000001	0.000002	0.000004	0.000006

Table 2. Simulations for the normal law $N(0, 1)$. n : sample size. $\bar{K}_{0.90}, \bar{K}_{0.95}, \bar{K}_{0.98}, \bar{K}_{0.99}$: mean values of estimated quantiles over 100 experiments. $S_{K_{0.90}}, S_{K_{0.95}}, S_{K_{0.98}}, S_{K_{0.99}}$: estimated standard deviations of the quantiles distributions.

n	$\bar{K}_{0.90}$	$\bar{K}_{0.95}$	$\bar{K}_{0.98}$	$\bar{K}_{0.99}$	$S_{K_{0.90}}$	$S_{K_{0.95}}$	$S_{K_{0.98}}$	$S_{K_{0.99}}$
3	0.206148	0.227731	0.240969	0.245439	0.001423	0.001022	0.000689	0.000456
4	0.176247	0.220096	0.260605	0.281725	0.002109	0.002444	0.002400	0.002368
5	0.153131	0.193065	0.238453	0.266296	0.001812	0.002268	0.003269	0.003323
6	0.136157	0.173063	0.215660	0.243214	0.001734	0.002212	0.003252	0.003874
7	0.123281	0.157455	0.197204	0.223840	0.001479	0.002202	0.002835	0.003373
8	0.111374	0.142651	0.180713	0.206476	0.001447	0.001952	0.003128	0.003745
9	0.102793	0.131712	0.167606	0.192213	0.001246	0.001594	0.002683	0.003199
10	0.094767	0.121944	0.155632	0.179397	0.001168	0.001666	0.002249	0.003116
11	0.088537	0.113730	0.145479	0.167983	0.001080	0.001417	0.002225	0.003192
12	0.082536	0.106016	0.136342	0.157983	0.001127	0.001471	0.002004	0.002733
13	0.077855	0.100089	0.128930	0.149682	0.000976	0.001377	0.002225	0.003116
14	0.073353	0.094355	0.121794	0.141797	0.001000	0.001486	0.002356	0.003088
15	0.069663	0.089547	0.115307	0.134313	0.000766	0.001277	0.001811	0.002797
16	0.065954	0.084925	0.109532	0.127891	0.000838	0.001189	0.001799	0.002688
17	0.063034	0.080967	0.104286	0.121876	0.000829	0.001105	0.001908	0.002463
18	0.060037	0.077133	0.100098	0.116555	0.000719	0.001091	0.001589	0.002402
19	0.057690	0.074178	0.095784	0.111988	0.000669	0.000995	0.001640	0.002282
20	0.055160	0.070931	0.091782	0.107161	0.000602	0.001065	0.001546	0.002367
21	0.053234	0.068194	0.088279	0.103222	0.000589	0.000972	0.001720	0.002399
22	0.051111	0.065554	0.084898	0.099353	0.000566	0.000842	0.001475	0.001799
23	0.049378	0.063333	0.082050	0.096044	0.000602	0.000925	0.001492	0.001675
24	0.047672	0.061163	0.079168	0.092479	0.000680	0.000976	0.001557	0.002070
25	0.046052	0.059073	0.076322	0.089272	0.000582	0.000838	0.001330	0.001855
26	0.044557	0.057052	0.073518	0.086034	0.000567	0.000768	0.001238	0.001484
27	0.043118	0.055299	0.071604	0.083819	0.000537	0.000781	0.001316	0.001738
28	0.041852	0.053532	0.069190	0.080993	0.000475	0.000709	0.001165	0.001732
29	0.040710	0.052073	0.067286	0.078917	0.000479	0.000726	0.001115	0.001827
30	0.039579	0.050613	0.065551	0.076793	0.000470	0.000632	0.001084	0.001740
31	0.038442	0.049161	0.063557	0.074347	0.000448	0.000621	0.001051	0.001446
32	0.037411	0.047811	0.061803	0.072384	0.000410	0.000621	0.001012	0.001540
33	0.036456	0.046615	0.060265	0.070494	0.000464	0.000623	0.001030	0.001400
34	0.035545	0.045467	0.058717	0.068982	0.000421	0.000602	0.000886	0.001413
35	0.034782	0.044340	0.057245	0.066953	0.000454	0.000595	0.001068	0.001458
36	0.033925	0.043316	0.055951	0.065549	0.000395	0.000564	0.000877	0.001153
37	0.033158	0.042346	0.054555	0.063907	0.000400	0.000642	0.000892	0.001175
38	0.032366	0.041311	0.053444	0.062713	0.000397	0.000632	0.000945	0.001404
39	0.031691	0.040411	0.052273	0.061247	0.000403	0.000630	0.000988	0.001345
40	0.030907	0.039486	0.051016	0.059674	0.000349	0.000507	0.000737	0.001080
41	0.030470	0.038804	0.050175	0.058859	0.000375	0.000551	0.000989	0.001338
42	0.029769	0.037979	0.048965	0.057236	0.000335	0.000551	0.000792	0.001158
43	0.029212	0.037180	0.047996	0.056155	0.000357	0.000489	0.000840	0.001196
44	0.028562	0.036398	0.047058	0.055077	0.000337	0.000538	0.000892	0.001235
45	0.028021	0.035723	0.046013	0.053733	0.000287	0.000457	0.000768	0.001158

Table 2. Continued.

46	0.027504	0.034977	0.045128	0.052973	0.000332	0.000496	0.000797	0.001238
47	0.026981	0.034399	0.044468	0.052066	0.000317	0.000456	0.000664	0.000909
48	0.026539	0.033824	0.043593	0.051103	0.000304	0.000475	0.000769	0.001128
49	0.026072	0.033144	0.042658	0.050046	0.000283	0.000427	0.000674	0.000952
50	0.025603	0.032559	0.041924	0.049126	0.000284	0.000385	0.000723	0.001093
51	0.025170	0.032012	0.041354	0.048368	0.000294	0.000463	0.000723	0.001073
52	0.024719	0.031493	0.040598	0.047669	0.000271	0.000391	0.000664	0.001032
53	0.024357	0.030968	0.039957	0.046826	0.000276	0.000425	0.000722	0.001012
54	0.023921	0.030456	0.039183	0.045895	0.000258	0.000393	0.000729	0.001036
55	0.023566	0.029956	0.038575	0.045169	0.000286	0.000427	0.000683	0.000984
56	0.023173	0.029461	0.038006	0.044487	0.000237	0.000376	0.000699	0.000900
57	0.022878	0.029069	0.037395	0.043870	0.000259	0.000385	0.000655	0.000927
58	0.022505	0.028573	0.036883	0.043208	0.000294	0.000424	0.000634	0.000909
59	0.022182	0.028209	0.036341	0.042576	0.000242	0.000373	0.000608	0.000847
60	0.021792	0.027644	0.035612	0.041752	0.000290	0.000375	0.000614	0.000861
61	0.021529	0.027330	0.035223	0.041278	0.000225	0.000363	0.000579	0.000775
62	0.021162	0.026857	0.034567	0.040376	0.000245	0.000377	0.000591	0.000847
63	0.020945	0.026563	0.034225	0.040067	0.000230	0.000340	0.000558	0.000868
64	0.020709	0.026250	0.033690	0.039466	0.000237	0.000397	0.000602	0.000782
65	0.020368	0.025842	0.033178	0.038883	0.000256	0.000344	0.000515	0.000722
66	0.020081	0.025505	0.032858	0.038502	0.000235	0.000342	0.000491	0.000801
67	0.019822	0.025140	0.032367	0.037828	0.000236	0.000350	0.000497	0.000671
68	0.019617	0.024870	0.032023	0.037465	0.000249	0.000334	0.000581	0.000828
69	0.019337	0.024498	0.031553	0.037002	0.000241	0.000342	0.000628	0.000860
70	0.019047	0.024098	0.030976	0.036225	0.000202	0.000316	0.000484	0.000656
71	0.018846	0.023871	0.030774	0.035986	0.000202	0.000339	0.000499	0.000801
72	0.018614	0.023588	0.030298	0.035473	0.000195	0.000276	0.000474	0.000691
73	0.018388	0.023307	0.029923	0.035058	0.000160	0.000289	0.000468	0.000762
74	0.018165	0.022995	0.029629	0.034691	0.000203	0.000322	0.000469	0.000785
75	0.017973	0.022792	0.029273	0.034279	0.000209	0.000302	0.000517	0.000867
76	0.017768	0.022478	0.028907	0.033877	0.000214	0.000307	0.000491	0.000662
77	0.017562	0.022266	0.028636	0.033480	0.000190	0.000294	0.000464	0.000601
78	0.017347	0.021952	0.028287	0.033076	0.000206	0.000304	0.000483	0.000657
79	0.017169	0.021757	0.027986	0.032782	0.000200	0.000306	0.000506	0.000681
80	0.016962	0.021429	0.027523	0.032168	0.000194	0.000301	0.000481	0.000632
81	0.016794	0.021259	0.027344	0.032046	0.000193	0.000333	0.000521	0.000711
82	0.016587	0.020979	0.026945	0.031550	0.000170	0.000271	0.000465	0.000628
83	0.016428	0.020782	0.026712	0.031312	0.000187	0.000260	0.000449	0.000712
84	0.016240	0.020571	0.026359	0.030807	0.000180	0.000275	0.000483	0.000707
85	0.016076	0.020330	0.026066	0.030547	0.000174	0.000291	0.000514	0.000823
86	0.015922	0.020125	0.025828	0.030202	0.000187	0.000278	0.000435	0.000642
87	0.015723	0.019908	0.025545	0.029892	0.000181	0.000286	0.000469	0.000645
88	0.015587	0.019710	0.025353	0.029600	0.000201	0.000265	0.000469	0.000680
89	0.015432	0.019527	0.025107	0.029337	0.000178	0.000264	0.000394	0.000573
90	0.015265	0.019309	0.024773	0.029009	0.000182	0.000267	0.000409	0.000632

Table 2. Continued.

91	0.015112	0.019104	0.024485	0.028674	0.000181	0.000262	0.000414	0.000590
92	0.014991	0.018947	0.024253	0.028369	0.000175	0.000261	0.000445	0.000672
93	0.014830	0.018752	0.023998	0.028147	0.000174	0.000260	0.000465	0.000606
94	0.014673	0.018556	0.023807	0.027843	0.000158	0.000244	0.000417	0.000627
95	0.014543	0.018352	0.023520	0.027501	0.000166	0.000251	0.000402	0.000581
96	0.014418	0.018171	0.023371	0.027314	0.000152	0.000232	0.000405	0.000613
97	0.014249	0.018026	0.023127	0.027003	0.000159	0.000234	0.000426	0.000662
98	0.014145	0.017878	0.022875	0.026742	0.000177	0.000241	0.000414	0.000590
99	0.014033	0.017695	0.022721	0.026638	0.000149	0.000240	0.000372	0.000532
100	0.013912	0.017552	0.022499	0.026394	0.000165	0.000248	0.000405	0.000634
110	0.012782	0.016139	0.020678	0.024151	0.000152	0.000215	0.000344	0.000458
120	0.011799	0.014855	0.019032	0.022210	0.000115	0.000196	0.000331	0.000437
130	0.010972	0.013802	0.017673	0.020649	0.000127	0.000186	0.000275	0.000473
140	0.010277	0.012932	0.016503	0.019234	0.000113	0.000192	0.000281	0.000394
150	0.009661	0.012134	0.015499	0.018072	0.000103	0.000155	0.000266	0.000393
160	0.009094	0.011421	0.014616	0.017020	0.000105	0.000158	0.000239	0.000381
170	0.008625	0.010829	0.013816	0.016068	0.000085	0.000134	0.000252	0.000366
180	0.008171	0.010249	0.013074	0.015266	0.000093	0.000146	0.000226	0.000306
190	0.007800	0.009781	0.012456	0.014526	0.000086	0.000130	0.000207	0.000306
200	0.007437	0.009308	0.011870	0.013828	0.000076	0.000116	0.000194	0.000287
210	0.007101	0.008911	0.011343	0.013223	0.000081	0.000115	0.000186	0.000258
220	0.006806	0.008513	0.010829	0.012609	0.000074	0.000118	0.000192	0.000279
230	0.006544	0.008183	0.010406	0.012120	0.000068	0.000097	0.000166	0.000244
240	0.006292	0.007861	0.009999	0.011630	0.000062	0.000101	0.000163	0.000232
250	0.006060	0.007573	0.009640	0.011220	0.000061	0.000105	0.000167	0.000236
260	0.005849	0.007294	0.009282	0.010826	0.000069	0.000094	0.000158	0.000219
270	0.005637	0.007041	0.008924	0.010367	0.000064	0.000102	0.000148	0.000191
280	0.005463	0.006825	0.008668	0.010077	0.000059	0.000085	0.000140	0.000210
290	0.005284	0.006594	0.008368	0.009739	0.000054	0.000089	0.000137	0.000194
300	0.005125	0.006385	0.008094	0.009432	0.000051	0.000085	0.000129	0.000171
310	0.004970	0.006199	0.007860	0.009136	0.000059	0.000082	0.000121	0.000176
320	0.004825	0.006012	0.007631	0.008856	0.000047	0.000071	0.000127	0.000199
330	0.004685	0.005832	0.007392	0.008598	0.000053	0.000076	0.000118	0.000168
340	0.004572	0.005692	0.007204	0.008396	0.000050	0.000074	0.000116	0.000159
350	0.004449	0.005532	0.007010	0.008137	0.000046	0.000071	0.000114	0.000171
360	0.004324	0.005383	0.006808	0.007920	0.000047	0.000064	0.000113	0.000157
370	0.004224	0.005259	0.006654	0.007745	0.000042	0.000067	0.000097	0.000156
380	0.004117	0.005117	0.006471	0.007535	0.000045	0.000061	0.000091	0.000131
390	0.004015	0.004990	0.006310	0.007345	0.000044	0.000064	0.000105	0.000150
400	0.003924	0.004889	0.006189	0.007190	0.000040	0.000061	0.000096	0.000148
410	0.003839	0.004768	0.006034	0.007003	0.000040	0.000057	0.000093	0.000158
420	0.003745	0.004653	0.005875	0.006841	0.000035	0.000060	0.000102	0.000136
430	0.003667	0.004559	0.005763	0.006691	0.000042	0.000056	0.000086	0.000129
440	0.003589	0.004456	0.005625	0.006535	0.000041	0.000060	0.000096	0.000132
450	0.003515	0.004373	0.005522	0.006427	0.000037	0.000056	0.000097	0.000129

Table 2. Continued.

460	0.003445	0.004278	0.005416	0.006302	0.000036	0.000054	0.000097	0.000129
470	0.003379	0.004189	0.005281	0.006133	0.000038	0.000056	0.000081	0.000105
480	0.003318	0.004113	0.005189	0.006031	0.000033	0.000048	0.000087	0.000112
490	0.003250	0.004033	0.005100	0.005910	0.000033	0.000048	0.000075	0.000111
500	0.003186	0.003945	0.004973	0.005791	0.000032	0.000053	0.000077	0.000119
510	0.003132	0.003885	0.004904	0.005699	0.000030	0.000053	0.000082	0.000120
520	0.003073	0.003812	0.004811	0.005572	0.000039	0.000054	0.000077	0.000123
530	0.003018	0.003737	0.004717	0.005480	0.000030	0.000045	0.000075	0.000109
540	0.002965	0.003676	0.004638	0.005388	0.000033	0.000050	0.000071	0.000101
550	0.002917	0.003615	0.004550	0.005297	0.000030	0.000043	0.000067	0.000098
560	0.002862	0.003550	0.004478	0.005192	0.000028	0.000049	0.000068	0.000098
570	0.002818	0.003488	0.004407	0.005125	0.000030	0.000045	0.000071	0.000110
580	0.002778	0.003440	0.004334	0.005025	0.000027	0.000040	0.000073	0.000100
590	0.002728	0.003378	0.004254	0.004932	0.000028	0.000043	0.000069	0.000099
600	0.002686	0.003326	0.004195	0.004869	0.000025	0.000043	0.000066	0.000095
610	0.002649	0.003280	0.004128	0.004790	0.000024	0.000039	0.000060	0.000091
620	0.002611	0.003231	0.004059	0.004722	0.000029	0.000042	0.000061	0.000091
630	0.002568	0.003175	0.003993	0.004633	0.000023	0.000041	0.000060	0.000087
640	0.002530	0.003126	0.003937	0.004564	0.000027	0.000041	0.000066	0.000095
650	0.002494	0.003087	0.003893	0.004519	0.000027	0.000041	0.000064	0.000085
660	0.002457	0.003038	0.003828	0.004430	0.000024	0.000038	0.000061	0.000083
670	0.002424	0.002999	0.003773	0.004371	0.000025	0.000033	0.000054	0.000087
680	0.002394	0.002961	0.003726	0.004318	0.000026	0.000036	0.000055	0.000082
690	0.002358	0.002912	0.003665	0.004247	0.000024	0.000035	0.000055	0.000088
700	0.002330	0.002878	0.003626	0.004201	0.000023	0.000032	0.000056	0.000082
710	0.002295	0.002839	0.003565	0.004130	0.000022	0.000032	0.000049	0.000074
720	0.002268	0.002804	0.003528	0.004087	0.000023	0.000034	0.000051	0.000081
730	0.002233	0.002758	0.003479	0.004033	0.000023	0.000033	0.000057	0.000083
740	0.002211	0.002734	0.003436	0.003984	0.000022	0.000034	0.000052	0.000079
750	0.002182	0.002697	0.003398	0.003936	0.000022	0.000035	0.000052	0.000088
760	0.002156	0.002661	0.003350	0.003884	0.000021	0.000028	0.000054	0.000075
770	0.002127	0.002624	0.003294	0.003811	0.000020	0.000027	0.000048	0.000072
780	0.002099	0.002596	0.003266	0.003780	0.000022	0.000031	0.000052	0.000071
790	0.002075	0.002563	0.003220	0.003730	0.000021	0.000033	0.000054	0.000077
800	0.002051	0.002535	0.003192	0.003695	0.000020	0.000032	0.000054	0.000075
810	0.002032	0.002508	0.003154	0.003655	0.000020	0.000032	0.000049	0.000072
820	0.002006	0.002476	0.003108	0.003589	0.000018	0.000028	0.000049	0.000069
830	0.001983	0.002452	0.003078	0.003567	0.000021	0.000033	0.000049	0.000078
840	0.001962	0.002418	0.003038	0.003520	0.000019	0.000028	0.000054	0.000083
850	0.001938	0.002387	0.003007	0.003472	0.000020	0.000027	0.000044	0.000057
860	0.001918	0.002367	0.002974	0.003446	0.000020	0.000026	0.000043	0.000069
870	0.001896	0.002340	0.002944	0.003412	0.000018	0.000029	0.000046	0.000069
880	0.001878	0.002316	0.002905	0.003368	0.000018	0.000027	0.000046	0.000064
890	0.001855	0.002288	0.002873	0.003327	0.000018	0.000026	0.000048	0.000064
900	0.001835	0.002263	0.002841	0.003298	0.000021	0.000028	0.000049	0.000067

Table 2. Continued.

910	0.001817	0.002241	0.002813	0.003257	0.000018	0.000027	0.000045	0.000069
920	0.001801	0.002220	0.002781	0.003222	0.000017	0.000027	0.000048	0.000066
930	0.001777	0.002195	0.002757	0.003196	0.000019	0.000026	0.000043	0.000059
940	0.001761	0.002174	0.002732	0.003162	0.000018	0.000026	0.000046	0.000066
950	0.001744	0.002149	0.002696	0.003115	0.000019	0.000025	0.000042	0.000062
960	0.001725	0.002128	0.002671	0.003100	0.000018	0.000027	0.000049	0.000062
970	0.001712	0.002110	0.002648	0.003072	0.000017	0.000028	0.000043	0.000062
980	0.001694	0.002089	0.002618	0.003036	0.000018	0.000025	0.000042	0.000063
990	0.001678	0.002070	0.002596	0.003001	0.000015	0.000024	0.000036	0.000057
1000	0.001664	0.002051	0.002572	0.002977	0.000015	0.000024	0.000050	0.000068
10000	0.000184	0.000224	0.000276	0.000317	0.000002	0.000003	0.000005	0.000006

4 Conclusions

Tables 1 and 2 have been generated respectively for the uniform $U(0, 1)$ and for the normal $N(0, 1)$ parent distributions. The sampling distribution of the chiral index statistic issued from an uniform law $U(a, b)$, $a < b$, does not depend on the interval $[a; b]$. The sampling distribution of the chiral index statistic issued from a normal law $N(m, s)$ depends neither from the expectation m nor on the standard deviation s . It means that the chiral index allows to perform location free and scale free symmetry tests.

It is expected that the tables will be used for symmetry testing. Nevertheless, asymptotic expressions of the sampling distribution of the chiral index remain desirable. The chiral index is easy to compute, even with a pocket calculator. So, beginners in programming should be able to generate more tables when desired. The following section of code in R [7] computes the chiral index of a sample:

```
x <- scan(file="") # import observations from keyboard; type is double
chiral_index <- (1+cor(sort(x,decreasing=FALSE),sort(x,decreasing=TRUE)))/2
chiral_index # dsplay the value of the chiral index
```

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